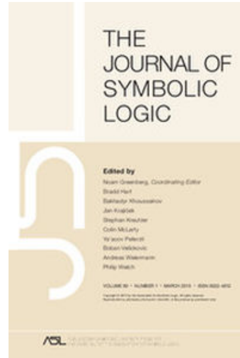

Peano arithmetic may not be interpretable in the monadic theory of linear orders

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PEANO ARITHMETIC MAY NOT BE INTERPRETABLE IN THE MONADIC THEORY OF LINEAR ORDERS

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Abstract. Gurevich and Shelah have shown that Peano Arithmetic cannot be interpreted in the monadic second-order theory of short chains (hence, in the monadic second-order theory of the real line). We will show here that it is consistent that the monadic second-order theory of no chain interprets Peano Arithmetic.

§0. Introduction. A *reduction* of a theory T to a theory T^* is an algorithm, associating a sentence φ^* in the language of T^*, to each sentence φ in the language of T, in such a way that: $T \vdash \varphi$ if and only if $T^* \vdash \varphi^*$.

Although reduction is a powerful method of proving undecidability results, it lacks in establishing any semantic relation between theories.

A (*semantic*) *interpretation* of a theory T in a theory T' is a special case of reduction in which models of T are defined inside models of T'.

It is known (via reduction) that the monadic theory of order and the monadic theory of the real line are at least as complicated as Peano Arithmetic, (In [10] this was proven from ZFC + MA and in [6] from ZFC), and even as second order logic ([7], [11] for the monadic theory of order). Moreover, second order logic was shown to be interpretable in the monadic theory of order ([8]) but this was done by using a weaker, non-standard form of interpretation: into a Boolean valued model. Using standard interpretation ([4]) it was shown that it is consistent that the second–order theory of ω_2 is interpretable in the monadic theory of ω_2 (hence in the monadic theory of well orders).

On the other hand, by [5], Peano Arithmetic is not interpretable in the monadic theory of short chains, and in particular in the monadic theory of the real line.

More details and historical background can be found in [5]).

The previous results leave a gap concerning the question whether it is provable from ZFC that Peano Arithmetic is interpretable in the monadic theory of order. In this paper we fill the gap and show that the previous results are the best possible, by proving:

THEOREM. *There is a forcing notion P such that in V^P, Peano Arithmetic (in fact a weaker theory) is not interpretable in the monadic second-order theory of chains.*

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From another point of view the theorem may be construed as presenting the strength of the interpretation method by showing that although Peano Arithmetic is recursive in the monadic theory of order, it is not interpretable in it.

The proof uses definitions and techniques from [10] and [5] but although we omitted some proofs we tried to make this paper as self contained as possible. We start by defining in §1 the notion of interpretation. Although this notion is not uniform in the literature, our notion of interpretation seems to follow from every reasonable definition. In §2 we define partial theories and present the relevant results about them from [10]. The theory T that is not interpretable in the monadic theory of order is presented in §3. We start by showing that if there is a chain C that interprets T, then the interpretation 'concentrates' on an initial segment $D \subseteq C$ called a major segment.

The main idea in the proof is that of shuffling subsets $X, Y \subseteq C$: Given a partition of C, $\langle S_j : j \in J\rangle$ and a subset $a \subseteq J$, the shuffling of X and Y with respect to J and a is the set: $\bigcup_{j\in a}(X \cap S_j) \cup \bigcup_{j\notin a}(Y \cap S_j)$. We show in §4 and §5 that under suitable conditions (in particular, if a is what we call a 'semi-club'), partial theories are preserved under shufflings. We use a simple class forcing P, defined in §5, to obtain a universe V^P in which generic semi–clubs are added to every suitable partition.

The contradiction to the assumption that an interpretation exists in V^P can be roughly described as follows: Assuming a chain C interprets T we choose a large enough number of subsets of C that represent pairwise different elements in a model of T. After some manipulations we are left with 3 ordered pairs of subsets of C and shuffle each pair $\langle U, V\rangle$ with respect to a generic semi-club a, added by the forcing. This results in a new subset which is equivalent to (i.e., represents the same element as) U. Here we use the preservation of partial theories under shufflings. However, a condition $p \in P$ that forces this, determines only a bounded subset of a, and it is shown that one could have gotten the same results by shuffling the pairs with respect to the complement of a. Thus for each pair U, V, p forces that the result of the 'inverse' shuffling is also equivalent to U. We conclude by showing that one of the shufflings is equivalent to V as well, and get a contradiction since U and V were not equivalent.

We would like to thank the referee for a careful reading of the paper and for suggesting many improvements to the representation.

§1. The notion of interpretation. The notion of semantic interpretation of a theory T in a theory T' is not uniform. Usually it means that models of T are defined inside models of T' but the definitions vary with context. The idea of our definition is that in some model of T' one can define (with parameters) a model of T. Alternative definitions could demand that every model of T is interpretable in a model of T' (As in [2]), or that in every model of T' there is a definable model of T (see [12]).

Our aim is to show that in no chain C there is a model of Peano Arithmetic that is definable by monadic formulas with parameters.

Definition 1.1. Let σ be a signature $\langle P_1, P_2, \ldots\rangle$ where each P_i is a predicate symbol of some arity r_i, in the language $\mathscr{L} = \mathscr{L}(\sigma)$. An *interpretation* of σ in an

$\mathcal{L}'$-theory T' is a sequence $\mathcal{I} = \langle \mathcal{M},\ d,\ U(\bar{x}_1, \bar{a}),\ E(\bar{x}_1, \bar{x}_2, \bar{a}),\ \psi_{P_1}(\bar{x}_1, \dots \bar{x}_{r_1}, \bar{a}),$ $\psi_{P_2}(\bar{x}_1, \dots \bar{x}_{r_2}, \bar{a}), \dots \rangle$ where:

(a) $\mathcal{M}$ is a T'-model;

(b) d is a positive integer (the *dimension of the interpretation*);

(c) $U(\bar{x}_1, \bar{a})$ and $E(\bar{x}_1, \bar{x}_2, \bar{a})$ are $\mathcal{L}'$-formulas (the *universe and the equality formulas*);

(d) each $\psi_{P_i}(\bar{x}_1, \dots \bar{x}_{r_i}, \bar{a})$ is an $\mathcal{L}'$-formula (the *interpretation* of P_i);

(e) $\bar{x}_1, \bar{x}_2 \dots$ are disjoint d-tuples of distinct variables of $\mathcal{L}'$;

(f) $\bar{a}$ is a finite sequence of elements of $\mathcal{M}$ (the parameters of the interpretation).

DEFINITION 1.2. Let σ and $\mathcal{I}$ be as in 1.1. Fix a function that associates each $\mathcal{L}$ variable x with a d-tuple $\bar{x}'$ of distinct $\mathcal{L}'$ variables in such a way that if x and y are different $\mathcal{L}$-variables then the tuples $\bar{x}'$ and $\bar{y}'$ are disjoint.

We define by induction the *$\mathcal{I}$-translation* φ' of an arbitrary $\mathcal{L}$-formula φ:

(a) $(x = y)' = E(\bar{x}', \bar{y}', \bar{a})$.

(b) If P is a predicate symbol of arity r in L, then $P(x_1 \dots x_r)' = \psi_P(\bar{x}_1' \dots \bar{x}_r', \bar{a})$.

(c) $(\neg\varphi)' = \neg(\varphi')$ and $(\varphi_1 \wedge \varphi_2)' = (\varphi_1' \wedge \varphi_2')$.

(d) $(\forall x)\varphi(x)' = (\forall \bar{x}')[U(\bar{x}', \bar{a}) \to \varphi'(\bar{x}', \bar{a})]$ and $(\exists x)\varphi(x)' = (\exists \bar{x}')[U(\bar{x}', \bar{a}) \wedge \varphi'(\bar{x}')]$.

(Of course the variables $\bar{x}'$, $\bar{y}'$, $\bar{x}_1'$, ... are not bound in the interpreting formulas).

DEFINITION 1.3. Let T and T' be first-order theories such that the signature of T consists of predicate symbols. Let $\mathcal{I}$ be an interpretation of the signature of T in T' and let $U(\bar{x}, \bar{a})$ be the universe formula of $\mathcal{I}$.

$\mathcal{I}$ is an *interpretation* of T in T' if:

(a) $\mathcal{M} \models (\exists x')U(x')$ and

(b) the $\mathcal{I}$ translation of every closed theorem of T is satisfied in $\mathcal{M}$.

T is *interpretable* in T' if there is an interpretation of T in T'.

REMARK. The definitions are easily generalized to the case that $\sigma(T)$ consists also of function symbols, see [5].

Being an $\mathcal{I}$ translation of the formula $x = y$, the equality formula $E(\bar{x}', \bar{y}', \bar{a})$ defines an equivalence relation between d-tuples of the interpreting model $\mathcal{M}$.

DEFINITION 1.4. Let $\sim$ be an equivalence relation on a nonempty set A, and let R be a relation of some arity r on A. We say that $\sim$ *respects* R if for all elements $a_1, \dots, a_r$, $b_1, \dots, b_r$ of A,

$$[R(a_1, \dots, a_r) \mathbin{\&} (a_1 \sim b_1) \mathbin{\&} \cdots \mathbin{\&} (a_r \sim b_r)] \text{ implies } R(b_1, \dots, b_r).$$

DEFINITION 1.5. Let σ, $\mathcal{M}$ and $\mathcal{I}$ be as in Definition 1.1.

(a) $U^{\mathcal{I}} := \{\bar{b} : \bar{b}$ is a d-tuple of elements of $\mathcal{M}$ and $\mathcal{M} \models U(\bar{b}, \bar{a})\}$;

(b) $E^{\mathcal{I}} := \{(\bar{b}, \bar{c}) :\ \bar{b}, \bar{c} \in U^{\mathcal{I}}$ and $\mathcal{M} \models E(\bar{b}, \bar{c}, \bar{a})\}$;

(c) $P_i^{\mathcal{I}} := \{(\bar{b}_1, \dots, \bar{b}_{r_i}) :$ each $\bar{b}_i$ belongs to $U^{\mathcal{I}}$ and $\mathcal{M} \models \psi_{P_i}(\bar{b}_1, \dots, \bar{b}_{r_i}, \bar{a})\}$, for every predicate symbol P_i of arity r_i in σ.

FACT 1.6. Let $\mathcal{I} = \langle \mathcal{M},\ d,\ U(\bar{x}_1, \bar{a}),\ E(\bar{x}_1, \bar{x}_2, \bar{a}),\ \psi_{P_1}(\bar{x}_1, \dots, \bar{x}_r, \bar{a}), \dots \rangle$ be an interpretation of a first-order theory T in a first-order theory T'. Then

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(1) The equivalence relation $\sim$ on $U^{\mathcal{I}}$ defined by $\bar{b} \sim \bar{c} \iff (\bar{b}, \bar{c}) \in E^{\mathcal{I}}$ respects each formula $\psi_{P_i}(\bar{x}_1, \dots, \bar{x}_{r_i}, \bar{a})$.

(2) $(U^{\mathcal{I}} / \sim;\ P_1^{\mathcal{I}}, \dots)$ is a model of T.

PROOF. Clear. ⊣

Although our notion of interpretation is not transitive the following clearly holds for any first-order theories, T_1, T_2 and T_3:

FACT 1.7. Suppose that for each model $\mathcal{M} \models T_2$ there is an interpretation $\mathcal{I}(\mathcal{M}) = \langle \mathcal{M}, 1, U(x), E(x,y), \psi_P(\bar{x}) \dots \rangle$ of T_1 in T_2 (without parameters). Then if T_3 interprets T_2, T_3 interprets T_1.

The next aim is to define an interpretation of a first-order theory T, in the monadic version of a first-order theory T'. This is done by associating a model of T to a model of the monadic version of T'. Rather than giving the general definitions we will restrict ourselves to the case that interests us – the monadic theory of linear orders.

DEFINITION 1.8. Let $(C, <)$ be a chain (i.e., a model of the first-order theory of linear orders). The *monadic second-order theory of C* is the first-order theory of the model

$$C^{\mathrm{mon}} = (\mathcal{P}(C);\ \subseteq,\ <^*,\ \mathrm{EM},\ \mathrm{SING})$$

where $\mathcal{P}(C)$ is the power set of C, $<$ and $\subseteq$ are binary relations, SING and EM are unary relations and:

(i) $C^{\mathrm{mon}} \models \mathrm{SING}(X)$ iff X is a singleton,

(ii) $C^{\mathrm{mon}} \models X <^* Y$ iff $X = \{x\}$, $Y = \{y\}$ (where $x, y \in C$) and $C \models x < y$,

(iii) $C^{\mathrm{mon}} \models \mathrm{EM}(X)$ iff $X = \emptyset$,

(iv) $\subseteq$ is interpreted as the usual inclusion relation between subsets of C.

DEFINITION 1.9. A first-order theory T is *interpreted in the monadic theory of linear orders* iff there is a chain C and an interpretation $\mathcal{I} = \langle C^{\mathrm{mon}}, d, U(\bar{x}_1, \bar{a}), E(\bar{x}_1, \bar{x}_2, \bar{a}) \dots \rangle$ of T in the first-order theory $\mathrm{Th}(C^{\mathrm{mon}})$.

DISCUSSION 1.10. The monadic version $\mathcal{L}^{\mathrm{mon}}$ of a first-order language $\mathcal{L}$ is usually described as enriching $\mathcal{L}$ by adding a new set of variables for sets of elements, atomic formulas of the form "$x \in Y$" and the quantifiers $(\exists Y)$ and $(\forall Y)$ ranging over subsets. There is a natural correspondence between monadic formulas (formulas in $\{<\}^{\mathrm{mon}}$) and first-order formulas in the language $\{\subseteq, <^*, \mathrm{EM}, \mathrm{SING}\}$ and a natural identification between the theory of a chain $(C, <)$ in $\{<\}^{\mathrm{mon}}$ and the first-order theory of C^{mon}.

We will think of an interpretation of a first-order theory T in the monadic theory of linear orders as a sequence $\langle (C, <), d, U(\bar{X}_1, \bar{A}), E(\bar{X}_1, \bar{X}_2, \bar{A}) \dots \rangle$ where E, U and the ψ_P's are monadic formulas with monadic parameters $\bar{A} \in {}^{\lg(\bar{A})}\mathcal{P}(C)$.

Abusing the notations we will often write $C \models X \subseteq Y$ and $C \models x \in Y$ instead of $C^{\mathrm{mon}} \models X \subseteq Y$ and $C^{\mathrm{mon}} \models (\mathrm{SING}(X) \wedge X \subseteq Y)$.

