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A NOTE ON CARDINAL EXPONENTIATION

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Abstract. Silver and subsequently Galvin and Hajnal, got bounds on $2^{\aleph_\alpha}$, for $\aleph_\alpha$ strong limit cardinal of cofinality $>\aleph_0$. We somewhat improve those results.

We discuss here bounds on $2^{\aleph_\alpha}$ for $\aleph_\alpha$ strong limit, a problem on which much work was done lately, and whose history is well known (see Silver [S], Galvin and Hajnal [GH], Magidor [Mg 1], Baumgartner and Prikry [BP], Jech and Prikry [JP] and, from other angles, Jensen's *Marginalia* (Devlin and Jensen [DJ]) and Magidor [Mg 2], [Mg 3]. We continue [GH].

For simplicity in the introduction we concentrate on the case cf $\aleph_\alpha = \aleph_1$. Let $D(st)$, $D(ub)$ be the filters of closed unbounded subsets of ω_1, cobounded subsets of ω_1 respectively. For $f \in \mathrm{ord}^{\omega_1}$ $\|f\|_{D(st)}$ is the rank of f, i.e., it is the minimal α such that $g <_{D(st)} f$ implies $\|g\|_{D(st)} < \alpha$. As D is $\aleph_1$ closed this is well defined. By Galvin and Hajnal [GH], $2^{\aleph_\alpha} \leq \aleph_{\alpha(*)}$, where $\alpha(*) \leq \|\alpha\|_{D(st)}$; now easily $\|\alpha\|_{D(st)} < (|\alpha|^{\aleph_1})^+$, so we get bounds on $2^{\aleph_\alpha}$. Three natural questions arise.

(α) Can we get any result when cf $\alpha = \aleph_0$?

(β) Can we improve the bound of $\|\alpha\|_{D(st)}$? on $2^{\aleph_\alpha}$?

(γ) The first α on which the theorem says nothing is the first $\alpha = \aleph_\alpha$, cf $\alpha = \omega_1$. Can we nevertheless prove something on this α?

Concerning (α), as stated we know nothing at present. The expected result is that, assuming the consistency of some large cardinal, $\aleph_\omega$ may be strong limit, and $2^{\aleph_\omega}$ can be any $\aleph_\alpha$, cf $\aleph_\alpha > \aleph_\omega$, but the present consistency results (Magidor [Mg 2], [Mg 3]) are far from this. It may be interesting to note that by [Sh 2], if cf $\alpha = \aleph_0$, $\alpha = \bigcup \alpha_n$, $\alpha_n < \alpha_{n+1}$, D a nonprincipal ultrafilter over ω, then the cofinality of $\prod_{n<\omega}(\aleph_{\alpha_n}, <)/D$ is $< \aleph_{\alpha(*)}$, $\alpha(*) < (2^{|\alpha|})^+$, and in $(\aleph_\alpha, <)^\omega/D$ there is no increasing sequence of length $\aleph_{\alpha(*)}$. Consequently there is $G \subseteq \prod_{n<\omega} \aleph_{\alpha_n}$, $|G| < \aleph_{\alpha(*)}$ such that for every $f \in \prod_{n<\omega} \aleph_{\alpha_n}$ for some $g \in G$, $\{n; f(n) \geq g(n)\}$ is finite.

Concerning question (β), Magidor [Mg 1] proves, assuming Chang-conjecture, that if $\alpha = \omega_1$ then $2^{\aleph_\alpha} < \aleph_{\omega_2}$. We notice that part of his argument implies immediately $\|\omega_1\|_{D(st)} = \omega_2$ (see 11), thus the conclusion follows from [GH]. This was noticed by Benda, too.

We get here (in conclusion 25) better bounds on $\|\alpha\|_{D(st)}$ for "large α", more exactly $\alpha \geq (2^{\aleph_1})^+$. E.g., if $|\alpha| = \beth_\omega$, then $\|\alpha\|_{D(st)} < \beth_\omega^+$. For this we have to use our main technical Lemma 19. Let for $g \in \mathrm{card}^{\omega_1}$, D an $\aleph_1$-complete filter on ω_1,

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$T_D(g) = \mathrm{Sup}\{|G|: G$ a family of functions $f \in \mathrm{ord}^{\omega_1}$, $(\forall i) f(i) < g(i)$, such that $f_1 \neq f_2 \in G \Rightarrow f_1 \neq_D f_2$ i.e., $\{i: f_1(i) \neq f_2(i) \in D\}$. The main technical lemma of [GH] is that if $g(i) = \aleph_{\alpha+f(i)}$, $\aleph_\alpha$ strong limit, then $T_D(g) \leq \aleph_{[\alpha+\|f\|_D]}$, and they also proved $T_D(\lambda) = 2^\lambda$ if $\lambda = \lambda^{\aleph_0}$, $2^\lambda = \lambda^{\aleph_1}$, which is the case for strong limit $\lambda = \aleph_\alpha$, cf $\aleph_\alpha = \aleph_1$. We prove a kind of converse (Lemma 19): if $\|f\|_D \geq \lambda > 2^{\aleph_1}$, λ regular, $g(i) = |f(i)|$, $g \in \mathrm{card}^{\omega_1}$, then for some $\aleph_1$-complete filter $D_1 \supseteq D$, $T_{D_1}(g) \geq \lambda$. As $T_{D_1}(\beth_\omega) = \beth_\omega$ for any $\aleph_1$-complete $D_1 \supseteq D$, we prove the above mentioned result. If, e.g., $\beth_\omega \equiv \aleph_\omega$ we can get similar results for $\|\gamma\|_D$, $\gamma < \aleph_{\alpha+\omega_1}$. However, for $\aleph_{\alpha+\omega_1}$ we can bound $T_D(\aleph_{\alpha+\omega_1})$ by Galvin-Hajnal result, and then use the above mentioned result to bound $\|\aleph_{\alpha+\omega_1}\|_D$. We are thus forced to prove results on a family $\boldsymbol{D}$ of $\aleph_1$-complete filters (sometimes-normal). We can in this way get bounds on $\|\gamma\|_D$, $T_D(\lambda)$ for γ, λ smaller than the first $\beta = \aleph_\beta$.

This leads us naturally to the third problem. We get a result only if we assume, e.g., Chang conjecture: if $\aleph_\lambda(\aleph_0)$ is the first cardinal $\alpha = \aleph_\alpha$ of confinality λ (λ a regular cardinal) and $\aleph_{\aleph_1}(\aleph_0)$ is strong limit, then $2^{\aleph_1(\aleph_0)} < \aleph_{\aleph_2}(\aleph_0)$.

The results in this paper are more elaborate (e.g, "$\aleph_\alpha$ strong limit" can be considerably weakened); they were announced in [Sh 1], and a preliminary version was [Sh 3], as remarks to the book of Erdös, Hajnal, Mate and Rado on partition calculus. We end by discussing whether Chang conjecture can be eliminated.

Added in proof. Further results were obtained and announced in the ***Notices of the American Mathematical Society***, vol. 25 (1978), p. A-599.

1. *Notation.* (A) κ will be a fixed regular cardinal $> \aleph_0$.

(B) I a family of sets of ordinals $\bigcup_{t \in I} t = \delta(I)$.

(C) D a κ-complete filter over I such that $\{t: \alpha \in t\} \in D$ for any $\alpha < \delta(I)$.

(D) $\mu(D)$ the maximal μ such that D is μ-complete (hence μ is regular).

