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A GRAPH WHICH EMBEDS ALL SMALL GRAPHS ON ANY LARGE SET OF VERTICES

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For certain cardinals λ and κ a colouring $P:[\lambda]^2 \to \lambda$ is constructed such that if $X \in [\lambda]^\lambda$ and $Q:[\kappa]^2 \to \lambda$, then there is a one-to-one function $i:\kappa \to X$ such that $P(i''A) = Q(A)$ for every $A \in [\kappa]^2$. Additional results are also obtained.

0. Introduction

The main objective of this paper will be to construct a graph on ω_1 which has the property that every finite graph appears as an induced subgraph in any uncountable set of vertices. Similar questions have been considered by Erdös and Hajnal [1, 2]. The construction to be presented was motivated by § 3 of [3] which follows Todorčević's proof that $\omega_1 \not\to [\omega_1]^2_{\aleph_1}$ in [6]. The same result was also obtained independently by Baumgartner using different techniques.

In Section 1 it will be shown that there is a colouring $P:[\omega_1]^2 \to \omega_1$ which satisfies the following property:

(0.1) If $X \in [\omega_1]^{\aleph_1}$, $n \in \omega$ and $h:[n]^2 \to \omega_1$, then there is a one-to-one function $i:n \to X$ s.t. $P(i''A) = h(A)$ for every $A \in [n]^2$.

Section 2 will explore some further properties of this colouring. In particular it will be shown that certain infinite graphs are also induced by every uncountable set of vertices. The question of colouring finite sets rather than pairs will also be discussed. In Section 3 it will be shown how to generalize the results of Section 1 to higher cardinals. The technical details here are more complicated than those in the case of ω_1 and, while they also apply to ω_1, it seems to be worthwhile to have a simpler proof in this important special case. Section 4 contains remarks pertaining to possible strengthenings of the counterexample. The final Section 5 is devoted to an application of this construction to Banach space theory. In particular a non-separable Banach space with few operators is constructed.

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1. The construction

For every limit ordinal $\alpha \in \omega_1$ choose an increasing sequence $\alpha^*: \omega \to \alpha$ cofinal in α. If $\alpha = \beta + 1$ let $\alpha^*: 1 \to \alpha$ be defined by $\alpha^*(0) = \beta$. The colouring of $[\omega_1]^2$ will be defined by comparing the sequences associated with pairs of ordinals. To facilitate this comparison define $k(\alpha, \beta)$ to be the least integer such that

$$\text{either} \quad \alpha^*(k(\alpha, \beta)) \neq \beta^*(k(\alpha, \beta)) \quad \text{or} \quad k(\alpha, \beta) \notin \operatorname{dom}(\alpha^*) \cap \operatorname{dom}(\beta^*).$$

An important point to notice is that if $\alpha^*(k(\alpha, \beta))$ and $\beta^*(k(\alpha, \beta))$ are both defined, then it makes sense to look at $k(\alpha^*(k(\alpha, \beta)), \beta^*(k(\alpha, \beta)))$. Since this process may be continued it is worthwhile introducing notation to describe it. The ordinals $\psi^\alpha_\beta(n)$ and $\Psi^\alpha_\beta(n)$ will be defined by induction on the integer n. Let $\psi^\alpha_\beta(0) = \min\{\alpha, \beta\}$ and $\Psi^\alpha_\beta(0) = \max\{\alpha, \beta\}$. Then define

$$\psi^\alpha_\beta(n+1) = (\psi^\alpha_\beta(n))^*(k(\psi^\alpha_\beta(n), \Psi^\alpha_\beta(n))) \quad \text{and}$$
$$\Psi^\alpha_\beta(n+1) = (\Psi^\alpha_\beta(n))^*(k(\psi^\alpha_\beta(n), \Psi^\alpha_\beta(n))).$$

Notice that $\Psi^\alpha_\beta(n+1) < \Psi^\alpha_\beta(n)$. Hence there is a least integer, $\Sigma(\alpha, \beta)$, such that $\Psi^\alpha_\beta(\Sigma(\alpha, \beta)) < \psi^\alpha_\beta(0)$ or else one of $\Psi^\alpha_\beta(\Sigma(\alpha, \beta))$ and $\psi^\alpha_\beta(\Sigma(\alpha, \beta))$ is not defined. Since $\Sigma(\alpha, \beta) > 0$ it will always make sense to talk about $\Sigma(\alpha, \beta) - 1$.

Before the colouring of $[\omega_1]^2$ can be defined it is necessary to introduce a mechanism for coding all finite colourings. For each ordinal $\alpha \in \omega_1$ choose a distinct subset of ω and call it $X(\alpha)$. Let $\{h_\alpha : \alpha \in \omega_1\}$ enumerate all functions $h: [\mathcal{P}(m)]^2 \to \omega_1$ where $m \in \omega$. Let $m(\alpha)$ be that integer such that $\operatorname{dom}(h_\alpha) = [\mathcal{P}(m(\alpha))]^2$.

The way the coding works is that for every pair of ordinals, α and β, there is some integer, m, such that $X(\alpha)$ and $X(\beta)$ restricted to m are distinct. The colour of the pair $\{\alpha, \beta\}$ will be determined by choosing a certain ordinal, γ, and evaluating $h_\gamma(X(\alpha) \cap m, X(\beta) \cap m)$.

To make this more precise let $\{S_\delta : \delta \in \omega_1\}$ partition ω_1 into stationary sets. For $\sigma \in \omega_1$ define $\delta(\sigma)$ to be the unique ordinal such that $\sigma \in S_{\delta(\sigma)}$. Now if α and β are distinct ordinals let $\xi = \delta(\Psi^\alpha_\beta(\Sigma(\alpha, \beta) - 1))$. Then define

$$P(\alpha, \beta) = h_\xi(X(\alpha) \cap m(\xi), X(\beta) \cap m(\xi))$$

if the right-hand side is defined. Otherwise let $P(\alpha, \beta) = 0$. It must now be shown that if Λ is an uncountable subset of ω_1 and $h: [t]^2 \to \omega_1$, then there is $T \in [\Lambda]^t$ such that h and $P \restriction [T]^2$ are isomorphic. Part of the proof of this fact will involve finding a countable elementary submodel, M, and an ordinal, α, outside of M. It will then be argued that there is an ordinal β in M which is in the shadow of α in the sense that $\alpha^* \cap M \subseteq \beta^*$, $\Psi^\alpha_\beta(1)^* \cap M \subseteq \psi^\alpha_\beta(1)^*$, $\Psi^\alpha_\beta(2)^* \cap M \subseteq \psi^\alpha_\beta(2)^*$ and so on. Of course in order to find something in the shadow of α we must define the shadow of α. This is done by defining $\Omega^\alpha_\beta(n)$ by induction on n in a manner similar to the definition of $\Psi^\alpha_\beta(n)$. In particular, let $\Omega^\alpha_\beta(0) = \max\{\alpha, \beta\}$ and let

