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# A PARTITION RELATION USING STRONGLY COMPACT CARDINALS

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Abstract. If $\kappa$ is strongly compact and $\lambda > \kappa$ and $\lambda$ is regular (or alternatively $\mathrm{cf}(\lambda) \geq \kappa$), then $\left(2^{<\lambda}\right)^+ \to (\lambda + \zeta)^2_\theta$ holds for $\zeta, \theta < \kappa$.

## §0. Introduction

The aim of this paper is to prove the following theorem.

**0.1 Theorem.** *If $\kappa$ is a strongly compact cardinal, $\lambda > \kappa$ is regular and $\zeta, \theta < \kappa$,* then *the partition relation $\left(2^{<\lambda}\right)^+ \to (\lambda + \zeta)^2_\theta$ holds.*

**0.2 Theorem.** *Assume the conditions in Theorem* 0.1 *hold, with "$\lambda$ regular". Then* $\mathrm{cf}(\lambda) > \kappa$ *suffices.*

We notice that our argument is valid in the case $\kappa = \omega$. As for the history of the problem we point out that Hajnal proved, in an unpublished work, that $(2^\omega)^+ \to (\omega_1 + n)^2_2$ holds for every $n < \omega$. Then it was shown in [Sh:26], §6, that for $\kappa > \omega$ regular and $2^{|\alpha|} < \kappa$, the relation $(2^{<\kappa})^+ \to (\kappa + \alpha)^2_2$ is true. More recently Baumgartner, Hajnal, and Todorčević in [BHT93] extended this to the case when the number of colors is arbitrarily finite. Earlier in [Sh:424], we have $(2^{<\lambda})^{+n} \to (\lambda \times m)^2_k$ for $n$ large enough (this was complimentary to the main result there that $\aleph_0 < \lambda = \lambda^{<\lambda} + 2^\lambda$ arbitrarily large does not imply $2^\lambda \to (\lambda \times \omega)^2_2$). Subsequently [BHT93] improves $n$. We hope that the way the strong compactness was used will be useful elsewhere; see [Sh:666] for a discussion of a possible consistency of failure. I also thank Peter Komjath for improving the presentation.

*Notation.* If $S$ is a set and $\kappa$ a cardinal, then $[S]^\kappa = \{a \subseteq S : |a| = \kappa\}$, $[S]^{<\kappa} = \{a \subseteq S : |a| < \kappa\}$. If $D$ is some filter over a set $S$, then $X \in D^+$ denotes that $S \setminus X \notin D$ and $X \subseteq S$. If $\kappa < \mu$ are regular cardinals, then $S^\mu_\kappa = \{\alpha < \mu : \mathrm{cf}(\alpha) = \kappa\}$, a stationary set. The notation $A = \{x_\alpha : \alpha < \gamma\}_<$, etc., means that $A$ is enumerated increasingly.

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## §1. The case of $\lambda$ regular

**1.1 Lemma.** *Assume $\mu = \mu^\theta$. Assume that $D$ is a normal filter on $\mu^+$ and $A^* \in D^+$ satisfies $\delta \in A^* \Rightarrow$ cf$(\delta) > \theta$, and $F'$ is a function with domain $[A^*]^2$ and range of cardinality $\theta$.* Then *there are a normal filter $D_0$ on $\mu^+$ extending $D, A_0 \in D_0$ with $A_0 \subseteq A^*$ and $C_0 \subseteq$ Rang$(F')$ satisfying Rang$(F' \restriction [A_0]^2) = C_0$ such that, if $X \in D_0^+$,* then *Rang$(F' \restriction [X]^2) \supseteq C_0$.*

We first prove a claim.

**1.2 Claim.** *Assume $\mu = \mu^\theta$ and $F' : [S^*]^2 \to \theta, D$ is a normal filter on $\mu^+, S^* \subseteq \mu^+$ belongs to $D^+$ and $\delta \in S^* \Rightarrow$ cf$(\delta) > \theta$. There is a set $A \in D^+$ such that $A \subseteq S^*$ and some $C \subseteq \theta$ satisfying Rang$(F' \restriction [A]^2) = C$ and, if $f : A \to \mu^+$ is a regressive function,* then *for some $\alpha < \mu^+$ we have Rang$(F' \restriction [f^{-1}(\alpha)]^2) = C$ and $f^{-1}(\alpha)$ is a subset of $\mu^+$ from $D^+$.*

