Sh:495

TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 349, Number 1, January 1997, Pages 103–128
S 0002-9947(97)01531-6

ON THE STRONG EQUALITY BETWEEN SUPERCOMPACTNESS AND STRONG COMPACTNESS

ARTHUR W. APTER AND SAHARON SHELAH

ABSTRACT. We show that supercompactness and strong compactness can be equivalent even as properties of pairs of regular cardinals. Specifically, we show that if $V \models$ ZFC + GCH is a given model (which in interesting cases contains instances of supercompactness), then there is some cardinal and cofinality preserving generic extension $V[G] \models$ ZFC + GCH in which, (a) (preservation) for $\kappa \le \lambda$ regular, if $V \models$ "κ is λ supercompact", then $V[G] \models$ "κ is λ supercompact" and so that, (b) (equivalence) for $\kappa \le \lambda$ regular, $V[G] \models$ "κ is λ strongly compact" iff $V[G] \models$ "κ is λ supercompact", except possibly if κ is a measurable limit of cardinals which are λ supercompact.

0. Introduction and Preliminaries

It is a well known fact that the notion of strongly compact cardinal represents a singularity in the hierarchy of large cardinals. The work of Magidor [Ma1] shows that the least strongly compact cardinal and the least supercompact cardinal can coincide, but also, the least strongly compact cardinal and the least measurable cardinal can coincide. The work of Kimchi and Magidor [KiM] generalizes this, showing that the class of strongly compact cardinals and the class of supercompact cardinals can coincide (except by results of Menas [Me] and [A] at certain measurable limits of supercompact cardinals), and the first n strongly compact cardinals (for n a natural number) and the first n measurable cardinals can coincide. Thus, the precise identity of certain members of the class of strongly compact cardinals cannot be ascertained vis à vis the class of measurable cardinals or the class of supercompact cardinals.

An interesting aspect of the proofs of both [Ma1] and [KiM] is that in each result, all "bad" instances of strong compactness are not obliterated. Specifically, in each model, since the strategy employed in destroying strongly compact cardinals which aren't also supercompact is to make them non-strongly compact after a certain point either by adding a Prikry sequence or a non-reflecting stationary set of ordinals of the appropriate cofinality, there may be cardinals κ and λ so that κ is λ strongly

Received by the editors May 2, 1994 and, in revised form, December 30, 1994.
1991 *Mathematics Subject Classification.* Primary 03E35; Secondary 03E55.
Key words and phrases. Strongly compact cardinal, supercompact cardinal.
The research of the first author was partially supported by PSC-CUNY Grant 662341 and a salary grant from Tel Aviv University. In addition, the first author wishes to thank the Mathematics Departments of The Hebrew University and Tel Aviv University for the hospitality shown him during his sabbatical in Israel.
The second author wishes to thank the Basic Research Fund of the Israeli Academy of Sciences for partially supporting this research, which is Publication 495 of the second author.

compact yet κ isn't λ supercompact. Thus, whereas it was proven by Kimchi and Magidor that the classes of strongly compact and supercompact cardinals can coincide (with the exceptions noted above), it was not known whether a "local" version of this were possible, i.e., if it were possible to obtain a model in which, for the class of pairs (κ, λ), κ is λ strongly compact iff κ is λ supercompact. This is more delicate.

The purpose of this paper is to answer the above question in the affirmative. Specifically, we prove the following

Theorem. *Suppose $V \models ZFC + GCH$ is a given model (which in interesting cases contains instances of supercompactness). There is then some cardinal and cofinality preserving generic extension $V[G] \models ZFC + GCH$ in which:*

(a) (Preservation) *For $\kappa \leq \lambda$ regular, if $V \models$ "κ is λ supercompact", then $V[G] \models$ "κ is λ supercompact". The converse implication holds except possibly when $\kappa = \sup\{\delta < \kappa : \delta$ is λ supercompact$\}$.*

(b) (Equivalence) *For $\kappa \leq \lambda$ regular, $V[G] \models$ "κ is λ strongly compact" iff $V[G] \models$ "κ is λ supercompact", except possibly if κ is a measurable limit of cardinals which are λ supercompact.*

Note that the limitation given in (b) above is reasonable, since trivially, if κ is measurable, $\kappa < \lambda$, and $\kappa = \sup\{\delta < \kappa : \delta$ is either λ supercompact or λ strongly compact$\}$, then κ is λ strongly compact. Further, it is a theorem of Menas [Me] that under GCH, for κ the first, second, third, or αth for $\alpha < \kappa$ measurable limit of cardinals which are κ^+ strongly compact or κ^+ supercompact, κ is κ^+ strongly compact yet κ isn't κ^+ supercompact. Thus, if there are sufficiently large cardinals in the universe, it will never be possible to have a complete coincidence between the notions of κ being λ strongly compact and κ being λ supercompact for λ a regular cardinal.

Note that in the statement of our Theorem, we do not mention what happens if $\lambda > \kappa$ is a singular cardinal. This is since the behavior when $\lambda > \kappa$ is a singular cardinal is provable in ZFC + GCH (which implies any limit cardinal is a strong limit cardinal). Specifically, if $\lambda > \kappa$ is so that $\text{cof}(\lambda) < \kappa$, then by a theorem of Magidor [Ma3], κ is λ supercompact iff κ is λ^+ supercompact, so automatically, by clause (a) of our Theorem, λ supercompactness is preserved between V and $V[G]$. Also, if $\lambda > \kappa$ is so that $\text{cof}(\lambda) < \kappa$, then by a theorem of Solovay [SRK], κ is λ strongly compact iff κ is λ^+ strongly compact, so by clause (b) of our Theorem, it can never be the case that $V[G] \models$ "κ is λ strongly compact" unless $V[G] \models$ "κ is λ supercompact" as well. Further, if $\lambda > \kappa$ is so that $\lambda > \text{cof}(\lambda) \geq \kappa$, then it is not too difficult to see (and will be shown in Section 2) that if κ is λ' strongly compact or λ' supercompact for all $\lambda' < \lambda$, then κ is λ strongly compact, and there is no reason to believe κ must be λ supercompact. In fact, it is a theorem of Magidor [Ma4] (irrespective of GCH) that if μ is a supercompact cardinal, there will always be many cardinals $\kappa, \lambda < \mu$ so that $\lambda > \kappa$ is a singular cardinal of cofinality $\geq \kappa$, κ is λ strongly compact, κ is λ' supercompact for all $\lambda' < \lambda$, yet κ isn't λ supercompact. Thus, there can never be a complete coincidence between the notions of κ being λ strongly compact and κ being λ supercompact if $\lambda > \kappa$ is an arbitrary cardinal, assuming there are supercompact cardinals in the universe.

The structure of this paper is as follows. Section 0 contains our introductory comments and preliminary material concerning notation, terminology, etc. Section 1 defines and discusses the basic properties of the forcing notion used in the iteration we employ to construct our final model. Section 2 gives a complete statement and proof of the theorem of Magidor mentioned in the above paragraph and proves our Theorem in the case for which there is one supercompact cardinal κ in the universe which contains no strongly inaccessible cardinals above it. Section 3 shows how the ideas of Section 2 can be used to prove the Theorem in the general case. Section 4 contains our concluding remarks.

Before beginning the material of Section 1, we briefly mention some preliminary information. Essentially, our notation and terminology are standard, and when this is not the case, this will be clearly noted. We take this opportunity to mention we will be assuming GCH throughout the course of this paper. For $\alpha < \beta$ ordinals, $[\alpha, \beta], [\alpha, \beta), (\alpha, \beta]$, and (α, β) are as in standard interval notation. If f is the characteristic function of a set $x \subseteq \alpha$, then $x = \{\beta : f(\beta) = 1\}$.

When forcing, $q \geq p$ will mean that q is stronger than p. For P a partial ordering, φ a formula in the forcing language with respect to P, and $p \in P$, $p \| \varphi$ will mean p decides φ. For G V-generic over P, we will use both $V[G]$ and V^P to indicate the universe obtained by forcing with P. If $x \in V[G]$, then $\dot{x}$ will be a term in V for x. We may, from time to time, confuse terms with the sets they denote and write x when we actually mean $\dot{x}$, especially when x is some variant of the generic set G.

If κ is a cardinal, then for P a partial ordering, P is (κ, ∞)-distributive if for any sequence $\langle D_\alpha : \alpha < \kappa \rangle$ of dense open subsets of P, $D = \bigcap\limits_{\alpha<\kappa} D_\alpha$ is a dense open subset of P. P is κ-closed if given a sequence $\langle p_\alpha : \alpha < \kappa \rangle$ of elements of P so that $\beta < \gamma < \kappa$ implies $p_\beta \leq p_\gamma$ (an increasing chain of length κ), then there is some $p \in P$ (an upper bound to this chain) so that $p_\alpha \leq p$ for all $\alpha < \kappa$. P is $< \kappa$-closed if P is δ-closed for all cardinals $\delta < \kappa$. P is κ-directed closed if for every cardinal $\delta < \kappa$ and every directed set $\langle p_\alpha : \alpha < \delta \rangle$ of elements of P (where $\langle p_\alpha : \alpha < \delta \rangle$ is directed if for every two distinct elements $p_\rho, p_\nu \in \langle p_\alpha : \alpha < \delta \rangle$, p_ρ and p_ν have a common upper bound) there is an upper bound $p \in P$. P is κ-strategically closed if in the two person game in which the players construct an increasing sequence $\langle p_\alpha : \alpha \leq \kappa \rangle$, where player I plays odd stages and player II plays even and limit stages, then player II has a strategy which ensures the game can always be continued. P is $< \kappa$-strategically closed if P is δ-strategically closed for all cardinals $\delta < \kappa$. P is $\prec \kappa$-strategically closed if in the two person game in which the players construct an increasing sequence $\langle p_\alpha : \alpha < \kappa \rangle$, where player I plays odd stages and player II plays even and limit stages, then player II has a strategy which ensures the game can always be continued. Note that trivially, if P is κ-closed, then P is κ-strategically closed and $\prec \kappa^+$-strategically closed. The converse of both of these facts is false.