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$\Omega^\alpha_\beta(n+1)=(\Omega^\alpha_\beta(n))^*(\theta^\alpha_\beta(n))$ where $\theta^\alpha_\beta(n)$ is the least integer such that $(\Omega^\alpha_\beta(n))^*(\theta^\alpha_\beta(n))\geq\min\{\alpha,\beta\}$. Let $L(\alpha,\beta)$ be the first integer such that $\Omega^\alpha_\beta(L(\alpha,\beta))$ is not defined (this will only happen because $\Omega^\alpha_\beta(L(\alpha,\beta)-1)=\min\{\alpha,\beta\}$). In order to keep track of the relevant initial segments of sequences for each $j\in L(\alpha,\beta)$ define $D^\alpha_\beta(j)=(\Omega^\alpha_\beta(j))^*\restriction\theta^\alpha_\beta(j)$.

Now let $\Lambda\in[\omega_1]^{\aleph_1}$ and $h:[t]^2\to\omega_1$ be given. Suppose, for the moment, that there is a countable elementary submodel $M<H=(H(\omega_2),\ \{\alpha^*:\alpha\in\omega_1\},\ \{S_\alpha:\alpha\in\omega_1\},\ \{h_\alpha:\alpha\in\omega_1\},\ \{X(\alpha):\alpha\in\omega_1\},\ \Lambda,\ h,\ \in)$ and there are L, $\{D(j):j\in L\}$, $\{Y(i):i\in t\}$ and γ in M such that, letting $\eta=M\cap\omega_1$, the following statements are true:

(1.1) $(\forall\{i,j\}\in[t]^2)(h(i,j)=h_\gamma(Y(i),Y(j)))$,

(1.2:n) $(\forall\alpha)(\exists\beta\in S_\gamma\setminus\alpha)(\beta^*\restriction(n+1)=\eta^*\restriction(n+1)$
$\quad\&\ (\forall\mu)(\exists\{\alpha_i:i\in t\}\subseteq\Lambda\setminus\mu)(\forall i\in t)(L(\alpha_i,\beta)=L$
$\quad\&\ X(\alpha_i)\cap m(\gamma)=Y(i)\ \&\ (\forall j\in L)(D^{\alpha_i}_\beta(j)=D(j))))$.

Under this assumption it is possible to choose $\{\sigma_i:i\in t\}$, $\{n_i:i\in t\}$ and $\{\alpha^i_j:i,j\in t\}$ such that

(1.3) $\sigma_i<\eta^*(n_i)<\alpha^i_0<\alpha^i_1<\cdots<\alpha^i_t<\sigma_{i+1}$

(1.4) $L(\alpha^i_j,\sigma_i)=L$ for $i,j\in t$

(1.5) $D^{\alpha^i_j}_{\sigma_i}(l)=D(l)$ for $i,j\in t$

(1.6) $\sigma_i\in S_\gamma$ and if $i>0$, then $\sigma^*_i\restriction n_{i-1}+1=\eta^*\restriction n_{i-1}+1$.

(See Fig. 1.)

To see that this is easily done by induction, suppose that $\{\alpha^m_i:i\in t\}$ have been chosen. Use the first existential quantifier of (1.2:n_m) to find $\sigma_{m+1}\in S_\gamma\setminus\alpha^m_t$. Find n_{m+1} such that $\eta^*(n_{m+1})>\sigma_{m+1}$ and then use the second existential quantifier of (1.2:n_m) to find $\{\alpha^{m+1}_i:i\in t\}$ satisfying (1.3), (1.4) and (1.5).

If it can now be shown that $P(\alpha^i_i,\alpha^j_j)=h(i,j)$, then T can simply be defined to be $\{\alpha^i_i:i\in t\}$. In order to calculate $P(\alpha^i_i,\alpha^j_j)$ let us assume that $\alpha^i_i>\alpha^j_j$ and let $\theta(l)=\mathrm{dom}(D(l))$. Then $(\alpha^i_i)^*\restriction\theta(0)=D(0)=(\alpha^j_j)^*\restriction\theta(0)$. Moreover, $(\alpha^i_i)^*(\theta(0))>\sigma_i>\alpha^j_j>(\alpha^j_j)^*(\theta(0))$ and so $k(\alpha^i_i,\alpha^j_j)=\theta(0)$. Hence

$$\Psi^{\alpha^i_i}_{\alpha^j_j}(1)=(\alpha^i_i)^*(\theta(0))=\Omega^{\alpha^i_i}_{\sigma_i}(1)\quad\text{and}\quad\psi^{\alpha^i_i}_{\alpha^j_j}(1)=(\alpha^j_j)^*(\theta(0))=\Omega^{\alpha^j_j}_{\sigma_j}(1)$$

neither of these is defined. (It is at this point that the fact that $\xi^*(0)=0$ if ξ is a limit ordinal is used. This ensures that α^i_i and α^j_j are either both limit ordinals or both successors.)

To calculate $\Psi^{\alpha^i_i}_{\alpha^j_j}(2)$ and $\psi^{\alpha^i_i}_{\alpha^j_j}(2)$ use the fact that

$$\Psi^{\alpha^i_i}_{\alpha^j_j}(2)=\Psi^{\alpha^i_i}_{\alpha^j_j}\restriction\theta(1)=D(1)=\psi^{\alpha^i_i}_{\alpha^j_j}\restriction\theta(1)=\psi^{\alpha^i_i}_{\alpha^j_j}(2).$$

Continuing this process L times one discovers that $\Psi^{\alpha^i_i}_{\alpha^j_j}(L-1)=\sigma_i$ and $\psi^{\alpha^i_i}_{\alpha^j_j}(L-1)=\sigma_j$ because $\Psi^{\alpha^i_i}_{\alpha^j_j}(l)>\alpha^i_i>\alpha^j_j$ for each $l\in L-1$ or, in other words, $L\leq\Sigma(\alpha^i_i,\alpha^j_j)$. If it turned out to be the case that we actually had $L=\Sigma(\alpha^i_i,\alpha^j_j)$ then, since $\sigma_j\in S_\gamma$,