§2. Partial theories. In this section we will define 3 kinds of partial theories following [10]: Th^n (Definition 2.2) which is the theory of formulas with monadic quantifier depth n, ATh^n (Definition 2.10) which is the n-theory of segments (and by Lemma 2.9 'many' segments have the same theory), and WTh^n which gives information about stationary subsets of the chain. The last two theories are naturally defined for well ordered chains only but will be modified later to apply to general chains as well.

The main result of this section states roughly that for every n there is an m such that WTh^m and ATh^m determine Th^n (Theorem 2.15).

CONVENTION 2.1. The monadic second-order theory of a chain C is the first-order theory of C^{mon} as described in Definition 1.8. We denote by $L(\mathrm{mon})$ the (first-order) language of C^{mon}.

$\mathscr{L}$ is the monadic second-order version of the first-order language of an order relation $\{<\}$ (described in 1.10).

NOTATION. We denote individual variables by x, y, z and set variables by X, Y, Z. a, b, c are elements and A, B, C are sets. $\bar{a}$ and $\bar{A}$ denote finite sequences having lengths $\lg(\bar{a})$ and $\lg(\bar{A})$. We will write $\bar{a} \in C$ and $\bar{A} \subseteq C$ instead of $\bar{a} \in {}^{\lg(\bar{a})}C$ or $\bar{A} \in {}^{\lg(\bar{A})}\mathscr{P}(C)$

The first definition is that of the partial monadic theory of a chain C:

DEFINITION 2.2. Let $(C, <)$ be a chain and $\bar{A} \subseteq C$. We define

$$t = \mathrm{Th}^n(C; \bar{A})$$

by induction on n:

for $n = 0$: $\quad t = \{\varphi(\bar{X}) : \varphi \in L(\mathrm{mon}),\ \varphi \text{ quantifier free, } C^{\mathrm{mon}} \models \varphi(\bar{A})\}$

for $n = m + 1$: $\quad t = \{\mathrm{Th}^m(C; \bar{A}^\wedge B) : B \subseteq C\}$.

LEMMA 2.3. (A) *For every formula $\psi(\bar{X}) \in \mathscr{L}$ there is an n such that from $\mathrm{Th}^n(C; \bar{A})$ we can decide effectively whether $C \models \psi(\bar{A})$. We call the minimal such n the depth of ψ and write* $\mathrm{dp}(\psi) = n$.

(B) *For every n and ℓ there is a finite set of monadic formulas (effectively computable from n and ℓ) $\Psi(n, \ell) = \{\psi_m(\bar{X}) : m < m^*,\ \lg(\bar{X}) = \ell\} \subseteq \mathscr{L}$ such that for any chains C, D and $\bar{A} \subseteq C$, $\bar{B} \subseteq D$ of length ℓ the following hold:*

(1) $\mathrm{dp}(\psi_m(\bar{X}) \leq n$ *for* $m < m^*$,

(2) $\mathrm{Th}^n(C; \bar{A})$ *can be computed from* $\{m < m^* : C \models \psi_m[\bar{A}]\}$,

(3) $\mathrm{Th}^n(C; \bar{A}) = \mathrm{Th}^n(D; \bar{B})$ *iff for any* $m < m^*$, $C \models \psi_m[\bar{A}] \iff D \models \psi_m[\bar{B}]$.

PROOF. In [10], Lemma 2.1. ⊣

DEFINITION 2.4. When $\Psi(n, \ell)$ is as in Lemma 2.3(B), for each chain C and $\bar{A} \subseteq C$ of length ℓ we can identify $\mathrm{Th}^n(C; \bar{A})$ with a subset of $\Psi(n, \ell)$. Denote by $T_{n,\ell}$ the collection of subsets of $\Psi(n, \ell)$ and call it *the set of formally possible (n, ℓ)-theories.*

LEMMA 2.5. *For given $n, \ell \in \mathbb{N}$, each $\mathrm{Th}^n(C; \bar{A})$ is hereditarily finite, (where $\lg(\bar{A}) = \ell$, C is a chain), and we can effectively compute the set of formally possible theories $T_{n,\ell}$.*

PROOF. In [10], Lemma 2.2. ⊣

DEFINITION 2.6. If $(C, <_C)$ and $(D, <_D)$ are chains then $(C + D, <)$ is the chain that is obtained by adding a copy of D after C (where $<$ is naturally defined).

If $(I, <)$ is a chain and $\langle (C_i, <_i) : i \in I \rangle$ is a sequence of chains then $\sum_{i \in I} (C_i, <_i)$ is the chain that is the concatenation of the C_i's along I equipped with the obvious order.

Next is the heavily used composition theorem for chains that states that the partial theory of a chain is determined by the partial theories of its convex parts.

Given $\bar{A} = \langle A_0, \dots, A_{\ell-1} \rangle$ and $\bar{B} = \langle B_0, \dots, B_{\ell-1} \rangle$ we denote by $\bar{A} \cup \bar{B}$ the sequence $\langle A_0 \cup B_0, \dots, A_{\ell-1} \cup B_{\ell-1} \rangle$.

THEOREM 2.7 (Composition theorem for chains). (1) *If C, C', D and D' are chains, $\bar{A} \subseteq C$, $\bar{A}' \subseteq C'$, $\bar{B} \subseteq D$ and $\bar{B}' \subseteq D'$ are of the same length then if*

$$\mathrm{Th}^m(C; \bar{A}) = \mathrm{Th}^m(C'; \bar{A}')$$

and

$$\mathrm{Th}^m(D; \bar{B}) = \mathrm{Th}^m(D'; \bar{B}')$$

then

$$\mathrm{Th}^m(C + D; \bar{A} \cup \bar{B}) = \mathrm{Th}^m(C' + D'; \bar{A}' \cup \bar{B}').$$

(2) *If I is a chain and $\mathrm{Th}^m(C_i; \bar{A}^i) = \mathrm{Th}^m(D_i; \bar{B}^i)$ for each $i \in I$ (with all sequences of subsets having the same length) then*

$$\mathrm{Th}^m\Big(\sum_{i \in I} C_i; \bigcup_i \bar{A}^i\Big) = \mathrm{Th}^m\Big(\sum_{i \in I} D_i; \bigcup_i \bar{B}^i\Big).$$

PROOF. By [10] Theorem 2.4 (where a more general theorem is proved), or directly by induction on m. $\dashv$

Using the composition theorem we can define a formal operation of addition of partial theories.

NOTATION 2.8. (1) When $t_1, t_2, t_3 \in T_{m,\ell}$ for some $m, \ell \in \mathbb{N}$, then $t_1 + t_2 = t_3$ means: there are chains C and D such that

$$\begin{aligned} t_1 &= \mathrm{Th}^m(C; A_0, \dots, A_{\ell-1}), \\ t_2 &= \mathrm{Th}^m(D; B_0, \dots, B_{\ell-1}), \\ t_3 &= \mathrm{Th}^m(C + D; \bar{A} \cup \bar{B}). \end{aligned}$$

(By the composition theorem, the choice of C and D is immaterial.)

(2) $\sum_{i \in I} \mathrm{Th}^m(C_i; \bar{A}^i)$ is $\mathrm{Th}^m\left(\sum_{i \in I} C_i; \bigcup_{i \in I} \bar{A}^i\right)$, (assuming $\lg(\bar{A}^i) = \lg(\bar{A}^j)$ for $i, j \in I$).

(3) If D is a sub-chain of C and $\bar{A} \subseteq C$ then $\mathrm{Th}^m(D; \langle A_0 \cap D, A_1 \cap D, \dots \rangle)$ is abbreviated by $\mathrm{Th}^m(D; \bar{A})$

(4) For C a chain, $a < b \in C$ and $\bar{P} \subseteq C$ we denote by $\mathrm{Th}^n(C; \bar{P}) \restriction_{[a,b)}$ the theory $\mathrm{Th}^n([a, b); \bar{P} \cap [a, b))$.

We will define now the partial theories ATh^n and WTh^n. The following definitions and results apply to well ordered chains (i.e., ordinals); we will modify them later.

For α an ordinal with $\mathrm{cf}(\alpha) > \omega$, let D_α denote the filter generated by the closed unbounded subsets of α.

The next lemma states that in a well ordered chain of uncountable cofinality, many convex segments have the same monadic theory.

LEMMA 2.9. *If the cofinality of α is $> \omega$, then for every $\bar{A} \subseteq \alpha$ there is a closed unbounded subset J of α such that: for each $\beta < \alpha$, all the models*

$$\{(\alpha; \bar{A})|_{[\beta,\gamma)} : \gamma \in J,\ \mathrm{cf}(\gamma) = \omega,\ \gamma > \beta\}$$

have the same monadic theory.

PROOF. In [10] Lemma 4.1. ⊣

DEFINITION 2.10. When $\beta < \alpha$ and $\mathrm{cf}(\alpha) > \omega$, $\mathrm{ATh}^n(\beta,\ (\alpha; \bar{A}))$ is $\mathrm{Th}^n(\alpha; \bar{A})|_{[\beta,\gamma)})$ for some (equivalently every) $\gamma \in J$, $\gamma > \beta$, $\mathrm{cf}(\gamma) = \omega$;
Here J is from Lemma 2.9.

REMARK. As D_α is a filter, the definition does not depend on the choice of J.

DEFINITION 2.11. For $a \in C$ and $\bar{A} \subseteq C$ let

$$\mathrm{th}(a; \bar{A}) = \{\text{“}x \in X_i\text{”} : a \in A_i\} \cup \{\text{“}x \notin X_i\text{”} : a \notin A_i\}.$$

So it is a finite set of formulas.

DEFINITION 2.12. We define $\mathrm{WTh}^n(\alpha; \bar{A})$ for an ordinal α and $\bar{A} \subseteq \alpha$:
(1) if α is a successor or has cofinality ω, it is $\emptyset$;
(2) otherwise $\mathrm{WTh}^n(\alpha; \bar{A})$ is defined by induction on n:
for $n = 0$: $\mathrm{WTh}^0(\alpha; \bar{A}) = \{t : \{\beta < \alpha : \mathrm{th}(\beta; \bar{A}) = t\}$ is a stationary subset of $\alpha\}$;
for $n + 1$: $\mathrm{WTh}^{n+1}(\alpha; \bar{A}) = \{\langle S_1^{\bar{A}}(B), S_2^{\bar{A}}(B)\rangle : B \subseteq \alpha\}$.
Where:
$S_1^{\bar{A}}(B) = \mathrm{WTh}^n(\alpha; \bar{A}{}^\wedge B)$,
$S_2^{\bar{A}}(\bar{B}) = \{\langle t, s\rangle : \{\beta < \alpha : \mathrm{WTh}^n(\alpha; \bar{A}{}^\wedge B) \restriction_\beta = t,\ th(\beta; \bar{A}{}^\wedge B) = s\}$ is stationary in $\alpha\}$.

REMARK. Clearly, if we replace $(\alpha; \bar{A})$ by a sub-model whose universe is a club subset of α, $\mathrm{WTh}^n(\alpha; \bar{A})$ will not change.

DEFINITION 2.13. Let $\mathrm{cf}(\alpha) > \omega$ and $\bar{A} \subseteq \alpha$ with $\lg(\bar{A}) = \ell$. We define a sequence $g^n(\alpha; \bar{A}) = g^n(\bar{A})$ of subsets of α:
for $s \in T_{n,\ell}$ let $g^n(\bar{A})_s := \{\beta < \alpha : s = \mathrm{ATh}^n(\beta, (\alpha; \bar{A}))\}$ and

$$g^n(\bar{A}) := \langle \dots, g^n(\bar{A})_s, \dots\rangle_{s \in T_{n,\ell}}.$$

LEMMA 2.14. (A) *$g^n(\alpha; \bar{A})$ is a partition of α.*
(B) *$g^n(\alpha; \bar{A}{}^\wedge \bar{B})$ is a refinement of $g^n(\alpha; \bar{A})$ and we can effectively correlate the parts.*
(C) *$g^{n+1}(\alpha; \bar{A})$ is a refinement of $g^n(\alpha; \bar{A})$ and we can effectively correlate the parts.*

PROOF. Easy. ⊣

The next theorem shows that the partial theory of a chain can be computed from the theories ATh and WTh and is the main tool for showing that monadic theories are preserved under shufflings of subsets.

THEOREM 2.15. *If* $\mathrm{cf}(\alpha) > \omega$ *then for each* $n, \ell \in \mathbb{N}$ *there is an* $m = m(n, \ell)$, *effectively computable from* n *and* ℓ, *such that if* $\bar{A} \subseteq C$ *and* $\lg(\bar{A}) = \ell$ *and if*

$$t_1 = \mathrm{WTh}^m\left(\alpha; g^m(\alpha; \bar{A})\right), \quad t_2 = \mathrm{ATh}^m\left(0, (\alpha; \bar{A})\right)$$

then we can effectively compute $\mathrm{Th}^n(\alpha; \bar{A})$ *from* $\langle t_1, t_2\rangle$.

PROOF. By [10], Theorem 4.4. ⊣

NOTATION 2.16. We will denote $\langle t_1, t_2\rangle$ from Theorem 2.15 by $\mathrm{WA}^m(\alpha; \bar{A})$.

§3. **Major segments.** Rather than working with Peano arithmetic we define a first order theory T such that any chain that interprets Peano arithmetic interprets T. Then, assuming a chain C interprets T we show that the interpretation 'concentrates' on a special initial (or final) segment $D \subseteq C$, called a minimal major segment.

DEFINITION 3.1. Let T be a first order theory with a signature consisting of one binary predicate p. The axioms of T are as follows:
(a) $\forall x \exists y \forall z [p(z, y) \leftrightarrow z = x]$
(b) $\forall x \forall y \exists u \forall z [p(z, u) \leftrightarrow (p(z, x) \vee p(z, y))]$
(c) $\exists x \forall y [\neg p(y, x)]$.

Intuitively (a) means that for every set x there exists the set $\{x\}$, (b) means that for every set x, y there exists the set $x \cup y$ and (c) means that the empty set (or an atom) exists.

DISCUSSION 3.2. By Remark 1.7 it is enough to show that there is no interpretation of T in the monadic theory of order since in every model of Peano arithmetic we can interpret T letting $U(x) :=$"$x = x$", $E(x, y) :=$"$x = y$" and (choosing our favorite way of coding finite sets) $\psi_p(x, y) :=$"y codes a finite set to which x belongs".