*Proof.* Toward contradiction assume that no such sets $A, C$ exist. We build a tree $T$ as follows. Every node $t$ of the tree will be of the form

$$\begin{aligned} t &= \langle \langle A_\alpha : \alpha \leq \varepsilon \rangle, \langle f_\alpha : \alpha < \varepsilon \rangle, \langle i_\alpha : \alpha < \varepsilon \rangle \rangle \\ &= \big\langle \langle A^t_\alpha : \alpha \leq \varepsilon \rangle, \langle f^t_\alpha : \alpha < \varepsilon \rangle, \langle i^t_\alpha : \alpha < \varepsilon \rangle \big\rangle \end{aligned}$$

for some ordinal $\varepsilon = \varepsilon(t)$ where $\langle A_\alpha : \alpha \leq \varepsilon \rangle$ is a decreasing, continuous sequence of subsets of $\mu^+$; for every $\alpha < \varepsilon, f_\alpha$ is a regressive function on $A_\alpha$; and $\langle i_\alpha : \alpha < \varepsilon \rangle$ is a sequence of distinct elements of $\theta$. It will always be true that if $t <_T t'$, then each of the three sequences of $t'$ extend the corresponding one of $t$.

To start, we make the node $t$ with $\varepsilon(t) = 0, A_0 = S^*$ the root of the tree.

At limit levels we extend (the obvious way) all cofinal branches to a node.

If we are given an element $t = \langle \langle A_\alpha : \alpha \leq \varepsilon \rangle, \langle f_\alpha : \alpha < \varepsilon \rangle, \langle i_\alpha : \alpha < \varepsilon \rangle \rangle$ of the tree and the set $A_\varepsilon$ is $= \emptyset$ mod $D$, then we leave $t$ as a terminal node. Otherwise, let $C = C_t =$ Rang$\big(F' \restriction [A_\varepsilon]^2\big)$ and notice that by hypothesis, toward contradiction, the pair $A_\varepsilon, C_t$ cannot be as required in the Claim. There is, therefore, a regressive function $f = f_t$ with domain $A_\varepsilon$, such that for every $x < \mu^+$ the set Rang$\big(F' \restriction [f^{-1}(x)]^2\big)$ is a proper subset of $C_t$ or $f^{-1}(x)$ is a $= \emptyset$ mod $D$ subset of $\mu^+$. We make as immediate extensions of $t$ the sequences of the form $t_x = \langle \langle A_\alpha : \alpha \leq \varepsilon + 1 \rangle, \langle f_\alpha : \alpha < \varepsilon + 1 \rangle, \langle i_\alpha : \alpha < \varepsilon + 1 \rangle \rangle$ where $A_{\varepsilon+1} = f^{-1}(x), f_\alpha = f_t$ and $i_\varepsilon \in C_t$ is some colour value such that if $A_{\varepsilon+1} \neq \emptyset$ mod $D$, then $i_\varepsilon$ is not in the range of $F' \restriction [A_\varepsilon]^2$.

Having constructed the tree, observe that every element $x \in S^* \subseteq \mu^+$ belongs to a set $A^{t(x)}_{\varepsilon(x)}$ for some (unique) terminal node $t(x)$ of $T$. Also, $\varepsilon(x) < \theta^+ (< \mu^+)$ holds by the selection of the $i_\beta$'s as $\langle i^{t(x)}_\alpha : \alpha < \varepsilon(x) \rangle$ is a sequence of members of $\theta$ with no repetitions while $\theta$, the set of colours, has $\theta$ members. For some set $S \subseteq S^*$ of ordinals $x < \mu^+$ which belong to $D^+$ (by the normality of $D$), the value of $\varepsilon(x)$ is the same, say $\varepsilon$. For $x \in S$ we let $g_\alpha(x) = f^{t(x)}_\alpha(x)$ where $f^{t(x)}_\alpha$ is the $\alpha$-th regressive function in the node $t(x) \in T$. Again, by $\mu^\theta = \mu$ and $(\forall \alpha \in S)[\mathrm{cf}(\alpha) > \theta]$ we have that $(\forall x \in S')(\forall \alpha < \varepsilon) g_\alpha(x) = \beta_\alpha$ holds for some sequence $\langle \beta_\alpha : \alpha < \varepsilon \rangle$ and subset $S' \subseteq S$ from $D^+$. But then we get that the set $S'$ satisfies $x, y \in S' \Rightarrow (A^{t(x)}_\alpha, f^{t(x)}_\alpha, i^{t(x)}_\alpha) = (A^{t(y)}_\alpha, f^{t(y)}_\alpha, i^{t(y)}_\alpha)$ for every $\alpha < \varepsilon$; we can prove this by induction on $\alpha$. We can then prove that $A^{t(x)}_\varepsilon = A^{t(y)}_\varepsilon$ for $x, y \in S'$.