(E) For $f, g \in \mathrm{ord}^I$, $f <_D g$ if $\{t \in I: f(t) < g(t)\} \in D$, and similarly for $\leq, \geq$, $=$, etc.

(F) For $J \subseteq I$, $f \restriction J <_D g \restriction J$ if $\{t \in J; f(t) < g(t)\} \cup (I - J) \in D$.

(G) $[E]$ is the filter generated by E, but $[D \cup \{A\}]$ is denoted by $D + A$.

(H) The function from I with constant value c is denoted by c or c_I.

(I) $D(st) = D^\kappa(st)$ is the filter over κ generated by the closed unbounded sets.

(J) $D(ub) = D^\kappa(ub)$ is the filter over κ generated by the sets with a bounded complement (both filters are κ-complete).

(K) D is μ-incomplete if there are μ sets in D with empty intersection which form a decreasing sequence (this is not negation of any kind of μ-complete).

(L) Let $\lambda^{<\kappa} = \sum_{\mu<\kappa} \lambda^\mu$.

2. Definition. For $f \in \mathrm{ord}^I$ we define the rank $\|f\|_D$ as the minimal ordinal such that

$$g <_D f \Rightarrow \|g\|_D < \|f\|_D$$

(as D is $\aleph_1$-complete, it is always well defined).

3. Definition. For every $f \in \mathrm{Card}^I$ we define $T_D(f)$ as the supremum of $|G|$, G a family of functions from I to ordinals such that;

$$g \in G \Rightarrow g <_D f; \qquad g_1 \neq g_2 \in G \Rightarrow g_1 \neq_D g_2$$

(clearly we can replace $g <_D f$ by $\{t: g(t) \in A_t\} \in D$ where $|A_t| = f(t)$.

4. Definition. D is called a normal, *if* when $I \supseteq A \neq \emptyset$ mod D (i.e., $I - A \notin D$) f a choice function on A (i.e., $(\forall t \in A)\ [f(t) \in t]$) *then* for some c, $A \cap f^{-1}(c) \neq \emptyset$ mod D.

5. *Fact.* (A) $f =_D g \Rightarrow \|f\|_D = \|g\|_D$, $T_D(f) = T_D(g)$.

(*B*) For $A \neq \emptyset$ mod D, $\mu(D + A) \geq \mu(D)$. If D is $\mu(D)$-incomplete, which holds in all our examples, equality holds.

(C) $A \neq \emptyset$ mod D, D normal implies $D + A$ is normal.

6. Lemma. (A) *If I, D_l $(l = 1, 2)$ are as in* 1, $D_1 \subseteq D_2$ *then for* $f \in \mathrm{ord}^I$, $\|f\|_{D_1} \leq \|f\|_{D_2}$ *and for* $g \in \mathrm{card}^I$: $T_{D_1}(g) \leq T_{D_2}(g)$.

(B) *If I_l, D_l $(l = 1, 2)$ are as in A, $h: I_2 \to I_1$ and $A \in D_1 \Rightarrow h^{-1}(A) \in D_2$ then for* $f \in \mathrm{ord}^{I_1}$ $\|f\|_{D_1} \leq \|hf\|_{D_2}$ *and for* $g \in \mathrm{card}^{I_2}$, $T_{D_1}(g) \leq T_{D_2}(hg)$. *If the hypothesis holds, we write* $D_1 \leq_{RK} D_2$.

Proof. Easy and known.

7. Lemma. (A) *If D is κ-incomplete, then for every α,* $\|\alpha\|_{D(ub)} \leq \|\alpha\|_D$, *in fact* $D(ub) \leq_{RK} D$. *If in addition D is κ-complete and normal then* $D(st) \leq_{RK} D$.

(B) *If $f \in \kappa^\kappa$ is monotone, i.e., $\alpha < \beta \Rightarrow f(\alpha) \leq f(\beta)$, then* $\|f\|_{D(st)} = \|f\|_{D(ub)}$.

Proof. (A) Use 6(B), for let $A_i \in D$, $\bigcup_{i<\kappa} A_i = \emptyset$, A_i decreasing: define $h: I \to \kappa$ by $h(t) = \{\min i: t \in A_i\}$. Now assume D is normal κ-complete.

Note that for each $\alpha < \delta(I)$, $\{t \in I: \alpha \in t\} \in D$, and D is κ-complete. So for every $i < \kappa$, $\{t \in I: i \subseteq t\} \in D$, so w.l.o.g. $t \in A_i \Rightarrow i \subseteq t$. We can also assume A_i is continuous, $A_0 = I$. So for every t for a unique $i = h(t)$, $t \in A_i - A_{i+1}$. So $h(t) \subseteq t$. Let $B = \{t: h(t) \in t\}$. So $t \in I - B$ implies $h(t) = t \cap \kappa$.

If $B \neq \emptyset$ mod D, h is a choice function on B: (it is a choice function by the definition of B). So as D is normal for some i, $B_1 = \{t \in B: h(t) = i\} \neq \emptyset$ mod D. So $B_1 \subseteq A_i - A_{i+1}$, contradicting the hypothesis $A_{i+1} \in D$. So $B = \emptyset$ mod D, and w.l.o.g. $B = \emptyset$. If $C \subseteq \kappa$ is closed unbounded we shall show that $h^{-1}(C) \in D$, this suffices by 6(B). So suppose $h^{-1}(C) \notin D$ so $A = I - h^{-1}(C) \neq \emptyset$ mod D. Define a choice function f on A: $f(t) = \max[C \cap h(t)]$; it exists as $C \cap h(t)$ is a bounded subset of $\kappa \cap h(t)$ by A's definition, and $\max(C \cap h(t))$ exists and is in t as C is closed. By the normality of D for some $\alpha < \kappa$, $f^{-1}(\{\alpha\}) \neq \emptyset$ mod D, but trivially $f^{-1}(\alpha) \subseteq \{t: t \cap C \subseteq (\alpha + 1)\} = \emptyset$ mod D, contradiction.

(B) By 6(A), $\|f\|_{D(ub)} \leq \|f\|_{D(st)}$; so we prove by induction on ξ, that $\|f\|_{D(st)} \geq \xi \Rightarrow \|f\|_{D(ub)} \geq \xi$, for monotonic f.

For $\xi = 0$, ξ limit, this is trivial; so let $\xi = \zeta + 1$; hence there is $g <_D f$, $\|g\|_{D(st)} \geq \zeta$; so for some closed unbounded $C \subseteq \kappa$, $t \in C \Rightarrow g(t) < f(t)$ and let $C = \{\alpha(\mathrm{i}): i < \kappa\}$. We can assume:

(I) $$\alpha \in C, \qquad \beta < \alpha \Rightarrow f(\beta) < \alpha.$$

If for some α, $g^{-1}(\alpha)$ is stationary, choose a minimal such α, so easily $\|g\|_{D(st)} = \alpha$; then $\alpha_\kappa \leq_{D(st)} g <_{D(st)} f$ and as f is monotonic $\alpha_\kappa <_{D(ub)} f$, so

$$\|f\|_{D(ub)} > \|\alpha_\kappa\|_{D(ub)} = \alpha = \|g\|_{D(st)} \geq \zeta.$$

So, by Fodour's theorem we can assume

(II) $$g(\alpha(i)) \geq \alpha(i).$$

Define

$$g^*(\beta) = \begin{cases} 0, & \beta < \alpha(0), \\ g(\alpha(i)), & \alpha(i) \leq \beta < \alpha(i+1). \end{cases}$$

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By (I) and (II), g^* is increasing and $g^* <_{D(ub)} f$ and $g^* =_{D(st)} g$ so the result follows.