For the remaining of the section we will assume that C is a chain, $\bar{Q} \subseteq C$ and

$$\mathcal{I} = \langle C, d, U(\bar{X}_1, \bar{Q}), E(\bar{X}_1, \bar{X}_2, \bar{Q}), P(\bar{X}_1, \bar{X}_2, \bar{Q})\rangle$$

is an interpretation of T in the monadic theory of chains.

We may assume by increasing d and adding dummy variables that $\lg(\bar{Q}) = d$. We also assume (by modifying E) that the interpretation is universal i.e., $C \models (\forall \bar{X}) U(\bar{X}, \bar{Q})$.

Therefore the interpretation defines a model of T:

$$\mathcal{M} = \langle \mathcal{P}(C)^d / E,\ P\rangle$$

CONVENTION. We will refer to (d-tuples of) subsets of C as 'elements'. If not otherwise mentioned, all the sequences appearing in the formulas have length d (= the dimension of the interpretation).

We will say that $\bar{A}$ is equivalent to $\bar{B}$ and write $\bar{A} \sim \bar{B}$ when $C \models E(\bar{A}, \bar{B}, \bar{Q})$.

DEFINITION 3.3. (1) A sub-chain $D \subseteq C$ is a *segment* if it is convex (i.e., $x < y < z \,\&\, x, z \in D \Rightarrow y \in D$).

(2) Let $\bar{A}, \bar{B} \subseteq C$. We will say that $\bar{A}, \bar{B}$ *coincide on* [*resp. outside*] a segment $D \subseteq C$ if $\bar{A} \cap D = \bar{B} \cap D$ [resp. $\bar{A} \cap (C \setminus D) = \bar{B} \cap (C \setminus D)$].

(3) The *bouquet size* of a segment $D \subseteq C$ is the supremum of cardinals $|S|$ where S ranges over collections of non-equivalent elements coinciding outside D.

(4) A *Dedekind cut* of C is a pair (L, R) where L is an initial segment of C, R is a final segment of C and $L \cap R = \emptyset$, $L \cup R = C$.

Our next step is to show that the bouquet sizes of initial segments are either infinite or uniformly bounded.

LEMMA 3.4. *There are monadic formulas* $\theta_1(\bar{X}, \bar{Z}, \bar{Q})$, *and* $\theta_2(\bar{X}, \bar{Y}, \bar{Z}, \bar{Q})$ *such that:*

(1) *For every finite, nonempty collection S of elements, there is an element $\bar{P}$ such that for an arbitrary element $\bar{A}$, $C \models \theta_1(\bar{A}, \bar{P}, \bar{Q})$ if and only if there is an element $\bar{B} \in S$ such that $\bar{B} \sim \bar{A}$.*

(2) *For every finite, nonempty collection S of pairs of elements, there is an element $\bar{P}$ such that for an arbitrary pair of elements $\langle \bar{A}_1, \bar{A}_2 \rangle$, $C \models \theta_2(\bar{A}_1, \bar{A}_2, \bar{P}, \bar{Q})$ if and only if there is a pair $\langle \bar{B}_1, \bar{B}_2 \rangle \in S$ such that $\bar{B}_1 \sim \bar{A}_1$ and $\bar{B}_2 \sim \bar{A}_2$.*

PROOF. Easy. ⊣

PROPOSITION 3.5. *Let $\theta_2(\bar{X}, \bar{Y}, \bar{Z}, \bar{Q})$ be from Lemma* 3.4 *and* $\mathrm{dp}(\theta_2) = m$, *let* $N_1 = |T_{m,4d}|$.

Then, for every Dedekind cut (L, R) of C, either the bouquet size of L is at most N_1 or the bouquet size of R is at most N_1.

PROOF. (By [5] Theorem 6.1). Assume that neither L nor R have bouquet size $\leq N_1$. Fix non-equivalent elements $\bar{A}_0, \dots, \bar{A}_{N_1}$ that coincide on R, and $\bar{B}_0, \dots, \bar{B}_{N_1}$ that coincide on L.

By Lemma 3.4(2) there is an element $\bar{P}$ that codes $\langle \bar{A}_0, \bar{B}_0 \rangle, \dots, \langle \bar{A}_{N_1}, \bar{B}_{N_1} \rangle$.

By the definition of N_1 there are $i < j \leq N_1$ such that

$$\mathrm{Th}^m(R; \bar{A}_i, \bar{B}_i, \bar{P}, \bar{Q}) = \mathrm{Th}^m(R; \bar{A}_j, \bar{B}_j, \bar{P}, \bar{Q}).$$

By the composition Theorem 2.7

$$\begin{aligned}
\mathrm{Th}^m(C; \bar{A}_i, \bar{B}_j, \bar{P}, \bar{Q}) &= \mathrm{Th}^m(L; \bar{A}_i, \bar{B}_j, \bar{P}, \bar{Q}) + \mathrm{Th}^m(R; \bar{A}_i, \bar{B}_j, \bar{P}, \bar{Q}) \\
&= \mathrm{Th}^m(L; \bar{A}_i, \bar{B}_i, \bar{P}, \bar{Q}) + \mathrm{Th}^m(R; \bar{A}_j, \bar{B}_j, \bar{P}, \bar{Q}) \\
&= \mathrm{Th}^m(R; \bar{A}_i, \bar{B}_i, \bar{P}, \bar{Q}) + \mathrm{Th}^m(R; \bar{A}_i, \bar{B}_i, \bar{P}, \bar{Q}) \\
&= \mathrm{Th}^m(C; \bar{A}_i, \bar{B}_i, \bar{P}, \bar{Q}).
\end{aligned}$$

Here the second equality holds since $\bar{B}_i \cap L = \bar{B}_j \cap L$ and $\bar{A}_i \cap R = \bar{A}_j \cap R$, and the third holds by the choice of $i < j$.

As $\mathrm{dp}(\theta_2) = m$ and $C \models \theta_2(\bar{A}_i, \bar{B}_i, \bar{P}, \bar{Q})$ we have $C \models \theta_2(\bar{A}_i, \bar{B}_j, \bar{P}, \bar{Q})$. By the properties of θ_2 and $\bar{P}$, $\bar{A}_i \sim \bar{A}_k$ and $\bar{B}_j \sim \bar{B}_k$ for some $k \leq N_1$.

As we started with sequences of non-equivalent elements, $i = k$ and $j = k$. Hence $i = j$ which is a contradiction. ⊣

DEFINITION 3.6. A segment $D \subseteq C$ is called *minor* if its bouquet size is at most N_1. A segment $D \subseteq C$ is called *major* if its bouquet size is infinite.

The following is Lemma 8.1 in [5]. Note that the first part is trivial as T has only infinite models.

CONCLUSION 3.7. C is major and for every Dedekind cut (L, R) of C either L is minor and R is major, or vice versa. ⊣

DEFINITION 3.8. An initial [final] segment D is called a *minimal major segment* if D is major and for every proper initial [final] segment $D' \subset D$, D' is minor.

LEMMA 3.9. *There is a chain C^* that interprets T and an initial segment $D \subseteq C^*$ (possibly $D = C^*$) such that D is a minimal major segment.*

PROOF. (By [5] Lemma 8.2). Let L be the union of all the minor initial segments (note that if L is minor and $L' \subseteq L$ then L' is minor as well). If L is major then set $L = D$, $C^* = C$ and we are done.

Otherwise, let $D = C \setminus L$, and by Conclusion 3.7 D is major. If there is a proper final segment $D' \subset D$ which is major then $C \setminus D'$ is minor. But, $(C \setminus D') \supset L$, and this is impossible by maximality of L.

Therefore D is a minimal major (final) segment. Now take C^* to be the inverse chain of C. By virtue of symmetry C^* interprets T and D is a minimal major initial segment of C^*. ⊣

NOTATION. By the previous lemma we may assume that C has a minimal major initial segment. Let D denote this segment.

DISCUSSION. Being the shortest initial segment such that there are at least $N_1 + 1$ non-equivalent elements coinciding outside it, D is definable in C. What about $\mathrm{cf}(D)$?

It's easy to see that D does not have a last point. On the other hand, it was proven in [5] that T is not interpretable in the monadic theory of short chains (where a chain C is short if every well ordered sub-chain of C or of the inverse chain C^{INV} is countable).

However, we don't need to assume that the interpreting chain is short in order to apply [5]'s argument. All we have to assume to get a contradiction is that $\mathrm{cf}(D) = \omega$ (which is of course the only possible case when C is short). So if C interprets T and $\mathrm{cf}(D) = \omega$, we can repeat the argument from [5] to get a contradiction. Therefore, we can conclude:

PROPOSITION 3.10. $\mathrm{cf}(D) > \omega$. ⊣

DISCUSSION (continued). Now, fix some $\bar{R} \subseteq (C \setminus D)$ witnessing the fact that D is major i.e., such that

$$\mathcal{S} := \{\bar{A} \subseteq C : \bar{A} \cap (C \setminus D) = \bar{R}\}$$

contains an infinite set of pairwise non-equivalent elements. Belonging to $\mathcal{S}$ is of course definable in C (using an additional parameter $\bar{R}$).

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Choose a finite set of pairwise non-equivalent elements $\mathscr{A} = \{\bar{A}_0, \dots, \bar{A}_{k-1}\} \subseteq \mathcal{S}$. Using the formula θ_1 from Lemma 3.4, for each sub-collection $\mathscr{A}_i \subseteq \mathscr{A}$ there is an element $\bar{B}_i$ that codes the elements of $\mathscr{A}_i$. Let $\mathscr{B} = \{\bar{B}_0, \dots, \bar{B}_{2^k-1}\}$ be a collection of such elements (they are of course pairwise non-equivalent). Repeating this another time we get a collection of "hypersets" $\mathscr{E} = \{\bar{E}_0, \dots, \bar{E}_{2^{2^k}-1}\}$ of elements coding subsets of $\mathscr{B}$, let $\bar{E}^*$ code $\mathscr{E}$.

Clearly the following are expressible (by monadic formulas with $\bar{E}^*$ as a parameter):

(a) "$\bar{X}$ is equivalent to a member of $\mathscr{A}$" ($:=$ Atom$(\bar{X})$).

(b) "$\bar{Y}$ is equivalent to a member of $\mathscr{B}$" ($:=$ Set$(\bar{Y})$).

(c) "$\bar{X}$ is equivalent to a member $\bar{A}_i$ of $\mathscr{A}$ and $\bar{Y}$ is equivalent to a member of $\mathscr{B}$ that codes a sub-collection to which $\bar{A}_i$ belongs" ($:=$ Code$(\bar{X}, \bar{Y})$).

(d) HSet$(\bar{Z})$ (meaning $\bar{Z}$ is a hyperset).

(e) HCode$(\bar{Y}, \bar{Z})$ (meaning the set $\bar{Y}$ is a member of the hyperset $\bar{Z}$).

Moreover, the depths of 'Atom', 'Set', 'Code' 'HSet' and 'HCode' are a function of the depth of θ_1 and d hence they are independent of $k = |\mathscr{A}|$.

NOTATION 3.11. Let T_k be the first-order theory of the model

$$\{\bar{A}_0, \dots, \bar{A}_{k-1}, \bar{B}_0, \dots, \bar{B}_{2^k-1}, \bar{E}_0, \dots, \bar{E}_{2^{2^k}-1}; \text{ Atom, Set, Code, HSet, HCode}\}.$$

DISCUSSION (continued). Clearly T_k is interpretable in C. Moreover, as for some $n \in \mathbb{N}$ the depths of the interpreting formulas are $\leq n$ regardless of the choice of k, and as there are only finitely many formulas (with a pre-fixed number of variables) with such depth we can conclude:

$(*)$ *There is an increasing and unbounded sequence of natural numbers* $\langle k_j : j < \omega \rangle$, *formulas* $U'(\bar{X}, \bar{W}, \bar{Q})$, $E'(\bar{X}, \bar{Y}, \bar{W}, \bar{Q})$, Atom$(\bar{X}, \bar{W}, \bar{Q})$, Set$(\bar{X}, \bar{W}, \bar{Q})$, Code$(\bar{X}, \bar{Y}, \bar{W}, \bar{Q})$, HSet$(\bar{Z}, \bar{W}, \bar{Q})$, *and* HCode$(\bar{Y}, \bar{Z}, \bar{W}, \bar{Q})$ *and a sequence of parameters* $\langle \bar{B}^*_j : j < \omega \rangle$ *such that for every* $j < \omega$

$$\langle C, d, U'(\bar{X}, \bar{B}^*_j, \bar{Q}), E'(\bar{X}, \bar{Y}, \bar{B}^*_j, \bar{Q}), \text{Atom}(\bar{X}, \bar{B}^*_j, \bar{Q}), \text{Set}(\bar{X}, \bar{B}^*_j, \bar{Q}) \dots \rangle$$

is an interpretation of T_{k_j} *in the monadic theory of chains.*

The next step is to show that T_k, for unboundedly many k's, is interpretable even in the minimal major initial segment D.

Choose some k_j and a parameter $\bar{B}^*_j$ as in $(*)$. Look at the formula Atom$(\bar{X}, \bar{B}^*_j, \bar{Q})$ and assume dp(Atom) $= n$. Let $\bar{A} \in \mathcal{S}$. By the composition theorem

$$\begin{aligned} t := {} & \mathrm{Th}^n(C; \bar{A}, \bar{B}^*_j, \bar{Q}) \\ = {} & \mathrm{Th}^n(D; \bar{A}, \bar{B}^*_j, \bar{Q}) + \mathrm{Th}^n(C \setminus D; \bar{A}, \bar{B}^*_j, \bar{Q}) \\ = {} & \mathrm{Th}^n(D; \bar{A}, \bar{B}^*_j, \bar{Q}) + \mathrm{Th}^n(C \setminus D; \bar{R}, \bar{B}^*_j, \bar{Q}). \end{aligned}$$

Where the second equality holds by $\bar{A} \cap C \setminus D = \bar{R}$.

Now $\bar{R}$ and $\bar{Q} \cap (C \setminus D)$ are fixed therefore $\mathrm{Th}^n(C \setminus D; \bar{R}, \bar{B}^*_j, \bar{Q})$ is constant for unboundedly many $j < \omega$. Call this theory t^*.

To determine if $C \models \text{Atom}(\bar{A}, \bar{B}^*_j, \bar{Q})$ one needs to ask if $\mathrm{Th}^n(D; \bar{A}, \bar{B}^*_j, \bar{Q})$ is such that $\mathrm{Th}^n(D; \bar{A}, \bar{B}^*_j, \bar{Q}) + t^* = t$. For unboundedly many k_j's a positive answer to the second question implies a positive answer to the first one.