We can conclude that $x, y \in S' \Rightarrow t(x) = t(y)$, so $S' \subseteq A^t_{\varepsilon(t)}$ for some terminal node $t$, but this latter set is in $D^+$, a contradiction. $\square_{1.2}$

*Proof of Lemma* 1.1. Apply Claim 1.2 with $S^* = A^*$ to get corresponding $(C, A)$. Define the ideal $I$ as follows. For $X \subseteq \mu^+$ we let $X \in I$ iff there are a member $E$ of $D$ and a regressive function $f : X \cap A \to \mu^+$ such that every $\mathrm{Rang}\big(F' \restriction [f^{-1}(\alpha)]^2\big)$ is a proper subset of $C$ or $f^{-1}(\alpha)$ is a $= \emptyset \bmod D$ subset of $\mu^+$.

Now:

**1.3 Claim.** *$I$ is a normal ideal on $\mu^+$ (and $A^* = \mu^+$ mod $I$).*

*Proof.* Straightforward.

Set $D_0$ to be the dual filter of $I$, let $A_0 = A$ and let $C_0 = C$; by Claim 1.2 we are done. $\square_{1.1}$

1.4 *Remark.* 1) If Lemma 1.1 holds for some $D_0, A_0, C_0$, then it holds for $D_1, A_1, C_0$ when the normal filter $D_1$ extends $D_0$, and $A_1 \in D_1$ satisfies $A_1 \subseteq A_0$.

2) If $D_0, A_0, C_0$ satisfy Lemma 1.5, and $X \in D_0^+$, then $X$ contains a homogeneous set of order type $\lambda + 1$ of color $\xi$ for every $\xi \in C_0$.

3) Lemma 1.1 is closely related to the proof in [Sh:26], i.e. 5.1 there.

*Proof of Theorem* 0.1. Let $\mu = 2^{<\lambda}$, and let $F : [\mu^+]^2 \to \theta$ be a colouring. We apply Lemma 1.1 for $A^* = S^{\mu^+}_{\mathrm{cf}(\lambda)}$, $(F = F, \theta, \mu = \mu)$ and $D$ the club filter. We shall write $F(\alpha, \beta)$ for $F(\{\alpha, \beta\})$ and 0 for $F(\alpha, \alpha)$.

We fix $A_0, D_0, C_0$ which we get by Lemma 1.1.

**1.5 Lemma.** *Almost every $\delta \in A_0$ (i.e. for all but a set $= \emptyset$ mod $D_0$) satisfies the following: if $s \in [A_0 \cap \delta]^{<\lambda}$ and $\{z_\alpha : \alpha < \gamma\}_< \subseteq A_0 \cap [\delta, \mu^+)$ with $\gamma < \kappa$,* then *there is $\{y_\alpha : \alpha < \gamma\}_< \subseteq A_0 \cap (\sup(s), \delta)$ such that:*

(a) $F(x, y_\alpha) = F(x, z_\alpha)$ *(for $x \in s, \alpha < \gamma$);*
(b) $F(y_\alpha, y_\beta) = F(z_\alpha, z_\beta)$ *(for $\alpha < \beta < \gamma$).*

*Proof.* By simple reflection (using the regularity of $\lambda$).

**1.6 Lemma.** *There*[1] *is $A_0' \subseteq A_0, A_0' \in D_0$ such that if $\delta \in A_0', s \in [\delta]^{<\lambda}$ and* $\xi \in C_0$, then *there exists a $\delta_1 \in A_0, \delta < \delta_1$ such that:*

(a) $F(x, \delta) = F(x, \delta_1)$ *(for $x \in s$);*
(b) $F(\delta, \delta_1) = \xi$.

*Proof.* Otherwise, there is some $X \subseteq A_0, X \in D_0^+$ such that for every $\delta \in X$ there are $s(\delta) \in [\delta]^{<\lambda}$ and $\xi(\delta) \in C_0$ such that there is no $\delta_1 > \delta$ satisfying (a) and (b). By normality and $\mu = \mu^{<\lambda}$ we can assume that $s(\delta) = s$ and $\xi(\delta) = \xi$ holds for $\delta \in X$. By Lemma 1.1, that is, the choice of $(A_0, D_0, C_0)$, there must exist $\delta < \delta_1$ in $X$ with $F(\delta, \delta_1) = \xi$, and this is a contradiction. $\square_{1.6}$

*Continuation of the proof of Theorem* 0.1. Let $A_0' \subseteq A_0$ satisfy Lemmas 1.1 and 1.6 and pick some $\delta_1 \in A_0'$. Then let $T = A_0' \backslash (\delta_1 + 1)$.

[1] In fact, if $A_1^* \in D_0^+$, then for some $A_0' \subseteq A_1 \cap A_0, A_1 \backslash A_0' = \emptyset$ modulo $D_0$ and the conclusion holds for every $\delta \in A_0'$.