For κ a regular cardinal, two partial orderings to which we will refer quite a bit are the standard partial orderings Q^0_κ for adding a Cohen subset to κ^+ using conditions having support κ and Q^1_κ for adding κ^+ many Cohen subsets to κ using conditions having support $< \kappa$. The basic properties and explicit definitions of these partial orderings may be found in [J].

Finally, we mention that we are assuming complete familiarity with the notions of strong compactness and supercompactness. Interested readers may consult [SRK] or [KaM] for further details. We note only that all elementary embeddings witnessing the λ supercompactness of κ are presumed to come from some fine, κ-complete, normal ultrafilter $\mathcal{U}$ over $P_\kappa(\lambda) = \{x \subseteq \lambda : |x| < \kappa\}$. Also, where appropriate, all ultrapowers via a supercompact ultrafilter over $P_\kappa(\lambda)$ will be confused with their transitive isomorphs.

1. The Forcing Conditions

In this section, we describe and prove the basic properties of the forcing conditions we shall use in our later iteration. Let $\delta < \lambda$, $\lambda \geq \aleph_1$ be regular cardinals in our ground model V.

We define three notions of forcing. Our first notion of forcing $P^0_{\delta,\lambda}$ is just the standard notion of forcing for adding a non-reflecting stationary set of ordinals of cofinality δ to λ^+. Specifically, $P^0_{\delta,\lambda} = \{p :$ For some $\alpha < \lambda^+$, $p : \alpha \to \{0, 1\}$ is a characteristic function of S_p, a subset of α not stationary at its supremum nor having any initial segment which is stationary at its supremum, so that $\beta \in S_p$ implies $\beta > \delta$ and $\text{cof}(\beta) = \delta\}$, ordered by $q \geq p$ iff $q \supseteq p$ and $S_p = S_q \cap \sup(S_p)$, i.e., S_q is an end extension of S_p. It is well-known that for G V-generic over $P^0_{\delta,\lambda}$ (see [Bu] or [KiM]), in $V[G]$, a non-reflecting stationary set $S = S[G] = \bigcup\{S_p : p \in G\} \subseteq \lambda^+$ of ordinals of cofinality δ has been introduced, the bounded subsets of λ^+ are the same as those in V, and cardinals, cofinalities, and GCH have been preserved. It is also virtually immediate that $P^0_{\delta,\lambda}$ is δ-directed closed.

Work now in $V_1 = V^{P^0_{\delta,\lambda}}$, letting $\dot S$ be a term always forced to denote the above set S. $P^2_{\delta,\lambda}[S]$ is the standard notion of forcing for introducing a club set C which is disjoint to S (and therefore makes S non-stationary). Specifically, $P^2_{\delta,\lambda}[S] = \{p :$ For some successor ordinal $\alpha < \lambda^+$, $p : \alpha \to \{0, 1\}$ is a characteristic function of C_p, a club subset of α, so that $C_p \cap S = \emptyset\}$, ordered by $q \geq p$ iff C_q is an end extension of C_p. It is again well-known (see [MS]) that for H V_1-generic over $P^2_{\delta,\lambda}[S]$, a club set $C = C[H] = \bigcup\{C_p : p \in H\} \subseteq \lambda^+$ which is disjoint to S has been introduced, the bounded subsets of λ^+ are the same as those in V_1, and cardinals, cofinalities, and GCH have been preserved.

Before defining in V_1 the partial ordering $P^1_{\delta,\lambda}[S]$ which will be used to destroy strong compactness, we first prove two preliminary lemmas.

Lemma 1. $\Vdash_{P^0_{\delta,\lambda}}$ *"$\clubsuit(\dot S)$", i.e., $V_1 \models$ "There is a sequence $\langle x_\alpha : \alpha \in S\rangle$ so that for each $\alpha \in S$, $x_\alpha \subseteq \alpha$ is cofinal in α, and for any $A \in [\lambda^+]^{\lambda^+}$, $\{\alpha \in S : x_\alpha \subseteq A\}$ is stationary".*

Proof of Lemma 1. Since $V \models$ GCH and V and V_1 contain the same bounded subsets of λ^+, we can let $\langle y_\alpha : \alpha < \lambda^+\rangle \in V$ be a listing of all elements $x \in ([\lambda^+]^\delta)^V = ([\lambda^+]^\delta)^{V_1}$ so that each $x \in [\lambda^+]^\delta$ appears on this list λ^+ times at ordinals of cofinality δ, i.e., for any $x \in [\lambda^+]^\delta$, $\lambda^+ = \sup\{\alpha < \lambda^+ : \text{cof}(\alpha) = \delta$ and $y_\alpha = x\}$. This then allows us to define $\langle x_\alpha : \alpha \in S\rangle$ by letting x_α be y_β for the least $\beta \in S - (\alpha + 1)$ so that $y_\beta \subseteq \alpha$ and y_β is unbounded in α. By genericity, each x_α is well-defined.

Now let $p \in P^0_{\delta,\lambda}$ be so that $p \Vdash$ "$\dot A \in [\lambda^+]^{\lambda^+}$ and $\dot K \subseteq \lambda^+$ is club". We show that for some $r \geq p$ and some $\zeta < \lambda^+$, $r \Vdash$ "$\zeta \in \dot K \cap \dot S$ and $\dot x_\zeta \subseteq \dot A$". To do this, we inductively define an increasing sequence $\langle p_\alpha : \alpha < \delta\rangle$ of elements of $P^0_{\delta,\lambda}$ and increasing sequences $\langle \beta_\alpha : \alpha < \delta\rangle$ and $\langle \gamma_\alpha : \alpha < \delta\rangle$ of ordinals $< \lambda^+$ so that $\beta_0 \leq \gamma_0 \leq \beta_1 \leq \gamma_1 \leq \cdots \leq \beta_\alpha \leq \gamma_\alpha \leq \cdots$ $(\alpha < \delta)$. We begin by letting $p_0 = p$ and $\beta_0 = \gamma_0 = 0$. For $\eta = \alpha + 1 < \delta$ a successor, let $p_\eta \geq p_\alpha$ and $\beta_\eta \leq \gamma_\eta$,

$\beta_\eta \geq \max(\beta_\alpha, \gamma_\alpha, \sup(\mathrm{dom}(p_\alpha))) + 1$ be so that $p_\eta \Vdash$ "$\beta_\eta \in \dot{A}$ and $\gamma_\eta \in \dot{K}$". For $\rho < \delta$ a limit, let $p_\rho = \bigcup_{\alpha<\rho} p_\alpha$, $\beta_\rho = \bigcup_{\alpha<\rho} \beta_\alpha$, and $\gamma_\rho = \bigcup_{\alpha<\rho} \gamma_\alpha$. Note that since $\rho < \delta$, p_ρ is well-defined, and since $\delta < \lambda^+$, $\beta_\rho, \gamma_\rho < \lambda^+$. Also, by construction, $\bigcup_{\alpha<\delta} \beta_\alpha = \bigcup_{\alpha<\delta} \gamma_\alpha = \bigcup_{\alpha<\delta} \sup(\mathrm{dom}(p_\alpha)) < \lambda^+$. Call ζ this common sup. We thus have that $q = \bigcup_{\alpha<\delta} p_\alpha \cup \{\zeta\}$ is a well-defined condition, so that $q \Vdash$ "$\{\beta_\alpha : \alpha \in \delta - \{0\}\} \subseteq \dot{A}$ and $\zeta \in \dot{K} \cap \dot{S}$".

To complete the proof of Lemma 1, we know that as $\langle \beta_\alpha : \alpha \in \delta - \{0\} \rangle \in V$ and as each $y \in \langle y_\alpha : \alpha < \lambda^+ \rangle$ must appear λ^+ times at ordinals of cofinality δ, we can find some $\eta \in (\zeta, \lambda^+)$ so that $\mathrm{cof}(\eta) = \delta$ and $\langle \beta_\alpha : \alpha \in \delta - \{0\} \rangle = y_\eta$. If we let $r \geq q$ be so that $r \Vdash$ "$\dot{S} \cap [\zeta, \eta] = \{\zeta, \eta\}$", then $r \Vdash$ "$\dot{x}_\zeta = y_\eta = \langle \beta_\alpha : \alpha \in \delta - \{0\} \rangle$". This proves Lemma 1. $\square$ Lemma 1

We fix now in V_1 a $\clubsuit(S)$ sequence $X = \langle x_\alpha : \alpha \in S \rangle$.