The same holds for the other formulas in the interpretation and as we have only 7 of them we can interpret T_{k_j} in D, for unboundedly many k_j's. This is done using $\bar{B}_j^* \cap D$ and $\bar{Q} \cap D$ as parameters and in fact for different k_j's the interpretations differ only in the first set of parameters.

Note also that for every proper initial segment $D' \subset D$ the bouquet size of D' (with respect to the interpretation of T_k in D) is at most N_1. This is because, putting it roughly, being an element with respect to the interpretation of T_k is stronger than being an element with respect to the interpretation of T. Therefore (for $k > N_1$) D is the minimal non-minor initial segment in D.

Summing up, we have proved:

THEOREM 3.12. *If there is an interpretation of T in the monadic theory of a chain C then, there is a chain D such that* $\mathrm{cf}(D) > \omega$, *and such that for unboundedly many $k < \omega$ there is an interpretation of T_k in the monadic theory of D. The interpretations do not "concentrate" on any proper initial segment of D (i.e., D itself is the minimal non-minor initial segment of D). Furthermore, the interpretations differ only in the set of parameters and in particular there is an $n < \omega$ which does not depend on k, such that all the interpreting formulas have depth $\leq n$.*

§4. Preservation of theories under shufflings.

We will define here shufflings of subchains and show that the partial theories defined in §2 are preserved under them. In fact what we really shuffle are sequences of partial theories, this is a key observation in passing from well ordered chains to general chains. We will elaborate on that later.

CONVENTION. Unless otherwise said, all the chains mentioned in this section are well ordered (i.e., ordinals). Throughout this section, δ will denote an ordinal, $\lambda > \aleph_0$ a regular cardinal and usually $\lambda = \mathrm{cf}(\delta)$.

DEFINITION 4.1. (1) Let $a \subseteq \lambda$. We say that a is a *semi-club subset of* λ if for every $\alpha < \lambda$ with $\mathrm{cf}(\alpha) > \omega$:

(a) if $\alpha \in a$ then there is a club subset of α, C_α such that $C_\alpha \subseteq a$, and

(b) if $\alpha \notin a$ then there is a club subset of α, C_α such that $C_\alpha \cap a = \emptyset$.

Note that λ and $\emptyset$ are semi-clubs and that a club $J \subseteq \lambda$ is a semi-club provided that the first and the successor points of J are of cofinality $\leq \omega$.

(2) Let $X, Y \subseteq \delta$, $J = \{\alpha_i : i < \lambda\}$ a club subset of δ, and let $a \subseteq \lambda$ be a semi-club of λ. We will define the *shuffling of X and Y with respect to a and J*, denoted by $[X, Y]_a^J$, as:

$$[X, Y]_a^J = \bigcup_{i \in a} \left(X \cap [\alpha_i, \alpha_{i+1})\right) \cup \bigcup_{i \notin a} \left(Y \cap [\alpha_i, \alpha_{i+1})\right).$$

(3) When J is fixed (which is usually the case), we will denote the shuffling of X and Y with respect to a and J, by $[X, Y]_a$.

(4) When $\bar{X}, \bar{Y} \subseteq \delta$ are of the same length, we define $[\bar{X}, \bar{Y}]_a$ naturally.

(5) We can define shufflings naturally when $J \subseteq \delta$ is a club, and $a \subseteq \mathrm{otp}(J)$ is a semi-club.

NOTATION 4.2. (1) Let $\bar{P}_i \subseteq \delta$ and $J \subseteq \delta$ a club subset of δ witnessing $\mathrm{ATh}(\delta, \bar{P}_i)$ as in Lemma 2.9. For $n < \omega$, and $\beta < \gamma$ with $\gamma \in J$, $\mathrm{cf}(\gamma) = \omega$, we denote

$\mathrm{Th}^n(\delta;\bar P_i)\restriction_{[\beta,\gamma)}=\mathrm{ATh}^n\left(\beta,(\delta;\bar P_i)\right)$ by $s^n_{\bar P_0}(\beta)$. (Of course, this does not depend on the choice of J and γ.)

(2) When n is fixed and using the parameters $\bar P_0,\bar P_1$, we will just write $s_0(\beta)$ and $s_1(\beta)$ instead of $s^n_{\bar P_0}(\beta)$ and $s^n_{\bar P_1}(\beta)$.

(3) Recall (2.13), $g^n(\bar P_0)_s$ is the set $\{\beta<\delta : s^n_{\bar P_0}(\beta)=s\}$.

(4) S^δ_0 is the set $\{\gamma<\delta : \mathrm{cf}(\gamma)=\omega\}$.

DEFINITION 4.3. Let $\bar P_0,\bar P_1\subseteq\delta$ be of the same length and $J\subseteq\delta$ be a club. We will say that J is *n-suitable for* $\bar P_0,\bar P_1$ if the following hold:

(a) J witnesses $\mathrm{ATh}(\delta;\bar P_\ell)$ for $\ell=0,1$,

(b) $J=\{\alpha_i : i<\lambda\}$, $\alpha_0=0$ and $\mathrm{cf}(\alpha_{i+1})=\omega$,

(c) for every theory s, $J\cap g^n(\bar P_\ell)_s\cap S^\delta_0$ is either a stationary subset of δ or is empty.

When $n\geq 1$ and $\mathrm{WA}^n(\delta;\bar P_0)=\mathrm{WA}^n(\delta;\bar P_1)$ (see Notation 2.16) we require also that:

(d) If $\alpha_j\in J$, $\mathrm{cf}(\alpha_j)\leq\omega$ and $s_\ell(\alpha_j)=s$ then there are $k_1,k_2<\omega$ such that $s_\ell(\alpha_{j+k_1})=s$, and $s_{1-\ell}(\alpha_{j+k_2})=s$.

REMARK 4.4. It is easy to see that for every finite sequence $\langle\bar P_0,\bar P_1,\dots,\bar P_k\rangle\subseteq\delta$ with equal lengths, there is a club $J\subseteq\delta$ which is n-suitable for every pair of the $\bar P_i$'s.

We will show now that ATh is preserved under 'suitable' shufflings.

THEOREM 4.5. *Suppose that $\bar P_0,\bar P_1\subseteq\delta$ are of the same length, $n\geq 1$ and $\mathrm{WA}^n(\delta;\bar P_0)=\mathrm{WA}^n(\delta;\bar P_1)$. (In particular, $\mathrm{ATh}^n(0,(\delta;\bar P_0))=\mathrm{ATh}^n(0,(\delta;\bar P_1)):=t$.)*

Let $J\subseteq\delta$ be an n-suitable club for $\bar P_0,\bar P_1$ of order type $\lambda=\mathrm{cf}(\delta)$ and $a\subseteq\lambda$ a semi-club.

Then, $\mathrm{ATh}^n(0,(\delta;[\bar P_0,\bar P_1]^J_a))=t$.

PROOF. Denote $\bar X:=[\bar P_0,\bar P_1]^J_a$. We will prove the following facts by induction on $0<j<\lambda$:

$(*)$ For every $i<j<\lambda$ with $\mathrm{cf}(j)\leq\omega$:

$$i\in a\Rightarrow \mathrm{Th}^n([\alpha_i,\alpha_j);\bar X)=\mathrm{Th}^n([\alpha_i,\alpha_j);\bar P_0)=s_0(\alpha_i),$$
$$i\notin a\Rightarrow \mathrm{Th}^n([\alpha_i,\alpha_j);\bar X)=\mathrm{Th}^n([\alpha_i,\alpha_j);\bar P_1)=s_1(\alpha_i).$$

$(**)$ For every $i<j<\lambda$ with $\mathrm{cf}(j)>\omega$:

$$i,j\in a\Rightarrow \mathrm{Th}^n([\alpha_i,\alpha_j);\bar X)=\mathrm{Th}^n([\alpha_i,\alpha_j);\bar P_0),$$
$$i,j\notin a\Rightarrow \mathrm{Th}^n([\alpha_i,\alpha_j);\bar X)=\mathrm{Th}^n([\alpha_i,\alpha_j);\bar P_1).$$

In particular, by choosing $i=0$ we get (remember $\alpha_0=0$), $\mathrm{Th}^n([0,\alpha_j);\bar X)=t$ whenever $\mathrm{cf}(\alpha_j)=\omega$.

$\underline{j=1}$ (so $i=0$): Let $\ell=0$ if $i\in a$ and $\ell=1$ if $i\notin a$. So $\bar X\cap[0,\alpha_j)=\bar P_\ell\cap[0,\alpha_j)$ and so $\mathrm{Th}^n([0,\alpha_j);\bar X)=\mathrm{Th}^n([0,\alpha_j);\bar P_\ell)=t$

$\underline{j=k+1<\omega}$: There are 4 cases. Let us check for example the case $i\in a$, $k\notin a$. By the composition theorem and the induction hypothesis we have:

$$\begin{aligned}\mathrm{Th}^n([\alpha_i,\alpha_j);\bar X)&=\mathrm{Th}^n([\alpha_i,\alpha_k);\bar X)+\mathrm{Th}^n([\alpha_k,\alpha_{k+1});\bar X)\\&=s_0(\alpha_i)+\mathrm{Th}^n([\alpha_k,\alpha_j);\bar P_1)=s_0(\alpha_i)+s_1(\alpha_k).\end{aligned}$$

So we have to prove $s_0(\alpha_i) + s_1(\alpha_k) = s_0(\alpha_i)$.

Since J is n-suitable there is an $m < \omega$ such that $s_0(\alpha_{i+m}) = s_1(\alpha_k)$ hence,

$$\begin{aligned} s_0(\alpha_i) &= \mathrm{Th}^n([\alpha_i, \alpha_{i+m+1}); \bar{P}_0) \\ &= \mathrm{Th}^n([\alpha_i, \alpha_{i+m}); \bar{P}_0) + \mathrm{Th}^n([\alpha_{i+m}, \alpha_{i+m+1}); \bar{P}_0) \\ &= s_0(\alpha_i) + s_0(\alpha_{i+m}) = s_0(\alpha_i) + s_1(\alpha_k). \end{aligned}$$

So $s_0(\alpha_i) + s_1(\alpha_k) = s_0(\alpha_i)$ as required. The other cases are proven similarly.

$\underline{j = \omega}$: Suppose $i < \omega$, $i \in a$. We have to prove that $\mathrm{Th}^n([\alpha_i, \alpha_\omega); \bar{X}) = s_0(i)$. Now either $(\lambda \setminus a) \cap \omega$ is unbounded or $a \cap \omega$ is unbounded and suppose the first case holds. Let $i < i_0 < i_1 \dots$ be a strictly increasing sequence in $(\lambda \setminus a) \cap \omega$. By the induction hypothesis we have:

$$\begin{aligned} \mathrm{Th}^n([\alpha_i, \alpha_\omega); \bar{X}) &= \mathrm{Th}^n([\alpha_i, \alpha_{i_1}); \bar{X}) + \sum_{0<m<\omega} \mathrm{Th}^n([\alpha_{i_m}, \alpha_{i_{m+1}}); \bar{X}) \\ &= s_0(\alpha_i) + \sum_{0<m<\omega} s_1(\alpha_{i_m}). \end{aligned}$$

Now choose (using the suitability of J), a strictly increasing sequence $\beta_{i_0} < \beta_{i_1} \cdots \subseteq \lambda$ such that $\beta_{i_m} = \alpha_{j_m+1}$ for some $j_m < \omega$, $\beta_{i_1} > \alpha_i$ and for every $0 < m < \omega$, $s_0(\beta_{i_m}) = s_1(\alpha_{i_m})$. We will get:

$$\begin{aligned} s_0(\alpha_i) = \mathrm{Th}^n([\alpha_i, \alpha_\omega); \bar{P}_0) &= \mathrm{Th}^n([\alpha_i, \beta_{i_1}); \bar{P}_0) + \sum_{0<m<\omega} \mathrm{Th}^n([\beta_{i_m}, \beta_{i_{m+1}}); \bar{P}_0) \\ &= s_0(\alpha_i) + \sum_{0<m<\omega} s_0(\beta_{i_m}) = s_0(\alpha_i) + \sum_{0<m<\omega} s_1(\alpha_{i_m}). \end{aligned}$$

So we have $s_0(\alpha_i) = \mathrm{Th}^n([\alpha_i, \alpha_\omega); \bar{X})$ as required. When only the other case holds (i.e., only $a \cap \omega$ is unbounded) the proof is easier. When $i \notin a$ we prove similarly that $\mathrm{Th}^n([\alpha_i, \alpha_\omega); \bar{X}) = s_1(\alpha_i)$.

$\underline{\mathrm{cf}(j) = \omega}$: Choose a sequence (in a or $\lambda \setminus a$), $i < i_0 < i_1 \dots$ with $\mathrm{Sup}_m\, i_m = j$, i_m non limit, and continue as in the case $j = \omega$.

$\underline{\mathrm{cf}(j) > \omega}$: Now we have to check $(**)$. Suppose $i, j \in a$ and we have to show $\mathrm{Th}^n([\alpha_i, \alpha_j); \bar{X}) = \mathrm{Th}^n([\alpha_i, \alpha_j); \bar{P}_0)$.

Let $\{\beta_\gamma : \gamma < \mathrm{cf}(j)\} \subseteq a$ be a club subset of $j \cap J$ with $\beta_0 = i$ and with $\mathrm{cf}(\beta_{\gamma+1}) = \omega$. By the induction hypothesis we have as required:

$$\begin{aligned} \mathrm{Th}^n([\alpha_i, \alpha_j); \bar{X}) &= \sum_{\gamma<\mathrm{cf}(j)} \mathrm{Th}^n([\beta_\gamma, \beta_{\gamma+1}); \bar{X}) = \sum_{\gamma<\mathrm{cf}(j)} s_0(\beta_\gamma) \\ &= \sum_{\gamma<\mathrm{cf}(j)} \mathrm{Th}^n([\beta_\gamma, \beta_{\gamma+1}), \bar{P}_0) = \mathrm{Th}^n([\alpha_i, \alpha_j), \bar{P}_0). \end{aligned}$$

The case $i, j \notin a$ is similar.

$\underline{j = k + 2}$: Easy.

$\underline{j = k + 1, \mathrm{cf}(k) = \omega}$: Easy.

$\underline{j = k + 1, \mathrm{cf}(k) > \omega}$: There are 4 cases. We will check for example the case: $i \in a, k \notin a$.