8. *Question.* Is $\|f\|_{D^\kappa(st)} = \|f\|_{D^\kappa(ub)}$ always; for f constant? Does $A \neq \emptyset \bmod D_{(st)} \Rightarrow \|\kappa\|_{D(st)} = \|\kappa\|_{D(st)+A}$?

9. DEFINITION. $I_{\mathrm{Mg}}(\delta, \rho)$ is the family of subsets of δ of order type $\geq \rho$ but $< |\rho|^+$. (So $\delta(I_{\mathrm{Mg}}(\delta, \rho)) = \delta$.) $D_{\mathrm{Mg}}(\delta, \rho)$ is the filter over $I_{\mathrm{Mg}}(\delta, \rho)$ generated by the following sets. For any model M with universe δ and language $<\kappa$, $A_M = \{|N| \in I_{\mathrm{Mg}}(\delta, l) : N \prec M,\ |N| \cap \kappa$ a proper initial segment of $\kappa\}$ (the last condition is not really necessary, in our results for successor κ we got the same filter) (Mg for Magidor). Theorems 10, 11 are rephrasings of, or obvious from [Mg 1] who uses $D_{\mathrm{Mg}}(\omega_2, \omega_1)$. Omitting ρ we mean order type $<\kappa$.

10. THEOREM. (A) *$D_{\mathrm{Mg}}(\lambda, \rho)$ is nontrivial if $\lambda = \aleph_2$, $\rho = \kappa = \aleph_1$ and Chang conjecture holds (this is in fact equivalent) or $\lambda > \rho \geq \kappa$, λ a Ramsey Cardinal (i.e., $\lambda \to (\lambda)_2^{<\omega}$). Using $D_{\mathrm{Mg}}(\lambda, \rho)$ we assume implicitly it is nontrivial.*

(B) *$D_{\mathrm{Mg}}(\delta, \rho)$ is normal, κ-complete and κ-incomplete, $I_{\mathrm{Mg}}(\delta, \rho) = \delta$.*

11. THEOREM. *Suppose*

(*) *D is a normal filter over I, $f_D \in \mathrm{ord}^I$, $f_D(t)$ is the order-type of t. Then*

(A) *$\|f_D\|_D = \delta(I)$, so if $\rho_I \leq_D f_D$, then $\|\rho\|_D \leq \delta(I)$ (this occurs if $D_{\mathrm{Mg}}(\lambda, \rho)$ is not trivial).*

(B) *For $f \in \mathrm{ord}^I$ let $\hat{f}$ be defined by: $\hat{f}(t)$ is the $f(t)$th element of $t \cup \{\mathrm{Sup}\ t\}$ if it exists, and $\min t$ otherwise (where $\mathrm{Sup}\ t = \min\{\alpha : (\forall \beta \in t)(\beta < \alpha)\}$. Then for every $A \neq \emptyset \bmod D$, $f \leq_D f_D$, $\|f\|_{D+A} = v(\hat{f}, A) = v_D(\hat{f}, A) = \min\{\alpha : \alpha \geq \delta(I)$ or $\{t : \hat{f}(t) = \alpha\} \neq \emptyset \bmod D + A\}$.*

(C) *For any $f \leq_D f_D$, for some $A \neq \emptyset \bmod D$, for any normal D_1, if $D + A \subseteq D_1$ then $\|f\|_D = \|f\|_{D_1}$ (it is $\delta(I)$ if $f =_{D_1} f_D$, and $v(\hat{f}, I)$ otherwise).*

PROOF. (A) Follows by (B).

(B) Let

$$S_1 = \{f : f \in \delta(I)^I, f \leq f_D \text{ (i.e., } t \in I \Rightarrow f(t) \leq f_D(t))\},$$

$$S_2 = \{f : \mathrm{Dom}\, f = I, \text{ and } f(t) \in t \cup \{\mathrm{Sup}\ t\} \text{ for } t \in I\}.$$

Note that $S_1, S_2 \neq \emptyset$ and if $f \leq_D f_D$ then for some $f' =_D f$, $f' \in S_1$, and if $\{t : t \in I,\ f(t) \in t \cup \{\mathrm{Sup}\ t\}\} \in D$ then for some $f' =_D f$, $f' \in S_2$. Also $f =_D f'$, $f \leq_D f_D$ implies $\hat{f} =_D \hat{f}'$.

It is easy to check that $f \mapsto \hat{f}$ is a one-to-one mapping from S_1 onto S_2. Also $f_1 <_D f_2$ iff $\hat{f}_1 <_D \hat{f}_2$. For $f \in S_2$, $A \subseteq I$, $A \neq \emptyset \bmod D$, let

$$v(f, A) = \min\{\alpha : \{t \in A : f(t) = \alpha\} \neq \emptyset \bmod D \text{ or } \alpha \geq \delta(I)\}.$$

As D is normal, $v(f, A)$ is well defined; and, if $f \neq_{D+A} f_D$ then

(0) $$v(f, A) = \max\{\alpha : \{t \in A : f(t) < \alpha\} = \emptyset \bmod D\}.$$

Now we prove for $f \in S_1$, $A \subseteq I$, $A \neq \emptyset \bmod D$

(1) $$\|f\|_{D+A} = v(\hat{f}, A).$$

For one inequality, we prove by induction on α, that

(2) $$\alpha \leq v(\hat{f}, A) \text{ implies } \alpha \leq \|f\|_{D+A}$$

For $\alpha = 0$, α limit, this is immediate, for $\alpha = \beta + 1$, clearly $\beta < \delta(I)$, so let $g_\beta \in \mathrm{ord}^I$ be defined by

$$g_\beta(t) = \begin{cases} \beta & \text{if } \beta \in t, \\ \sup t & \text{if } \beta \notin t. \end{cases}$$

Clearly $g_\beta \in S_2$, so for some $f_\beta \in S_1, \hat{f}_\beta = g_\beta$. It is also clear that $g_\beta <_{D+A} \hat{f}$ (by the definition of $\nu(\hat{f}, A)$), hence $f_\beta <_{D+A} f$, so $\|f\|_{D+A} > \|f_\beta\|_{D+A}$, but clearly $\nu(g_\beta, A) = \beta$, hence by the induction hypothesis (as $\hat{f}_\beta = g_\beta$) $\|f_\beta\|_{D+A} \geq \beta$; together we get $\|f\|_{D+A} \geq \beta + 1 = \alpha$.

To complete the proof of (1) we have to prove the other inequality, i.e., we prove by induction on α that

(3) $$\alpha \leq \|f\|_{D+A} \quad \text{implies} \quad \alpha \leq \nu(\hat{f}, A).$$

For $\alpha = 0$, α limit, this is obvious (see the definition of $\nu(f, A)$). So assume $\alpha = \beta + 1$, so there is $g \in \mathrm{ord}^I$, $g <_{D+A} f$, $\beta \leq \|g\|_{D+A}$, there is $g_1 \in S_1$, $g_1 =_{D+A} g$, so $g_1 <_{D+A} f$, $\beta \leq \|g\|_{D+A}$, and so $\hat{g}_1$ is well defined and $\hat{g}_1 <_{D+A} \hat{f}$. By the induction hypothesis, as $\beta \leq \|g_1\|_{D+A}$, $\beta \leq \nu(\hat{f}, A)$ so by (0), $\{t \in A: \hat{g}_1(t) < \beta\} = \emptyset$ mod D, but

$$\{t \in A: \hat{f}(t) < \alpha\} = \{t \in A: \hat{f}(t) \leq \beta\} \subseteq \{t \in A: \hat{g}_1(t) < \beta\} \mod D$$

(because $\hat{g}_1 <_D \hat{f}$). So combining $\{t \in A: \hat{f}(t) < \alpha\} = \emptyset$ mod D, but this implies $\nu(\hat{f}, A) \geq \alpha$, as required.

