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ON TREE IDEALS

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ABSTRACT. Let l^0 and m^0 be the ideals associated with Laver and Miller forcing, respectively. We show that $\mathbf{add}(l^0) < \mathbf{cov}(l^0)$ and $\mathbf{add}(m^0) < \mathbf{cov}(m^0)$ are consistent. We also show that both Laver and Miller forcing collapse the continuum to a cardinal $\leq \mathfrak{h}$.

INTRODUCTION AND NOTATION

In this paper we investigate the ideals connected with the classical tree forcings introduced by Laver [La] and Miller [Mi]. Laver forcing $\mathbb{L}$ is the set of all trees p on ${}^{<\omega}\omega$ such that p has a stem and whenever $s \in p$ extends $stem(p)$ then $Succ_p(s) := \{n : s^\frown n \in p\}$ is infinite. Miller forcing $\mathbb{M}$ is the set of all trees p on ${}^{<\omega}\omega$ such that p has a stem and for every $s \in p$ there is $t \in p$ extending s such that $Succ_p(t)$ is infinite. We denote the set of all these splitting nodes in p by $Split(p)$. For any $t \in Split(p)$, $Split_p(t)$ is the set of all minimal (with respect to extension) members of $Split(p)$ which properly extend t. For both $\mathbb{L}$ and $\mathbb{M}$ the order is inclusion.

The Laver ideal l^0 is the set of all $X \subseteq {}^\omega\omega$ with the property that for every $p \in \mathbb{L}$ there is $q \in \mathbb{L}$ extending p such that $X \cap [q] = \varnothing$. Here $[q]$ denotes the set of all branches of q. The Miller ideal m^0 is defined analogously, using conditions in $\mathbb{M}$ instead of $\mathbb{L}$. By a fusion argument one easily shows that l^0 and m^0 are σ-ideals.

The additivity (**add**) of any ideal is defined as the minimal cardinality of a family of sets belonging to the ideal whose union does not. The covering number (**cov**) is defined as the least cardinality of a family of sets from the ideal whose union is the whole set on which the ideal is defined— ${}^\omega\omega$ in our case. Clearly $\omega_1 \leq \mathbf{add}(l^0) \leq \mathbf{cov}(l^0) \leq \mathfrak{c}$ and $\omega_1 \leq \mathbf{add}(m^0) \leq \mathbf{cov}(m^0) \leq \mathfrak{c}$ hold.

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The main result in this paper says that there is a model of ZFC where $\mathbf{add}(l^0) < \mathbf{cov}(l^0)$ and $\mathbf{add}(m^0) < \mathbf{cov}(m^0)$ hold. The motivation was that by a result of Plewik [P1] it was known that the additivity and the covering number of the ideal connected with Mathias forcing are the same and they are equal to the cardinal invariant $\mathfrak{h}$—the least cardinality of a family of maximal antichains of $\mathscr{P}(\omega)/fin$ without a common refinement. On the other hand, in [JuMiSh] it was shown that $\mathbf{add}(s^0) < \mathbf{cov}(s^0)$ is consistent, where s^0 is Marczewski's ideal—the ideal connected with Sacks forcing $\mathbb{S}$. Intuitively, $\mathbb{L}$ and $\mathbb{M}$ sit somewhere between Mathias forcing and $\mathbb{S}$. In [GoJoSp] it was shown that under Martin's axiom $\mathbf{add}(l^0) = \mathbf{add}(m^0) = \mathfrak{c}$, whereas this is false for s^0 (see [JuMiSh]).

The method of proof for $\mathbf{add}(s^0) < \mathbf{cov}(s^0)$ in [JuMiSh] is the following: For a forcing P denote by $\kappa(P)$ the least cardinal to which forcing with P collapses the continuum. In [JuMiSh] it is shown that $\mathbf{add}(s^0) \le \kappa(\mathbb{S})$. In [BaLa] it was shown that in $V^{\mathbb{S}_{\omega_2}} \kappa(\mathbb{S}) = \omega_1$ holds, where $\mathbb{S}_{\omega_2}$ is the countable support iteration of length ω_2 of $\mathbb{S}$. Hence $V^{\mathbb{S}_{\omega_2}} \models \mathbf{add}(s^0) = \omega_1$. On the other hand, a Löwenheim-Skolem argument shows that $V^{\mathbb{S}_{\omega_2}} \models \mathbf{cov}(s^0) = \omega_2$.