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Choose $\{i_\gamma : \gamma < \mathrm{cf}(k)\} \subseteq \lambda \backslash a$ a club of $k \cap J$ such that $i < i_0, \mathrm{cf}(\alpha_{i_0}), \mathrm{cf}(\lambda_{i_{\gamma+1}}) = \omega$. Using $(**)$ we get

$$\begin{aligned}\mathrm{Th}^n([\alpha_{i_0}, \alpha_j); \bar{X}) &= \sum_{\gamma<\mathrm{cf}(k)} \mathrm{Th}^n([\alpha_{i_\gamma}, \alpha_{i_{\gamma+1}}); \bar{X}) + \mathrm{Th}^n([\alpha_k, \alpha_j); \bar{X}) \\ &= \sum_{\gamma<\mathrm{cf}(k)} \mathrm{Th}^n([\alpha_{i_\gamma}, \alpha_{i_{\gamma+1}}); \bar{P}_1) + \mathrm{Th}^n([\alpha_k, \alpha_j); \bar{P}_1) = s_1(\alpha_{i_0}).\end{aligned}$$

Therefore $\mathrm{Th}^n([\alpha_i, \alpha_j); \bar{X}) = s_0(\alpha_i) + s_1(\alpha_{i_0})$, and all there is left is to show that

$$s_0(\alpha_i) + s_1(\alpha_{i_0}) = s_0(\alpha_i).$$

This follows from clause (d) in Definition 4.3: for some $m < \omega$ we have $s_1(\alpha_{i_0}) = s_0(\alpha_{i+m})$ so $s_0(\alpha_i) + s_1(\alpha_{i_0}) = s_0(\alpha_i) + s_0(\alpha_{i+m}) = s_0(\alpha_i)$ as required. The other cases are similar.

So we have gone through all the cases and proven $(*)$ and $(**)$. ⊣

Conclusion 4.6. Let $\bar{P}_0, \bar{P}_1 \subseteq \delta$, $WA^n(\delta; \bar{P}_0) = WA^n(\delta; \bar{P}_1)$ and let $J \subseteq \delta$ be an n-suitable club for $\bar{P}_0$, $\bar{P}_1$. Assume that $a \subseteq \lambda$ is a semi-club that is generic enough i.e., that for $\ell \in \{0, 1\}$ if $g^n(\bar{P}_\ell)_s \cap S_0^\delta$ is stationary then its intersection with a is stationary and co-stationary. Then:

(1) J is an n-suitable club subset for the pair $\bar{P}_0$, $[\bar{P}_0, \bar{P}_1]_a^J$ (except that maybe k_1 and k_2 from clause (d) in Definition 4.3 are guaranteed only to be $< \delta$).

(2) $[g^n(\bar{P}_0), g^n(\bar{P}_1)]_a^J \cap J = g^n([\bar{P}_0, \bar{P}_1]_a^J) \cap J$ (so $[g^n(\bar{P}_0), g^n(\bar{P}_1)]_a^J$ and $g^n([\bar{P}_0, \bar{P}_1]_a^J)$ have the same WTh^n).

Proof. Use $(*), (**)$ from the last theorem. ⊣

Our next aim is to show that WTh (hence, by 2.15 and 4.6(2), also Th) is preserved under shufflings.

Definition 4.7. Let $a, \bar{P} \subseteq \lambda$. We define a-$\mathrm{WTh}^n(\lambda; \bar{P})$ by induction on n:
for $n = 0$: a-$\mathrm{WTh}^0(\lambda; \bar{P}) = \{t : th(\lambda; (\bar{P}, a))$ is stationary in $\lambda\}$ (see Def. 2.11)
for $n+1$: a-$\mathrm{WTh}^{n+1}(\lambda; \bar{P}) = \{\langle S_1^{\bar{P},a}(Q), S_2^{\bar{P},a}(Q), S_3^{\bar{P},a}(Q)\rangle : Q \subseteq \lambda\}$ Where:

$$\begin{aligned} S_1^{\bar{P},a}(Q) &= a\text{-}\mathrm{WTh}^n(\lambda; \bar{P}, Q) \\ S_2^{\bar{P},a}(Q) &= \left\{\begin{matrix}\langle t, s\rangle : \{\beta \in a : \mathrm{WTh}^n(\lambda; \bar{P}, Q) \restriction_\beta = t, \\ \mathrm{th}(\beta; \bar{P}, Q) = s\} \text{ is stationary in } \lambda\end{matrix}\right\} \\ S_3^{\bar{P},a}(Q) &= \left\{\begin{matrix}\langle t, s\rangle : \{\beta \in \lambda\backslash a : \mathrm{WTh}^n(\lambda; \bar{P}, Q) \restriction_\beta = t, \\ \mathrm{th}(\beta; \bar{P}, Q) = s\} \text{ is stationary in } \lambda\end{matrix}\right\}. \end{aligned}$$

Remark 4.8. (0) Remember that if $\bar{P} \subseteq \delta$ and $J \subseteq \delta$ is a club, then $\mathrm{WTh}^n(\delta; \bar{P} \cap J) = \mathrm{WTh}^n(\delta; \bar{P})$. Moreover, if $J \subseteq \delta$ club of order type λ and $h \colon J \to \lambda$ is the isomorphism between J and λ, then for every $\bar{P} \subseteq \delta$, $\mathrm{WTh}^n(\delta; \bar{P}) = \mathrm{WTh}^n(\lambda; h(\bar{P} \cap J))$.

(1) $\mathrm{WTh}^n(\lambda; \bar{P})$ tells us if certain sets are stationary. a-$\mathrm{WTh}^n(\lambda; \bar{P})$ tells us if their intersections with a and $\lambda \setminus a$ are stationary.

(2) We could have defined a-$\mathrm{WTh}^n(\lambda; \bar{P})$ by $\mathrm{WTh}^n(\lambda; \bar{P}, a)$, which gives us the same information. We preferred the original definition because it seems to be more friendly in proving the preservation under shufflings.

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FACT 4.9. (1) $\mathrm{WTh}^n(\lambda;\bar{P})$ is effectively computable from a-$\mathrm{WTh}^n(\lambda;\bar{P})$, so if $\bar{P},\bar{Q}\subseteq\lambda$ and a-$\mathrm{WTh}^n(\lambda;\bar{P}) = a$-$\mathrm{WTh}^n(\lambda;\bar{Q})$ then $\mathrm{WTh}^n(\lambda;\bar{P}) = \mathrm{WTh}^n(\lambda;\bar{Q})$.

(2) From a-$\mathrm{WTh}^n(\lambda;\bar{P}\cap a)$ and a-$\mathrm{WTh}^n(\lambda;\bar{P}\cap(\lambda\setminus a))$ we can effectively compute a-$\mathrm{WTh}^n(\lambda;\bar{P})$.

PROOF. (1) is trivial. We prove (2) by induction on n:

$n = 0$: To compute a-$\mathrm{WTh}^0(\lambda;\bar{P})$ all we need to know is which Boolean combinations of the elements of $\bar{P}$, a and $\lambda\setminus a$ are stationary. This clearly can be computed from a-$\mathrm{WTh}^0(\lambda;\bar{P}\cap a)$ and a-$\mathrm{WTh}^0(\lambda;\bar{P}\cap(\lambda\setminus a))$ because a subset of λ is stationary if and only if it is either stationary on a or on $\lambda\setminus a$.

$n+1$: We need to compute the set of possibilities $\langle S_1^{\bar{P},a}(Q), S_2^{\bar{P},a}(Q), S_3^{\bar{P},a}(Q)\rangle$ for $Q\subseteq\lambda$. For $S_2^{\bar{P},a}(Q)$ all we need to know is $\bar{Q}\cap a$ and for $S_3^{\bar{P},a}(Q)$ all we need to know is $\bar{Q}\cap\lambda\setminus a$.

Now from a-$\mathrm{WTh}^{n+1}(\lambda;\bar{P}\cap a)$ we can compute

$$\mathcal{T}_1 = \{\langle S_1^{\bar{P}\cap a,a}(Q), S_2^{\bar{P}\cap a,a}(Q)\rangle : Q\cap(\lambda\setminus a)\text{ nonstationary}\}$$

and from a-$\mathrm{WTh}^{n+1}(\lambda;\bar{P}\cap(\lambda\setminus a))$ we can compute

$$\mathcal{T}_2 = \{\langle S_1^{\bar{P}\cap(\lambda\setminus a),a}(Q), S_3^{\bar{P}\cap(\lambda\setminus a),a}(Q)\rangle : Q\cap a\text{ nonstationary}\}.$$

Let f_n be the recursive function (given by the induction hypothesis), that computes a-$\mathrm{WTh}^n(\lambda;\bar{A},B)$ from a-$\mathrm{WTh}^n(\lambda;\bar{A}\cap a, B\cap a)$ and a-$\mathrm{WTh}^n(\lambda;\bar{A}\cap(\lambda\setminus a), B\cap(\lambda\setminus a))$. Then $\langle S_1^{\bar{P},a}(Q), S_2^{\bar{P},a}(Q), S_3^{\bar{P},a}(Q)\rangle\in a$-$\mathrm{WTh}^{n+1}(\lambda;\bar{P})$ if and only if there are $\langle t_1,s_1\rangle\in\mathcal{T}_1$ and $\langle t_2,s_2\rangle\in\mathcal{T}_2$ such that $S_2^{\bar{P},a}(Q)=s_1$, $S_3^{\bar{P},a}(Q)=s_2$ and $S_1^{\bar{P},a}(Q)$ is $f_n(t_1,t_2)$. ⊣

THEOREM 4.10. *Suppose $a, J, \bar{P}_0, \bar{P}_1\subseteq\lambda$, a a semi-club, J a club, $\bar{X} := [\bar{P}_0,\bar{P}_1]_a^J$ and a-$\mathrm{WTh}^n(\lambda;\bar{P}_0) = a$-$\mathrm{WTh}^n(\lambda;\bar{P}_1)$.*

Then a-$\mathrm{WTh}^n(\lambda;\bar{P}_0) = a$-$\mathrm{WTh}^n(\lambda;\bar{X})$. (Hence $\mathrm{WTh}^n(\lambda;\bar{P}_0) = \mathrm{WTh}^n(\lambda;\bar{X}) = \mathrm{WTh}^n(\lambda;\bar{P}_1)$.)

PROOF BY INDUCTION ON n (FOR EVERY $a', J', \bar{X}', \bar{Y}'$). $n = 0$: Check.

$n+1$: Suppose $Q_0\subseteq\lambda$ and (by the equality of the theories) let $Q_1\subseteq\lambda$ satisfy

$$\langle S_1^{\bar{P}_0,a}(Q_0), S_2^{\bar{P}_0,a}(Q_0), S_3^{\bar{P}_0,a}(Q_0)\rangle = \langle S_1^{\bar{P}_1,a}(Q_1), S_2^{\bar{P}_1,a}(Q_1), S_3^{\bar{P}_1,a}(Q_1)\rangle.$$

Define $Q_X := [Q_0,Q_1]_a^J$. Now a-$\mathrm{WTh}^n(\lambda;\bar{P}_0,Q_0) = a$-$\mathrm{WTh}^n(\lambda;\bar{X},Q_X)$ by the induction hypothesis so $S_1^{\bar{P}_0,a}(Q_0) = S_1^{\bar{X},a}(Q_X)$.

Now suppose $\langle t,s\rangle\in S_2^{\bar{P}_0,a}(Q_0)$, $t\neq\emptyset$. Let

$$B_{t,s}^{\bar{P}_0} := \{\beta\in a : \mathrm{WTh}^n(\lambda;\bar{P}_0,Q_0)\restriction_\beta = t,\ th(\beta;\bar{P}_0,Q_0) = s\}$$

and this is a stationary subset of λ. For each such β, since $t\neq\emptyset\Rightarrow\mathrm{cf}(\beta)>\omega$, a contains a club $C_\beta\subseteq\beta$ and, remembering a previous remark, we can restrict ourselves to $(\bar{P}_0,Q_0)\cap C_\beta$.

Suppose further that $a=\langle i_\gamma:\gamma<\lambda\rangle$ (note that a has to be stationary otherwise S_2 is empty) and $J=\langle\alpha_\gamma:\gamma<\lambda\rangle$. Look at the club $J'=\langle\alpha_\gamma:\alpha_\gamma=\gamma\rangle$ and let $J''=$ the accumulation points of J'. Now $B_{t,s}^{\bar{P}_0}\cap J''$ is also stationary, and choose

β in this set, and a club $C_\beta \subseteq a \cap J'$. By the choice of C_β we get $(\bar{P}_0, Q_0) \cap C_\beta = (\bar{X}, Q_X) \cap C_\beta$, and this implies: $\mathrm{WTh}^n(\lambda; \bar{X}, Q_X) \restriction_\beta = t$, and $th(\beta; \bar{X}, Q_X) = s$. So, (since β was random) $B^{\bar{X}}_{t,s}$ is also stationary.

The case $t = \emptyset$ is left to the reader. We deal with S_3 symmetrically, replacing a with $\lambda \setminus a$.

So we have proved that $a\text{-WTh}^{n+1}(\lambda; \bar{P}_0) \subseteq a\text{-WTh}^{n+1}(\lambda; \bar{X})$.

Now, for the inverse inclusion suppose $Q_X \subseteq \lambda$ and $\langle S_1^{\bar{X},a}(Q_X), S_2^{\bar{X},a}(Q_X), S_3^{\bar{X},a}(Q_X)\rangle$ is in $a\text{-WTh}^{n+1}(\lambda; \bar{X})$. Let $R_0 = Q_X \cap a$ and $R_1 = Q_X \cap (\lambda \setminus a)$.

Now choose T_0 (w.l.o.g $\subseteq \lambda \setminus a$) such that

$$\langle S_1^{\bar{P}_0,a}(T_0), S_2^{\bar{P}_0,a}(T_0), S_3^{\bar{P}_0,a}(T_0)\rangle = \langle S_1^{\bar{P}_1,a}(R_1), S_2^{\bar{P}_1,a}(R_1), S_3^{\bar{P}_1,a}(R_1)\rangle$$

and choose T_1 (w.l.o.g $\subseteq a$) such that

$$\langle S_1^{\bar{P}_0,a}(T_1), S_2^{\bar{P}_0,a}(T_1), S_3^{\bar{P}_0,a}(T_1)\rangle = \langle S_1^{\bar{P}_1,a}(R_0), S_2^{\bar{P}_1,a}(R_0), S_3^{\bar{P}_1,a}(R_0)\rangle.$$

Let Q_0 be equal to R_0 on a and to T_0 on $\lambda \setminus a$, let Q_1 be equal to T_1 on a and to R_1 on $\lambda \setminus a$.