Lemma 2. *Let S' be an initial segment of S so that S' is not stationary at its supremum nor has any initial segment which is stationary at its supremum. There is then a sequence $\langle y_\alpha : \alpha \in S' \rangle$ so that for every $\alpha \in S'$, $y_\alpha \subseteq x_\alpha$, $x_\alpha - y_\alpha$ is bounded in α, and if $\alpha_1 \neq \alpha_2 \in S'$, then $y_{\alpha_1} \cap y_{\alpha_2} = \emptyset$.*

Proof of Lemma 2. We define by induction on $\alpha \leq \alpha_0 = \sup S' + 1$ a function h_α so that $\mathrm{dom}(h_\alpha) = S' \cap \alpha$, $h_\alpha(\beta) < \beta$, and $\langle x_\beta - h_\alpha(\beta) : \beta \in S' \cap \alpha \rangle$ is pairwise disjoint. The sequence $\langle x_\beta - h_{\alpha_0}(\beta) : \beta \in S' \rangle$ will be our desired sequence.

If $\alpha = 0$, then we take h_α to be the empty function. If $\alpha = \beta + 1$ and $\beta \notin S'$, then we take $h_\alpha = h_\beta$. If $\alpha = \beta + 1$ and $\beta \in S'$, then we notice that since each $x_\gamma \in X$ has order type δ and is cofinal in γ, for all $\gamma \in S' \cap \beta$, $x_\beta \cap \gamma$ is bounded in γ. This allows us to define a function h_α having domain $S' \cap \alpha$ by $h_\alpha(\beta) = 0$, and for $\gamma \in S' \cap \beta$, $h_\alpha(\gamma) = \min(\{\rho : \rho < \gamma,\ \rho \geq h_\beta(\gamma), \text{ and } x_\beta \cap \gamma \subseteq \rho\})$. By the next to last sentence and the induction hypothesis on h_β, $h_\alpha(\gamma) < \gamma$. And, if $\gamma_1 < \gamma_2 \in S' \cap \alpha$, then if $\gamma_2 < \beta$, $(x_{\gamma_1} - h_\alpha(\gamma_1)) \cap (x_{\gamma_2} - h_\alpha(\gamma_2)) \subseteq (x_{\gamma_1} - h_\beta(\gamma_1)) \cap (x_{\gamma_2} - h_\beta(\gamma_2)) = \emptyset$ by the induction hypothesis on h_β. If $\gamma_2 = \beta$, then $(x_{\gamma_1} - h_\alpha(\gamma_1)) \cap (x_{\gamma_2} - h_\alpha(\gamma_2)) = (x_{\gamma_1} - h_\alpha(\gamma_1)) \cap x_{\gamma_2} = \emptyset$ by the definition of $h_\alpha(\gamma_1)$. The sequence $\langle x_\gamma - h_\alpha(\gamma) : \gamma \in S' \cap \alpha \rangle$ is thus as desired.

If α is a limit ordinal, then as S' is non-stationary at its supremum nor has any initial segment which is stationary at its supremum, we can let $\langle \beta_\gamma : \gamma < \mathrm{cof}(\alpha) \rangle$ be a strictly increasing, continuous sequence having sup α so that for all $\gamma < \mathrm{cof}(\alpha)$, $\beta_\gamma \notin S'$. Thus, if $\rho \in S' \cap \alpha$, then $\{\beta_\gamma : \beta_\gamma < \rho\}$ is bounded in ρ, meaning we can find some largest γ so that $\beta_\gamma < \rho$. It is also the case that $\rho < \beta_{\gamma+1}$. This allows us to define $h_\alpha(\rho) = \max(\{h_{\beta_{\gamma+1}}(\rho), \beta_\gamma\})$ for the γ just described. It is still the case that $h_\alpha(\rho) < \rho$. And, if $\rho_1, \rho_2 \in (\beta_\gamma, \beta_{\gamma+1})$, then $(x_{\rho_1} - h_\alpha(\rho_1)) \cap (x_{\rho_2} - h_\alpha(\rho_2)) \subseteq (x_{\rho_1} - h_{\beta_{\gamma+1}}(\rho_1)) \cap (x_{\rho_2} - h_{\beta_{\gamma+1}}(\rho_2)) = \emptyset$ by the definition of $h_{\beta_{\gamma+1}}$. If $\rho_1 \in (\beta_\gamma, \beta_{\gamma+1})$, $\rho_2 \in (\beta_\sigma, \beta_{\sigma+1})$ with $\gamma < \sigma$, then $(x_{\rho_1} - h_\alpha(\rho_1)) \cap (x_{\rho_2} - h_\alpha(\rho_2)) \subseteq x_{\rho_1} \cap (x_{\rho_2} - \beta_\sigma) \subseteq \rho_1 - \beta_\sigma \subseteq \rho_1 - \beta_{\gamma+1} = \emptyset$. Thus, the sequence $\langle x_\rho - h_\alpha(\rho) : \rho \in S' \cap \alpha \rangle$ is again as desired. This proves Lemma 2. $\square$ Lemma 2

At this point, we are in a position to define in V_1 the partial ordering $P^1_{\delta,\lambda}[S]$ which will be used to destroy strong compactness. $P^1_{\delta,\lambda}[S]$ is now the set of all 4-tuples $\langle w, \alpha, \bar{r}, Z \rangle$ satisfying the following properties.

1. $w \in [\lambda^+]^{<\lambda}$.
2. $\alpha < \lambda$.
3. $\bar{r} = \langle r_i : i \in w \rangle$ is a sequence of functions from α to $\{0, 1\}$, i.e., a sequence of subsets of α.
4. $Z \subseteq \{x_\beta : \beta \in S\}$ is a set so that if $z \in Z$, then for some $y \in [w]^\delta$, $y \subseteq z$ and $z - y$ is bounded in the β so that $z = x_\beta$.

Note that the definition of Z implies $|Z| < \lambda$.

The ordering on $P^1_{\delta,\lambda}[S]$ is given by $\langle w^1, \alpha^1, \bar{r}^1, Z^1 \rangle \le \langle w^2, \alpha^2, \bar{r}^2, Z^2 \rangle$ iff the following hold.

1. $w^1 \subseteq w^2$.
2. $\alpha^1 \le \alpha^2$.
3. If $i \in w^1$, then $r^1_i \subseteq r^2_i$.
4. $Z^1 \subseteq Z^2$.
5. If $z \in Z^1 \cap [w^1]^\delta$ and $\alpha_1 \le \alpha < \alpha_2$, then
$$|\{i \in z : r^2_i(\alpha) = 0\}| = |\{i \in z : r^2_i(\alpha) = 1\}| = \delta.$$

If $W = \langle \langle w^\beta, \alpha^\beta, \bar{r}^\beta, Z^\beta \rangle_{\beta<\gamma<\delta} \rangle$ is a directed set of elements of $P^1_{\delta,\lambda}[S]$, then since by the regularity of δ any δ sequence from $\bigcup_{\beta<\gamma} w^\beta$ must contain a δ sequence from w^β for some $\beta < \gamma$, it can easily be verified that $\langle \bigcup_{\beta<\gamma} w^\beta, \bigcup_{\beta<\gamma} \alpha^\beta, \bigcup_{\beta<\gamma} \bar{r}^\beta, \bigcup_{\beta<\gamma} Z^\beta \rangle$ is an upper bound for each element of W. (Here, if $\bar{r}^\beta = \langle r^\beta_i : i \in w^\beta \rangle$, then $r_i \in \bigcup_{\beta<\gamma} \bar{r}^\beta$ if $i \in \bigcup_{\beta<\gamma} w^\beta$ and $r_i = \bigcup_{\beta<\gamma} r^\beta_i$, taking $r^\beta_i = \emptyset$ if $i \notin w^\beta$.) This means $P^1_{\delta,\lambda}[S]$ is δ-directed closed.

At this point, a few intuitive remarks are in order. If κ is λ strongly compact for $\lambda \ge \kappa$ regular, then it must be the case (see [SRK]) that λ carries a κ-additive uniform ultrafilter. If $\delta < \kappa < \lambda$, the forcing $P^1_{\delta,\lambda}[S]$ has specifically been designed to destroy this fact. It has been designed, however, to destroy the λ strong compactness of κ "as lightly as possible", making little damage. In the case of the argument of [KiM], the non-reflecting stationary set S is added directly to λ in order to kill the λ strong compactness of κ. In our situation, the non-reflecting stationary set S, having been added to λ^+ and not to λ, does not kill the λ strong compactness of κ by itself. The additional forcing $P^1_{\delta,\lambda}[S]$ is necessary to do the job. The forcing $P^1_{\delta,\lambda}[S]$, however, has been designed so that if necessary, we can resurrect the λ supercompactness of κ by forcing further with $P^2_{\delta,\lambda}[S]$.

Lemma 3. $V_1^{P^1_{\delta,\lambda}[S]} \models$ "*κ is not λ strongly compact*" *if* $\delta < \kappa < \lambda$.

Remark. Since we will only be concerned in general with the case when κ is strongly inaccessible and $\delta < \kappa < \lambda$, we assume without loss of generality that this is the case throughout the rest of the paper.

Proof of Lemma 3. Assume to the contrary that $V_1^{P^1_{\delta,\lambda}[S]} \models$ "κ is λ strongly compact", and by our earlier remarks, let $p \Vdash$ "$\dot{\mathcal{D}}$ is a κ-additive uniform ultrafilter over λ". We show that p can be extended to a condition q so that for some ordinal $\alpha^q < \lambda$ and some δ sequence $\langle s_i : i < \delta \rangle$ of $\mathcal{D}$ measure 1 sets, $q \Vdash$ "$\bigcap_{i<\delta} \dot{s}_i \subseteq \alpha^q$", an immediate contradiction.