Our method of proof is similar. Denoting by P_{ω_2} a countable support iteration of length ω_2 of $\mathbb{L}$ and $\mathbb{M}$ (each occurring on a stationary set), in §2 we prove the following:

Theorem.

$$V^{P_{\omega_2}} \models \omega_1 = \mathbf{add}(l^0) = \mathbf{add}(m^0) < \mathbf{cov}(l^0) = \mathbf{cov}(m^0) = \omega_2.$$

The crucial steps in the proof are to show that $\kappa(\mathbb{L})$, $\kappa(\mathbb{M})$ equal ω_1 and $\mathbf{add}(l^0) \le \kappa(\mathbb{L})$, $\mathbf{add}(m^0) \le \kappa(\mathbb{M})$ hold.

We will use the standard terminology for set theory and forcing. By $\mathfrak{b}$ we denote the least cardinality of a family of functions in ${}^\omega\omega$ which is unbounded with respect to eventual dominance and $\mathfrak{d}$ will be the least cardinality of a dominating family in ${}^\omega\omega$. Moreover, $\mathfrak{p}$ is the least cardinality of a filter base on $([\omega]^\omega, \subseteq^*)$ without any lower bound, and $\mathfrak{t}$ is the least cadinality of a decreasing chain in $([\omega]^\omega, \subseteq^*)$ without any lower bound. It is easy to see that $\omega_1 \le \mathfrak{p} \le \mathfrak{t} \le \mathfrak{b} \le \mathfrak{d} \le \mathfrak{c}$.

1. Upper and lower bounds

Theorem 1.1. (1) $\mathfrak{t} \le \mathbf{add}(l^0) \le \mathbf{cov}(l^0) \le \mathfrak{b}$.

(2) $\mathfrak{p} \le \mathbf{add}(m^0) \le \mathbf{cov}(m^0) \le \mathfrak{d}$.

Proof of Theorem 1.1(1). We have to prove the first and the third inequality. For the third inequality, let $\langle f_\alpha : \alpha < \mathfrak{b}\rangle$ be an unbounded family. Define

$$X_\alpha := \{f \in {}^\omega\omega : (\exists^\infty k) f(k) < f_\alpha(k)\}.$$

Clearly $\bigcup\{X_\alpha : \alpha < \mathfrak{b}\} = {}^\omega\omega$. We claim $X_\alpha \in l^0$. Let $p \in \mathbb{L}$. We define $q \in \mathbb{L}$ as follows: $stem(q) := stem(p)$, and for any s extending $stem(q)$ we have $s \in q$ if and only if $s \in p$ and $(\forall k)$ if $|stem(q)| \le k < |s|$, then $s(k) \ge f_\alpha(k)$. Then clearly $q \in \mathbb{L}$, q extends p, and $[q] \cap X_\alpha = \varnothing$.

In order to prove the first inequality we use the following notation from [JuMiSh]: Let $Q := \{\overline{A} = \langle A_s : s \in {}^{<\omega}\omega\rangle : (\forall s) A_s \in [\omega]^\omega\}$. For $\overline{A} \in Q$ we define a sequence of Laver trees $\langle p_s(\overline{A}) : s \in {}^{<\omega}\omega\rangle$ as follows: $p_s(\overline{A})$ is the unique Laver tree such that $stem(p_s(\overline{A})) = s$ and if $t \in p_s(\overline{A})$ extends s, then $Succ_{p_s(\overline{A})}(t) = A_t$.

For $\overline{A}, \overline{B} \in Q$ we define:

$$\overline{A} \subseteq \overline{B} \Leftrightarrow (\forall s) A_s \subseteq B_s,$$
$$\overline{A} \subseteq^* \overline{B} \Leftrightarrow (\forall s) A_s \subseteq^* B_s,$$
$$\overline{A} \leq^* \overline{B} \Leftrightarrow (\forall s) A_s \subseteq^* B_s \wedge (\forall^\infty s) A_s \subseteq B_s.$$

Here $\leq^*$ is a slight but important modification of $\subseteq^*$ from [JuMiSh].