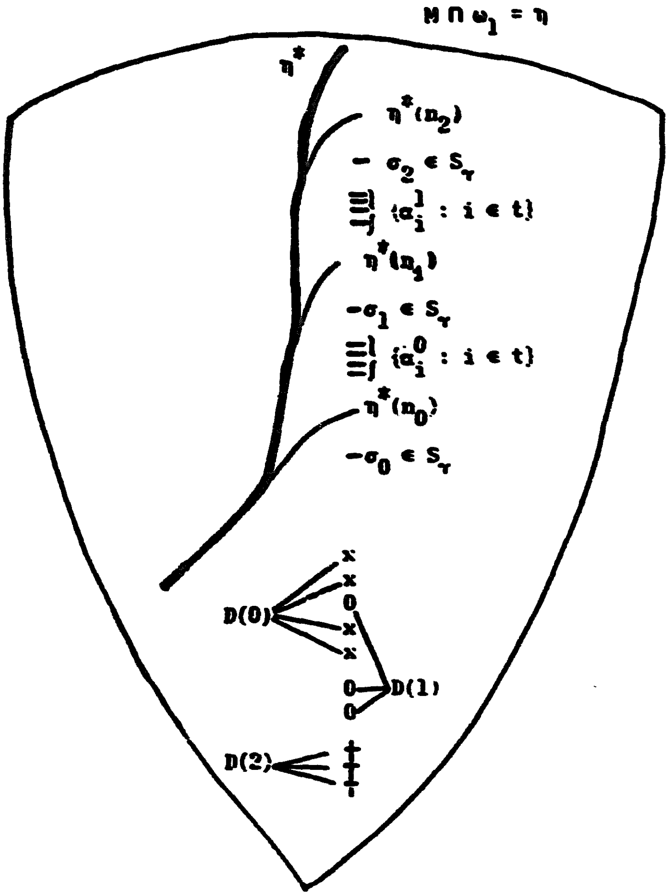

Fig. 1

it follows that $\delta(\sigma_j) = \gamma$ and, hence, that

$$P(\alpha_i^i, \alpha_j^j) = h_\gamma(X(\alpha_i^i) \cap m(\gamma), X(\alpha_j^j) \cap m(\gamma)) = h_\gamma(Y(i), Y(j)) = h(i, j).$$

But why should equality hold? This is where the second half of clause (1.6) is used. Since $\Psi^{\alpha_i^i}_{\alpha_j^j}(L-1) = \sigma_i$ and $\sigma_i^* \restriction (n_{i-1}+1) \supseteq \sigma_i^* \restriction (n_j + 1) = \eta^* \restriction (n_j + 1)$ and since $\sigma_j < \eta^*(n_j)$ it follows that $\sigma_j^*(n_j) \neq \sigma_i^*(n_j)$. Moreover, since $\sigma_i^*(n_j) = \eta^*(n_j) < \alpha_j^j$ it follows that $\Psi^{\alpha_i^i}_{\alpha_j^j}(L) < \alpha_j^j$. Hence $L = \Sigma(\alpha_i^i, \alpha_j^j)$.

All that remains to be shown now is that the supposition upon which the preceding discussion was based is valid. To do this first choose, for each $\sigma \in \omega_1$, $\{\sigma(i) : i \in t\} \subseteq \Lambda \setminus \sigma$, $\bar{L}(\sigma)$, $\{\bar{D}(j, \sigma) : j \in \bar{L}(\sigma)\}$ and $m(\sigma)$ such that

(1.7) $L(\sigma(i), \sigma) = \bar{L}(\sigma)$ for $i \in t$,

(1.8) $D_\sigma^{\sigma(i)}(j) = \bar{D}(j, \sigma)$ for $i \in t$ and $j \in \bar{L}(\sigma)$,

(1.9) $|\{X(\sigma(i)) \cap m(\sigma) : i \in t\}| = t$.

Using Fodor's Lemma it is easy to find $W \in [\omega_1]^{\aleph_1}$, $\bar{L}$, $\{\bar{D}(j) : j \in L\}$, m and

$\{Y(i) : i \in t\}$ such that for $\sigma \in W$ and $i \in t$ the following hold:

(1.10) $\quad \bar{L}(\sigma(i), \sigma) = \bar{L}$,

(1.11) $\quad D_\sigma^{\sigma(i)}(j) = \bar{D}(j)$,

(1.12) $\quad X(\sigma(i)) \cap m = Y(i) \quad$ and $\quad m(\sigma) = m$.

Now find a continuous increasing sequence of elementary submodels $\langle M_\alpha : \alpha \in \omega_1\rangle$ of H (recall the original statement of the supposition) such that M_0 contains $\{\bar{D}(j) : j \in \bar{L}\}$. Choose $\gamma \in \omega_1$ such that $\{Y(i) : i \in t\} \subseteq \mathcal{P}(m(\gamma))$ and (1.1) is satisfied. We would like to find η and ξ such that $M_\eta \cap \omega_1 = \sigma \in W \cap S_\gamma$ and $\{\sigma(i) : i \in t\} \cap M_\xi = 0$ but of course there is no reason why this should be the case. Instead, choose $\bar{\eta}$ such that $M_{\bar{\eta}} \cap \omega_1 \in S_\gamma$ and choose $\sigma \in W \setminus M_{\bar{\eta}+1}$. Let $\eta = M_{\bar{\eta}} \cap \omega_1$. Notice that $\Omega_\sigma^{\sigma(i)}(j) \in M_{\bar{\eta}+1}$ for $j \in \bar{L}$ and that the range of $(\Omega_\sigma^{\sigma(i)}(j))^*$ is disjoint from $M_{\bar{\eta}+1} \setminus \eta$ since $\{\bar{D}(j) : j \in \bar{L}\} \in M_\eta$. Hence $L(\sigma(i), \eta) > \bar{L}$. Moreover $D_\eta^{\sigma(i)}(j) = \bar{D}(j)$ for $j \in \bar{L}$. But even more is true; since $\Omega_\eta^{\sigma(i)}(\bar{L} - 1) = \sigma$ for every $i \in t$ it follows that $\Omega_\eta^{\sigma(i)}(l) = \Omega_\eta^{\sigma(j)}(l)$ for $l \in L(\sigma(i), \eta)$.

If we now let $L = L(\sigma(0), \eta)$ and $D(j) = D_\eta^{\sigma(0)}(j)$ for $j \in L$, then:

(1.13) $\quad \{\sigma(i) : i \in t\} \cap M_{\bar{\eta}+1} = 0$,

(1.14) $\quad L(\sigma(i), \eta) = L \quad$ for $i \in t$,

(1.15) $\quad D_\eta^{\sigma(i)}(j) = D(j) \quad$ for $i \in t$ and $j \in L$,

(1.16) $\quad \{D(j) : j \in L\} \in M_{\bar{\eta}}$.