(C) Easy by (B).

12. *Conclusion.* If D is normal and κ-incomplete and each $t \in I$ has order type $\geq \rho$ then $\|\rho\|_{D(st)} \leq \delta(I)$, so if $D_{\mathrm{Mg}}(\delta, \rho)$ is nontrivial, $\|\rho\|_{D(st)} \leq \delta$.

PROOF. By 6(B), 7(A) and last lemma.

13. LEMMA. (A) *If* $\mu < \mu(D)$, $I = \bigcup_{j<\mu} I_j$, $f, f_j \in \mathrm{ord}^I$, $f \restriction I_j \geq_D f_j \restriction I_j$, *then* $\|f\|_D \geq \min_{j<\mu} \|f_j\|_D$.

(B) *If* $(*)D$ *is normal*, $I\alpha \subseteq I(\alpha < \delta(I))$, $\bigcup_\alpha I_\alpha = I$, (*so w.l.o.g. they form a partition of* I) *and* $t \in I_\alpha \Rightarrow \alpha \in t$, *and* $f \restriction I_\alpha \geq_D f_\alpha \restriction I_\alpha$ *then* $\|f\|_D \geq \min_\alpha \|f_\alpha\|_D$.

PROOF. We prove (B) only, as the proof is the same (in fact the property of the I_α's we need is $A \neq \emptyset$ mod $D \Rightarrow (\exists\alpha)\, I_\alpha \cap A \neq \emptyset$ mod D).

We prove by induction on ξ that if for all α, $\|f_\alpha\|_D \geq \xi$ then $\|f\|_D \geq \xi$. For $\xi = 0$, ξ limit, it is immediate. So let $\xi = \zeta + 1$; so there are $f^*_\alpha <_D f_\alpha$, $\|f^*_\alpha\|_D \geq \zeta$. Define $f^* = \bigcup_\alpha f^*_\alpha \restriction I_\alpha$. By the induction hypothesis $\|f^*\|_D \geq \zeta$, so it suffices to show $f^* <_D f$. Otherwise let $J = \{t \in I: f(t) \leq f^*(t)\} \neq \emptyset$ mod D. On J we define a choice function h: $h(t) = \alpha \Leftrightarrow t \in I_\alpha$. By normality h is constant on some $J_1 \subseteq J$, $J_1 \neq \emptyset$ mod D, with value α_*, so $J_1 \subseteq I_{\alpha_*}$ so

$$f^* \restriction J_1 = (f^*_{\alpha_*} \restriction J_1) <_D (f_{\alpha_*} \restriction J_1) \leq f \restriction J_1$$

contradiction to J's definition.

13(C). REMARK. We can replace $\min_j \|f_j\|_D$ by $\min_j \|f_j\|_{D+I_j}$. (Also in 14(A)).

14. LEMMA. (A) *If in* 13(A) (*or* (B)) $f, f_j \in \mathrm{card}^I$, *then* $T_D(f) \geq \min_j T_D(f_j)$.

(B) *Suppose* $g \in \mathrm{Card}^I$, $(\forall t)[\mathrm{cf}\, g(t) = \mu]$, $\mu < \mu(D)$, *and let* $g(t) = \sum_{\alpha<\mu} g_\alpha(t)$, $g_\alpha(t) < g(t)$. *Then*

$$T_D(g) \leq \chi = 2^{|I|} + \sup\{T_{D+A}(g_\alpha): \alpha < \mu, A \neq \emptyset \bmod D\}.$$

PROOF. (A) If $\{g^i_j : i < \chi^+\}$ exemplifies $T_D(f_j) > \chi$, let $g_i = \bigcup_j (g^j_i \upharpoonright I_j)$ and $\{g_i : i < \chi^+\}$ exemplify $T_D(f) > \chi$.

(B) Suppose $\{f_i : i < \chi^+\}$ contradicts $T_D(g) \le \chi$. For each f_i let $A^\alpha_i = \{t \in I: f_i(t) \le g_\alpha(t)\}$, so $I = \bigcup_{\alpha<\mu} A^\alpha_i$ so for some $\alpha(i)$, $A^{\alpha(i)}_i \neq \emptyset$ mod D and as $2^{|I|} \le \chi$ for some α^*, A: $|\{i : \alpha(i) = \alpha^*, A^{\alpha(i)}_i = A\}| = \chi^+$. Then $\{f_i : i < \chi^+, \alpha(i) = \alpha^*, A^{\alpha(i)}_i = A\}$, show $T_{D+A}(g_{\alpha^*}) > \chi$, contradiction.

15. THEOREM (GALVIN AND HAJNAL [GH]). (A) $T_{D(st)}(\lambda^{<\kappa}) = \lambda^\kappa$ (*Note. If* $\aleph_\alpha$ *is strong limit of cofinality* κ, $\aleph_\alpha = \aleph_\alpha^{<\kappa} < \aleph_\alpha^\kappa = 2^{\aleph_\alpha}$) [*in fact* $T_{D(ub)}(\lambda^{<\kappa}) = \lambda^\kappa$, *similarly in B*].

(B) *If* $\alpha = \bigcup_{i<\kappa} \alpha_i$; $\alpha_i < \alpha$ *increasing* $\prod_{j<i} \aleph_{\alpha_j} = g(i)$ *then* $T_{D(st)}(g) = \aleph_\alpha^\kappa$.

16. THEOREM (GALVIN AND HAJNAL [GH]). *If* $g(i) = \aleph_{[\alpha_i + f(i)]}$, α_i $(i < \kappa)$ *increasing and continuous*, $g \in \mathrm{card}^\kappa$, *then* $T_{D(st)}(g) \le \aleph_{[\alpha + \|f\|_{D(x)}]}$ *where* $\aleph_\alpha = \sup_i T_{D(st)}(\aleph_{\alpha_i})$, $2^\kappa \le \aleph_\alpha$.

Combining 15, 16, they get bounds on λ^κ, if we have bounds of $\|f\|_{D(st)}$. They use: (for $D = D(st)$) the following bound on $\|f\|_{D(st)}$:

17. LEMMA [GH]. (A) $\|f\|_D < |\prod_s f(s)|^+$, *so* $\|\alpha\|_D < (|\alpha|^{|I|})^+$.

(B) $T_D(g) \le \prod_{s \in I} g(s)$ (*when* $g(s) \neq 0$).

We can generalize 16 trivially to, e.g.:

18. LEMMA. (A) *If* $g(i) = \aleph_{\alpha+f(i)}$, $(i < \kappa)$, $g \in \mathrm{card}^\kappa$, *then* $T_{D(ub)}(g) \le \aleph_{[\beta + \|f\|_{D(ub)}]}$ *when* $T_{D(ub)}(\aleph_\alpha) = \aleph_\beta$, $2^\kappa \le \aleph_\beta$. *So if* $\beta\omega < f(\rho)$ *for* $\rho < \kappa$ *then* $T_{D(ub)}(g) \le \aleph_{\|f\|_{D(ub)}}$ (*provided that* $2^\kappa \le \aleph_\beta$).