Fact 1.2. $(Q, \leq^*)$ *is* $\mathfrak{t}$*-closed.*

Proof of Fact 1.2. Suppose $\langle \overline{A}_\alpha : \alpha < \gamma\rangle$, where $\gamma < \mathfrak{t}$ is a decreasing sequence in $(Q, \leq^*)$. Let $\overline{A}_\alpha := \langle A^\alpha_s : s \in {}^{<\omega}\omega\rangle$. Since $\gamma < \mathfrak{t}$, there is $\overline{B}' = \langle B'_s : s \in {}^{<\omega}\omega\rangle \in Q$ such that $(\forall \alpha < \gamma)\overline{B}' \subseteq^* \overline{A}_\alpha$. Define $f_\alpha : {}^{<\omega}\omega \to \omega$ such that $(\forall s) B'_s \setminus f_s(\alpha) \subseteq A^\alpha_s$. Since $\mathfrak{t} \leq \mathfrak{b}$, there exists $f : {}^{<\omega}\omega \to \omega$ such that $(\forall \alpha)(\forall^\infty s) f_\alpha(s) \leq f(s)$. Now let $B_s := B'_s \setminus f(s)$ and $\overline{B} := \langle B_s : s \in {}^{<\omega}\omega\rangle$. It is easy to check that $(\forall \alpha < \gamma)\overline{B} \leq^* \overline{A}_\alpha$.

Fact 1.3. *Suppose* $X \in l^0$ *and* $\overline{A} \in Q$. *There exists* $\overline{B} \in Q$ *such that* $\overline{B} \subseteq \overline{A}$ *and* $(\forall s \in {}^{<\omega}\omega)[p_s(\overline{B})] \cap X = \varnothing$.

Proof of Fact 1.3. First note that if $D := \{p \in \mathbb{L} : [p] \cap X = \varnothing\}$, then D is open dense and even 0-dense, i.e., for every $p \in \mathbb{L}$ there exists $q \in D$ extending p such that $stem(q) = stem(p)$. The proof of this is similar to Laver's proof in [La] that the set of Laver trees deciding a sentence in the language of forcing with $\mathbb{L}$ is 0-dense: Suppose $p \in \mathbb{L}$ has no 0-extension whose branches are not in X. Then inductively we can construct $q \in \mathbb{L}$ extending p such that every extension of q has a branch in X, contradicting $X \in l^0$.

Using this it is straightforward to construct $\overline{B}$ as desired.

Fact 1.4. *Suppose* $X \subseteq {}^\omega\omega$, $\overline{A}, \overline{B} \in Q$, $\overline{B} \leq^* \overline{A}$, *and* $(\forall s)[p_s(\overline{A})] \cap X = \varnothing$. *Then* $(\forall s)[p_s(\overline{B})] \cap X = \varnothing$.

Proof of Fact 1.4. Clearly, if $F \subseteq p_s(\overline{B})$ is finite, then

$$[p_s(\overline{B})] = \bigcup\{[p_t(\overline{B})] : t \in p_s(\overline{B}) \setminus F\}.$$

But for almost all $t \in p_s(\overline{B})$, $p_t(\overline{B})$ extends $p_t(\overline{A})$. So clearly $[p_s(\overline{B})] \subseteq [p_s(\overline{A})]$ and hence $[p_s(\overline{B})] \cap X = \varnothing$.

End of the proof of Theorem 1.1(1). Suppose we are given $\langle X_\alpha : \alpha < \gamma\rangle$ and $q \in \mathbb{L}$, where $\gamma < \mathfrak{t}$ and $(\forall \alpha) X_\alpha \in l^0$. Choose $\overline{A} \in Q$ such that $p_{stem(q)}(\overline{A}) = q$, and let $\overline{B}_0$ be the $\overline{B}$ given by Fact 1.3 for $\overline{A}$ and X_0. If $\langle \overline{B}_\alpha : \alpha < \beta\rangle$ is constructed for $\beta \leq \gamma$ and β is a successor, then choose $\overline{B}_\beta$ as given by Fact 1.3 for $\overline{A} = \overline{B}_{\beta-1}$ and $X = X_\beta$. If β is a limit, then by Fact 1.2 choose first $\overline{A}$ such that $(\forall \alpha < \beta)\overline{A} \leq^* \overline{B}_\alpha$ and then find $\overline{B}_\beta \subseteq \overline{A}$ as given by Fact 1.3 for $\overline{A}$ and $X = X_\beta$. Finally, if we have constructed $\overline{B}_\gamma = \langle B^\gamma_s : s \in {}^{<\omega}\omega\rangle$, define $\overline{B} := \langle B_s : s \in {}^{<\omega}\omega\rangle$ by $B_s := B^\gamma_s \cap Succ_q(s)$ if $s \in q$ extends $stem(q)$, and $B_s := B^\gamma_s$ otherwise. It is easy to check that $\overline{B} \in Q$, $p_{stem(q)}(\overline{B})$ extends q and $(\forall \alpha < \gamma)[p_{stem(q)}(\overline{B})] \cap X_\alpha = \varnothing$.