We use a Δ-system argument to establish this. First, for G_1 V_1-generic over $P^1_{\delta,\lambda}[S]$ and $i < \lambda^+$, let $r^*_i = \bigcup\{r^p_i : \exists p = \langle w^p, \alpha^p, \bar{r}^p, Z^p\rangle \in G_1[r^p_i \in \bar{r}^p]\}$. It is the case that $\Vdash_{P^1_{\delta,\lambda}[S]}$ "$\dot{r}^*_i : \lambda \to \{0,1\}$ is a function whose domain is all of λ". To see this, for $p = \langle w^p, \alpha^p, \bar{r}^p, Z^p\rangle$, since $|Z^p| < \lambda$, $w^p \in [\lambda^+]^{<\lambda}$, and $z \in Z^p$ implies $z \in [\lambda^+]^\delta$, the condition $q = \langle w^q, \alpha^q, \bar{r}^q, Z^q\rangle$ given by $\alpha^q = \alpha^p$, $Z^q = Z^p$, $w^q = w^p \cup \bigcup\{z : z \in Z^p\}$, and $\bar{r}^q = \langle r'_i : i \in w^q\rangle$ defined by $r'_i = r_i$ if $i \in w^p$ and r'_i is the empty function if $i \in w^q - w^p$ is a well-defined condition. (This just means we may as well assume that for $p = \langle w^p, \alpha^p, \bar{r}^p, Z^p\rangle$, $z \in Z^p$ implies $z \subseteq w^p$.) Further, since $|Z^q| < \lambda$, $\bigcup\{\beta : \exists z \in Z^q[z = x_\beta]\} = \gamma < \lambda^+$. Therefore, if $\gamma' \in (\gamma, \lambda^+)$ and $S' \subseteq \gamma'$ is so that $\sup S' = \gamma'$ and S' is an initial segment of S so that S' is not stationary at its supremum nor has any initial segment which is stationary at its supremum, then by Lemma 2, there is a sequence $\langle y_\beta : \beta \in S'\rangle$ so that for every $\beta \in S'$, $y_\beta \subseteq x_\beta$, $x_\beta - y_\beta$ is bounded in β, and if $\beta_1 \neq \beta_2 \in S'$, then $y_{\beta_1} \cap y_{\beta_2} = \emptyset$. This means that if $z \in Z^q$ and $z = x_\beta$ for some β, then $y_\beta \subseteq w$.

Choose now for $\beta \in S'$ sets y^1_β and y^2_β so that $y_\beta = y^1_\beta \cup y^2_\beta$, $y^1_\beta \cap y^2_\beta = \emptyset$, and $|y^1_\beta| = |y^2_\beta| = \delta$. If $\rho \in (\alpha^q, \lambda)$, then for each β so that $x_\beta \in Z^q$ and for each $r'_i \in \bar{r}^q$ such that $i \in y_\beta$, we can extend r'_i to $r''_i : \rho \to \{0,1\}$ by letting $r''_i|\alpha^q = r'_i|\alpha^q$, and for $\alpha \in [\alpha^q, \rho)$, $r''_i(\alpha) = 0$ if $i \in y^1_\beta$ and $r''_i(\alpha) = 1$ if $i \in y^2_\beta$. For $i \in w^q$ so that there is no β with $x_\beta \in Z^q$ and $i \in y_\beta$, we extend r'_i to $r''_i : \rho \to \{0,1\}$ by letting $r''_i|\alpha^q = r'_i|\alpha^q$, and for $\alpha \in [\alpha^q, \rho)$, $r''_i(\alpha) = 0$. If we let $\bar{s} = \langle r''_i : i \in w^q\rangle$, then $t = \langle w^q, \rho, \bar{s}, Z^q\rangle$ can be verified to be such that t is well-defined and $t \geq q \geq p$. We have therefore shown by density that $\Vdash_{P^1_{\delta,\lambda}[S]}$ "$\dot{r}^*_i \to \{0,1\}$ is a function whose domain is all of λ". Thus, we can let $r^\ell_i = \{\alpha < \lambda : r^*_i(\alpha) = \ell\}$ for $\ell \in \{0,1\}$.

For each $i < \lambda^+$, pick $p_i = \langle w^{p_i}, \alpha^{p_i}, \bar{r}^{p_i}, Z^{p_i}\rangle \geq p$ so that $p_i \Vdash$ "$\dot{r}^{\ell(i)}_i \in \dot{\mathcal{D}}$" for some $\ell(i) \in \{0,1\}$. This is possible since $\Vdash_{P^1_{\delta,\lambda}[S]}$ "For each $i < \lambda^+$, $\dot{r}^0_i \cup \dot{r}^1_i = \lambda$". Without loss of generality, by extending p_i if necessary, we can assume that $i \in w^{p_i}$. Thus, since each $w^{p_i} \in [\lambda^+]^{<\lambda}$, we can find some stationary $A \subseteq \{i < \lambda^+ : \mathrm{cof}(i) = \lambda\}$ so that $\{w^{p_i} : i \in A\}$ forms a Δ-system, i.e., so that for $i \neq j \in A$, $w^{p_i} \cap w^{p_j}$ is some constant value w which is an initial segment of both. (Note we can assume that for $i \in A$, $w_i \cap i = w$, and for some fixed $\ell(*) \in \{0,1\}$, for every $i \in A$, $p_i \Vdash$ "$\dot{r}^{\ell(*)}_i \in \dot{\mathcal{D}}$".) Also, by clause 4) of the definition of the forcing, $|Z^{p_i}| < \lambda$ for each $i < \lambda^+$. Therefore, $Z^{p_i} \in [[\lambda^+]^\delta]^{<\lambda}$, so as $|[\lambda^+]^\delta| = \lambda^+$ by GCH, the same sort of Δ-system argument allows us to assume in addition that for all $i \in A$, $Z^{p_i} \cap \mathcal{P}(w)$ is some constant value Z. Further, since each $\alpha^{p_i} < \lambda$, we can assume that α^{p_i} is some constant α^0 for $i \in A$. Then, since any $\bar{r}^{p_i} = \langle r_j : j \in w^{p_i}\rangle$ for $i \in A$ is composed of a sequence of functions from α_0 to 2, $\alpha_0 < \lambda$, and $|w| < \lambda$, GCH allows us to conclude that for $i \neq j \in A$, $\bar{r}^{p_i}|w = \bar{r}^{p_j}|w$. And, since $i \in w^{p_i}$, we know that we can also assume (by thinning A if necessary) that $B = \{\sup(w^{p_i}) : i \in A\}$ is so that $i < j \in A$ implies $i \leq \sup(w^{p_i}) < \min(w^{p_j} - w) \leq \sup(w^{p_j})$. We know in addition by the choice of $X = \langle x_\beta : \beta \in S\rangle$ that for some $\gamma \in S$, $x_\gamma \subseteq A$. Let $x_\gamma = \{i_\beta : \beta < \delta\}$.

We are now in a position to define the condition q referred to earlier. We proceed by defining each of the four coordinates of q. First, let $w^q = \bigcup_{\beta<\delta} w^{p_{i_\beta}}$. As λ and λ^+ are regular, $\delta < \lambda$, and each $w^{p_{i_\beta}} \in [\lambda^+]^{<\lambda}$, w^q is well-defined and in $[\lambda^+]^{<\lambda}$. Second, let $\alpha^q = \alpha^0$. Third, let $\bar{r}^q = \langle r^q_i : i \in w^q\rangle$ be defined by $r^q_i = r^{p_{i_\beta}}_i$ if $i \in w^{p_{i_\beta}}$. The property of the Δ-system that $i \neq j \in A$ implies $\bar{r}^{p_i}|w = \bar{r}^{p_j}|w$ tells us $\bar{r}^q$ is well defined. Finally, to define Z^q, let $Z^q = \bigcup_{\beta<\delta} Z^{i_\beta} \cup \{\{i_\beta : \beta < \delta\}\}$. By the last three sentences in the preceding paragraph and our construction, $\{i_\beta : \beta < \delta\}$ generates a new set which can be included in Z^q, and Z^q is well-defined.

We claim now that $q \geq p$ is so that $q \Vdash$ "$\bigcap_{\beta<\delta} \dot{r}^{\ell(*)}_{i_\beta} \subseteq \alpha^q$". To see this, assume the claim fails. This means that for some $q^1 \geq q$ and some $\alpha^q \leq \eta < \lambda$, $q^1 \Vdash$ "$\eta \in \bigcap_{\beta<\delta} \dot{r}^{\ell(*)}_{i_\beta}$". Without loss of generality, since q^1 can always be extended if necessary, we can assume that $\eta < \alpha^{q^1}$. But then, by the definition of $\leq$, for δ many $\beta < \delta$, $q^1 \Vdash$ "$\eta \notin \dot{r}^{\ell(*)}_{i_\beta}$", an immediate contradiction. Thus, $q \Vdash$ "$\bigcap_{\beta<\delta} \dot{r}^{\ell(*)}_{i_\beta} \subseteq \alpha^q$", which, since $\delta < \kappa$, contradicts that $q \Vdash$ "$\bigcap_{\beta<\delta} \dot{r}^{\ell(*)}_{i_\beta} \in \dot{\mathcal{D}}$ and $\dot{\mathcal{D}}$ is a κ-additive uniform ultrafilter over λ". This proves Lemma 3. $\square$ Lemma 3

Recall we mentioned prior to the proof of Lemma 3 that $P^1_{\delta,\lambda}[S]$ is designed so that a further forcing with $P^2_{\delta,\lambda}[S]$ will resurrect the λ supercompactness of κ, assuming the correct iteration has been done. That this is so will be shown in the next section. In the meantime, we give an idea of why this will happen by showing that the forcing $P^0_{\delta,\lambda} * (P^1_{\delta,\lambda}[\dot{S}] \times P^2_{\delta,\lambda}[\dot{S}])$ is rather nice. Specifically, we have the following lemma.