(B) *If* $g(t) = \aleph_{h(t)+f(t)}$ (*for* $t \in I$), $h, f \in \mathrm{ord}^I$, $g \in \mathrm{card}^I$ *then* $T_D(g) \le \aleph_{[\beta + \|f\|_D]} + 2^{|I|}$ *where* $\aleph_\beta = \sup\{T_{D+A}(h) : A \neq \emptyset \bmod D\}$.

18(C) REMARK. See 7(A), (B); 18(A) will be really interesting if 8 is answered negatively but we use 18 for the induction in 26.

19. MAIN LEMMA. *Suppose* $\|f\|_D \ge \lambda > 2^{|I|}$, λ *regular*, $|f(s)| = g(s)$, $g \in \mathrm{card}^I$ *then for some filter* D_1 *over* I:

(i) $\mu(D) \le \mu(D_1)$ *and* $D \subseteq D_1$,

(ii) D_1 *is normal if* D *is normal*,

(iii) $T_{D_1}(g) \ge \lambda$.

PROOF. Clearly there is $f^* \le_D f$, such that $\|f^*\|_D = \lambda$, and for each $\alpha < \lambda$ there is $f_\alpha <_D f^*$, $\|f_\alpha\|_D = \alpha$. We define a filter D_* extending D: its generators are intersections of a member of D with $< \mu(D)$ set $I - J$ where $(\exists \alpha < \lambda)\ (\exists^\lambda \beta > \alpha)$ $(f_\beta \upharpoonright J = f_\alpha \upharpoonright J)$. Clearly D_* is closed under supersets and intersection of $< \mu(D)$ sets. We prove it is nontrivial. So suppose $J \in D$, $J_i \subseteq I$, $i < \alpha^* < \mu(D)$ and $\alpha_i < \lambda$, $\gamma \le \beta_{i,\gamma} < \lambda$ for $i < \alpha^*$, $\gamma < \lambda$ and $f_{\beta_{i,\gamma}} \upharpoonright J_i =_D f_{\alpha_i} \upharpoonright J_i$ and we have to prove $J \cap \bigcap_{i<\alpha_*} (I - J_i) \neq \emptyset$. Suppose not, let $J'_0 = J_0 \cup (I - J)$ and $J'_i = J_i - \bigcup_{j<i} J'_j$ for $i < \alpha^*$. Then J'_i $(i < \alpha^*)$ form a partition of I and still $f_{\beta_{i,\gamma}} \upharpoonright J'_i =_D f_{\alpha_i} \upharpoonright J'_i$. So without loss of generality the J_i $(i < \alpha^*)$ form a partition of I.

Let $f^+ = \bigcup_i (f_{\alpha_i} \upharpoonright J_i)$, using $f_{\beta_{i,\gamma}}$ $(i < \alpha^*)$ we see, by 13(A), that $\|f^+\|_D \ge \gamma$; as this holds for every γ $\|f^+\|_D \ge \lambda$; but $f^+ <_D f^*$ by its definition as $f_{\alpha_i} <_D f^*$ contradiction; so D_* is nontrivial, $\mu(D_*) \ge \mu(D)$ (i.e., (i) holds). Let us prove (iii). For each $i < \lambda$, let $h(i) = \{\beta : \{t \in I : f_\beta(t) = f_i(t)\} \neq \emptyset \bmod D_*\}$, $h(i)$ is bounded. (Otherwise as λ is regular $> 2^{|I|}$, there is $A \neq \emptyset$ mod D_* such that for λ β's, $A = \{t : f_\beta(t) = f_i(t)\}$. Hence $I - A \in D_*$ contradiction). So there are distinct $i_\beta < \lambda$ such that $\gamma < \beta \Rightarrow i_\beta \notin h_i(i_\gamma)$, so $\{f_{i_\beta} : \beta < \lambda\}$ exemplify $T_{D_*}(g) \ge \lambda$ by the remark to Definition 3 and as $f_{i_\beta}(t) < f^*(t) \le f(t)$.

The only thing left to be proved is (ii), so assume D is normal. Let

$$D_+ = \{A \subseteq I;\ \text{there is a choice function } f \text{ on } I - A, \text{ such that for every } \alpha\ f^{-1}(\alpha) = \emptyset \bmod D_*\}.$$

D_+ is nontrivial by 13(B).

Now $D_* \subseteq D_+$ (for $A \in D_*$ any choice function f on $I - A$ exemplifies $A \in D_+$), D_+ is $\mu(D)$-complete. [If $\alpha < \mu(D)$, $A_i \in D_+$, f_i exemplifies it, define f on $I - \bigcap_{i<\alpha} A_i$ by $f(t) = f_{i(t)}(t)$, $i(t) = \min\{i\colon t \notin A_i\}$, now for $\gamma < \delta(I)$ $f^{-1}(\gamma) \subseteq \bigcup_{i<\alpha} f_i^{-1}(\gamma)$. As D_* is $\mu(D)$ complete, and for $i < \alpha$, $f_i^{-1}(\gamma) = \emptyset \bmod D_*$ clearly $f^{-1}(\gamma) = \emptyset \bmod D_*$ so f exemplifies $\bigcap_{i<\alpha} A_i \in D_+$.] Also D_+ is normal, for suppose $A \subseteq I$, $A \neq \emptyset \bmod D_+$, f a choice function on A, but for every $\gamma < \delta(I)$, $f^{-1}(\gamma) = \emptyset \bmod D_+$, so there is a choice function f_γ on $f^{-1}(\gamma)$, so that for every $\beta < \delta$, $f_\gamma^{-1}(\beta) = \emptyset \bmod D_*$. Now define a choice function f_+ on A: $f_+(t) = \langle f(t), f_{f(t)}(t)\rangle$, where $\langle\ ,\ \rangle$ is a pairing function on $\delta(I)$ such that $B = \{t \in I\colon t$ closed under $\langle =, \text{ing}\rangle\} \in D$.

Now f_+ clearly exemplifies $A = \emptyset \bmod D_*$, contradiction provided we find a suitable pairing function $\langle\ ,\ \rangle$. However, any pairing function $\langle\ ,\ \rangle$ satisfies this. Otherwise for every $t \in I - B$, choose $\beta_1^{(t)}, \beta_2^{(t)} \in t$, $\langle \beta_1, \beta_2\rangle \notin t$. As D is normal, $B \notin D \Rightarrow I - B \neq \emptyset \bmod D \Rightarrow$ for some $\beta_1, B_1 = \{t \in I - B\colon \beta_1(t) = \beta_1\} \neq \emptyset \bmod D$, hence for some β_2, $B_2 = \{t \in B_1\colon \beta_2(t) = \beta_2\} \neq \emptyset \bmod D$. So $\{t\colon \langle \beta_1, \beta_2\rangle \in t\} \notin D$, contradiction.

20. Remark. (A) Instead of "λ regular," "cf $\lambda > 2^{|I|}$" was sufficient.

(B) For any f, D if we add to D the sets J, $\|f\|_{D+J} > \|f\|_D$ and closed under intersection of $< \mu(D)$, we get a filter D_f, nontrivial, $\mu(D)$-complete, normal if D was normal. Is $\|f\|_D = \|f\|_{D_f}$?