Proof of Theorem 1.1(2). The proof is similar to (1). For the third inequality, let $\langle f_\alpha : \alpha < \mathfrak{d} \rangle$ be a dominating family. Define

$$X_\alpha := \{f \in {}^\omega\omega \colon (\forall^\infty k) f(k) < f_\alpha(k)\}.$$

Then $\bigcup\{X_\alpha : \alpha < \mathfrak{d}\} = {}^\omega\omega$ and in an analogous way as in (1) it can be seen that $X_\alpha \in m^0$.

In order to prove the first inequality we need the following concept from [GoJoSp]. Let R be the set of all $\overline{P} = \langle P_s : s \in {}^{<\omega}\omega \rangle$ where each $P_s \subseteq {}^{<\omega}\omega$ is infinite, $t \in P_s$ implies $s \subset t$, and if $t, t' \in P_s$ are distinct, then $t(|s|) \neq t'(|s|)$. Given $\overline{P} \in R$ we can define $\langle p_s(\overline{P}) : s \in {}^{<\omega}\omega \rangle$ as follows: $p_s(\overline{P})$ is the unique Miller tree with stem s such that if $t \in Split(p_s(\overline{P}))$, then $Split_{p_s(\overline{P})}(t) = P_t$.

Define the following relations on R:

$$\overline{P} \leq \overline{Q} \Leftrightarrow (\forall s) p_s(\overline{P}) \leq p_s(\overline{Q}),$$
$$\overline{P} \approx \overline{Q} \Leftrightarrow (\forall s) P_s =^* Q_s \wedge (\forall^\infty s) P_s = Q_s,$$
$$\overline{P} \leq^* \overline{Q} \Leftrightarrow (\exists \overline{P}') \overline{P} \approx \overline{P}' \wedge \overline{P}' \leq \overline{Q}.$$

Fact 1.5 [GoJoSp, 4.14]. *Assume $MA_\kappa(\sigma$-centered$)$. If $\langle \overline{P}_\alpha : \alpha < \kappa \rangle$ is a $\leq^*$-decreasing sequence in R, then there exists $\overline{Q} \in R$ such that $(\forall \alpha < \kappa) \overline{Q} \leq^* \overline{P}_\alpha$.*

The following two facts have proofs similar to those of Facts 1.3 and 1.4.

Fact 1.6. *Suppose $X \in m^0$ and $\overline{P} \in R$. There exists $\overline{Q} \leq \overline{P}$ such that $(\forall s)[p_s(\overline{Q})] \cap X = \varnothing$.*

Fact 1.7. *Suppose $X \in m^0, \overline{P}, \overline{Q} \in R, \overline{P} \leq^* \overline{Q}$, and $(\forall s)[p_s(\overline{Q})] \cap X = \varnothing$. Then $(\forall s)[p_s(\overline{P})] \cap X = \varnothing$.*

Now using, Facts 1.5, 1.6, 1.7 and the well-known result that for all $\kappa < \mathfrak{p}$ $MA_\kappa(\sigma$-centered) holds, a similar construction as in Theorem 1.1(1) shows that $\mathfrak{p} \leq \mathbf{add}(m^0)$.

2. ADD AND COV ARE DISTINCT

Definition 2.1. A set $A \subseteq {}^\omega\omega$ is called *strongly dominating* if and only if

$$(\forall f \in {}^\omega\omega)(\exists \eta \in A)(\forall^\infty k) f(\eta(k-1)) < \eta(k).$$

Definition 2.2. For any set $A \subseteq {}^\omega\omega$, we define the domination game $D(A)$ as follows:

There are two players, GOOD and BAD. GOOD plays first. The game lasts ω moves.

GOOD	BAD
s	
	n_0
m_0	
	n_1
m_1	
$\vdots$	$\vdots$

The rules are: s is a sequence in ${}^{<\omega}\omega$, and the n_i and m_i are natural numbers. (Whoever breaks these rules first, loses immediately.)