**1.7 Lemma.** *There exists a function* $G : T \times T \to C_0$ *such that if* $s \in [\delta_1]^{<\lambda}, \gamma < \kappa$, *and* $Z = \{z_\alpha : \alpha < \gamma\}_< \subseteq T$, then *there is* $\{y_\alpha : \alpha < \gamma\}_< \subseteq (\sup(s), \delta_1)$ *such that:*

(a) $F(x, y_\alpha) = F(x, z_\alpha)$ *(for* $x \in s, \alpha < \gamma$*);*
(b) $F(y_\alpha, y_\beta) = F(z_\alpha, z_\beta)$ *(for* $\alpha < \beta < \gamma$*);*
(c) $F(y_\alpha, z_\beta) = G(z_\alpha, z_\beta)$ *(for* $\alpha, \beta < \gamma$*).*

*Proof.* As $\kappa$ is strongly compact, it suffices to show that for every $Z \in [T]^{<\kappa}$ there exists a function $G : Z \times Z \to \theta$ as required. Clauses (a) and (b) are obvious by Lemma 1.5, and it is clear that, if we fix $Z$, then for every $s \in [\delta_1]^{<\lambda}$ there is an appropriate $G : Z \times Z \to \theta$. We show that there is some $G : Z \times Z \to \theta$ that works for every $s$. Assume otherwise, that is, for every $G : Z \times Z \to \theta$ there is some $s_G \in [\delta_1]^{<\lambda}$ such that $G$ is not appropriate for $s_G$. Notice that the number of these functions $G$ is less than $\kappa$. Then no $G$ could be right for $s = \bigcup\{s_G : G$ a function from $Z \times Z$ to $\theta\} \in [\delta_1]^{<\lambda}$, a contradiction. $\square_{1.7}$

*Continuation of the proof of Theorem* 0.1. We now apply Lemma 1.1 to the colouring $\bar{G}\{x, y\} = \bar{G}(x, y) = \langle F(x, y), G(x, y)\rangle$ for $x < y$ in $T$ and 0 otherwise, and the filter $D_0$ and the set $T$ to get the normal filter $D_1 \supseteq D_0$, the set $A_1 \subseteq T \subseteq A'_0$ such that $A_1 \in D_1$ and the colour set $C_1 \subseteq \theta \times \theta$. Notice that actually $C_1 \subseteq C_0 \times C_0$. We can also apply Lemmas 1.5 and 1.6 to get some set $A'_1 \subseteq A_1$.

**1.8 Lemma.** *There is a set* $a \in [A'_1]^{<\kappa}$ *such that for every decomposition* $a = \bigcup\{a_{\bar{\xi}} : \bar{\xi} \in C_1\}$ *there is some* $\bar{\xi} \in C_1$ *such that:*

($\alpha$) *for every* $\bar{\varepsilon} \in C_1$ *there is an* $\bar{\varepsilon}$*-homogeneous subset for the colouring* $\bar{G}$ *of order type* $\zeta$ *in* $a_{\bar{\xi}}$*;*
($\beta$) *similarly for every* $\varepsilon \in C_0$ *and* $F$*.*

*Proof.* This follows from the strong compactness of $\kappa$, as $A'_1$ itself has this partition property (see Claim 2.8 for more details). $\square_{1.8}$

*Continuation of the proof of Theorem* 0.1. Fix a set $a$ as in Lemma 1.8.

We now describe the construction of the required homogeneous subset. Let $\delta_2 \in A'_1$ be some element with $\delta_2 > \sup(a)$. For $\bar{\xi} = (\xi_1, \xi_2) \in C_1 \subseteq \theta \times \theta$ let $a_{\bar{\xi}}$ be the following set:

$$a_{\bar{\xi}} = \{x \in a : \bar{G}(x, \delta_2) = \bar{\xi}\}.$$

By Lemma 1.8, there is some $\bar{\xi} = (\xi_1, \xi_2) \in C_1$ for which the statement in that lemma is true and necessarily (as $a \cup \{\delta_2\} \subseteq A'_1 \subseteq A_0$ and $a_{\bar{\xi}} \neq \emptyset$) we have $\xi_1, \xi_2 \in C_0$. Select some $b \subseteq a_{\bar{\xi}}$, $\text{otp}(b) = \zeta$ such that $F$ is constantly $\xi_2$ on $b$; this is possible by clause ($\beta$) of Lemma 1.8. This set $b$ will be the $\zeta$ part of our homogeneous set of ordinals of order type $\lambda + \zeta$, so we will have to construct a set of order type $\lambda$ below $b$. By induction on $\alpha$ we will choose $x_\alpha$ such that the set $\{x_\alpha : \alpha < \lambda\}_< \subseteq \delta_1$ satisfies the following conditions:

$(*)_1$ $F(x_\beta, x_\alpha) = \xi_2$ (for $\beta < \alpha$),
$(*)_2$ $F(x_\alpha, b \cup \{\delta_2\}) = \xi_2$, i.e. $F(x_\alpha, y) = \xi_2$ when $y \in b \cup \{\delta_2\}$.