Clause (1.13) ensures that the formula

$$(\forall \mu)(\exists \{\alpha_i : i \in t\} \subseteq \Lambda \setminus \mu)(\forall i \in t)(L(\alpha_i, \beta) = L$$
$$\& \ X(\alpha_i) \cap m(\gamma) = Y(i) \ \& \ (\forall j \in L)(D_\beta^{\alpha_i}(j) = D(j)))$$

is satisfied by η in $M_{\bar{\eta}+1}$. Now, since η^* obviously contains every one of its initial segments, clause $(1.2{:}n)$ is satisfied for every n. This completes the proof.

2. Other properties of the colouring

The reader of Section 1 will no doubt have noticed that the argument there can be strengthened and generalized. Perhaps the most obvious part of the argument which can be strengthened is in the selection of the t-element set $\{\alpha_j^i : i \in t\}$. The inductive selection of these sets need not have stopped after t steps but could have been carried on infinitely often. Having done this, of course, it is no longer possible to choose the ith element of the ith set. However the only part used in the calculation of $P(\alpha_i^i, \alpha_j^j)$ was that $i \neq j$. Hence the calculation of $P(\alpha_i^i, \alpha_j^j)$ can be carried out as before provided that $i \neq j$. The consequence of this is that given any equivalence relation, E, on ω with only finitely many equivalence classes, $\{[m_i]_E : i \in k\}$, a colouring $h : [\{m_i : i \in k\}]^2 \to \omega_1$ and $X \in [\omega_1]^{\aleph_1}$ it is possible to find $\{x_i : i \in \omega\} \in [X]^{\aleph_0}$ such that

(2.1) if iEi' and jEj', then $P(x_i, x_j) = P(x_{i'}, x_{j'})$;

(2.2) $P(x_{m_i}, x_{m_j}) = h(m_i, m_j)$ if $i \neq j$.

Whenever a colouring of pairs is constructed with certain properties there is always a temptation to construct a colouring of triples with similar properties. The example of Section 1 is easily modified to accomplish this. In fact a colouring $P^* : [\omega_1]^{<\aleph_0} \to \omega_1$ with the following property can be constructed:

(2.3) If $X \in [\omega_1]^{\aleph_1}$ and $h : \mathcal{P}(n) \to \omega_1$ where $n \in \omega$, then there is a one-to-one function $i : n \to X$ such that $h(A) = P^*(i''A)$ for every $A \in [n]^{\geq 2}$.

To do this simply choose a one-to-one $H_h : [n(h)]^2 \to \omega_1$ for every $h : \mathcal{P}(n(h)) \to \omega_1$ such that if $h \neq g$, then the range of H_h is disjoint from the range of H_g. Now, given any $P : [\omega_1]^2 \to \omega_1$ define $P^* : [\omega_1]^{<\aleph_0} \to \omega_1$ by $P^*(A) = h(i''A)$ if there is $i : A \to n(h)$ such that i is an isomorphism of $P \restriction [A]^2$ and $H_h \restriction [i''A]^2$. If there is no such isomorphism define $P^*(A) = 0$. It is easy to check that P^* works. Having seen that it is possible to construct a colouring of all finite sets with property (2.3), it is reasonable to ask whether it is possible to construct a colouring of pairs which will induce a property similar to (2.3) on n-tuples. In particular there is a colouring $P : [\omega_1]^2 \to \omega_1$ of Section 1 which satisfies:

(2.4) If $p \in \omega$ and $\{x_\alpha \in \omega_1; \alpha \in \omega_1\}$ are disjoint p-tuples and $h : [p \times k]^2 \to \omega_1$ is given, then there are $\{\alpha_i : i \in k\}$ such that $P(x_{\alpha_i}(m), x_{\alpha_j}(n)) = h((i, m), (j, n))$ provided that $m = n$.

To see this suppose that disjoint p-tuples $\{x_\alpha \in {}^p\omega_1 : \alpha \in \omega_1\}$ and $h : [p \times k]^k \to \omega_1$ are given. To modify the proof change (1.2:n) to

$$\begin{aligned}(2.2{:}n)\quad & (\forall\alpha)(\exists\beta \in S_\gamma \setminus \alpha)(\beta^* \restriction (n+1) = \eta^* \restriction (n+1) \\ & \& (\forall\mu)(\exists\{x_i : i \in t\} \subseteq {}^p(\Lambda\setminus\mu))(\forall i \in t)(\forall i' \in p) \\ & (L(x_i(i'), \beta) = L(i') \;\&\; X(x_i(i')) \cap m(\gamma) = Y(i, i') \\ & \& (\forall j \in L(i'))_j(D_\beta^{x_i(i')}(j) = D(i', j)))).\end{aligned}$$

Of course (1.1) must now be changed to

$$(\forall\{(i, i')(j, j')\} \in [p \times k]^2)(h(\{(i, i'), (j, j')\}) = h_\gamma(Y(i, i'), Y(j, j'))).$$

The rest of the proof requires only obvious modifications. For a fixed m the calculation of $P(x_i(m), x_j(m))$ is exactly the same as before.

However if $m \neq n$ we do not have enough information to calculate $P(x_i(m), x_j(n))$. Is this an essential difficulty with the proof or is it in fact possible to obtain a colouring P which satisfies (2.4) *without* the final proviso that $m = n$? Perhaps, but if so then this would also provide a counterexample to the partition relation $\omega_1 \to [K_{\aleph_1, \aleph_2}]^2_{\aleph_1}$ (in other words, we would have a colouring of the complete graph in $\aleph_1$ colours such that no large bipartite graph misses any colour). What is known is that it is easy to modify the example of Section 1 to obtain the exact opposite in behaviour namely

(2.5) If $p \in \omega$ and $\{x_\alpha \in {}^p\omega_1; \alpha \in \omega_1\}$ are disjoint p-tuples, then there is $X \in [\omega_1]^{\aleph_1}$ and $h: p \times p \to \omega_1$ such that if $\{\alpha, \beta\} \in [X]^2$, then $P(x_\alpha(n), x_\beta(m)) = h(n, m)$ provided that $n \neq m$.

So, without loss of generality, there is no freedom at all in the colour of $\{x_\alpha(n), x_\beta(m)\}$ unless $m = n$. For an application of (2.5) see [5]. The modifications necessary to achieve (2.5) are discussed in the next section.