Lemma 4. $P^0_{\delta,\lambda} * (P^1_{\delta,\lambda}[\dot{S}] \times P^2_{\delta,\lambda}[\dot{S}])$ *is equivalent to* $Q^0_\lambda * \dot{Q}^1_\lambda$.

Proof of Lemma 4. Let G be V-generic over $P^0_{\delta,\lambda} * (P^1_{\delta,\lambda}[\dot{S}] \times P^2_{\delta,\lambda}[\dot{S}])$, with $G^0_{\delta,\lambda}$, $G^1_{\delta,\lambda}$, and $G^2_{\delta,\lambda}$ the projections onto $P^0_{\delta,\lambda}$, $P^1_{\delta,\lambda}[S]$, and $P^2_{\delta,\lambda}[S]$ respectively. Each $G^i_{\delta,\lambda}$ is appropriately generic. So, since $P^1_{\delta,\lambda}[S] \times P^2_{\delta,\lambda}[S]$ is a product in $V[G^0_{\delta,\lambda}]$, we can rewrite the forcing in $V[G^0_{\delta,\lambda}]$ as $P^2_{\delta,\lambda}[S] \times P^1_{\delta,\lambda}[S]$ and rewrite $V[G]$ as $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}][G^1_{\delta,\lambda}]$.

It is well-known (see [MS]) that the forcing $P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]$ is equivalent to Q^0_λ. That this is so can be seen from the fact that $P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]$ is non-trivial, has cardinality λ^+, and is such that $D = \{\langle p, q\rangle \in P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]$: For some α, $\mathrm{dom}(p) = \mathrm{dom}(q) = \alpha + 1$, $p \Vdash$ "$\alpha \notin \dot{S}$", and $q \Vdash$ "$\alpha \in \dot{C}$"$\}$ is dense in $P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]$ and is λ-closed. This easily implies the desired equivalence. Thus, V and $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}]$ have the same cardinals and cofinalities, and the proof of Lemma 4 will be complete once we show that in $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}]$, $P^1_{\delta,\lambda}[S]$ is equivalent to Q^1_λ.

To this end, working in $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}]$, we first note that as $S \subseteq \lambda^+$ is now a non-stationary set all of whose initial segments are non-stationary, by Lemma 2, for the sequence $\langle x_\beta : \beta \in S\rangle$, there must be a sequence $\langle y_\beta : \beta \in S\rangle$ so that for every $\beta \in S$, $y_\beta \subseteq x_\beta$, $x_\beta - y_\beta$ is bounded in β, and if $\beta_1 \neq \beta_2 \in S$, then $y_{\beta_1} \cap y_{\beta_2} = \emptyset$. Given this fact, it is easy to observe that $P^1 = \{\langle w, \alpha, \bar{r}, Z\rangle \in P^1_{\delta,\lambda}[S]$: For every $\beta \in S$, either $y_\beta \subseteq w$ or $y_\beta \cap w = \emptyset\}$ is dense in $P^1_{\delta,\lambda}[S]$. To show this, given $\langle w, \alpha, \bar{r}, Z\rangle \in P^1_{\delta,\lambda}[S]$, $\bar{r} = \langle r_i : i \in w\rangle$, let $Y_w = \{y \in \langle y_\beta : \beta \in S\rangle : y \cap w \neq \emptyset\}$. As $|w| < \lambda$ and $y_{\beta_1} \cap y_{\beta_2} = \emptyset$ for $\beta_1 \neq \beta_2 \in S$, $|Y_w| < \lambda$. Hence, as $|y| = \delta < \lambda$ for $y \in Y_w$, $|w'| < \lambda$ for $w' = w \cup (\bigcup Y_w)$. This means $\langle w', \alpha, \bar{r}', Z\rangle$ for $\bar{r}' = \langle r'_i : i \in w'\rangle$ defined by $r'_i = r_i$ if $i \in w$ and r'_i is the empty function if $i \in w' - w$ is a well-defined condition extending $\langle w, \alpha, \bar{r}, Z\rangle$. Thus, P^1 is dense in $P^1_{\delta,\lambda}[S]$, so to analyze the forcing properties of $P^1_{\delta,\lambda}[S]$, it suffices to analyze the forcing properties of P^1.

For $\beta \in S$, let $Q_\beta = \{\langle w, \alpha, \bar{r}, Z\rangle \in P^1 : w = y_\beta\}$, and let $Q' = \{\langle w, \alpha, \bar{r}, Z\rangle \in P^1 : w \subseteq \lambda^+ - \bigcup_{\beta \in S} y_\beta\}$. Let Q'' be those elements of $\prod_{\beta \in S} Q_\beta \times Q'$ of support $< \lambda$ under the product ordering. Adopting the notation of Lemma 3, given $p = \langle\langle q_\beta : \beta \in A\rangle, q\rangle \in Q''$ where $A \subseteq S$ and $|A| < \lambda$, as $|A| < \lambda$ and λ is regular, $\alpha = \sup\{\alpha^{q_\beta} : \beta \in A\} \cup \alpha^q < \lambda$, so without loss of generality, each q_β and q can be extended to conditions q'_β and q' so that α occurs in q'_β and q'. This means $Q = \{p = \langle q_\beta : \beta < \gamma < \lambda\rangle \in Q'' : \alpha^{q_\beta} = \alpha^{q_{\beta'}}$ for β and β' different coordinates of $p\}$ is dense in Q'', so Q and Q'' are forcing equivalent. Then, for $p = \langle\langle q_\beta : \beta \in A\rangle, q\rangle \in Q$ where $A \subseteq S$ and $|A| < \lambda$, as $w^{q_{\beta_1}} \cap w^{q_{\beta_2}} = \emptyset$ for $\beta_1 \neq \beta_2 \in A$ ($y_{\beta_1} \cap y_{\beta_2} = \emptyset$), $w^{q_{\beta_1}} \cap w^q = \emptyset$, $\alpha^{q_{\beta_1}} = \alpha^{q_{\beta_2}} = \alpha^q$ for $\beta_1 \neq \beta_2 \in A$, the domains of any two $\bar{r}^{q_{\beta_1}}$, $\bar{r}^{q_{\beta_2}}$ are disjoint for $\beta_1 \neq \beta_2 \in A$, $Z^{q_{\beta_1}} \cap Z^{q_{\beta_2}} = \emptyset$ for $\beta_1 \neq \beta_2 \in A$, the domains of $\bar{r}^{q_\beta}$ and $\bar{r}^q$ are disjoint for $\beta \in A$, and $Z^{q_\beta} \cap Z^q = \emptyset$ for $\beta \in A$, the function $F(p) = \langle \bigcup_{\beta \in A} w^{q_\beta} \cup w^q, \alpha, \bigcup_{\beta \in A} \bar{r}^{q_\beta} \cup \bar{r}^q, \bigcup_{\beta \in A} Z^{q_\beta} \cup Z^q\rangle$ can easily be seen to yield an isomorphism between Q and P^1. Thus, over $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}]$, forcing with P^1, $P^1_{\delta,\lambda}[S]$, Q, and Q'' are all equivalent.

We examine now in more detail the exact nature of Q''. For $\beta \in S$, GCH shows $|Q_\beta| = \lambda$. It quickly follows from the definition of Q_β that Q_β is $< \lambda$-closed, so Q_β is forcing equivalent to adding a Cohen subset to λ. Since the definitions of $P^1_{\delta,\lambda}[S]$ and P^1 ensure that for $\langle w, \alpha, \bar{r}, Z\rangle \in Q'$, $Z = \emptyset$ (for every $\beta \in S$, $w \cap y_\beta = \emptyset$, $y_\beta \subseteq x_\beta$, and $x_\beta - y_\beta$ is bounded in δ), Q' can easily be seen to be a re-representation of the Cohen forcing where instead of working with functions whose domains have cardinality $< \lambda$ and are subsets of $\lambda \times \lambda^+$, we work with functions whose domains have cardinality $< \lambda$ and are subsets of $\lambda \times (\lambda^+ - \bigcup_{\beta \in S} y_\beta)$. Thus, Q'' is isomorphic to a Cohen forcing using functions having domains of cardinality $< \lambda$ which adds λ^+ many Cohen subsets to λ. By the last sentence of the last paragraph, this means that over $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}]$, the forcings $P^1_{\delta,\lambda}[S]$ and Q^1_λ are equivalent. This proves Lemma 4. $\square$ Lemma 4

As we noted in the proof of Lemma 4, without the last coordinate Z^p of a condition $p \in P^1_{\delta,\lambda}[S]$ and the associated condition on the ordering, $P^1_{\delta,\lambda}[S]$ is just a re-representation of Q^1_λ. This last coordinate and change in the ordering are necessary to destroy the λ strong compactness of κ when forcing with $P^1_{\delta,\lambda}[S]$. Once the fact S is stationary has been destroyed by forcing with $P^2_{\delta,\lambda}[S]$, Lemma 4 shows that this last coordinate Z^p of a condition $p \in P^1_{\delta,\lambda}[S]$ and change in the ordering in a sense become irrelevant.