We remark:

21. *Claim*. (A) cf $\lambda > |I| \Rightarrow T_D(\lambda) \leq \sum_{\mu<\lambda} T_D(\mu)$.

(B) Similarly if $g(s) = \sum_{\alpha<\lambda} g_\alpha(s)$, cf $\lambda > |I|$, $g_\alpha(s) < g(s)$, $g_\alpha(s)$ $(\alpha < \lambda)$ $<_D$ increasing then $T_D(g) \leq \sum_{\alpha<\lambda} T_D(g_\alpha)$.

(C) If cf $\alpha > |I|$, $\|\alpha\|_D = \sup_{\beta<\alpha} \|\beta\|_D$.

(D) If $1 <$ cf $\alpha < \mu(D)$ then $\|\alpha\| \leq \sup\{\|\beta\|_{D+A}\colon \beta < \alpha, A \neq \emptyset \bmod A\}$.

(We can generalize (C) and (D) to function as in (B) for (A) and solve $X/D = Y/C = B/A$.)

22. *Claim* (folk). (1) $\|\kappa\|_{D(st)} < \delta(\kappa)$ (= where $\beth_{\delta(\kappa)}$ is the Hanf number for $L_{\kappa^+,\omega}$) and $\|\alpha\|_{D(st)} \leq \delta(|\kappa + \alpha|)$.

(2) $\|\kappa\|_{D(ub)} \geq \kappa^+$.

(3) If there are $f_\alpha \in \kappa^\kappa$ $(\alpha < \delta)$, $\alpha < \beta \Rightarrow f_\alpha <_{D(ub)} f_\beta$ then $\delta \leq \|\kappa\|_{D(ub)}$.

23. *Conclusion*. Suppose cf $\aleph_\alpha < \mu(D)$, and $(\forall \chi < \aleph_\alpha)\ \chi^{|I|} < \aleph_\alpha$.

(A) When $\beta < \mu(D)$, $T_D(\aleph_{\alpha+\beta}) \leq \aleph_{\alpha+\beta}$.

(B) When $\beta < \aleph_\alpha$, $T_D(\aleph_{\alpha+\beta}) \leq \aleph_{\alpha+\|\beta\|_D} < \aleph_{\alpha+\aleph_\alpha}$.

(C) If $\gamma < \aleph_{\alpha+\beta}$, $\beta < \mu(D)$ then $\|\gamma\|_D < \aleph_{\alpha+\beta}$ (hence $\|\aleph_{\alpha+\beta}\|_D = \aleph_{\alpha+\beta}$).

Proof. (A) By induction on β for all the D's over I (which are μ-complete), for a fixed I. (In fact, it suffices to do it for all $D + A$, $A \neq \emptyset \bmod D$.)

Case I. $\beta = 0$ by 14(B). Let $\aleph_\alpha = \Sigma\{\aleph_{\alpha_i}\colon i <$ cf $\aleph_\alpha\}$. α_i increasing and continuous, $\alpha_i < \alpha$; so by 17(B), $T_{D+A}(\aleph_{\alpha_i}) \leq \aleph_{\alpha_i}^{|I|} < \aleph_\alpha$, for $A \subseteq I$, $A \neq \emptyset \bmod D$, and 14(B) gives the conclusion.

Case II. β limit. As $\beta < \mu(D)$, cf $\beta < \mu(D)$, so the proof is as in Case I.

Case III. $\beta = \gamma + 1$. Clearly cf $\aleph_{\alpha+\beta} = \aleph_{\alpha+\beta} > |I|$, so 21(A) applies.

(B) By 18(B), $T_D(\aleph_{\alpha+\beta}) \leq \aleph_{\alpha+\|\beta\|_D}$; by 17(A), $\|\beta\|_D < \aleph_\alpha$ (in fact, this implies (A)).

(C) By 23(A) and 19 (for w.l.o.g., $\beta = \zeta + 1$, by 19, $T_{D_1}(\aleph_{\alpha+\zeta}) \geq \aleph_{\alpha+\zeta+1}$ for some D_1; contradiction by 23(A)).

24. *Notation.* Let us define $\aleph_\beta(\aleph_\alpha)$ by induction on β: $\aleph_0(\aleph_\alpha) = \aleph_\alpha$, $\aleph_1(\aleph_\alpha) = \aleph_{\alpha+\aleph_\alpha}$ [so when $\alpha < \aleph_\alpha$, $\aleph_1(\aleph_\alpha) = \aleph_{\aleph_\alpha}$], $\aleph_{\beta+1}(\aleph_\alpha) = \aleph_1(\aleph_\beta(\aleph_\alpha))$, and for limit $\beta = \delta, \aleph_\delta(\aleph_\alpha) = \bigcup_{\gamma<\delta} \aleph_\gamma(\aleph_\alpha)$. Note that cf $\aleph_{\beta+1}(\aleph_\alpha) =$ cf $\aleph_\beta(\aleph_\alpha)$: cf $\aleph_\delta(\aleph_\alpha) =$ cf δ (for limit δ) and $\beta < \gamma \Rightarrow \aleph_\beta(\lambda) < \aleph_\gamma(\lambda)$; $\lambda = \aleph_\delta(\aleph_\alpha) \Rightarrow \lambda = \aleph_\lambda$ (δ limit).

25. *First main conclusion.* Suppose (i) $\aleph_\alpha \geq 2^{|I|}$;

(ii) $\boldsymbol{D}$ a family of (μ-complete) filters over I closed under Lemma 19, and $I - A \notin D \in \boldsymbol{D} \Rightarrow D + A \in \boldsymbol{D}$; $\mu > \aleph_0$, μ regular;

(iii) For every $D \in \boldsymbol{D}$ and $\chi < \aleph_\alpha$, $T_D(\chi) < \aleph_{\alpha(*)}$;

(iv) $\beta(*)$ is the first ordinal $> \alpha$ of cofinality μ, $\beta(*) = \aleph_{\beta(*)}$ [so $\beta(*) = \aleph_\mu(\aleph_\alpha)$];

(v) cf $\aleph_\alpha < \mu$ or cf $\aleph_\alpha > |I|$.

Then for every $D \in \boldsymbol{D}$,

(A) $\gamma < \beta(*) \Rightarrow T_D(\aleph_\gamma) < \aleph_\mu(\aleph_{\alpha(*)})$, $\|\aleph_\gamma\|_D < \aleph_\mu(\aleph_{\alpha(*)})$.

(B) For $\gamma < \mu$, $T_D(\aleph_\gamma(\aleph_\alpha)) \leq \aleph_\gamma(\aleph_{\alpha(*)})$ and

$$\lambda < \aleph_\gamma(\aleph_\alpha) \Rightarrow T_D(\lambda) < \aleph_\gamma(\aleph_{\alpha(*)}).$$

(C) For $\gamma < \mu$, $\|\aleph_\gamma(\aleph_\alpha)\|_D \leq \aleph_\gamma(\aleph_{\alpha(*)})$ and

$$\zeta < \aleph_\gamma(\aleph_\alpha) \Rightarrow \|\zeta\|_D < \aleph_\gamma(\aleph_{\alpha(*)}).$$

26. REMARK. (A) The natural case is cf $\aleph_\alpha < \mu$, $(\forall \chi < \aleph_\alpha)\chi^{|I|} < \aleph_\alpha$ as in 2.3 so $\alpha(*) = \alpha$.