3. Generalizations to larger cardinals

Until now the discussion has concentrated entirely on colouring the pairs of ω_1 but it is not difficult to imagine similar colourings on larger cardinals. Can these colourings be constructed by trivially generalizing the construction of Section 1? The answer appears to be negative. To see this begin by considering the sequences μ^*. The obvious choice of μ^* for the construction on higher cardinals would be to let $\mu^*: \mathrm{cof}(\mu) \to \mu$ be increasing and cofinal. However notice that the fact that if $\mathrm{cof}(\mu) = \omega$ and η is a limit ordinal below μ, then $(\mu^{*}{}''\omega) \cap \eta$ is bounded below η was often relied upon (see (1.8), (1.15) and (1.16) for example). One way of ensuring that this remains true when $\mathrm{cof}(\mu) > \omega$ is to insist that μ^* be continuous and that η does not belong to the range of μ^*. Of course we cannot insist that no η belongs to the range of μ^* but we need sufficiently many such ordinals to intersect every closed unbounded set so that, for example, we can find $\bar{\eta}$ as in Section 1. Hence if we are to construct a colouring on some higher cardinal λ we must assume the following hypothesis.

(3.1) There is $S \subseteq \lambda$ which is stationary but such that $S \cap \alpha$ is not stationary for every $\alpha \in \lambda$.

The reason is that now we can choose a closed set $C_\alpha \subseteq \alpha$ which is unbounded in α and disjoint from S. We will let α^* be the increasing enumeration of C_α.

The next difficulty arises when one considers why it was possible to get (1.16) to hold. The reason of course was that each $F(j)$ is finite but if $\mathrm{cof}(\alpha) > \omega$, then initial segments of α^* will not be finite. In order to have an analog of (1.16) hold it would be necessary to have the elementary submodels closed under certain subsets and this would require a hypothesis on cardinal arithmetic. To avoid making this hypothesis we will alter the definition of $P(\alpha, \beta)$ so that it does not depend on the initial segment of α^* below β but only on the maximal element of this initial segment.

For $0 < \alpha < \beta < \lambda$ define $\Gamma_l^+(\alpha, \beta)$ and $\Gamma_l^-(\alpha, \beta)$ by induction on l as follows:

(3.2) $\Gamma_0^+(\alpha, \beta) = \beta$ and $\Gamma_0^-(\alpha, \beta) = 0$;

(3.3) if $\Gamma_l^+(\alpha, \beta)$ is defined and greater than α let $\Gamma_{l+1}^+(\alpha, \beta) = (\Gamma_l^+(\alpha, \beta))^*(\theta_l(\alpha, \beta))$ where $\theta_l(\alpha, \beta)$ is the least ordinal such that $(\Gamma_l^+(\alpha, \beta))^*(\theta_l(\alpha, \beta)) \geq \alpha$;

(3.4) define $\Gamma_{l+1}^-(\alpha, \beta) = \sup\{(\Gamma_l^+(\alpha, \beta))^*(\xi) : \xi \in \theta_l(\alpha, \beta)\}$.

Notice that if $\alpha \in S$, then $\Gamma_l^-(\alpha, \beta) < \alpha$ and so if we let $\mu_m(\alpha, \beta) = \max\{\Gamma_l^-(\alpha, \beta) + 1 : l \in m\}$, then $\mu_m(\alpha, \beta) < \alpha$ provided that $\alpha \in S$ and α is a limit ordinal. What is also true is that $\alpha \leq \Gamma_{l+1}^+(\alpha, \beta) < \Gamma_l^+(\alpha, \beta)$ and hence there is some least integer k such that $\Gamma_k^+(\alpha, \beta)$ is not defined because $\Gamma_{k+1}^+(\alpha, \beta) = \alpha$. Let $k(\alpha, \beta) = k$.

The next hypothesis required on λ is that there is some $\tau < \lambda$ such that $2^\tau \geq \lambda$. If this is the case, then we can let $\{X(\alpha) : \alpha \in \lambda\}$ list distinct subsets of τ and let $\{h_\alpha : \alpha \in \lambda\}$ list all functions $h : [\mathcal{P}(M)]^2 \to \lambda$ where $M \in [\tau]^{<\aleph_0}$. As before let $\{S_\alpha : \alpha \in \lambda\}$ partition S into λ many stationary sets. Given $\zeta \in S$ let $\delta(\zeta)$ be the unique ordinal such that $\zeta \in S_{\delta(\zeta)}$ and let $M(\zeta)$ be the finite subset of τ such that $\mathrm{dom}(h_{\delta(\zeta)})$ is $[\mathcal{P}(M(\zeta))]^2$.

The function $P : [S]^2 \to \lambda$ can now be defined. Let $\{\alpha, \beta\} \in [S]^2$, $\alpha < \beta$ and let $i \leq k(\alpha, \beta)$ be maximal such that

(3.5) $\Gamma_l^-(\mu_i(\alpha, \beta), \alpha) = \Gamma_l^-(\mu_i(\alpha, \beta), \beta)$ for $l \in i$,

(3.6) $M(\Gamma_l^+(\mu_i(\alpha, \beta), \alpha)) = M(\Gamma_l^+(\mu_i(\alpha, \beta), \beta))$ for $l \in i$,

(3.7) $X(\alpha) \cap M(\Gamma_l^+(\mu_i(\alpha, \beta), \alpha)) = X(\beta) \cap M(\Gamma_l^+(\mu_i(\alpha, \beta), \beta))$ for $l \in i$.

If $i = 0$ let $P(\{\alpha, \beta\}) = 0$. Otherwise let $\gamma = \Gamma_{i-1}^+(\mu_i(\alpha, \beta), \beta)$ and define $P(\{\alpha, \beta\}) = h_\gamma(\{X(\beta) \cap M(\gamma),\ X(\alpha) \cap M(\gamma)\})$. If $X(\beta) \cap M(\gamma) = X(\alpha) \cap M(\gamma)$ let $P(\{\alpha, \beta\}) = 0$.

Techniques very similar to those of Section 1 can now be used to show:

(3.8) If λ is a regular cardinal which has a non-reflecting stationary set and such that there is $\tau < \lambda$ such that $2^\tau \geq \lambda$, then there is $P : [\lambda]^2 \to \lambda$ such that if $X \in [\lambda]^\lambda$, $n \in \omega$ and $h : [n]^2 \to \lambda$ are given, then there is a one-to-one function $i : n \to \lambda$ such that $P(i''A) = h(A)$ for every $A \in [n]^2$.

The details are left to the reader. Moreover, there is no difficulty in replacing ω by κ provided that $\rho^\kappa < \lambda$ for every $\rho < \lambda$.