Assume that we have reached step $\alpha$, that is, we are given the set of ordinals with $\{x_\beta : \beta < \alpha\}_<$ and call this set $s$. Applying Lemma 1.6 for $A_1, A'_1, \delta_2$ and $s \cup b$ and the colouring $\bar{G}$ here standing for $A_0, A'_0, \delta, s$ and the colouring $F$ there (that is, the choice of $A'_1$) we get that there exists some $\delta_3 > \delta_2$ (standing for $\delta_1$ there) such that:

(i) $\delta_3 \in A_1$;
(ii) $\bar{G}(x, \delta_3) = \bar{G}(x, \delta_2)$ for $x \in s \cup b$;
(iii) $\bar{G}(\delta_2, \delta_3) = (\xi_1, \xi_2)$.

Hence

$(*)_3$ $F(x_\beta, \delta_3) = \xi_2$ (for $\beta < \alpha$).
[Why? As $F(x_\beta, \delta_3) = F(x_\beta, \delta_2)$ by (ii) and the choice of $\bar{G}$ and $F(x_\beta, \delta_2) = \xi_2$ by $(*)_2$ from the induction hypothesis.]

$(*)_4$ $G(b \cup \{\delta_2\}, \delta_3) = \xi_2$, i.e. $G(y, \delta_3) = \xi_2$ when $y \in b \cup \{\delta_2\}$.
[Why? If $y \in b$, then by (ii) and the definition of $\bar{G}$ we have $G(y, \delta_3) = G(y, \delta_2)$, but $b \subseteq a_{\bar{\xi}}$ so by the choice of $a_{\bar{\xi}}$ we have $G(y, \delta_2) = \xi_2$. For $y = \delta_2$ use clause (iii), that is, $(\xi_1, \xi_2) = \bar{G}(\delta_2, \delta_3) = (F(\delta_2, \delta_3), G(\delta_2, \delta_3))$.]

By the choice of $G$ this implies that there is some $x_\alpha$ as required; that is, by the choice of $\bar{G}$ (see Lemma 1.7) applied to $Z = \{z_i : i < \gamma\}$, enumerating the set $b \cup \{\delta_2, \delta_3\}$ and $s$ as above, we get $\{y_i : i < \gamma\}$, now necessarily $\delta_3 = z_{\gamma-1}$, and we can choose $y_{\gamma-1}$ as $x_\alpha$. $\square_{1.1}$

## §2. The case of $\lambda$ singular

We prove version 0.2 of the main theorem.

*Proof of Theorem* 0.2. Let $\sigma = \text{cf}(\lambda)$. Let $\lambda = \sum_{\varepsilon<\sigma} \lambda_\varepsilon$ with $\lambda_\varepsilon > \sigma \geq \kappa > \theta$ strictly increasing. Let $\mu_\varepsilon = 2^{\lambda_\varepsilon}$ and $\mu = \Sigma\{\mu_\varepsilon : \varepsilon < \sigma\} = 2^{<\lambda}$. We also fix $F : [\mu^+]^2 \to \theta$.

**2.1 Claim.** *For some $\bar{\mathcal{C}}$ we have:*

(a) $\bar{\mathcal{C}} = \langle \mathcal{C}_\alpha : \alpha \in S \rangle$;
(b) $S \subseteq \mu^+, \mathcal{C}_\delta \subseteq \delta$;
(c) $\text{otp}(\mathcal{C}_\delta) \leq \sigma$;
(d) $S^* = \{\delta < \lambda : \text{otp}(\mathcal{C}_\delta) = \sigma\}$ *is stationary;*
(e) $\mathcal{C}_\delta$ *unbounded in* $\delta$ *if* $\text{otp}(\mathcal{C}_\delta) = \sigma$;
(f) $\alpha \in \mathcal{C}_\delta \Rightarrow \alpha \in S$ *and* $\mathcal{C}_\alpha = \mathcal{C}_\delta \cap \alpha$.

$\square_{2.1}$

*Proof.* By [Sh:420, §1] as $\sigma^+ < \mu^+, \sigma = \text{cf}(\sigma)$.