The GOOD player wins if and only if:

(a) For all i, $m_i > n_i$.

(b) The sequence $s\frown m_0 \frown m_1 \frown \cdots$ is in A.

Lemma 2.3. *Let $A \subseteq {}^{\omega}\omega$ be a Borel set. Then the following are equivalent:*

(1) *There exists a Laver tree p such that $[p] \subseteq A$.*

(2) *A is strongly dominating.*

(3) *GOOD has a winning strategy in the game $D(A)$.*

Remark. Strongly dominating is not the same as dominating. For example, the closed set

$$A := \{\eta \in {}^{\omega}\omega\colon (\forall k)\eta(2k) = \eta(2k+1)\}$$

is dominating but is not strongly dominating.

Proof of Lemma 2.3. We consider the following condition:

(4) (For all $F\colon {}^{<\omega}\omega \times \omega \to \omega)(\exists \eta \in A)(\forall^{\infty} k)(\forall i \leq k)\eta(k) > F(\eta \restriction k, i)$. We will show $(1) \to (2) \to (4) \to (3) \to (1)$.

$(1) \to (2)$ is clear.

$(2) \to (4)$: Given F, define f by

$$f(m) := \max\{F(s, i) : i \leq m, s \in m^{\leq m+1}\} + m;$$

f is increasing, $f(m) \geq m$ for all m.

Find η such that $(\forall^{\infty} k)\eta(k) > f(\eta(k-1))$. Then η is increasing. For almost all k we have, letting $m := \eta(k-1)$: $m \geq k-1$, so $\eta \restriction k \in m^{\leq m+1}$, so by the definition of f we get $f(m) \geq F(\eta \restriction k, i)$ for any $i \leq k$. So $\eta(k) > f(\eta(k-1) \geq F(\eta \restriction k, i)$.

$(4) \to (3)$: Assume that GOOD has no winning strategy. Then BAD has a winning strategy σ (since the game $D(A)$ is Borel, hence determined).

We can find a function $F\colon {}^{<\omega}\omega \times \omega \to \omega$ such that for all $s, m_0, \ldots, m_k$ we have

$$\sigma(s, m_0, \ldots, m_k) = F(s\frown m_0 \frown \cdots \frown m_k, |s|).$$

Find $\eta \in A$ as in (4). So there is k_0 such that $\forall k \geq k_0 \;\; \eta(k) \geq F(\eta \restriction k, k_0)$. So in the play

GOOD	BAD
$s := \eta \restriction k_0$	
	$n_0 := \sigma(s) = F(\eta \restriction k_0, k_0)$
$m_0 := \eta(k_0+1)$	
	$n_1 := \sigma(s, m_0) = F(\eta \restriction (k_0+1), k_0)$
$m_1 := \eta(k_0+2)$	
$\vdots$	$\vdots$

player BAD followed the strategy σ, but player GOOD won, a contradiction.

$(3) \to (1)$: Let B be the set of all sequences $s\frown m_0 \frown m_1 \frown \cdots$ that can be played when GOOD follows a specific winning strategy. Clearly $B \subseteq A$, and for some Laver tree p, $B = [p]$.

Lemma 2.4 [Ke]. *Let $A \subseteq {}^\omega\omega$ be an analytic set. Then the following are equivalent*:

(1) *There exists a Miller tree p such that $[p] \subseteq A$.*

(2) *A is unbounded in $({}^\omega\omega, \leq^*)$.*

Lemma 2.5. (1) *Suppose $\mathfrak{b} = \mathfrak{c}$. For every dense open $D \subseteq \mathbb{L}$ there exists a maximal antichain $A \subseteq D$ such that*

$$(*) \qquad \forall q \in \mathbb{L}([q] \subseteq \bigcup\{[p] : p \in A\} \Rightarrow \exists A' \in [A]^{<\mathfrak{c}} \forall p \in A \setminus A' \, p \perp q).$$

(2) *The same is true for $\mathbb{M}$.*

Proof. Let $\mathbb{L} = \{q_\alpha : \alpha < \mathfrak{c}\}$. Inductively we will define a set $S \subseteq \mathfrak{c}$ and sequences $\langle x_\gamma : \gamma < \mathfrak{c}\rangle$ and $\langle p_\gamma : \gamma \in S\rangle$. Finally we will let $A = \{p_\gamma : \gamma \in S\}$.

Let $0 \in S$ and choose $x_0 \in [q_0]$ arbitrarily.