It is clear from Lemma 4 that $P^0_{\delta,\lambda} * (P^1_{\delta,\lambda}[\dot{S}] \times P^2_{\delta,\lambda}[\dot{S}])$, being equivalent to $Q^0_\lambda * \dot{Q}^1_\lambda$, preserves GCH, cardinals, and cofinalities, and has a dense subset which is $< \lambda$-closed and satisfies λ^{++}-c.c. Our next lemma shows that the forcing $P^0_{\delta,\lambda} * P^1_{\delta,\lambda}[\dot{S}]$ is also rather nice.

Lemma 5. *$P^0_{\delta,\lambda} * P^1_{\delta,\lambda}[\dot{S}]$ preserves GCH, cardinals, and cofinalities, is $< \lambda$-strategically closed, and is λ^{++}-c.c.*

Proof of Lemma 5. Let $G' = G^0_{\delta,\lambda} * G^1_{\delta,\lambda}$ be V-generic over $P^0_{\delta,\lambda} * P^1_{\delta,\lambda}[\dot{S}]$, and let $G^2_{\delta,\lambda}$ be $V[G']$-generic over $P^2_{\delta,\lambda}[S]$. Thus, $G' * G^2_{\delta,\lambda} = G$ is V-generic over $P^0_{\delta,\lambda} * (P^1_{\delta,\lambda}[\dot{S}] * P^2_{\delta,\lambda}[\dot{S}]) = P^0_{\delta,\lambda} * (P^1_{\delta,\lambda}[\dot{S}] \times P^2_{\delta,\lambda}[\dot{S}])$. By Lemma 4, $V[G] \models$ GCH and has the same cardinals and cofinalities as V, so since $V[G'] \subseteq V[G]$, forcing with $P^0_{\delta,\lambda} * P^1_{\delta,\lambda}[\dot{S}]$ over V preserves GCH, cardinals, and cofinalities.

We next show the $< \lambda$-strategic closure of $P^0_{\delta,\lambda} * P^1_{\delta,\lambda}[\dot{S}]$. We first note that as $(P^0_{\delta,\lambda} * P^1_{\delta,\lambda}[\dot{S}]) * P^2_{\delta,\lambda}[\dot{S}] = P^0_{\delta,\lambda} * (P^1_{\delta,\lambda}[\dot{S}] * P^2_{\delta,\lambda}[\dot{S}])$ has by Lemma 4 a dense subset which is $< \lambda$-closed, the desired fact follows from the more general fact that if $P * \dot{Q}$ is a partial ordering with a dense subset R so that R is $< \lambda$-closed, then P is $< \lambda$-strategically closed. To show this more general fact, let $\gamma < \lambda$ be a cardinal. Suppose I and II play to build an increasing chain of elements of P, with $\langle p_\beta : \beta \le \alpha + 1\rangle$ enumerating all plays by I and II through an odd stage $\alpha + 1$ and $\langle \dot{q_\beta} : \beta < \alpha + 1$ and β is even or a limit ordinal$\rangle$ enumerating a set of auxiliary plays by II which have been chosen so that $\langle\langle p_\beta, \dot{q_\beta}\rangle : \beta < \alpha + 1$ and β is even or a limit ordinal$\rangle$ enumerates an increasing chain of elements of the dense subset $R \subseteq P * \dot{Q}$. At stage $\alpha + 2$, II chooses $\langle p_{\alpha+2}, \dot{q}_{\alpha+2}\rangle$ so that $\langle p_{\alpha+2}, \dot{q}_{\alpha+2}\rangle \in R$ and so that $\langle p_{\alpha+2}, \dot{q}_{\alpha+2}\rangle \ge \langle p_{\alpha+1}, \dot{q_\alpha}\rangle$; this makes sense, since inductively, $\langle p_\alpha, \dot{q}_\alpha\rangle \in R \subseteq P * \dot{Q}$, so as I has chosen $p_{\alpha+1} \ge p_\alpha$, $\langle p_{\alpha+1}, \dot{q}_\alpha\rangle \in P * \dot{Q}$. By the $< \lambda$-closure of R, at any limit stage $\eta \le \gamma$, II can choose $\langle p_\eta, \dot{q_\eta}\rangle$ so that $\langle p_\eta, \dot{q_\eta}\rangle$ is an upper bound to $\langle\langle p_\beta, \dot{q_\beta}\rangle : \beta < \eta$ and β is even or a limit ordinal$\rangle$. The preceding yields a winning strategy for II, so P is $< \lambda$-strategically closed.

Finally, to show $P^0_{\delta,\lambda} * P^1_{\delta,\lambda}[\dot{S}]$ is λ^{++}-c.c., we simply note that this follows from the general fact about iterated forcing (see [Ba]) that if $P * \dot{Q}$ satisfies λ^{++}-c.c., then P satisfies λ^{++}-c.c. (Here, $P = P^0_{\delta,\lambda} * P^1_{\delta,\lambda}[\dot{S}]$ and $Q = P^2_{\delta,\lambda}[\dot{S}]$.) This proves Lemma 5. $\square$ Lemma 5

We remark that $\Vdash_{P^0_{\delta,\lambda}}$ "$P^1_{\delta,\lambda}[\dot{S}]$ is λ^+-c.c.", for if $\mathcal{A} = \langle p_\alpha : \alpha < \lambda^+\rangle$ were a size λ^+ antichain of elements of $P^1_{\delta,\lambda}[S]$ in $V[G^0_{\delta,\lambda}]$, then as $V[G^0_{\delta,\lambda}]$ and $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}]$ have the same cardinals, $\mathcal{A}$ would be a size λ^+ antichain of elements of $P^1_{\delta,\lambda}[S]$ in $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}]$. By Lemma 4, in this model, a dense subset of $P^1_{\delta,\lambda}[S]$ is isomorphic to Q^1_λ, which has the same definition in either $V[G^0_{\delta,\lambda}]$ or $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}]$ (since $P^0_{\delta,\lambda}$ is λ-strategically closed and $P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]$ is λ-closed) and so is λ^+-c.c. in either model.

We conclude this section with a lemma which will be used later in showing that it is possible to extend certain elementary embeddings witnessing the appropriate degree of supercompactness.

Lemma 6. *For $V_1 = V^{P^0_{\delta,\lambda}}$, the models $V_1^{P^1_{\delta,\lambda}[S] \times P^2_{\delta,\lambda}[S]}$ and $V_1^{P^1_{\delta,\lambda}[S]}$ contain the same λ sequences of elements of V_1.*

Proof of Lemma 6. By Lemma 4, since $P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]$ is equivalent to the forcing Q^0_λ and $V \subseteq V^{P^0_{\delta,\lambda}} \subseteq V^{P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]}$, the models V, $V^{P^0_{\delta,\lambda}}$, and $V^{P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]}$ all contain the same λ sequences of elements of V. Thus, since a λ sequence of elements of $V_1 = V^{P^0_{\delta,\lambda}}$ can be represented by a V-term which is actually a function $h : \lambda \to V$, it immediately follows that $V^{P^0_{\delta,\lambda}}$ and $V^{P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]}$ contain the same λ sequences of elements of $V^{P^0_{\delta,\lambda}}$.

Now let $f : \lambda \to V_1$ be so that $f \in (V^{P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]})^{P^1_{\delta,\lambda}[S]} = V_1^{P^1_{\delta,\lambda}[S] \times P^2_{\delta,\lambda}[S]}$, and let $g : \lambda \to V_1$, $g \in V^{P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]}$ be a term for f. By the previous paragraph, $g \in V^{P^0_{\delta,\lambda}}$. Since Lemma 4 shows that $P^1_{\delta,\lambda}[S]$ is λ^+-c.c. in $V^{P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]}$, for each $\alpha < \lambda$, the antichain $\mathcal{A}_\alpha$ defined in $V^{P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]}$ by $\{p \in P^1_{\delta,\lambda}[S] : p$ decides a value for $g(\alpha)\}$ is so that $V^{P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]} \models$ "$|\mathcal{A}_\alpha| \leq \lambda$". Hence, by the preceding paragraph, since $\mathcal{A}_\alpha$ is a set of elements of $V^{P^0_{\delta,\lambda}}$, $\mathcal{A}_\alpha \in V^{P^0_{\delta,\lambda}}$ for each $\alpha < \lambda$. Therefore, again by the preceding paragraph, the sequence $\langle \mathcal{A}_\alpha : \alpha < \lambda \rangle \in V^{P^0_{\delta,\lambda}}$. This just means that the term $g \in V^{P^0_{\delta,\lambda}}$ can be evaluated in $V_1^{P^1_{\delta,\lambda}[S]}$, i.e., $f \in V_1^{P^1_{\delta,\lambda}[S]}$. This proves Lemma 6. $\Box$ Lemma 6

2. The Case of One Supercompact Cardinal with No Larger Inaccessibles

In this section, we give a proof of our Theorem, starting from a model V for "ZFC + GCH + There is one supercompact cardinal κ and no $\lambda > \kappa$ is inaccessible". Before defining the forcing conditions used in the proof of this version of our Theorem, we first give a proof of the theorem of Magidor mentioned in Section 0 which shows that if there is a supercompact cardinal, then there always must be cardinals $\delta < \lambda$ so that δ is λ strongly compact yet δ isn't λ supercompact.