*Continuation of the proof of Theorem* 0.2. Let $D_0, A_0, C_0$ be as given by Lemma 1.1 with the club filter of $\mu^+, S^*$ (from clause (d) of Claim 2.1 above) here standing for $D, A^*$ there, so $A_0 \subseteq S^*$.

*Notation.* $\varepsilon(\alpha) = \text{otp}(\mathcal{C}_\alpha)$.

**2.2 Claim.** *Let $\chi > 2^\mu, <^*_\chi$ a well ordering of $\mathcal{H}(\chi)$). For any $x \in \mathcal{H}(\chi)$ we can find $\bar{\mathfrak{B}} = \langle \mathfrak{B}_\alpha : \alpha < \lambda \rangle$ such that:*

(a) $\mathfrak{B}_\alpha \prec (\mathcal{H}(\chi), \in, <^*_\chi)$;
(b) $\bar{\lambda}, \mu, F, \langle \lambda_\varepsilon : \varepsilon < \sigma \rangle, \bar{\mathcal{C}}, A_0, C_0, D_0$ *belong to* $\mathfrak{B}_\alpha$;
(c) $\langle \mathfrak{B}_\beta : \beta < \alpha \rangle \in \mathfrak{B}_\alpha$ *if* $\alpha \notin S^*$;
(d) $\|\mathfrak{B}_\beta\| = \mu_{\varepsilon(\beta)}$ *and* $[\mathfrak{B}_\beta]^{\leq \lambda_{\varepsilon(\beta)}} \subseteq \mathfrak{B}_\beta$ *and* $\mu_{\varepsilon(\beta)} + 1 \subseteq \mathfrak{B}_\beta$ *(actually follows);*
(e) $\mathfrak{B}_\alpha = \bigcup\{\mathfrak{B}_\beta : \beta \in \mathcal{C}_\alpha\}$ *if* $\alpha \in S^*$.

*Proof.* Straightforward.

2.3 *Observation.* 1) We have $\varepsilon(\alpha) < \varepsilon(\beta)$, and $\mathfrak{B}_\alpha \in \mathfrak{B}_\beta$ and $\mathfrak{B}_\alpha \prec \mathfrak{B}_\beta$ if $\alpha \in \mathcal{C}_\beta$.

**2.4 Claim.** *There is a set $A'_0 \subseteq A_0$ such that:*

($\alpha$) $A'_0 \in D_0$ *and* $\alpha < \delta \in A'_0 \Rightarrow \sup(\mathfrak{B}_\alpha \cap \mu^+) < \delta$;
($\beta$) *if* $\xi \in C_0$ *and* $\delta \in A'_0$ *and* $s \in \bigcup\{[\delta \cap \mathfrak{B}_\alpha]^{\leq \lambda_{\varepsilon(\alpha)}} : \alpha \in \mathcal{C}_\delta\}$, then *there is* $\delta_1 \in A_0$ *such that* $\delta < \delta_1$ *and*
  (a) $F(x, \delta) = F(x, \delta_1)$ *for* $x \in s$,
  (b) $F(\delta, \delta_1) = \xi$.

*Proof.* Requirement ($\alpha$) holds for all but a nonstationary set of $\delta \in A_0$. Requirement ($\beta$) is proved as in Lemma 1.6. $\square_{2.4}$

Now fix $A'_0 \subseteq A_0$ as in Claim 2.4, and fix $\delta_1 \in A'_1$ and let $T = A'_0 \backslash (\delta_1 + 1)$. Recall $\delta_1 \in A'_0 \subseteq S^* = \{\delta : \mathrm{otp}(\mathcal{C}_\delta) = \sigma, \delta = \sup(\mathcal{C}_\delta)\} \subseteq \{\delta < \mu^+ : \mathrm{cf}(\delta) = \sigma\}$.

**2.5 Claim.** *There is a function $G_\varepsilon : T \times T \to C_0$ such that:*

$\boxdot$ *if* $s \in [\delta \cap \mathfrak{B}_\alpha]^{\leq \lambda_\varepsilon}$ *and* $\varepsilon = \varepsilon(\alpha)$ *and* $\alpha \in \mathcal{C}_{\delta_1}$ *and* $\gamma < \kappa$ *and* $Z = \{z_\beta : \beta < \gamma\}_< \subseteq T$, then *there is* $\{y_\beta : \beta < \gamma\}_< \subseteq \delta \cap \mathfrak{B}_\alpha = \mu^+ \cap \mathfrak{B}_\alpha, y_0 > \sup(s)$ *such that:*
  (a) $F(x, y_\beta) = F(x, z_\beta)$ *for* $x \in s, \beta < \delta$;
  (b) $F(z_{\beta_1}, y_{\beta_2}) = G(y_{\beta_1}, y_{\beta_2})$;
  (c) $F(z_{\beta_1}, z_{\beta_2}) = F(y_{\beta_1}, y_{\beta_2})$ *for* $\beta_1 < \beta_2 < \gamma$.