It can easily be seen that every Laver tree contains $\mathfrak{c}$ extensions such that every two of them do not contain a common branch. So clearly we may find $p_0 \in D$ such that $x_0 \notin [p_0]$.

Now suppose that $\langle x_\gamma : \gamma < \alpha\rangle$ and $\langle p_\gamma : \gamma \in S \cap \alpha\rangle$ have been constructed for $\alpha < \mathfrak{c}$.

First choose $x_\alpha \in [q_\alpha]$ arbitrarily, but such that, if $[q_\alpha] \not\subseteq \bigcup\{[p_\gamma] : \gamma < \alpha\}$, then $x_\alpha \notin \bigcup\{[p_\gamma] : \gamma < \alpha\}$.

In order to decide whether $\alpha \in S$ or not we distinguish the following two cases:

Case 1. q_α is compatible with some p_γ, $\gamma < \alpha$. In this case $\alpha \notin S$.

Case 2. q_α is incompatible with all p_γ, $\gamma < \alpha$. Now we let $\alpha \in S$, and we define p_α as follows:

By Lemma 2.3 for each $\gamma \in \alpha$ we may find $f_\gamma : \omega \to \omega$ such that

$$(**) \qquad (\forall \eta \in [p_\gamma] \cap [q_\alpha])(\exists^\infty k)\eta(k) \leq f_\gamma(\eta(k-1)).$$

By our assumption on $\mathfrak{b}$ there exists a strictly increasing f which dominates all the f_γ's. Now define $p'_\alpha \in \mathbb{L}$ as follows: $stem(p'_\alpha) = stem(q_\alpha)$, and for $t \in p'_\alpha$, if $t \supseteq stem(p'_\alpha)$ and $|t| =: n$, we require

$$Succ_{p'_\alpha}(t) = Succ_{q_\alpha}(t) \cap [f(t(n-1)), \infty).$$

Clearly $p'_\alpha \in \mathbb{L}$, $p'_\alpha \subseteq q_\alpha$, and by $(**)$ and our assumption on f we conclude $[p_\gamma] \cap [p'_\alpha] = \varnothing$ for every $\gamma < \alpha$.

By the remark above that every Laver tree contains $\mathfrak{c}$ extensions such that every two of them do not contain a common branch, we may find $p_\alpha \in D$ such that p_α extends p'_α and $[p_\alpha]$ and $\{x_\gamma : \gamma \leq \alpha\}$ are disjoint.

This finishes the construction. Now let $A := \{p_\gamma : \gamma \in S\}$.

Since every q_α is either compatible with some p_γ, $\gamma < \alpha$ (Case 1) or contains the condition p_α (Case 2), and for $\alpha \neq \gamma$ with $\alpha, \gamma \in S$ we have $[p_\alpha] \cap [p_\gamma] = \varnothing$, we conclude that A is a maximal antichain.

A also satisfies condition $(*)$: Let $q = q_\alpha$. By construction, if $[q_\alpha] \not\subseteq \bigcup\{[p_\gamma] : \gamma \in S \cap \alpha\}$, then $[q_\alpha] \not\subseteq \bigcup\{[p_\gamma] : \gamma \in S\}$.

The proof of (2) is analogous, but instead of Lemma 2.3 we use Lemma 2.4.

Lemma 2.6. *Suppose $\mathfrak{b} = \mathfrak{c}$. Then $\mathbf{add}(l^0) \leq \kappa(\mathbb{L})$ and $\mathbf{add}(m^0) \leq \kappa(\mathbb{M})$.*

Proof. We may assume $\kappa(\mathbb{L}) < \mathfrak{c}$. Let $\dot{f}$ be a $\mathbb{L}$-name such that $\Vdash_{\mathbb{L}}$ "$\dot{f} : \kappa(\mathbb{L}) \to \mathfrak{c}$ is onto". For $\alpha < \kappa(\mathbb{L})$ let

$$D_\alpha := \{p \in \mathbb{L} : (\exists \beta) p \Vdash_{\mathbb{L}} \dot{f}(\alpha) = \beta\}.$$

For $p \in D_\alpha$ we write $\beta_p = \beta_p(\alpha)$ for the unique β satisfying $p \Vdash_{\mathbb{L}} \dot{f}(\alpha) = \beta$.