By Claim 4.9(2) it can be easily checked that

$$\langle S_1^{\bar{P}_0,a}(Q_0), S_2^{\bar{P}_0,a}(Q_0), S_3^{\bar{P}_0,a}(Q_0)\rangle = \langle S_1^{\bar{P}_1,a}(Q_1), S_2^{\bar{P}_1,a}(Q_1), S_3^{\bar{P}_1,a}(Q_1)\rangle.$$

But $Q_X = [Q_0, Q_1]^J_a$ on a club of λ hence these are, by the same arguments as in the first part of the proof, equal to $\langle S_1^{\bar{X},a}(Q_X), S_2^{\bar{X},a}(Q_X), S_3^{\bar{X},a}(Q_X)\rangle$.

This proves the inverse inclusion: $a\text{-WTh}^{n+1}(\lambda; \bar{P}_0) \supseteq a\text{-WTh}^{n+1}(\lambda; \bar{X})$, hence the equality $a\text{-WTh}^{n+1}(\lambda; \bar{P}_0) = a\text{-WTh}^{n+1}(\lambda; \bar{P}_1) = a\text{-WTh}^{n+1}(\lambda; \bar{X})$. $\dashv$

NOTATION 4.11. Suppose $\bar{P}, J \subseteq \delta$, J a club of order type λ and $a \subseteq \lambda$ a semi-club.

Let $t_1 := \mathrm{ATh}^m(0, (\delta; \bar{P}))$ and (keeping in mind Remark 4.8.(0)), let $h\colon J \to \lambda$ be the isomorphism between J and λ and let $t_2 := a\text{-WTh}^m(\lambda; h(g^m(\delta; \bar{P}) \cap J))$.

We denote $\langle t_1, t_2\rangle$ by $a\text{-WA}^m(\delta; \bar{P})$ (assuming J is fixed).

Collecting the last results we can conclude:

THEOREM 4.12. *Let $J, \bar{P}_0, \bar{P}_1 \subseteq \delta$, $\lg(\bar{P}_0) = \lg(\bar{P}_1)$, J an m-suitable club for $\bar{P}_0, \bar{P}_1$ of order type λ and $a \subseteq \lambda$ a semi-club and set $\bar{X} := [\bar{P}_0, \bar{P}_1]^J_a$.*

Then: $a\text{-WA}^m(\delta; \bar{P}_0) = a\text{-WA}^m(\delta; \bar{P}_1) \Rightarrow a\text{-WA}^m(\delta; \bar{P}_0) = a\text{-WA}^m(\delta; \bar{X})$, and in particular, if $m = m(n, \ell)$ then: $\mathrm{Th}^n(\delta; \bar{P}_0) = \mathrm{Th}^n(\delta, \bar{P}_1) = \mathrm{Th}^n(\delta; \bar{X})$.

PROOF. The first statement follows directly from Theorm 4.5, Conclusion 4.6(2) and Theorem 4.10. For the second, by the definition of a-WA, and by Remark 4.8(0), Fact 4.9(1), the equality of $a\text{-WA}^{m(n,\ell)}$ implies equality of $\mathrm{WA}^{m(n,\ell)}$ from Definition 2.16. But by Theorem 2.15 this implies the equality of Th^n. $\dashv$

§5. Formal shufflings. The purpose of this section is to overcome two difficulties:

1. We want to generalize the definitions and results of §2 which apply to well ordered chains, to the case of a general chain of uncountable cofinality.

2. It could happen that the interpreting chain is of cofinality λ but of a larger cardinality. Still, we want to shuffle objects of cardinality $\leq \lambda$. The reason is that the

contradiction we want to reach depends on shufflings of elements along a generic semi-club added by the forcing, and a semi-club of cardinality λ will be generic only with respect to objects of cardinality $\leq \lambda$. What we want to show is that we can shuffle sets of partial theories rather than subsets of our given chain.

PROVISO. We are working with a chain C, with a first element c_0 and with cofinality $\lambda > \omega$

Note that in view of §3 these are the chains that interest us, except maybe for the demand of having a first element. However, it is clear that adding a first element to the minimal major initial segment will cost us at most an additional parameter (alternatively a larger depth of the interpreting formulas) and Theorem 3.12 still holds.

The first task is to generalize the definitions of ATh and WTh.

NOTATION 5.1. Let C be a chain as above. Fix a partition $C = \sum_{i<\lambda} C_i$ and denote by $C_{[\alpha,\beta)}$ the sub-chain $\sum_{\alpha\leq i<\beta} C_i$. ($C_{(\alpha,\beta)}$, $C_{[\alpha,\beta]}$ are defined naturally.) Note that if $C = \sum_{i<\lambda} D_i$ is another partition of C then on a club $J \subseteq \lambda$, if $\alpha < \beta$ are in J then $C_{[\alpha,\beta)} = D_{[\alpha,\beta)}$.

All the clubs that are mentioned below have c_0 as their first element.

LEMMA 5.2 (2.9*). *For every $\bar{A} \subseteq C$ there is a club $J \subseteq \lambda$ such that if $\alpha < \lambda$ is in J then $s(\alpha) = \mathrm{Th}^n(C_{[\alpha,\beta)}; \bar{A})$ does not depend on the particular choice of $\beta > \alpha$ of cofinality ω in J.*

PROOF. Replace $[\alpha, \beta)$ with $C_{[\alpha,\beta)}$ in the proof of Lemma 4.1 in [10]. ⊣

NOTATION 5.3 (2.10*). Let C, $\bar{A}$ and J be as above. $\mathrm{ATh}^n(\beta, (C; \bar{A}))$ for $\beta \in J$ is $\mathrm{Th}^n(C_{[\beta,\gamma)}; \bar{A})$ for every $\gamma \in J$ with $\gamma > \beta$, $\mathrm{cf}(\gamma) = \omega$.

(The choice of J and $\gamma \in J$ is as usual immaterial.)

DEFINITION 5.4 (2.13*). Let C, λ and $\bar{A} \subseteq C$ with $\lg(\bar{A}) = \ell$ be as above. Then
(1) $g^n(\bar{A})_s$ is (for $s \in T_{n,\ell}$) the set $\{\alpha < \lambda : \mathrm{ATh}^n(\alpha, (C; \bar{A})) = s\}$ and
(2) $g^n(\bar{A}) := \langle \dots, g^n(\bar{A})_s, \dots \rangle_{s \in T_{n,\ell}}$.

THEOREM 5.5 (2.15*). *For every $n, \ell \in \mathbb{N}$ there is an $m = m(n, \ell)$, effectively computable from n and ℓ, such that whenever C is as above, $\bar{A} \subseteq C$ and $\lg(\bar{A}) = \ell$, if*

$$t_1 = \mathrm{WTh}^m\,(\lambda; g^m(\bar{A})), \quad t_2 = \mathrm{ATh}^m(0, (C; \bar{A}))$$

then we can effectively compute $\mathrm{Th}^n(C; \bar{A})$ from $\langle t_1, t_2 \rangle$.

PROOF. Immediate from the proof of Theorem 4.4 in [10]. Note that t_1 and t_2 do not depend on the partition of C nor on the choice of the ATh-club J. ⊣

NOTATION 5.6 (2.16*). $\langle t_1, t_2 \rangle$ from above is denoted by $\mathrm{WA}^m(C; \bar{A})$.

In the second part of the section we formulate the results of §4 in terms of shuffling λ-sequences of theories. Given a chain $C = \sum_{i<\lambda} C_i$ and $\bar{A} \subseteq C$ we set $s_i := \mathrm{Th}^n(C_i; \bar{A})$. Clearly, by the composition theorem, $\mathrm{Th}^n(C; \bar{A}) = \sum_{i<\lambda} s_i$ and the sequence $\langle s_i : i < \lambda \rangle$ is all the information that we need.

Moreover, letting $H(\lambda^+) := \{x : x \text{ is hereditarily of cardinality smaller than } \lambda^+\}$, the equation $\sum_{i<\lambda} s_i = t$ can be checked in $H(\lambda^+)$ regardless of the cardinality of C.

In view of Lemma 5.2 we can even choose the partition of C in a way such that for every $i < j \in \lambda$ with $\mathrm{cf}(j) \leq \omega$, $s_i = \sum_{i \leq k < j} s_k$. This motivates our next definitions.

DEFINITION 5.7. (1) $\mathcal{S} = \langle s_i : i < \lambda \rangle$ is an (n, ℓ)*-formally possible set of theories* if each s_i is a member of $T_{n,\ell}$ and for every $i < j < \lambda$ with $\mathrm{cf}(j) \leq \omega$ we have $s_i = \sum_{i \leq k < j} s_k$.

(2) The (n, ℓ)-formally possible set of theories $\mathcal{S}$ is *realized* in a model N if there is a chain $C = \sum_{i<\lambda} C_i$ and $\bar{A} \subseteq C$ of length ℓ such that $\mathrm{Th}^n(C_i; \bar{A}) = s_i$ for every $i < \lambda$.

(3) Let $\mathcal{S} = \langle s_i : i < \lambda \rangle$ and $\mathcal{T} = \langle t_i : i < \lambda \rangle$ be (n, ℓ)-formally possible sets of theories, and $a \subseteq \lambda$ a semi-club. We define the *formal shuffling of $\mathcal{S}$ and $\mathcal{T}$ with respect to a as*: $[\mathcal{S}, \mathcal{T}]_a := \langle u_i : i < \lambda \rangle$ *where*

$$u_i = \begin{cases} s_i & \text{if } i \in a \\ t_i & \text{if } i \notin a. \end{cases}$$

FACT 5.8. Let C be of cofinality λ and $\bar{A}, \bar{B} \subseteq C$ of length ℓ.

(1) There is a partition $C = \sum_{i<\lambda} C_i$ such that letting $s_i = \mathrm{Th}^n(C_i; \bar{A})$, $\mathcal{S} = \langle s_i : i < \lambda \rangle$, $t_i = \mathrm{Th}^n(C_i; \bar{B})$, $\mathcal{T} = \langle t_i : i < \lambda \rangle$ we get $\mathcal{S}$ and $\mathcal{T}$ are (n, ℓ)-formally possible sets of theories. Moreover, for any semi-club $a \subseteq \lambda$ $[\mathcal{S}, \mathcal{T}]_a = \langle \mathrm{Th}^n(C_i, [\bar{A}, \bar{B}]_a) : i < \lambda \rangle$. (Here $[\bar{A}, \bar{B}]_a \cap C_i$ is $\bar{A} \cap C_i$ if $i \in a$ and $\bar{B} \cap C_i$ if $i \notin a$.)

(2) If in addition $\mathrm{WA}^{m(n,\ell)}(C; \bar{A}) = \mathrm{WA}^{m(n,\ell)}(C; \bar{B})$, then we can choose a partition such that $[\mathcal{S}, \mathcal{T}]_a$ is an (n, ℓ)-formally possible set of theories.

(3) If in addition $a\text{-}\mathrm{WA}^{m(n,\ell)}(C; \bar{A}) = a\text{-}\mathrm{WA}^{m(n,\ell)}(C; \bar{B})$, then we can choose a partition such that $\sum_{i<\lambda} s_i = \sum_{i<\lambda} t_i = \sum_{i<\lambda} u_i$.

($a\text{-}\mathrm{WA}^{m(n,\ell)}(C; \bar{A})$ is $\langle a\text{-}\mathrm{WTh}^{m(n,\ell)}\left(\lambda; g^m(\bar{A})\right), \mathrm{ATh}^{m(n,\ell)}(0, (C; \bar{A}))\rangle$.)

(4) Given a finite sequence $\langle \bar{A}_0, \dots, \bar{A}_{k-1} \rangle$ of sequences of length ℓ, we can choose a partition such that the above properties hold for each pair.

PROOF. Part 1 is obvious, part 2 follows from the proof of Theorem 4.5 (choosing an "n-suitable partition"), part 3 follows from Theorem 4.12, and part 4 is Remark 4.4. ⊣

We can define in a natural way the partial theories WTh^m and $a\text{-}\mathrm{WTh}^m$ of an (n, ℓ)-formally possible set of theories.

DEFINITION 5.9. For $\mathcal{S} = \langle s_i : i < \lambda \rangle$ an (m, ℓ)-formally possible set of theories, denote $g^m(\mathcal{S})_s := \langle i < \lambda : s_i = s \rangle$ and $g^m(\mathcal{S}) := \langle g^m(S)_s : s \in T_{m,\ell} \rangle$. We define $\mathrm{WTh}^m(\mathcal{S})$ to be $\mathrm{WTh}^m(\lambda; g^m(\mathcal{S}))$, and for $a \subseteq \lambda$ a semi-club, $a\text{-}\mathrm{WTh}^m(\mathcal{S})$ is $a\text{-}\mathrm{WTh}^m(\lambda; g^m(\mathcal{S}))$.

Finally we define $a\text{-}\mathrm{WA}^m(\mathcal{S})$ to be the pair $\langle s_0, a\text{-}\mathrm{WTh}^m(\mathcal{S}) \rangle$.

THEOREM 5.10. *If $C = \sum_{i<\lambda} C_i$, (where the partition is as in Fact 5.8(1)) $\bar{A} \subseteq C$ and $\mathcal{S} = \langle \mathrm{Th}^{m(n,\ell)}(C_i; \bar{A}) \rangle$ are given then we can compute $\mathrm{Th}^n(C; \bar{A})$ from $\mathrm{WA}^{m(n,\ell)}(\mathcal{S})$. Moreover, the computation can be done in $H(\lambda^+)$ even if $|C| > \lambda$.*

PROOF. The first claim is exactly Theorem 2.15. The second is trivial. ⊣

§6. The forcing. To contradict the existence of an interpretation we will need generic semi-clubs in every regular cardinal. To obtain that we use a simple class forcing.

CONTEXT. $V \models$ G.C.H.

DEFINITION 6.1. Let $\lambda > \aleph_0$ be a regular cardinal

(1) $SC_\lambda := \{f : f : \alpha \to \{0,1\},\ \alpha < \lambda,\ \mathrm{cf}(\alpha) \leq \omega\}$ where each f, considered to be a subset of α (or λ), is a semi-club. The order is inclusion. (So SC_λ adds a generic semi-club to λ.)