Lemma 7 (Magidor [Ma4]). *Suppose κ is a supercompact cardinal. Then $B = \{\delta < \kappa : \delta$ is λ_δ strongly compact for λ_δ the least singular strong limit cardinal $> \delta$ of cofinality δ, δ is not λ_δ supercompact, yet δ is α supercompact for all $\alpha < \lambda_\delta\}$ is unbounded in κ.*

Proof of Lemma 7. Let $\lambda_\kappa > \kappa$ be the least singular strong limit cardinal of cofinality κ, and let $j : V \to M$ be an elementary embedding witnessing the λ_κ supercompactness of κ with $j(\kappa)$ minimal. As $j(\kappa)$ is least, $M \models$ "κ is not λ_κ supercompact". As $M^{\lambda_\kappa} \subseteq M$ and λ_κ is a strong limit cardinal, $M \models$ "κ is α supercompact for all $\alpha < \kappa$".

Let $\mu \in V$ be a κ-additive measure over κ, and let $\langle \lambda_\alpha : \alpha < \lambda_\kappa \rangle$ be a sequence of cardinals cofinal in λ_κ in both V and M. As $M^{\lambda_\kappa} \subseteq M$ and λ_κ is a strong limit cardinal, $\mu \in M$. Also, as $M \models$ "κ is α supercompact for all $\alpha < \lambda_\kappa$", the closure properties of M allow us to find a sequence $\langle \mu_\alpha : \alpha < \kappa \rangle \in M$ so that $M \models$ "μ_α is a fine, normal, κ-additive ultrafilter over $P_\kappa(\lambda_\alpha)$". Thus, we can define in M the collection μ^* of subsets of $P_\kappa(\lambda_\kappa)$ by $A \in \mu^*$ iff $\{\alpha < \kappa\colon A|\lambda_\alpha \in \mu_\alpha\} \in \mu$, where for $A \subseteq P_\kappa(\lambda_\kappa)$, $A|\lambda_\alpha = \{p \cap P_\kappa(\lambda_\alpha) : p \in A\}$. It is easily checked that μ^* defines in M a κ-additive fine ultrafilter over $P_\kappa(\lambda_\kappa)$. Thus, $M \models$ "κ is α supercompact for all $\alpha < \lambda_\kappa$, κ is not λ_κ supercompact, yet κ is λ_κ strongly compact", so by reflection, the set B of the hypothesis is unbounded in κ. This proves Lemma 7. $\Box$ Lemma 7

We note that the proof of Lemma 7 goes through if λ_δ becomes the least singular strong limit cardinal $> \delta$ of cofinality δ^+, of cofinality δ^{++}, etc. To see this, observe that the closure properties of M and the strong compactness of κ ensure that κ^+, κ^{++}, etc. each carry κ-additive measures μ_{κ^+}, $\mu_{\kappa^{++}}$, etc. which are elements of M. These measures may then be used in place of the μ of Lemma 7 to define the strongly compact measure μ^* over $P_\kappa(\lambda_\kappa)$.

We return now to the proof of our Theorem. Let $\bar{\delta} = \langle \delta_\alpha : \alpha \leq \kappa \rangle$ enumerate the inaccessibles $\leq \kappa$, with $\delta_\kappa = \kappa$. Note that since we are in the simple case in which κ is the only supercompact cardinal in the universe and has no inaccessibles above it, we can assume each δ_α isn't $\delta_{\alpha+1}$ supercompact and for the least regular cardinal $\lambda_\alpha \geq \delta_\alpha$ so that $V \models$ "δ_α isn't λ_α supercompact", $\lambda_\alpha < \delta_{\alpha+1}$. (If δ were the least cardinal so that δ is $< \beta$ supercompact for β the least inaccessible $> \delta$ yet δ isn't β supercompact, then V_β would provide the desired model.)

We are now in a position to define the partial ordering P used in the proof of the Theorem. We define a κ stage Easton support iteration $P_\kappa = \langle\langle P_\alpha, \dot{Q}_\alpha\rangle : \alpha < \kappa\rangle$, and then define $P = P_{\kappa+1} = P_\kappa * \dot{Q}_\kappa$ for a certain class partial ordering Q_κ definable in V^{P_κ}. The definition is as follows:

1. P_0 is trivial.
2. Assuming P_α has been defined for $\alpha < \kappa$, $P_{\alpha+1} = P_\alpha * \dot{Q}_\alpha$, with $\dot{Q}_\alpha$ a term for the full support iteration $\langle P^0_{\omega,\lambda} * (P^1_{\omega,\lambda}[\dot{S}_\lambda] \times P^2_{\omega,\lambda}[\dot{S}_\lambda]) : \delta^+_\alpha \le \lambda < \lambda_\alpha$ and λ is regular$\rangle * \langle P^0_{\omega,\lambda_\alpha} * P^1_{\omega,\lambda_\alpha}[\dot{S}_{\lambda_\alpha}]\rangle$, where $\dot{S}_\lambda$ is a term for the non-reflecting stationary subset of λ^+ introduced by $P^0_{\omega,\lambda}$ for $\lambda < \lambda_\alpha$ and $\dot{S}_{\lambda_\alpha}$ is a term for the non-reflecting stationary subset of λ^+_α introduced by $P^0_{\omega,\lambda_\alpha}$.
3. $\dot{Q}_\kappa$ is a term for the Easton support iteration of $\langle P^0_{\omega,\lambda} * (P^1_{\omega,\lambda}[\dot{S}_\lambda] \times P^2_{\omega,\lambda}[\dot{S}_\lambda]) :$ $\lambda > \kappa$ is a regular cardinal$\rangle$, where as before, $\dot{S}_\lambda$ is a term for the non-reflecting stationary subset of λ^+ introduced by $P^0_{\omega,\lambda}$.

The intuitive motivation behind the above definition is that below κ at any inaccessible, we must first destroy and then resurrect all "good" instances of strong compactness, i.e., those which also witness supercompactness, but then destroy the least regular "bad" instance of strong compactness, thus destroying all "bad" instances of strong compactness beyond the least "bad" instance. Since κ is supercompact, it has no "bad" instances of strong compactness, so all instances of κ's supercompactness are destroyed and then resurrected.

Lemma 8. *For G a V-generic class over P, V and $V[G]$ have the same cardinals and cofinalities, and $V[G] \models$ ZFC + GCH.*

Proof of Lemma 8. Write $G = G_\kappa * H$, where G_κ is V-generic over P_κ, and H is a $V[G_\kappa]$-generic class over Q_κ. We show $V[G_\kappa][H] \models$ ZFC, and by assuming for the time being that $V[G_\kappa] \models$ GCH and has the same cardinals and cofinalities as V, we show $V[G_\kappa][H] \models$ GCH and has the same cardinals and cofinalities as $V[G_\kappa]$ (and hence as V).

To do this, note that Q_κ is equivalent in $V[G_\kappa] = V_1$ to the Easton support iteration of $\langle Q^0_\lambda * \dot{Q}^1_\lambda : \lambda > \kappa$ is a regular cardinal$\rangle$, so we assume without loss of generality that Q_κ is in fact this ordering. Note also that as we are assuming κ has no inaccessibles above it, Q_κ is in fact equivalent to the Easton support iteration of $\langle Q^0_\lambda * \dot{Q}^1_\lambda : \lambda > \kappa$ is a successor cardinal$\rangle$. We first show inductively that for any successor cardinal $\delta^+ > \kappa$, forcing over V_1 with the iteration of $\langle Q^0_\lambda * \dot{Q}^1_\lambda : \kappa < \lambda < \delta^+$ and λ is a successor cardinal$\rangle$ preserves cardinals, cofinalities, and GCH. If δ is regular (meaning δ is a successor cardinal since κ has no inaccessibles above it), then this iteration can be written as $Q_{<\delta} * (\dot{Q}^0_\delta * \dot{Q}^1_\delta)$, where $Q_{<\delta}$ is the iteration of $\langle Q^0_\lambda * \dot{Q}^1_\lambda : \kappa < \lambda < \delta$ and λ is a successor cardinal$\rangle$. By induction, forcing over V_1 with $Q_{<\delta}$ preserves cardinals, cofinalities, and GCH, so since forcing over $V_1^{Q_{<\delta}}$ with $\dot{Q}^0_\delta * \dot{Q}^1_\delta$ will preserve GCH and the cardinals and cofinalities of $V_1^{Q_{<\delta}}$, forcing over V_1 with $Q_{<\delta} * (\dot{Q}^0_\delta * \dot{Q}^1_\delta)$ preserves cardinals, cofinalities, and GCH. If δ is singular, let $\gamma < \delta$ be a cardinal in V_1, and write the iteration of $\langle Q^0_\lambda * \dot{Q}^1_\lambda : \kappa < \lambda < \delta^+$ and λ is a successor cardinal$\rangle$ as $Q_{<\gamma^+} * \dot{Q}^{\ge\gamma^+}$, where $Q_{<\gamma^+}$ is as above and $\dot{Q}^{\ge\gamma^+}$ is a term in V_1 for the rest of the iteration; if $\gamma < \kappa$, then $Q_{<\gamma^+}$ is trivial and $\dot{Q}^{\ge\gamma^+}$ is a term for the whole iteration. By induction, $V_1^{Q_{<\gamma^+}} \models$ "γ is a cardinal, $2^\gamma = \gamma^+$, and $\mathrm{cof}(\gamma) = \mathrm{cof}^{V_1}(\gamma)$", so as $V_1^{Q_{<\gamma^+}} \models$ "$Q^{\ge\gamma^+}$ is γ-closed", $V_1^{Q_{<\gamma^+} * \dot{Q}^{\ge\gamma^+}} \models$ "γ is a cardinal, $2^\gamma = \gamma^+$, and $\text{cof}(\gamma) = \text{cof}^{V_1}(\gamma)$" , i.e., GCH, cardinals, and cofinalities below δ are preserved when forcing over V_1 with $Q_{<\gamma^+} * \dot{Q}^{\geq\gamma^+}$. In addition, since the last sentence shows any $f : \gamma \to \delta$ or $f : \gamma \to \delta^+$, $f \in V^{Q_{<\gamma^+} * \dot{Q}^{\geq\gamma^+}}$, is so that $f \in V_1^{Q_{<\gamma^+}}$ for arbitrary $\gamma < \delta$, the fact $V_1^{Q_{<\gamma^+}}$ and V_1 have the same cardinals and cofinalities, together with the fact $V_1^{Q_{<\gamma^+} * \dot{Q}^{\geq\gamma^+}} \models$ "δ is a singular limit of cardinals satisfying GCH" yields that forcing over V_1 with $Q_{<\gamma^+} * \dot{Q}^{\geq\gamma^+}$ preserves δ is a singular cardinal of the same cofinality as in V_1, $2^\delta = \delta^+$, and δ^+ is a regular cardinal. Finally, as GCH in V_1 tells us $|Q_{<\gamma^+} * \dot{Q}^{\geq\gamma^+}| = \delta^+$, forcing with $Q_{<\gamma^+} * \dot{Q}^{\geq\gamma^+}$ over V_1 preserves cardinals and cofinalities $\geq \delta^{++}$ and GCH $\geq \delta^+$.