Clearly D_α is dense and open. So we may choose a maximal antichain $A_\alpha \subseteq D_\alpha$ as in Lemma 2.5. Let

$$X_\alpha := {}^\omega\omega \setminus \bigcup\{[p] : p \in A_\alpha\}.$$

Then $X_\alpha \in l^0$. We claim that $X = \bigcup_{\alpha<\kappa(\mathbb{L})} X_\alpha \notin l^0$. Suppose on the contrary $X \in l^0$. So we may find $q \in \mathbb{L}$ such that $[q] \cap X = \varnothing$ and hence $[q] \subseteq \bigcup\{[p] : p \in A_\alpha\}$ for each α. By the choice of A_α each of the sets

$$B_\alpha := \{\beta_p(\alpha) : p \in A_\alpha, p \text{ compatible with } q\}$$

is bounded in $\mathfrak{c}$. Since $\mathfrak{c}$ is regular by our assumption $\mathfrak{b} = \mathfrak{c}$, we can find $\nu < \mathfrak{c}$ such that for all $\alpha < \kappa(\mathbb{L})$, $B_\alpha \subseteq \nu$. So easily conclude that

$$q \Vdash_{\mathbb{L}} \text{"ran}(\dot{f}) \subseteq \nu < \mathfrak{c}\text{"}.$$

This is a contradiction.

The proof for $\mathbb{M}$ is similar.

Theorem 2.7. $\kappa(\mathbb{L}) \leq \mathfrak{h}$ *and* $\kappa(\mathbb{M}) \leq \mathfrak{h}$.

Proof. We prove it only for $\mathbb{L}$. The proof for $\mathbb{M}$ is very similar. We work in V. Let $\langle \mathscr{A}_\alpha : \alpha < \mathfrak{h}\rangle$ be a family of maximal almost disjoint families such that:

(1) if $\alpha < \beta < \mathfrak{c}$, then $\mathscr{A}_\beta$ refines $\mathscr{A}_\alpha$;
(2) there exists no maximal almost disjoint family refining all the $\mathscr{A}_\alpha$;
(3) $\bigcup\{\mathscr{A}_\alpha : \alpha < \mathfrak{h}\}$ is dense in $([\omega]^\omega, \subseteq^*)$.

That such a sequence exists was shown in [BaPeSi].

Since $\mathfrak{h}$ is regular, for every $p \in \mathbb{L}$ there exists $\alpha < \mathfrak{h}$ such that for each $s \in Split(p)$ there is $A \in \mathscr{A}_\alpha$ with $A \subseteq^* Succ_p(s)$. Hence, writing $\mathbb{L}_\alpha$ for the set of those $p \in \mathbb{L}$ for which α has the property just stated, we conclude $\mathbb{L} = \bigcup\{\mathbb{L}_\alpha : \alpha < \mathfrak{h}\}$.

For each $A \in \mathscr{A}_\alpha$ choose $\mathscr{B}_A = \{B^A(p) : p \in \mathbb{L}\}$, a maximal almost disjoint family on A.

Now we will define $\mathbb{L}'_\alpha := \{q^\alpha(p) : p \in \mathbb{L}_\alpha\}$ such that $q^\alpha(p)$ extends p for every $p \in \mathbb{L}_\alpha$ and $p_1 \neq p_2$ implies $q^\alpha(p_1) \perp q^\alpha(p_2)$. For $p \in \mathbb{L}_\alpha$, $q^\alpha(p)$ will be defined as follows:

> For each $s \in Split(p)$ let $C_s^\alpha(p) := Succ_p(s) \cap B^A(p)$ where $A \in \mathscr{A}_\alpha$ is such that $A \subseteq^* Succ_p(s)$. So clearly $C_s^\alpha(p)$ is infinite. Now $q^\alpha(p)$ is the unique Laver tree $\leq p$ satisfying $stem(q^\alpha(p)) = stem(p)$ and for each $s \in Split(q^\alpha(p))$ we have $Succ_{q^\alpha(p)}(s) = C_s^\alpha(p)$.

It is not difficult to see that $\mathbb{L}'_\alpha$ has the stated properties.