(2) Q_λ will be an iteration of the forcing SC_λ with length λ^+ and with support $< \lambda$.

(3) $P := \langle P_\mu, \underset{\sim}{Q}_\mu : \mu$ a cardinal $> \aleph_0 \rangle$ where $\underset{\sim}{Q}_\mu$ is forced to be Q_μ if μ is regular, otherwise it is $\emptyset$. The support of P is Easton's: each condition $p \in P$ is a function from the class of cardinals to names of conditions where the class S of cardinals that are matched to non-trivial names is a set. Moreover, when κ is an inaccessible cardinal, $S \cap \kappa$ has cardinality $< \kappa$.

(4) $P_{<\lambda}, P_{>\lambda}, P_{\leq\lambda}$ are defined naturally. For example $P_{<\lambda}$ is $\langle P_\mu, \underset{\sim}{Q}_\mu : \aleph_0 < \mu < \lambda\rangle$.

DISCUSSION 6.2. Assuming G.C.H it is standard to see that Q_λ satisfies the λ^+ chain condition and that Q_λ and $P_{\geq\lambda}$ do not add subsets of λ with cardinality $< \lambda$. Hence, P does not collapse cardinals and does not change cofinalities, so V and V^P have the same regular cardinals.

Moreover, for a regular $\lambda > \aleph_0$ we can split the forcing into 3 parts, $P = P_0 * P_1 * P_2$ where P_0 is $P_{<\lambda}$, P_1 is a P_0-name of the forcing Q_λ and P_2 is a $P_0 * P_1$-name of the forcing $P_{>\lambda}$ such that V^P and $V^{P_0 * P_1}$ have the same $H(\lambda^+)$.

In the next section, when we restrict ourselves to $H(\lambda^+)$ it will suffice to look only in $V^{P_0 * P_1}$.

§7. The contradiction. Collecting the results from the previous sections we will reach a contradiction from the assumption that there is, in V^P, an interpretation of T in the monadic theory of a chain C. For the moment we will assume that the minimal major initial segment D is (isomorphic to) a regular cardinal, later we will dispose of this by using formal shufflings. So we may assume the following:

ASSUMPTIONS.

(1) $C \in V^P$ interprets T by $\langle U_C(\bar{X}, \bar{Q}^*), E_C(\bar{X}, \bar{Y}, \bar{Q}^*), P(\bar{X}, \bar{Y}, \bar{Q}^*)\rangle$, $\bar{Q}^* \subseteq C$, $d = \lg(\bar{X}) = \lg(\bar{Y}) = \lg(\bar{Q}^*)$.

(2) $D = \lambda$ is the minimal major initial segment of C, $\mathrm{cf}(\lambda) = \lambda > \omega$.

(3) $\bar{R} \subseteq (C \setminus D)$ and $S := \{\bar{A} \subseteq C : \bar{A} \cap (C \setminus D) = \bar{R}\}$ contains an infinite number of pairwise non-equivalent representatives of E_C-equivalence classes.

(4) There are monadic formulas $U(\bar{X}, \bar{Z})$, $E(\bar{X}, \bar{Y}, \bar{Z})$, $\mathrm{Atom}(\bar{X}, \bar{Z})$, $\mathrm{Set}(\bar{X}, \bar{Z})$, $\ldots$, such that for infinitely many $k < \omega$ there is a sequence $\bar{Q}_k \subseteq D$ such that

$$\mathcal{I} = \langle D, d, U(\bar{X}, \bar{Q}_k), E(\bar{X}, \bar{Y}, \bar{Q}_k), \mathrm{Atom}(\bar{X}, \bar{Q}_k), \mathrm{Set}(\bar{Y}, \bar{Q}_k), \mathrm{Code}(\bar{X}, \bar{Y}, \bar{Q}_k) \ldots\rangle$$

is an interpretation of T_k in D. D is the minimal non-minor initial segment for these interpretations.

(5) The depths of all the mentioned formulas are $< n - 10$, $m = m(n + d, 4d)$ is as in Theorem 2.15.

(6) N_1 is the maximal bouquet size of a minor segment.

Fixing k, a sequence $\bar{A} \subseteq D$ satisfying $U(\bar{A}, \bar{Q}_k)$ will be called an element.

DEFINITION 7.1. The *vicinity* $[\bar{A}]$ of an element $\bar{A}$ is the collection $\{\bar{B}$: some element $\bar{E} \sim \bar{B}$ coincides with $\bar{A}$ outside some proper initial segment of $D\}$.

REMARK. By the choice of n, $\bar{A} \in [\bar{B}]$ is determined by $\mathrm{Th}^{n+d}(D; \bar{A}, \bar{B}, \bar{Q}_k)$.

LEMMA 7.2. *Every vicinity $[\bar{A}]$ is the union of at most N_1 different equivalence classes.*

PROOF. See [5] Lemma 9.1. ⊣

Using Ramsey theorem we define the following functions.

NOTATION 7.3. (1) Let M be the number of possibilities for $a\text{-WA}^m(C; \bar{B}_1, \bar{B}_2, \bar{B}_3, \bar{A})$ where $\lg(\bar{B}_1) = \lg(\bar{B}_2) = \lg(\bar{B}_3) = \lg(\bar{A}) = d$, C a chain and a a semi-club of $\mathrm{cf}(C)$. Given $k < \omega$ let $t(k)$ be such that for every coloring of $\{(i, j) : i < j < t(k)\}$ into M colors, there is a subset I of $\{0, \dots, t(k) - 1\}$ such that $|I| \geq k$ and all the pairs from $\{(i, j) : i < j,\ i, j \in I\}$ have the same color.

(2) Given $k < \omega$, let $h(k)$ be such that for every coloring of $\{(i, j, \ell) : i < j < \ell < h(k)\}$ into 32 colors, there is a subset I of $\{0, 1, \dots, h(k) - 1\}$ such that $|I| > k$ and all the members of $\{(i, j, \ell) : i < j < \ell,\ i, j, \ell \in I\}$ have the same color.

We are ready now to prove the main theorem:

THEOREM 7.4. *The above assumptions lead to a contradiction.*

PROOF. The proof will be split into several steps.

Step 1. Let $\bar{R} \subseteq (C \setminus D)$ and $S := \{\bar{A} \subseteq C : \bar{A} \cap (C \setminus D) = \bar{R}\}$ be as in Assumption 3. Choose $0 \ll K_1 \ll K < \omega$ and interpret T_K on D using parameters $\bar{Q}_K = \bar{Q} \subseteq D$. ($K_1$ and K depend only on n and d and their actual size is obtained from repeated applications of the Ramsey functions.)

Choose sequences of non-equivalent elements from S, $\mathscr{B} := \langle \bar{U}_i : i < K \rangle$, $\mathscr{B}_1 := \langle \bar{V}_s : s < 2^K \rangle$ and $\mathscr{B}_2 := \langle \bar{W}_t : t < 2^{2^K} \rangle$ such that $\mathscr{B}$ is the family of "atoms" of the interpretation, $\mathscr{B}_1$ the family of "sets" of the interpretation. and $\mathscr{B}_2$ the family of "hypersets" of the interpretation.

Step 2. Fix $J := \{\alpha_j : j < \lambda\} \subseteq \lambda$ an m-suitable club for every combination one can think of from the $\bar{U}_i$'s, the $\bar{V}_s$'s, the $\bar{W}_t$'s and $\bar{Q}$.

Now, everything mentioned happens in $H(\lambda^+)^{V^P}$ which is equal (using a previous remark and notations) to $H(\lambda^+)^{V^{P_0 * P_1}}$. P_1 is an iteration of length λ^+ and it follows that all the mentioned subsets of λ are added to $H(\lambda^+)^{V^{P_0 * P_1}}$ after a proper initial segment of the forcing which we denote by $P_0 * (P_1 \restriction_\beta)$. So there is a semi-club $a \subseteq \lambda$ in $H(\lambda^+)^{V^{P_0 * P_1}}$ which is added after all the mentioned sets, say at stage β of P_1. Fix a.

Step 3. We will begin now to shuffle the elements with respect to a and J. Let for $i < j < K$,

$$k(i, j) := \min\{k : [\bar{U}_i, \bar{U}_j]^J_a \sim \bar{U}_k, \text{ or } k = K\}.$$

Assuming that $K > h(K_2)$ for some $K_2 \gg t(2K_1 + 2N_1 + 2)$ there is a subset $s \subseteq \{0, \dots, K-1\}$ of cardinality at least K_2 such that for every $\bar{U}_i, \bar{U}_j, \bar{U}_\ell$ with $i < j < \ell$ in s the following five statements have the same truth value:

$$k(j, \ell) = i,\ k(i, \ell) = j,\ k(i, j) = i,\ k(i, j) = j,\ k(i, j) = \ell.$$

Moreover, by [5] Lemma 10.2, if there is a pair $i < j$ in s such that $k(i, j) \in s$ (i.e., not all the statements are false on s) then, either for every pair $i < j$ in s, $k(i, j) = i$ or for every $i < j$ in s, $k(i, j) = j$.

Step 4. Let us show that for some $i < j$ in s we have $k(i, j) \in s$. Let $\bar{V}_s$ be the set that codes $\langle \bar{U}_i : i \in s\rangle$. By the definition of t there is a set $s' \subseteq s$ with at least $K_3 \gg 2K_1 + 2N_1 + 2$ elements and a sequence $\langle \bar{U}_i : i \in s'\rangle$ such that for every $r < \ell$ in s', a-WA$^m(D; \bar{U}_r, \bar{U}_\ell, \bar{V}_s, \bar{Q})$ is constant.

It follows that for every $r < \ell$ in s', a-WA$^m(D; \bar{U}_r, \bar{V}_s, \bar{Q}) = a$-WA$^m(D; \bar{U}_\ell, \bar{V}_s, \bar{Q})$, and by the preservation theorem both theories are equal to a-WA$^m(D; [\bar{U}_r, \bar{U}_\ell]^J_a, \bar{V}_s, \bar{Q})$.

But $D \models \text{Code}(\bar{U}_r, \bar{V}_s, \bar{Q})$, and since a-WA^m decides if 'Code' holds, the equality of the theories implies that $D \models \text{Code}([\bar{U}_r, \bar{U}_\ell]^J_a, \bar{V}_s, \bar{Q})$.

By the definition of 'Code' there is $k \in s$ such that $[\bar{U}_r, \bar{U}_\ell]^J_a \sim \bar{U}_k$. Therefore there are r, ℓ in s with $k(r, \ell) \in s$ and by Step 3 we can conclude that, without loss of generality,

$$i, j \in s,\ i < j \Rightarrow [\bar{U}_i, \bar{U}_j]^J_a \sim \bar{U}_i.$$

Step 5. By applying the function t again choose $s'' \subseteq s'$ of size $> 2K_1 + 2N_1 + 2$ such that for every $r < \ell$ in s'', a-WA$^m(D; \bar{U}_r, \bar{U}_\ell, \bar{V}_{s'}, \bar{Q})$ is constant.

Note that if a is a semi-club then $\lambda \setminus a$ is also a semi-club. We will use the fact that a is generic with respect to the other sets for finding a pair $i < j \in s''$ such that $[\bar{U}_i, \bar{U}_j]^J_{\lambda \setminus a} \sim \bar{U}_i$ holds as well. Let $p \in P_0 * P_1$ be a condition that forces the (equal) value of all the theories a-WA$^m(D; \bar{U}_r, \bar{U}_\ell, \bar{V}_{s'}, \bar{Q})$ for $r < \ell \in s''$. The condition p is a pair (q_1, q_2) where $q_1 \in P_0$ and q_2 is a P_0-name of a function from λ^+ to conditions in the forcing SC_λ. $q_2(\beta)$ is forced by p to be an initial segment of a of height $\gamma < \lambda$ and w.l.o.g. we can assume that $\gamma = \alpha_{j+1} \in J$ (So $\text{cf}(\gamma) = \omega$.) As $\gamma < \lambda = D$, γ is a minor segment. Recall that $|s''| > 2K_1 + 2N_1 + 2$ and define $s^* \subseteq s''$ with $|s^*| > 2K_1$ by

$$s^* := \{i \in s'' : |\{j \in s'' : j < i\}| > N_1 \text{ and } |\{j \in s'' : j > i\}| > N_1\}.$$

Denote by $\bar{A} \frown \bar{B}$ the sequence $(\bar{A} \cap \gamma) \cup (\bar{B} \cap (D \setminus \gamma))$. We claim that for every i, j, k in s^*, $\bar{U}_k \sim [\bar{U}_i, \bar{U}_j]_a \frown \bar{U}_k$.

Note that by the definition of s'' and the preservation theorem for ATh, p forces for i, j, k in s^*:

$$
\begin{aligned}
&\mathrm{Th}^{n+d}(D;[\bar U_i,\bar U_j]_a{}^\frown \bar U_k,\bar V_{s'},\bar Q)\\
&\quad=\mathrm{Th}^{n+d}([0,\gamma);[\bar U_i,\bar U_j]_a,\bar V_{s'},\bar Q)+\mathrm{Th}^{n+d}([\gamma,\lambda);\bar U_k,\bar V_{s'},\bar Q)\\
&\quad=(\text{by } \gamma\in J, \text{the equality of the } a\text{-WA's and the preservation theorem})\\
&\qquad\qquad \mathrm{Th}^{n+d}([0,\gamma);\bar U_i,\bar V_{s'},\bar Q)+\mathrm{Th}^{n+d}([\gamma,\lambda);\bar U_k,\bar V_{s'},\bar Q)\\
&\qquad\qquad =(\text{by } \gamma\in J \text{ and the equality of the ATh's})\\
&\qquad\qquad \mathrm{Th}^{n+d}([0,\gamma);\bar U_k,\bar V_{s'},\bar Q)+\mathrm{Th}^{n+d}([\gamma,\lambda);\bar U_k,\bar V_{s'},\bar Q)\\
&\qquad\qquad\qquad =\mathrm{Th}^{n+d}(D,\bar U_k,\bar V_{s'},\bar Q).
\end{aligned}
$$

Hence, since $\bar V_{s'}$ codes s', $[\bar U_i,\bar U_j]_a{}^\frown\bar U_k\sim\bar U_\ell$ for some $\ell\in s'$.