(B) In 23(A), (B) instead of "$(\forall \chi < \aleph_\alpha)$ $(\chi^{|I|} < \aleph_\alpha)$" we could assume $2^{|I|} \leq \aleph_{\alpha(*)}$) and

$$(\forall A)\ (I - A \notin D \Rightarrow (\forall \chi < \aleph_\alpha)\ [T_{D+A}(\chi) < \aleph_\alpha]).$$

PROOF OF 2.5. (A) follows from (B) (in fact (B) gives sharper bounds; use the monotonicity of $T_D(-)$, $\|-\|_D$).

(C) (B) we prove by induction on γ (for all $D \in \boldsymbol{D}$).

Case I. $\gamma = 0$. Note $\aleph_0(\aleph_\alpha) = \aleph_\alpha$, so (B) holds by assumption (iii) and 21(A) when cf $\aleph_\alpha > |I|$ and 14(B) when cf $\aleph_\alpha < \mu(D)$ (at least one occurs by (v)) and (C) holds by 19 and (B).

Case II. $\gamma = \zeta + 1$. (B) *second phrase*; by 18(B); let $\aleph_\zeta(\aleph_\alpha) = \aleph_\eta$; for $\lambda < \aleph_\eta$ this is trivial by the induction hypothesis. For $\lambda = \aleph_{\eta+\xi}$, $\xi < \aleph_\eta$, by 18(B), $T_D(\lambda) \leq \aleph_{\eta(*)+\|\xi\|_D}$ [where $\aleph_{\eta(*)} = \aleph_\eta(\aleph_{\alpha(*)})$]. By induction hypothesis, $\|\xi\|_D < \aleph_{\eta(*)}$ so $T_D(\xi) < \aleph_{\eta(*)+\aleph_{\eta*}} = \aleph_1(\aleph_{\eta(*)}) = \aleph_{\zeta+1}(\aleph_{\alpha(*)}) = \aleph_\gamma(\aleph_{\alpha(*)})$.

(B) *first phrase*: follows from the second by 14(B) or 21(A), as cf $\aleph_\gamma(\aleph_\alpha) =$ cf $\aleph_\zeta(\aleph_\alpha)$ is $< \mu$ or $> |I|$, (for let $\zeta = \delta + n$, δ limit or zero, then cf $\aleph_\zeta(\aleph_\alpha) =$ cf $\aleph_\delta(\aleph_\alpha)$ so it is cf δ, which is $\leq \delta \leq \gamma < \mu$, or cf $\aleph_\alpha$ which is $< \mu$ or $> |I|$ by 25(v).

(C) *second phrase*: by (B)'s second phrase; and 19 (as $\aleph_\gamma(\aleph_\alpha)$ is a limit card).

(C) *first phrase*: follows from (C) second phrase by 21(C), (D).

Case III. γ limit. As $\gamma < \mu$, cf $\aleph_\gamma(\aleph_\alpha) = \text{cf}\,\gamma < \mu$; the second phrase of (B) and (C) follows by the induction hypothesis, the first phrase of (B) and (C) follows from the second; as in Case II.

27(A). REMARK. There are quite a few α's such that $\|\alpha\|_D = \alpha$ for all $D \in \boldsymbol{D}$ (all μ-complete).

(i) $\alpha < \mu(D)$.

(ii) $(\lambda^{|I|})^+$.

(iii) $\aleph_{\alpha+\beta}$ if $T_D(\aleph_\alpha) = \aleph_\alpha$, $\|\beta\|_D = \beta > 0$, so D is cf β-incomplete hence 21 essentially holds.

(iv) $\alpha = \lambda$, cf $\lambda < \mu(D)$, $(\forall \chi < \lambda)\chi^{|I|} < \lambda$.

(v) $\aleph_\beta(\aleph_\alpha)$ when $T_D(\aleph_\alpha) < \aleph_\gamma(\aleph_\alpha)$, $\gamma < \mu$, $\gamma + \beta = \beta$.

27 (B). REMARK. We can improve 25 by defining inductively h; $\beta(*) \to \beta(*)$ such that $\|\gamma\|_D \le h(\gamma)$ for every $D \in \boldsymbol{D}$, $\gamma < \beta(*)$.

28. NOTE. That always

(A) $\|\alpha\|_D \ge \alpha$, $T_D(\lambda) \ge \lambda$;

(B) if D is cf λ-incomplete, $\|\lambda\|_D \ge \lambda^+$, $T_D(\lambda) \ge \lambda^+$, so in many instances the results are best possible.

Unfortunately 25 does not say anything for $T_D(\aleph_{\mu(D)}(\aleph_0))$. However if $D_{\text{Mg}}(\delta, \rho)$ is nontrivial, we can continue the induction from the proof of 25 through ρ rather than just through $\mu(D)$.

29. *Second main conclusion.* Let $\boldsymbol{D}$ be a family of $\mu = \kappa$-complete filters over I which are normal. Suppose also (i)—(v) from 25. We use 11's notation.

For any $g \in \text{ord}^I$ define g^0 by $g^0(t) = \aleph_{g(t)}(\aleph_\alpha)$. Let $G = \{f\colon f \le_D f_D$ for any $D \in \boldsymbol{D}\}$. *Then* for any $g \in G$, $D \in \boldsymbol{D}$,

(A) $T_D(g^0) \le \aleph_{\|g\|_D}(\aleph_{\alpha(*)})$.

(B) $f^0 <_D g^0 \Rightarrow T_D(f^0) < \aleph_{\|g\|_D}(\aleph_{\alpha(*)})$ when $g >_D 0$. Otherwise $T_D(f^0) \le \aleph_{\alpha(*)}$.

(C) $\|g^0\|_D \le \aleph_{\|g\|_D}(\aleph_{\alpha(*)})$.

PROOF. We prove by induction on γ for all $D \in \boldsymbol{D}$, that (A) + (B) + (C) holds when $\|\hat{g}\|_D = \gamma$. Remember that by 11(C) if $D_1 \in \boldsymbol{D}$, for some $A \ne \varnothing$ mod D_1, for any D_2, $D + A \subseteq D_2 \in \boldsymbol{D}$, $\|\hat{g}\|_{D_1} = \|\hat{g}\|_{D_2}$. Remember also that by 6(A) if $D_1 \subseteq D_2$ then $T_{D_1}(g) \le T_{D_2}(g)$. So during the proof we can replace $D \in \boldsymbol{D}$ by any extension (which is in $\boldsymbol{D}$) as this only increases the left side of our inequalities, and does not change the right side.

Case I. $\gamma = 0$. Easy checking.

Case II. $\{t \in I\colon g(t)$ is a successor$\} \ne \varnothing$ mod D. Then w.l.o.g. $g(t) = f(t) + 1$ for every t. Clearly $\hat{f} < \hat{g}$ hence $\|\hat{f}\|_D < \|\hat{g}\|_D$. So on f the induction hypothesis works and we can continue as in 25, Case II.

Case III. $g(t)$ is limit for every t.