Now we are ready to define a $\mathbb{L}$-name $\dot{f}$ such that $\Vdash_{\mathbb{L}}$ "$\dot{f} : \mathfrak{h}^V \to \mathfrak{c}^V$ is onto": For each $p \in \mathbb{L}_\alpha$, let $\{r_\xi^\alpha(p) : \xi < \mathfrak{c}\} \subseteq \mathbb{L}$ be a maximal antichain below $q^\alpha(p)$, and define $\dot{f}$ in such a way that $r_\xi^\alpha(p) \Vdash_{\mathbb{L}}$ "$\dot{f}(\alpha) = \xi$". As $\bigcup\{\mathbb{L}'_\alpha : \alpha < \mathfrak{h}\}$ is dense in $\mathbb{L}$, it is easy to check that $\dot{f}$ is as desired.

Theorem 2.8. *Let $\omega_2 = S_{\mathbb{M}} \dot\cup S_{\mathbb{L}}$, where the sets $S_{\mathbb{M}}$ and $S_{\mathbb{L}}$ are disjoint and stationary. Let $(P_\alpha, Q_\alpha : \alpha < \omega_2)$ be a countable support iteration of length ω_2 such that for all α we have $\Vdash_{P_\alpha} Q_\alpha = \mathbb{M}$ whenever $\alpha \in S_{\mathbb{M}}$, and $\Vdash_{P_\alpha} Q_\alpha = \mathbb{L}$ otherwise. Also suppose that V satisfies CH. Then in V^P, $\mathfrak{h} = \omega_1$ holds.*

Proof. Both $\mathbb{M}$ and $\mathbb{L}$ have the property $(*)_1$ of [JuSh]. (For $\mathbb{L}$, this was proved in [JuSh] and for $\mathbb{M}$ this was proved in [BaJuSh].) [JuSh] also showed that this property is preserved under countable support iterations, so also P_{ω_2} has this property. Hence, the reals of V do not have measure zero in V^P, so from $\mathfrak{h} \le \mathfrak{s} \le \mathbf{unif}(\mathfrak{L})$ (where $\mathfrak{s}$ is the splitting number and $\mathbf{unif}(\mathfrak{L})$ is the smallest cardinality of a set of reals which is not null) we get the desired conclusion.

Theorem 2.9. *Let P_{ω_2} be as in Theorem 2.8. Then*

$$V^{P_{\omega_2}} \models \omega_1 = \mathbf{add}(l^0) = \mathbf{add}(m^0) < \mathbf{cov}(l^0) = \mathbf{cov}(m^0) = \omega_2.$$

Proof. Since $\mathbb{L}$ adds a dominating real, we have $V^{P_{\omega_2}} \models \mathfrak{b} = \mathfrak{c}$; so by Lemma 2.6 and Theorems 2.7 and 2.8 it suffices to prove that the covering coefficients are ω_2 in the respective models. The proof of this is similar to the proof of [JuMiSh, Theorem 1.2] that **cov** of the Marczewski ideal is ω_2 in the iterated Sacks's forcing model.

We give the proof only for l^0. Suppose $\langle X_\alpha : \alpha < \omega_1\rangle \in V^{P_{\omega_2}}$ is a sequence of l^0-sets. In $V^{P_{\omega_2}}$ let $f_\alpha : \mathbb{L} \to \mathbb{L}$ be such that for every $p \in \mathbb{L}$, $f_\alpha(p)$ extends p and $[f_\alpha(p)] \cap X_\alpha = \varnothing$. Since P_{ω_2} has the ω_2-chain condition, by a Löwenheim-Skolem argument it is possible to find $\gamma < \omega_2$ such that

$$\langle f_\alpha \restriction \mathbb{L}^{V_\gamma} : \alpha < \omega_1\rangle \in V^{P_\gamma}$$

where $V_\gamma := V^{P_\gamma}$. Moreover, it is possible to find such a γ in $S_{\mathbb{L}}$. We claim that the Laver real x_γ (which is added by $Q_\gamma = \mathbb{L}^{V_\gamma}$) is not in $\bigcup_{\alpha<\omega_1} X_\alpha$, which will finish the proof. Otherwise, for some $p \in \mathbb{L}_{\gamma\omega_2}$ where $\mathbb{L}_{\gamma\omega_2} := \mathbb{L}_{\omega_2}/G_\gamma$ and some $\alpha < \omega_1$ we would have $p \Vdash x_\gamma \in X_\alpha$. But letting $q := p(\gamma) \in \mathbb{L}$ and letting $r(\gamma) := f_\alpha(q)$ and $r(\beta) := p(\beta)$ for $\beta > \gamma$ we see that $r \Vdash x_\gamma \notin X_\alpha$, a contradiction.

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