*Proof.* As in Claim 1.7.

**2.6 Claim.** *There exists a function $G : T \times T \to C_0$ such that if $s \in [T]^{<\kappa}$,* then *for arbitrarily large $\varepsilon < \sigma$ we have $G \restriction (s \times s) = G_\varepsilon \restriction (s \times s)$.*

*Proof.* Let $D^*$ be a uniform $\kappa$-complete ultrafilter on $\sigma$ and define $G$ by $G(\alpha, \beta)$ is the unique $\xi \in C_0$ such that $\{\varepsilon < \sigma : G_\varepsilon(\alpha, \beta) = \xi\} \in D^*$. $\square_{2.6}$

*Continuation of the proof of Theorem* 0.2. Now we apply Lemma 1.1 to the colouring $\bar{G}$ where $\bar{G}\{x, y\} = \bar{G}(x, y) = (F(x, y), G(x, y))$ for $x < y$ in $T$ and zero otherwise and to the filter $D_0$ and the set $T$. We get a normal filter $D_1$ and a set $A_1 \subseteq T \subseteq A'_0$ and a set of colours $C_1$. As $A_1 \subseteq A_0$ necessarily $C_1 \subseteq C_0 \times C_0$.

**2.7 Claim.** *There is $A'_1 \subseteq A_1$ such that:*

($\alpha$) $A_1 \backslash A'_1 = \emptyset$ *mod* $D_1$;
($\beta$) *if* $\delta \in A'_1, \alpha \in \mathcal{C}_\delta$ *and* $s \in [\delta \cap \mathfrak{B}_\alpha]^{\leq \lambda_{\varepsilon(\alpha)}}$ *and* $\bar{\xi} \in C_1$, then *for some* $\delta_*$ *we have* $\delta < \delta_* \in A_1$ *and*
  (a) $\bar{G}(x, \delta) = \bar{G}(x, \delta_1)$ *for every* $x \in s$,
  (b) $\bar{G}(\delta, \delta_*) = \bar{\xi}$.

*Proof.* As in the proof of Lemma 1.6. $\square_{2.7}$

**2.8 Claim.** *There is a set $a \in [A'_1]^{<\kappa}$ such that:*

$\boxdot$ *for every decomposition of $a$ as $\bigcup\{a_{\bar{\xi}} : \bar{\xi} \in C_1\}$ there is $\bar{\xi} \in C_1$ such that:*
  ($\alpha$) *for every $\bar{\varepsilon} \in C_1$ there is $b \subseteq a_{\bar{\xi}}$ of order type $\zeta$ such that $\bar{G} \restriction [b]^2$ is constantly $\bar{\varepsilon}$;*
  ($\beta$) *for every $\varepsilon \in C_0$ there is $b \subseteq a_{\bar{\xi}}$ of order type $\zeta$ such that $F \restriction [b]^2$ is constantly $\varepsilon$.*

*Proof.* The claim holds since $A_1'$ has this property and $\kappa$ is strongly compact. If $A_1' = \cup\{a_{\bar{\xi}} : \bar{\xi} \in C_1\}$ for some $\bar{\xi}, a_{\bar{\xi}} \in D_1^+$ hence clause $(\alpha)$ holds by the choice of $D_1, C_1$; and clause $(\beta)$ holds as $D_1^+ \subseteq D_0^+$ (as $D_0 \subseteq D_1$) and the choice of $D_0, C_0$. $\square_{2.8}$

*Continuation of the proof of Theorem* 0.2. Now choose $\delta_2 \in A_1'$ such that $\delta_2 > \sup(a)$ and for $\bar{\xi} = (\xi_1, \xi_2) \in C_1 \subseteq \theta \times \theta$ define $a_{\bar{\xi}}$ as

$$\bar{a}_{\bar{\xi}} = \{x \in a : \bar{G}(x, \delta_2) = \bar{\xi}\}.$$

Clearly $\langle a_{\bar{\xi}} : \bar{\xi} \in C_1\rangle$ is a decomposition of $a$ and so there is $\bar{\xi} = (\xi_1, \xi_2) \in C_1$ as guaranteed by $\boxdot$ of Claim 2.8. In particular, there is $b \subseteq a_{\bar{\xi}}$ of order type $\zeta$ such that $F \restriction [b]^2$ is constantly $\xi_2$ (note that $(\xi_1, \xi_2) \in C_1 \subseteq C_0 \times C_0$ so $\xi_2 \in C_0$). Now let $E = \{\varepsilon < \sigma : G_\varepsilon(\alpha, \delta_2) = G(\alpha, \delta_2)$ for every $\alpha \in b\}$. By the definition of $G$ this is an unbounded subset of $\sigma$ and clearly

$(*)$ if $\varepsilon \in E$ and $\alpha \in b$, then $G_\varepsilon(\alpha, \delta_2) = G(\alpha, \delta_2) = (\xi_1, \xi_2)$.