Finally it will be shown that this version of the colouring P satisfies (2.5). To see this suppose that $\{y_\zeta : \zeta \in \lambda\} \subseteq {}^n\lambda$ are one-to-one functions with disjoint ranges. To see that (2.5) is satisfied choose $M_\zeta \in [\tau]^{<\aleph_0}$ such that $|\{X_{y_\zeta(i)} \cap M_\zeta : i \in n\}| = n$, without loss of generality $M_\zeta = M$ and $X_{y_\zeta(i)} \cap M = a_i$ for $\zeta \in \lambda$ and $i \in n$. Now choose σ such that the domain of h_σ is $[\mathcal{P}(M)]^2$. Next, for $\delta \in S_\sigma$ choose $\theta(\delta)$ such that $(y''_{\theta(\delta)}n) \cap \delta = 0$. Since S_σ is stationary there is $\beta \in \lambda$ and $S \in [S_\sigma]^\lambda$ such that $\mu_{k(\delta, y_{\theta(\delta)}(i))}(\delta, y_{\theta(\delta)}(i)) = \beta$ for $i \in n$ and $\delta \in S$. Let Σ_δ be the closure of $\{\delta\} \cup y''_{\theta(\delta)}n$ under Γ_1^+ and Γ_1^-. Notice that this closure is finite because the functions Γ_1^+ and Γ_1^- are regressive in the second variable. Let X_δ be the

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function defined on Σ_δ by $X_\delta(\xi) = X_\xi \cap M(\xi)$. Let H_δ be defined by $H_\delta(\alpha, \beta) = h_\beta(X_\alpha \cap M(\beta))$. It follows that there is $\Lambda \in [S]^\lambda$ such that if $\{\zeta, \xi\} \subseteq \Lambda$ and $\zeta \leqslant \xi$, then there is an isomorphism $I_{\zeta,\xi}$ of the two structures $(\Sigma_\zeta, \Gamma_1^+, \Gamma_1^-, X_\zeta, H_\zeta, \in)$ and $(\Sigma_\xi, \Gamma_1^+, \Gamma_1^-, X_\xi, H_\xi, \in)$ such that $I_{\zeta,\xi}$ is the identity below ζ. Moreover it can be arranged that if $\zeta \in \xi$ and $\{\zeta, \xi\} \subseteq \Lambda$ then $\Sigma_\zeta \subseteq \xi$.

Now let $\{i, j\} \in [n]^2$ and $\zeta \in \xi$ such that $\{\zeta, \xi\} \subseteq \Lambda$. It will be shown that $P(y_{\theta(\zeta)}(i), y_{\theta(\xi)}(j))$ depends only on i, j and the unique isomorphism type of the structures indexed by Λ. First note that since $\zeta \in y_{\theta(\zeta)}(i) \in \xi \in y_{\theta(\xi)}(j)$ it follows that $k(y_{\theta(\zeta)}(i), y_{\theta(\xi)}(j)) > k(\xi, y_{\theta(\xi)}(j))$. Now let $q \leqslant k(y_{\theta(\zeta)}(i), y_{\theta(\xi)}(j))$ be maximal satisfying (3.5), (3.6) and (3.7) in the definition of P.

Now note that for $l \leqslant k(\xi, y_{\theta(\xi)}(j))$, $\mu_l(y_{\theta(\zeta)}(i), y_{\theta(\xi)}(j)) = \mu_l(\xi, y_{\theta(\xi)}(j)) < \xi$. Hence the sequences

$$\{\Gamma_t^-(\mu_l(y_{\theta(\zeta)}(i), (y_{\theta(\xi)}(j)), y_{\theta(\xi)}(j)) : t \in l\},$$
$$\{\Gamma_t^-(\mu_l(y_{\theta(\zeta)}(i), (y_{\theta(\xi)}(j)), y_{\theta(\xi)}(i)) : t \in l\},$$
$$\{X_{y_{\theta(\xi)}(j)} \cap M(\Gamma_t^+(\mu_l(y_{\theta(\zeta)}(i), y_{\theta(\xi)}(j)), y_{\theta(\xi)}(j))) : t \in l\},$$
$$\{X_{y_{\theta(\xi)}(i)} \cap M(\Gamma_t^+(\mu_l(y_{\theta(\zeta)}(i), y_{\theta(\xi)}(j)), y_{\theta(\zeta)}(i))) : t \in l\},$$
$$\{M(\Gamma_t^+(\mu_l(y_{\theta(\zeta)}(i), y_{\theta(\xi)}(j)), y_{\theta(\xi)}(j))) : t \in l\},$$
$$\{M(\Gamma_t^+(\mu_l(y_{\theta(\zeta)}(i), y_{\theta(\xi)}(j)), y_{\theta(\xi)}(i))) : t \in l\}$$

depend only on the unique isomorphism type of the structures indexed by Λ if $l \leqslant k(\xi, y_{\theta(\xi)}(j))$. Since $P(y_{\theta(\zeta)}(i), y_{\theta(\xi)}(j))$ is determined by these sequences and by the function H_ξ it suffices to show that $q \leqslant k(\xi, y_{\theta(\xi)}(j))$.

To see this let $K = k(\xi, y_{\theta(\xi)}(j))$ and notice that

$$\Gamma_{K-1}^+(\mu_K(y_{\theta(\zeta)}(i), y_{\theta(\xi)}(j)), y_{\theta(\xi)}(j)) = \xi \quad \text{and} \quad M(\xi) = M.$$

Then either $M(\Gamma_{K-1}^+(\mu_K(y_{\theta(\zeta)}(i), y_{\theta(\xi)}(j)), y_{\theta(\zeta)}(i))) \neq M$ in which case (3.6) in the definition of P fails or else equality holds in which case

$$X_{y_{\theta(\xi)}(i)} \cap M(\Gamma_{K-1}^+(\mu_K(y_{\theta(\zeta)}(i), y_{\theta(\xi)}(j)), y_{\theta(\zeta)}(i)))$$
$$= X_{y_{\theta(\xi)}(i)} \cap M = a_i \neq a_j = X_{y_{\theta(\xi)}(j)} \cap M(\xi)$$

and so (3.5) fails.

4. Remarks

One possible strengthening of the construction of Section 1 would be:

(4.1) There is a colouring $P : [2^\omega]^2 \to \omega_1$ s.t. if $X \in [2^\omega]^{\aleph_1}$, $n \in \omega$ and $h : [n]^2 \to \omega_1$, then there is $i : n \to X$, one-to-one, s.t. $P(i''A) = h(A)$ for $A \in [n]^2$.

The proof of the Erdös–Rado theorem shows that 2^ω cannot be replaced by $(2^\omega)^+$ since then there is always $X \in [2^\omega]^{\aleph_1}$ s.t. the colour of pairs is determined by the first element of the pair. Also it is at least consistent that (4.1) is false since it is shown in [3] that $2^{\aleph_0}\to[\aleph_1]^2_3$ is consistent assuming the existence of a Mahlo cardinal.