For (C): let $f <_D g$. Let $\beta = \|f\|_D$. Then over some $A \ne \varnothing$ mod Df, is constantly β; clearly $\beta < \gamma$, by induction $\|f^0\|_D \le \aleph_\beta(\aleph_{\alpha(*)}) < \aleph_\gamma(\aleph_{\alpha(*)})$; and note that if $f_1 <_D g^0$ ($f_1 \in \text{ord}^I$) then for some $f \in G$, $f <_D g$, $f_1 <_D f^0$; so clearly $\|g^0\|_D \le \aleph_\gamma(\aleph_{\alpha(*)})$. Similarly (A) follows by the induction hypothesis. For (B) note that the number of $f <_D g$, $f \in G$ is (up to $=_D$) $\le |I|^{|I|} \le 2^{|I|} \le \aleph_\alpha$; so we shall have no problem too by (i) of 25.

30. *Conclusion.* Suppose $D_{\text{Mg}}(\delta, \rho)$ is not trivial, then, e.g.,

(A) $\|\rho\|_{D(st)} \le \delta(I)$ (remember $D(st)$ is the filter generated by the closed unbounded subsets of κ).

(B) If $\aleph_\rho$ is strong limit, $\delta < \aleph_\rho$ then $2^{\aleph_\rho} < \aleph_\delta$.

(C) If $\aleph_\rho(\aleph_0)$ is strong limit, $\delta < \aleph_\rho$ then $2^{\aleph_\rho(\aleph_0)} < \aleph_\delta(\aleph_0)$.

PROOF. By 15(A) we can in (B) and (C) replace $2^{\aleph_\rho}$, $2^{\aleph_\rho(\aleph_0)}$ by $T_{D(st)}(\aleph_\rho)$, $T_{D(st)}(\aleph_\rho(\aleph_0))$ resp. By 7(A), $D(st) \leq_{RK} D_{\mathrm{Mg}}(\delta, \rho)$ hence by 6(B) we can replace $D(st)$ by $D_{\mathrm{Mg}}(\delta, \rho)$. Now (A) follows by 11, (B) by 16 and (C) by 29.

31. REMARK. If we have a nontrivial $D_{\mathrm{Mg}}(\delta, \rho)$ for a ρ, we can get one for ρ^n and more (see Gaifman [G]). Essentially, to get to ρ_1 we need a model M with universe $\delta_1 > \delta$, $|\delta_1| = |\delta|$, language $< \kappa$, such that for any $N \prec M$, $\operatorname{otp}(N \cap \delta) \geq \rho \Rightarrow \operatorname{otp}(N) \geq \rho_1$ (otp—order type).

32. REMARK. In 29, the essential property of G is that $f_1 \leq f_2, f_2 \in G \Rightarrow f_1 \in G$, and that for $f \in G$, $\|f\|_D$ is equal for all $D \in \boldsymbol{D}$, or even that for each $f \in G$, $D \in \boldsymbol{D}$ an ordinal $r(f, D)$ and filter $D(f)$, $D \subseteq D(f) \in \boldsymbol{D}$ are attached, such that: $D(f_1) \subseteq D \in \boldsymbol{D}$, $f_2 \in G$, $f_1 <_D f_2$ implies $r(f_1, D) < r(f_2, D)$. This may be formulated as a game.

33. LEMMA. (A) *If* $(\forall t \in I) g(t) > 2^{|I|}$ *or even* $T_D(g) > 2^{|I|}$, *where* $g \in \mathrm{card}^I$ *then* $\|g\|_D \geq T_D(g)$.

(B) *If* $T_D(g) \geq \lambda$, $\lambda > 2^{|I|}$, λ *regular*, D *a filter over* I, *then for some filter* D_1 *over* I:

(i) $\mu(D) \leq \mu(D_1)$ *and* $D \subseteq D_1$,

(ii) D_1 *is normal if* D *is normal*,

(iii) *there are* $f_\alpha \in \mathrm{ord}^I$ *such that* $\alpha < \beta < \lambda \Rightarrow f_\alpha <_{D_1} f_\beta <_{D_1} g$.

PROOF. (A) Suppose $\|g\|_D < T_D(g)$, let $\lambda = |\|g\|_D| + 2^{|I|}$ so some $\{f_i : i < \lambda^+\}$ exemplifies $\lambda^+ \leq T_D(g)$. Now clearly $\|f_i\|_D < \|g\|_D < \lambda^+$, so we can assume $\|f_i\|_D$ is fixed. But necessarily for some $i < j < (2^{|I|})^+ \leq \lambda$, $(\forall t) f_i(t) \leq f_j(t)$. (We can find $\alpha < (2^{|I|})^+$ such that: for every $A \subseteq \alpha$, $|A| \leq |I|$, there is β such that $\beta < \alpha$ $(\forall \gamma \in A)(\forall t \in I)\, [(f_\gamma(t) > f_\alpha(t) \equiv f_\gamma(t) > f_\beta(t)) \wedge f_\gamma(t) < f_\alpha(t) \equiv f_\gamma(t) < f_\beta(t))]$. For each t there is a finite $A_t \subseteq \alpha$ such that: for each appropriate β, $f_\beta(t) \leq f_\alpha(t)$ otherwise define inductively β_n such that $f_{\beta_n}(t) > f_\alpha(t)$ so $f_{\beta_n}(t)$ is decreasing, contradiction. Let $A = \bigcup_t A_t$, β as above, then $(\forall t) f_\beta(t) \leq f_\alpha(t))$. But $f_i \neq_D f_j$ hence $f_i <_D f_j$ hence $\|f_i\|_D < \|f_j\|_D$, contradiction.

(B) Like 19.

34. *Discussion.* Can we eliminate Chang conjecture from the bound on $2^{\aleph_{\omega_1}(\aleph_0)}$? (We concentrate on $\kappa = \aleph_1$.) Let

$$(*)_\lambda \qquad \text{for every } f \in \omega_1^{\omega_1}\ \{A \in I_{\mathrm{Mg}}(\lambda, \omega): \operatorname{otp}(A) \geq f(\operatorname{otp}(A \cap \omega_1)\} \neq \varnothing \bmod D_{\mathrm{Mg}}(\lambda, \omega)$$

(so this is a weakening of Chang conjecture).

Now $D_{Mg}(\lambda, \omega_1)$ nontrivial $\Rightarrow (*)_\lambda \Rightarrow 2^{\aleph_{\omega_1}(\aleph_0)} < \aleph_\lambda(\aleph_0)$ provided that $\aleph_{\omega_1}(\aleph_0)$ is strong limit. Note that if λ is a Ramesy cardinal $(*)_\chi$ for many $\chi < \lambda$ but $(*)_\lambda \Rightarrow V \neq L$.

But we can hope that a variant of Jensen's Marginalia will give, together with the above, an absolute bound.

35. REMARK. Seeing this manuscript, Galvin shows:

THEOREM. *For every uncountable regular* κ *the following statements are equivalent*:

(1) *for every* $f \in \kappa^\kappa$, $\|f\|_{D(st)} < \kappa^+$,

(2) *for every function F from the finite subsets of κ^+ to κ, there is $\xi < \kappa$ such that for every $\alpha < \kappa$ there is $A \subseteq \kappa^+$, $\{F(X): X \subseteq A$ finite$\} \subseteq \xi$,* otp $A \geq \alpha$,

(3) $(*)_{\kappa^+}$ (*replacing ω_1 by κ*).

36. REMARK. We can improve Theorem 2.5, by defining a function $h: \beta(*) \to \beta(*)$, and proving $\|\gamma\|_D \leq h(\gamma)$ for $\gamma < \beta(*)$, $D \in \boldsymbol{D}$. We can define h by induction, using the proof, with no problem.

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