If $\ell=k$ we are done so assume w.l.o.g that $\ell<k$. Now $\bar U_l\in[U_k]$ and this is determined by a previous remark by $\mathrm{Th}^{n+d}(D;\bar U_\ell,\bar U_k,\bar Q)$. Since $\ell<k$ are in the homogeneous s', for every $m<k$ in s' we have $\mathrm{Th}^{n+d}(D;\bar U_\ell,\bar U_k,\bar Q)=\mathrm{Th}^{n+d}(D;\bar U_m,\bar U_k,\bar Q)$. By the choice of s^* and as $k\in s^*$ we have more than N_1 such m's in s'. This means that $\|[\bar U_k]\|>N_1$ and this contradicts Lemma 7.2. It follows that $\ell=k$ after all and the claim is verified.

Therefore for a set of $|s^*|$ elements it is possible to replace an initial segment with a shuffling without changing equivalence classes.

Step 6. We are ready to prove that for every $i<j$ in s^*, $[\bar U_i,\bar U_j]_a\sim[\bar U_i,\bar U_j]_{\lambda\setminus a}$.

By Step 4 $p\Vdash[\bar U_i,\bar U_j]_a\sim\bar U_i$. Remember that p 'knows' only an initial segment of a, namely only $a\cap(j+1)$ where $\gamma=\alpha_{j+1}$. Since our forcing is homogeneous $b:=\big(a\cap[0,j+1)\big)\cup\big((\lambda\setminus a)\cap[j+1,\lambda)\big)$ is also generic for all the mentioned sets and parameters, and everything p forces for a it forces for b. Therefore $p\Vdash$ "$[\bar U_i,\bar U_j]_b\sim\bar U_i$". Note that by the preservation theorem

$$
\begin{aligned}
&\mathrm{Th}^n([0,\gamma);[\bar U_i,\bar U_j]_{\lambda\setminus a},\bar Q)=\mathrm{Th}^n([0,\gamma);[\bar U_j,\bar U_i]_a,\bar Q)\\
&\quad=\mathrm{Th}^n([0,\gamma);[\bar U_i,\bar U_j]_a,\bar Q)|=\mathrm{Th}^n([0,\gamma);\bar U_i,\bar Q)=\mathrm{Th}^n([0,\gamma);\bar U_j,\bar Q).
\end{aligned}
$$

It follows that

$$\mathrm{Th}^n([0,\gamma);[\bar U_i,\bar U_j]_a,[\bar U_i,\bar U_j]_a,\bar Q)=\mathrm{Th}^n([0,\gamma);[\bar U_i,\bar U_j]_{\lambda\setminus a},[\bar U_i,\bar U_j]_{\lambda\setminus a},\bar Q).$$

By Step 5 (where we used only the fact that $i,j\in s^*$), $[\bar U_i,\bar U_j]_{\lambda\setminus a}{}^\frown\bar U_i\sim\bar U_i\sim[\bar U_i,\bar U_j]_b$. Now

$$
\begin{aligned}
&\mathrm{Th}^n(D;[\bar U_i,\bar U_j]_{\lambda\setminus a}{}^\frown\bar U_i,[\bar U_i,\bar U_j]_{\lambda\setminus a},\bar Q)\\
&=\mathrm{Th}^n([0,\gamma);[\bar U_i,\bar U_j]_{\lambda\setminus a},[\bar U_i,\bar U_j)]_{\lambda\setminus a},\bar Q)+\mathrm{Th}^n([\gamma,\lambda);\bar U_i,[\bar U_i,\bar U_j]_{\lambda\setminus a},\bar Q)\\
&=\mathrm{Th}^n([0,\gamma);[\bar U_i,\bar U_j]_a,[\bar U_i,\bar U_j)]_a,\bar Q)+\mathrm{Th}^n([\gamma,\lambda);\bar U_i,[\bar U_i,\bar U_j]_{\lambda\setminus a},\bar Q)\\
&=\mathrm{Th}^n(D;[\bar U_i,\bar U_j]_a{}^\frown\bar U_i,[\bar U_i,\bar U_j]_b,\bar Q).
\end{aligned}
$$

But $[\bar U_i,\bar U_j]_a{}^\frown\bar U_i\sim\bar U_i\sim[\bar U_i,\bar U_j]_b$ and it follows, by the equality of the theories, that $[\bar U_i,\bar U_j]_{\lambda\setminus a}\sim[\bar U_i,\bar U_j]_a\sim\bar U_i$ as required.

Step 7. By renaming $\langle\bar U_i:i\in s^*\rangle$ we get a sequence $\langle\bar A_i:i<2K_1\rangle$ such that for every $i<j<2K_1$, $r<\ell<2K_1$ the following hold:

(i) $a\text{-WA}^m(D;\bar A_i,\bar A_j,\bar Q)=a\text{-WA}^m(D;\bar A_r,\bar A_\ell,\bar Q)$.

(ii) $[\bar A_i,\bar A_j]_a\sim[\bar A_i,\bar A_j]_{\lambda\setminus a}\sim\bar A_i$.

For $i < K_1$ denote by $\bar{B}_i$ the element that codes the set $\{\bar{A}_i, \bar{A}_{2K_1-i-1}\}$ and look at the sequence $\langle \bar{B}_i : i < K_1 \rangle$. As K_1 is large enough by repeating Steps 1,2 and 3 (using sets and hypersets for atoms and sets) one is left with $i < j < K_1$ such that :
(iii) $a\text{-WA}^m(D; \bar{A}_i, \bar{A}_{2K_1-i-1}, \bar{B}_i, \bar{Q}) = a\text{-WA}^m(D; \bar{A}_j, \bar{A}_{2K_1-j-1}, \bar{B}_j, \bar{Q})$.
(iv) $[\bar{B}_i, \bar{B}_j]_a \sim \bar{B}_i$ or $[\bar{B}_i, \bar{B}_j]_a \sim \bar{B}_j$.
(Note that in (iv) choosing one of the possibilities will cause a loss of generality.)

Now let's shuffle with respect to a and J using clause (iii):

$$\begin{aligned}&\mathrm{Th}^n(D; \bar{A}_i, \bar{A}_{2K_1-i-1}, \bar{B}_i, \bar{Q})\\ &= \mathrm{Th}^n(D; [\bar{A}_i, \bar{A}_j]_a, [\bar{A}_{2K_1-i-1}, \bar{A}_{2K_1-j-1}]_a, [\bar{B}_i, \bar{B}_j]_a, \bar{Q})\\ &= \mathrm{Th}^n(D; [\bar{A}_i, \bar{A}_j]_a, [\bar{A}_{2K_1-j-1}, \bar{A}_{2K_1-i-1}]_{\lambda\setminus a}, [\bar{B}_i, \bar{B}_j]_a, \bar{Q}).\end{aligned}$$

But $[\bar{A}_i, \bar{A}_j]_a \sim \bar{A}_i$, and by Step 6, $[\bar{A}_{2K_1-j-1}, \bar{A}_{2K_1-i-1}]_{\lambda\setminus a} \sim \bar{A}_{2K_1-j-1}$ and by clause (iv) $[\bar{B}_i, \bar{B}_j]_a \sim \bar{B}_i$ or $[\bar{B}_i, \bar{B}_j]_a \sim \bar{B}_j$.

So we have, as implied by the equality of Th^n, either

$$D \models \mathrm{Code}(\bar{A}_i, \bar{B}_i, \bar{Q}) \ \&\ \mathrm{Code}(\bar{A}_{2K_1-j-1}, \bar{B}_i, \bar{Q})$$

or

$$D \models \mathrm{Code}(\bar{A}_i, \bar{B}_j, \bar{Q}) \ \&\ \mathrm{Code}(\bar{A}_{2K_1-j-1}, \bar{B}_j, \bar{Q}).$$

Both cases are impossible!

We have reached a contradiction assuming, in V^P, that a chain C interprets T with a minimal major initial segment D which is a regular cardinal. $\dashv$

We still have to prove that there is no interpretation in the case D is not a regular cardinal. For that we will use formal shufflings as in Section 5.

LEMMA 7.5. *The assumption "D is a regular cardinal" is not necessary.*

PROOF. The only place where we used genericity is Step 6 so the difficulty is to find 2 elements $\bar{A}, \bar{B}$ and a semi-club a such that $[\bar{A}, \bar{B}]_a \sim [\bar{B}, \bar{A}]_a$. Since it is possible that $|a| < |D|$, a will be generic not with respect to $\bar{A}$ and $\bar{B}$ but with respect to sequences of theories of length λ. We will repeat Steps 1 to 7 from the previous proof modifying and translating them to the language of formal shufflings.

Step 1. We assume D interprets T_K, and choose $\bar{Q}$, K atoms $\langle \bar{U}_i : i < K \rangle$, sets $\bar{V}_s$ and hypersets $\bar{W}_t$ as before.

Step 2. As in Fact 5.8(3) fix an "m-suitable" partition $D = \sum_{i<\lambda} D_i$. Let τ_i be the theory $\mathrm{Th}^m(D_i; \bar{E}, \bar{Q})$ where $\bar{E}$ is the sequence of all atoms, sets and hypersets. Let $\mathcal{T} = \langle \tau_i : i < \lambda \rangle$, so every relevant theory can be computed from $\mathcal{T}$. Now $\mathcal{T} \in H(\lambda^+)^{V^P}$ and fix some $a \in H(\lambda^+)^{V^P} = H(\lambda^+)^{V^{P_1 * P_2}}$, generic with respect to $\mathcal{T}$.

STEPS 3–4: Repeat Steps 3 and 4 in V^P (there is no need of referring to $\mathcal{T}$ at this point.)

We are left with a sequence $\langle \bar{U}_j : j \in s \rangle$ such that w.l.o.g $j_1 < j_2 \Rightarrow [\bar{U}_{j_1}, \bar{U}_{j_2}]_a \sim \bar{U}_{j_1}$. ($[\bar{U}_{j_1}, \bar{U}_{j_2}]_a \cap D_i$ is of course $\bar{U}_{j_1} \cap D_i$ if $i \in a$ and $\bar{U}_{j_2} \cap D_i$ if $i \notin a$).

Step 5. Go down to $H(\lambda^+)^{V^P}$ and translate everything we have achieved so far to a statement about $\mathcal{T}$. For example, the 'formal' meaning of $[\bar{U}_\ell, \bar{U}_r]_a^J \sim \bar{U}_k$ is as follows:

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"if $s_i^1 = \mathrm{Th}^n(D_i; \bar{U}_\ell, \bar{U}_k, \bar{Q})$ and $s_i^2 = \mathrm{Th}^n(D_i; \bar{U}_r, \bar{U}_k, \bar{Q})$ then letting $\mathcal{S}_1 := \langle s_i^1 : i < \lambda\rangle$, $\mathcal{S}_2 := \langle s_i^2 : i < \lambda\rangle$ and $\mathcal{U} = [\mathcal{S}_1, \mathcal{S}_2]_a = \langle u_i : i < \lambda\rangle$ we get that $u = \sum_{i<\lambda} u_i$ is a theory that satisfies: if C is a chain, $\bar{A}, \bar{B}, \bar{E} \subseteq C$ then $[\mathrm{Th}^n(C; \bar{A}, \bar{B}, \bar{E}) = u \Rightarrow C \models E(\bar{A}, \bar{B}, \bar{E})]$." ($\mathcal{S}_1$ and $\mathcal{S}_2$ are of course computable from $\mathcal{T}$.)

Now find a condition $p \in P_{\leq\lambda}$ that forces each such formal statement (we are talking about events occurring in $H(\lambda^+)$) and define $\mathcal{S}_1 {}^\frown \mathcal{S}_2$. Repeat Step 5 to get $s^* \subseteq s$ as there. So the 'formal' version of $j_1, j_2, k \in s^* \Rightarrow [\bar{U}_{j_1}, \bar{U}_{j_2}]_a {}^\frown \bar{U}_k \sim \bar{U}_k$ holds.

Step 6. We have to prove for $\ell, r \in s^*$:

"if $s_i^1 = \mathrm{Th}^n(D_i; \bar{U}_\ell, \bar{U}_r, \bar{Q})$ and $s_i^2 = \mathrm{Th}^n(D_i; \bar{U}_r, \bar{U}_\ell, \bar{Q})$ then both $\sum_{i<\lambda} u_i$ and $\sum_{i<\lambda} v_i$ imply $E(\bar{X}, \bar{Y}, \bar{Z})$ where for $u_i = s_i^1$ and $v_i = s_i^2$ for $i \in a$ and $u_i = s_i^2$, $v_i = s_i^1$ for $i \notin a$". This is easily achieved using the genericity of a.

Step 7. Go back to V^P and shuffle elements. No genericity is required. ⊣

Combining Theorem 7.4 and Lemma 7.5 we conclude:

THEOREM 7.6. *There is a forcing notion P such that in V^P, Peano Arithmetic is not interpretable in the monadic second-order theory of chains.*

REFERENCES

[1] J. BALDWIN, *Definable second-order quantifiers*, ***Model theoretic logics*** (J. Barwise and S. Feferman, editors), Springer-Verlag, Berlin, 1985, pp. 445–477.

[2] J. BALDWIN and S. SHELAH, *Classification of theories by second order quantifiers*, ***Notre Dame Journal of Formal Logic***, vol. 26 (1985), pp. 229–303.

[3] Y. GUREVICH, *Monadic second-order theories*, (J. Barwise and S. Feferman, editors), Springer-Verlag, Berlin, 1985, pp. 479–506.

[4] Y. GUREVICH, M. MAGIDOR, and S. SHELAH, *The monadic theory of ω_2*, this JOURNAL, vol. 48 (1983), pp. 387–398.

[5] ———, *On the strength of the interpretation method*, this JOURNAL, vol. 54 (1989), pp. 305–323.

[6] Y. GUREVICH and S. SHELAH, *Monadic theory of order and topology in ZFC*, ***Annals of Mathematical Logic***, vol. 23 (1982), pp. 179–182.

[7] ———, *Interpreting second-order logic in the monadic theory of order*, this JOURNAL, vol. 48 (1983), pp. 816–828.

[8] ———, *The monadic theory and the 'next world'*, ***Israel Journal of Mathematics***, vol. 49 (1984), pp. 55–68.

[9] S. LIFSCHES and S. SHELAH, *Random graphs in the monadic theory of order*, in preparation.

[10] S. SHELAH, *The monadic theory of order*, ***Annals of Mathematics***, vol. 102 (1975), pp. 379–419.

[11] ———, *Notes on monadic logic, Part B: Complexity of linear orders*, ***Israel Journal of Mathematics***, vol. 69 (1990), pp. 99–116.

[12] A. TARSKI, A. MOSTOWSKI, and R. M. ROBINSON, ***Undecidable theories***, North-Holland Publishing Company, 1953, pp. *xi* and 98.

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