However it is also consistent that (4.1) is true. To see this start with a model where there is a Kurepa tree $(T,\leqslant)$ with $\kappa\geqslant 2^{\aleph_0}$ branches and $\mathrm{cof}(\kappa)>\omega$. Enumerate the branches as $\{b_\alpha:\alpha\in\kappa\}$. Let $\mathbb{P}$ consist of functions $h:[D]^2\to\omega_1$, where $D\in[\kappa]^{<\aleph_0}$, satisfying:

(4.2) If $\{\alpha,\beta\}\in[X]^2$, then b_α agrees with b_β at least until level $h(\{\alpha,\beta\})$.

The partial order is trivially c.c.c. since after applying a Δ-system any two conditions can be amalgamated by colouring new pairs 0. To see that the generic function will satisfy (4.1) suppose not and that $1\Vdash$ "$X\in[\kappa]^{\aleph_1}$ and h witnesses that (4.1) fails". Choose $\{p_\xi:[D_\xi]^2\to\omega_1;\xi\in\omega_1\}$ and $\{x_\xi:\xi\in\omega_1\}$ such that $p_\xi\Vdash$ "$x_\xi\in X$". Without loss of generality $\{D_\xi:\xi\in\omega_1\}$ forms a Δ-system with root D and $x_\xi\in D_\xi\setminus D$ and $p_\xi\restriction[D]^2=p$ for all $\xi\in\omega_1$. Let γ be the maximal ordinal mentioned by h. Then choose $\{\xi(i):i\in|h|\}$ such that $\{b_{x_{\xi(i)}}:i\in|h|\}$ all agree below $\gamma+1$. It is now trivial to amalgamate to get a contradiction.

Suppose $P:[\omega_1]^2\to\omega_1$ establishes that $\omega_1\to[\omega_1;\omega_1]^2_{\aleph_1}$ (in other words, whenever $\{a_\alpha:\alpha\in\omega_1\}$ and $\{b_\alpha:\alpha\in\omega_1\}$ are disjoint uncountable sets and $\eta\in\omega_1$, then there exist $\alpha<\beta$ such that $P(\{a_\alpha,b_\beta\})=\eta$). Then

(4.3) If $h:[\omega]^2\to\omega_1$ and $X\in[\omega_1]^{\aleph_1}$ there is $i:\omega\to X$, one-to-one, such that $P(i''A)=h(A)$ for every $A\in[\omega]^2$.

To see this suppose that $i:n\to X$ has been defined so that $P(i''A)=h(A)$ for every $A\in[n]^2$. Suppose further that for every $f:n\to h''[\omega]^2$ there is $Z(f)\in[X]^{\aleph_1}$ such that $P(\{\alpha,i(j)\})=f(j)$ for every $j\in n$ and $\alpha\in Z(f)$. We now wish to extend $i:n+1\to X$ so that the above hypotheses are satisfied.

To do this let $F(j)=h(\{j,n\})$ for $j\in n$. Let A and B be disjoint uncountable subsets of $Z(F)$ and redefine $Z(F)=B$. It now suffices to show that for every $f:n\to h''[\omega]^2$ and $\alpha\in h''[\omega]^2$ there is $\xi(f,\alpha)\in\omega_1$ such that:

(4.4) If $\sigma\in A\setminus\xi(f,\alpha)$, then $|X(f)^\sigma_\alpha|=\aleph_1$ where $X(f)^\sigma_\alpha=\{\delta\in X(f):P(\{\sigma,\delta\})=\alpha\}$.

If this is the case, then we simply choose $i(n)\in A\setminus\bigcup\{\xi(f,\alpha):f:n\to h''[\omega]^2$ and $\alpha\in h''[\omega]^2\}$ and define $X(f)=X(f\restriction n)^{i(n)}_{f(n)}$ for $f:(n+1)h''[\omega]^2$.

But clearly if (4.4) fails for some f and α, then there is $\bar{A}\in[A]^{\aleph_1}$ and $X\in[X(f)]^{\aleph_1}$ such that $P(\{\rho,\mu\})=\alpha$ for $\rho\in\bar{A}$ and $\mu\in X$ and $M>\rho$. This contradicts the fact that P witnesses $\omega_1\not\to[\omega_1;\omega_1]^2_{\aleph_1}$. This is similar to Theorem 1.1 from [1].

Since certain colourings automatically have the strong embedding property (4.3) it is reasonable to ask if there is a simple class of graphs which will have the weaker property (0.1). In light of the fact that the construction in Section 1 was motivated by Todorčević's proof that $\omega_1\not\to[\omega_1]^2_{\aleph_1}$ it might be conjectured that any colouring which witnesses that $\omega_1\not\to[\omega_1]^2_{\aleph_1}$ will satisfy (0.1). It is easy to see that this is not the case but Baumgartner wondered whether such a colouring might not be obtained by identifying some of the colours. The weakest possible result in this direction one could hope for would be

(4.5) If $P:[\omega_1]^2\to\omega_1$ witnesses that $\omega_1\not\to[\omega_1]^2_{\aleph_1}$, then there is a partition of ω_1, $\omega_1=A^0\cup A^1$, such that the induced colouring $P^*[\omega_1]^2\to 2$ defined by $P^*(B)=i$ if and only if $P(B)\in A^i$ satisfies the following property: if $X\in[\omega_1]^{\aleph_1}$ and $h:[n]^2\to 2$, then there is $i:n\to X$, one-to-one such that $P^*(i''B)=h(B)$ for $B\in[n]^2$.

This is not a theorem of ZFC however, because of the following forcing construction. Define an order on graphs according to the following definition.

(4.6) $(V,E)\leqslant(V',E')$ if and only if there is a function $f:V'\to V$ which takes one set of vertices onto the other and such that if $f(x)\neq f(y)$, then $(x,y)\in E'$ if and only if $\{f(x),f(y)\}\in E$.

Now if $\mathcal{I}$ is an initital segment of the class of graphs with more than one vertex with respect to this ordering, let $\mathbb{P}(\mathcal{I})$ be the partial order consisting of functions $f:[X]^2\to\omega_1$, where $X\in[\omega_1]^{<\aleph_0}$ such that

(4.7) $f(\{\alpha,\beta\})<\max\{\alpha,\beta\}\in[X]^2$;

(4.8) if $f''[X]^2=A^0\cup A^1$ is a partition, then no induced subgraph of $(X,\{\{\alpha,\beta\}\in[X]^2:f(\{\alpha,\beta\})\in A^1\})$ belongs to $\mathcal{I}$.