For $\alpha < \lambda$ let $\Upsilon(\alpha) = \text{Min}\{\varepsilon \in E : \alpha < \lambda_\varepsilon\}$ and let $C_{\delta_1} = \{\gamma(\Upsilon) : \Upsilon < \sigma\}_<$. Now we try to choose by induction on $\alpha < \lambda$ a element $x_\alpha$ satisfying

$(*)_0$ $x_\alpha < \delta_1$ and moreover $x_\alpha \in \delta_1 \cap \mathfrak{B}_{\gamma(\Upsilon(\alpha))}$, and $\beta < \alpha \Rightarrow x_\beta < x_\alpha$,
$(*)_1$ $F(x_\beta, x_\alpha) = \xi_2$ for $\beta < \alpha$,
$(*)_2$ $F(x_\alpha, \beta) = \xi_2$ for $\beta \in b \cup \{\delta_2\}$.

At step $\alpha$, by Claim 2.7, that is, by the choice of $A_1'$ applying clause $(\beta)$ there with $\{x_\beta : \beta < \alpha\} \cup b, \delta_2, \bar{\xi}$ here standing for $s, \delta, \bar{\xi}$ there, we can find $\delta_3$ satisfying the requirement there on $\delta_1$, so

(i) $\delta_2 < \delta_3 \in A_1$,
(ii) $\bar{G}(x, \delta_3) = \bar{G}(x, \delta_2)$ for $x \in s \cup b$,
(iii) $\bar{G}(\delta_2, \delta_3) = (\xi_1, \xi_2)$.

Now

$(*)_3$ $F(x_\beta, \delta_3) = \xi_2$ for $\beta < \alpha$.
[Why? By (ii) we have $\bar{G}(x_\beta, \delta_3) = \bar{G}(x_\beta, \delta_2)$, hence $F(x_\beta, \delta_3) = F(x_\beta, \delta_2)$, but the latter by $(*)_2$ is equal to $\xi_2$.]
$(*)_4$ $G(\beta, \delta_3) = \xi_2$ for $\beta \in b$.
[Why? By (ii) and as $\beta \in b \Rightarrow \bar{G}(\beta, \delta_2) = (\xi_1, \xi_2) \Rightarrow G(\beta, \delta_2) = \xi_2$).]
$(*)_5$ $G(\delta_2, \delta_3) = \xi_2$.
[Why? By clause (iii).]
$(*)_6$ $\{x_\beta : \beta < \alpha\}$ is a subset of $\delta_1 \cap \mathfrak{B}_{\gamma(\Upsilon(\alpha))}$.

Let $\langle y_i : i < \zeta + 2\rangle$ list $b \cup \{\delta_2, \delta_3\}$ increasing order.

Now we use the choice of $G_{\Upsilon(\alpha)}$ to choose an increasing sequence $\langle z_i : i < \zeta + 2\rangle$ in $\delta_1 \cap \mathfrak{B}_{\gamma(\Upsilon(\alpha))}, z_0 > x_\beta$ for $\beta < \alpha$ such that $F(z_i, y_j) = G(y_i, y_j)$ for $i, j < \zeta + 2$ and $F(x_\beta, z_i) = F(x_\beta, y_i)$ for $i < \zeta + 2$. Let $x_\alpha = z_{\zeta+1}$ so $x_\alpha = \delta_1 \cap \mathfrak{B}_{\gamma(\Upsilon(\alpha))}$ is $> x_\beta$ for $\beta < \alpha$.

Also $x_\alpha$ satisfies $(*)_0$ of the recursive definition. Now $\beta < \alpha \Rightarrow F(x_\beta, x_\alpha) = F(x_\beta, z_{\zeta+1}) = F(x_\beta, y_{\zeta+1}) = F(x_\beta, \delta_3)$ which is $\xi_2$ by $(*)_3$ above, so for our choice of $x_\alpha$, $(*)_1$ holds. Next if $\beta \in b \cup \{\delta_2\}$, then $F(x_\alpha, x_\beta) = F(x_\beta, z_{\zeta+1}) = G(x_\beta, \delta_3)$ which is $\xi_2$ by $(*)_4$ or $(*)_5$. So $x_\alpha$ is as required. $\square_{0.2}$

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