The generic colouring, P_G, has the property that no partition of ω_1 will yield a graph which contains any member of $\mathcal{I}$ as an induced subgraph. However, if $h:[n]^2\to\omega_1$ is a colouring whose range can not be partitioned so that the resulting graph is in $\mathcal{I}$ and $1\Vdash$"$Z\in[\omega_1]^{\aleph_1}$", then $1\Vdash$ "there is $i:n\to Z$, one-to-one, such that $P_G(i''A)=h(A)$ for $A\in[n]^2$".

To see this suppose not and let $\{f_\xi:[X_\xi]^2\to\omega_1;\xi\in\omega_1\}$ and $\{z_\xi:\xi\in\omega_1\}$ be such that $z_\xi\in X_\xi$ and $f_\xi\Vdash$"$z_\xi\in Z$". The usual Δ-system argument allows us to find $\{C(i):i\in n\}$ such that $\max h''[n]^2<\min\{z_{\xi(i)}:i\in n\}$ and $(f_{\xi(i)},z_{\xi(i)})$ is isomorphic to $(f_{\xi(j)},z_{\xi(j)})$ for $i,j\in n$. Let $f=\bigcup\{f_{\xi(i)}:i\in n\}$. Then f is a partial function from $[\bigcup\{X_{\xi(i)}:i\in n\}]^2$ to ω_1.

Now suppose that neither α nor β belong to the root of the Δ-system and $\alpha\in X_{\xi(i)}$ and $\beta\in X_{\xi(j)}$ where $i\neq j$. Let $I:X_{\xi(i)}\to X_{\xi(j)}$ be the natural isomorphism. If $I(\alpha)\neq\beta$, then define $f(\{\alpha,\beta\})=f_{\xi(j)}(\{I(\alpha),\beta\})$. If $I(\alpha)=\beta$, then define $f(\{\alpha,\beta\})=h(\{i,j\})$. It is straightforward to check that f is a condition and hence we have a contradiction.

As an immediate corollary we can let $\mathcal{I}$ be the initial segment consisting of the 5-cycle (it is easy to see that this *is* an initial segment). Then $\mathbb{P}(\mathcal{I})$ yields a graph witnessing $\omega_1\not\to[\omega_1]^2_{\aleph_1}$ but which is a counterexample to (4.5) and this is witnessed by h being the 5-cycle.

5. An application

In this section an application of preceding results to the theory of Banach spaces will be discussed. It will be shown that there is a non-separable Banach space with the property that every operator from the space to itself is the sum of a diagonalizable operator and one with separable range. An operator T is diagonalizable if and only if there is a basis for the Banach space $\{b_\alpha : \alpha \in \lambda\}$ such that $T(b_\alpha) = \gamma_\alpha b_\alpha$ for every α.

To construct the Banach space let $P:[\omega_1]^2 \to \omega_1$ satisfy (2.4) (it will soon become apparent that a much weaker hypothesis is sufficient but since (2.4) is already available to us we will use it instead of introducing a new hypothesis). Now notice that if X is any set and $\mathcal{A} \subseteq [X]^{<\aleph_0}$ is closed under subsets, then it is possible to define a norm on ${}^X\mathbb{R}$ by

$$\|f\| = \sup\left\{\sum_{a \in A} |f(a)|; A \in \mathcal{A}\right\}$$

where ${}^X\mathbb{R}$ is a vector space under the pointwise operations. It is now possible to define a Banach space $l_1(X, \mathcal{A}) = \{f \in {}^X\mathbb{R}; \|f\| < \infty\}$ with this norm. Let $\hat{\alpha}: X \to \mathbb{R}$ be defined by $\hat{\alpha}(\alpha) = 1$ and $\hat{\alpha}(\beta) = 0$ if $\beta \neq \alpha$. Then $\{\hat{\alpha}: \alpha \in X\}$ is a basis for $l_1(X, \mathcal{A})$ provided that $\bigcup \mathcal{A} = X$. Let this subspace be denoted $l_1^+(X, \mathcal{A})$. The routine verifications are left to the reader who should also notice that if $X = V \cup W$, then

$$l_1^+(X, \mathcal{A}) = l_1^+(V, \mathcal{A} \cap [V]^{<\aleph_0}) \oplus l_1^+(W, \mathcal{A} \cap [W]^{<\aleph_0}).$$

Returning to the problem at hand, it will be shown that the Banach space $l_1^+(\omega_1, \mathcal{H})$ has the desired property where $\mathcal{H} = \{A \in [\omega_1]^{<\aleph_0}: P''[A]^2 = 0\}$. To see this suppose that $T: l_1^+(\omega_1, \mathcal{H}) \to l_1^+(\omega_1, \mathcal{H})$ is a linear mapping which can not be decomposed as $T = T_1 + T_2$ where T_1 has separable range and T_2 is diagonalizable. If there is some $\alpha \in \omega_1$ such that $T(\hat{\beta}) = V(\beta) \oplus \gamma_\beta \hat{\beta}$ for every $\beta > \alpha$ where $V(\beta) \in l_1^+(\alpha, \mathcal{H} \cap [\alpha]^{<\aleph_0})$, then we have the desired decomposition. Hence it can be supposed that for every α there is $\beta(\alpha) > \alpha$ and $\gamma(\alpha) > \alpha$ such that $\gamma(\alpha) \neq \beta(\alpha)$ and $T(\hat{\beta}(\alpha))(\gamma(\alpha)) \neq 0$. Choose $\varepsilon > 0$ and $X \in [\omega_1]^{\aleph_1}$ such that $|T(\hat{\beta}(\alpha))(\gamma(\alpha)))| > \varepsilon$ for $\alpha \in X$ and such that if $\{\eta, \zeta\} \in [X]^2$, then $T(\hat{\beta}(\eta))(\gamma(\zeta)) = 0$.

It will be shown that $\|T\| > M$ for any integer M. To see this choose $k \in \omega$ such that $k > M/\varepsilon$. Using (2.4) find $\{\alpha_i : i \in k\}$ such that $P(\{\beta(\alpha_i), \beta(\alpha_j)\}) = 1$ and $P(\{\gamma(\alpha_i), \gamma(\alpha_j)\}) = 0$ if $\{i, j\} \in [k]^2$. From the definition of $\mathcal{H}$ it follows that $\|\sum_{i \in k} \beta(\alpha_i)\| = 1$. However,

$$\left\|T\left(\sum_{i \in k} \hat{\beta}(\alpha_i)\right)\right\| \geqslant \left\|\sum_{i \in k} \gamma(\alpha_i)\right\| \geqslant \sum_{i \in k} \varepsilon > M.$$

Hence T is unbounded. A similar example using $\diamondsuit$ was found in [4]. The example can be modified to obtain the even stronger assertion that every bounded operator is the sum of a multiple of the identity and an operator with separable range. This will appear elsewhere.

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