It is now easy to show $V_2 = V[G_\kappa][H] \models \text{ZFC} + \text{GCH}$ and has the same cardinals and cofinalities as $V[G_\kappa] = V_1$. To show $V_2 \models$ GCH and has the same cardinals and cofinalities as V_1, let γ again be a cardinal in V_1, and write $Q_\kappa = Q_{<\gamma^+} * \dot{Q}$, where $\dot{Q}$ is a term in V_1 for the rest of Q_κ. As before, $V_1^{Q_{<\gamma^+}} \models$ "$2^\gamma = \gamma^+$ and $\text{cof}(\gamma) = \text{cof}^{V_1}(\gamma)$", so since $V_1^{Q_{<\gamma^+}} \models$ "Q is γ-closed", $V_2 \models$ "$2^\gamma = \gamma^+$ and $\text{cof}(\gamma) = \text{cof}^{V_1}(\gamma)$", i.e., by the arbitrariness of γ, $V_2 \models$ GCH, and all cardinals of V_1 are cardinals of the same cofinality in V_2. Finally, as all functions $f : \gamma \to \delta$, $\delta \in V_1$ some ordinal, $f \in V_2$ are so that $f \in V_1^{Q_{<\gamma^+}}$ by the last sentence, it is the case $V_2 \models$ Power Set, and since $V_2 \models$ AC and Q_κ is an Easton support iteration, by the usual arguments, the aforementioned fact implies $V_2 \models$ Replacement. Thus, $V_2 \models$ ZFC.

It remains to show that $V[G_\kappa] \models$ GCH and has the same cardinals and cofinalities as V. To do this, we first note that Easton support iterations of δ-strategically closed partial orderings are δ-strategically closed for δ any regular cardinal. The proof is via induction. If R_1 is δ-strategically closed and $\Vdash_{R_1}$ "$\dot{R}_2$ is δ-strategically closed", then let $p \in R_1$ be so that $p \Vdash$ "$\dot{g}$ is a strategy for player II ensuring that the game which produces an increasing chain of elements of $\dot{R}_2$ of length δ can always be continued for $\alpha \leq \delta$". If II begins by picking $r_0 = \langle p_0, \dot{q}_0 \rangle \in R_1 * \dot{R}_2$ so that $p_0 \geq p$ has been chosen according to the strategy f for R_1 and $p_0 \Vdash$ "$\dot{q}_0$ has been chosen according to $\dot{g}$", and at even stages $\alpha + 2$ picks $r_{\alpha+2} = \langle p_{\alpha+2}, \dot{q}_{\alpha+2} \rangle$ so that $p_{\alpha+2}$ has been chosen according to f and is so that $p_{\alpha+2} \Vdash$ "$\dot{q}_{\alpha+2}$ has been chosen according to $\dot{g}$", then at limit stages $\lambda \leq \delta$, the chain $r_0 = \langle p_0, \dot{q}_0 \rangle \leq r_1 = \langle p_1, \dot{q}_1 \rangle \leq \cdots \leq r_\alpha = \langle p_\alpha, \dot{q}_\alpha \rangle \leq \cdots (\alpha < \lambda)$ is so that II can find an upper bound p_λ for $\langle p_\alpha : \alpha < \lambda \rangle$ using f. By construction, $p_\lambda \Vdash$ "$\langle \dot{q}_\alpha : \alpha < \lambda \rangle$ is so that at limit and even stages, II has played according to $\dot{g}$", so for some $\dot{q}_\lambda$, $p_\lambda \Vdash$ "$\dot{q}_\lambda$ is an upper bound to $\langle \dot{q}_\alpha : \alpha < \lambda \rangle$", meaning the condition $\langle p_\lambda, \dot{q}_\lambda \rangle$ is as desired. These methods, together with the usual proof at limit stages (see [Ba, Theorem 2.5]) that the Easton support iteration of δ-closed partial orderings is δ-closed, yield that δ-strategic closure is preserved at limit stages of all of our Easton support iterations of δ-strategically closed partial orderings. In addition, the ideas of this paragraph will also show that Easton support iterations of $\prec \delta^+$-strategically closed partial orderings are $\prec \delta^+$-strategically closed for δ any regular cardinal.

For $\alpha < \kappa$ and $P_{\alpha+1} = P_\alpha * \dot{Q}_\alpha$, since $\lambda_\alpha < \delta_{\alpha+1}$, the definition of Q_α in V^{P_α} implies $V^{P_\alpha} \models$ "$|Q_\alpha| < \delta_{\alpha+1}$". This fact, together with Lemma 5 and the definition of Q_α in V^{P_α}, now yields the proof that $V^{P_{\alpha+1}} \models$ GCH and has the same cardinals and cofinalities as V is virtually identical to the proof given in the first part of this lemma that $V_2 \models$ GCH and has the same cardinals and cofinalities as V_1, replacing

γ-closure with γ-strategic closure, which also implies that the forcing adds no new functions from γ to the ground model.

If λ is a limit ordinal so that $\bar{\lambda} = \sup(\{\delta_\alpha : \alpha < \lambda\})$ is singular, then, again, the proof that $V^{P_\lambda} \models$ GCH and has the same cardinals and cofinalities as V is virtually the same as the just referred to proof of the first part of this lemma for virtually identical reasons as in the previous sentence, keeping in mind that since $|P_\alpha| < \delta_\alpha$ inductively for $\alpha < \lambda$, $|P_\lambda| = \bar{\lambda}^+$. If $\lambda \le \kappa$ is a limit ordinal so that $\bar{\lambda} = \lambda$, then for cardinals $\gamma \le \lambda$, the proof that $V^{P_\lambda} \models$ "γ is a cardinal and $\mathrm{cof}(\gamma) = \mathrm{cof}^V(\gamma)$" is once more as before, as is the proof that $V^{P_\lambda} \models$ "$2^\gamma = \gamma^+$" for $\gamma < \lambda$. As again $|P_a| < \delta_\alpha < \lambda$ for $\alpha < \lambda$, $|P_\lambda| = \lambda$, so $V^{P_\lambda} \models$ "γ is a cardinal, $\mathrm{cof}(\gamma) = \mathrm{cof}^V(\gamma)$, and $2^\gamma = \gamma^+$" for $\gamma \ge \lambda$ a cardinal. Thus, $V[G_\kappa] \models$ GCH and has the same cardinals and cofinalities as V. This proves Lemma 8. $\square$ Lemma 8

We now show that the intuitive motivation for the definition of P as set forth in the paragraph immediately preceding the statement of Lemma 8 actually works.

Lemma 9. *If $\delta < \gamma$ and $V \models$ "δ is γ supercompact and γ is regular", then for G V-generic over P, $V[G] \models$ "δ is γ supercompact".*

Proof of Lemma 9. Let $j : V \to M$ be an elementary embedding witnessing the γ supercompactness of δ so that $M \models$ "δ is not γ supercompact". For the α_0 so that $\delta = \delta_{\alpha_0}$, let $P = P_{\alpha_0} * \dot{Q}'_{\alpha_0} * \dot{T}_{\alpha_0} * \dot{R}$, where $\dot{Q}'_{\alpha_0}$ is a term for the full support iteration of $\langle P^0_{\omega,\lambda} * (P^1_{\omega,\lambda}[\dot{S}_\lambda] \times P^2_{\omega,\lambda}[\dot{S}_\lambda]) : \delta^+ \le \lambda \le \gamma$ and λ is regular$\rangle$, $\dot{T}_{\alpha_0}$ is a term for the rest of Q_{α_0}, and $\dot{R}$ is a term for the rest of P. We show that $V^{P_{\alpha_0} * \dot{Q}'_{\alpha_0}} \models$ "δ is γ supercompact". This will suffice, since $\Vdash_{P_{\alpha_0} * \dot{Q}'_\alpha}$ "$\dot{T}_{\alpha_0} * \dot{R}$ is γ-strategically closed", so as the regularity of γ and GCH in $V^{P_{\alpha_0} * \dot{Q}'_{\alpha_0}}$ imply $V^{P_{\alpha_0} * \dot{Q}'_{\alpha_0}} \models$ "$|[\gamma]^{<\delta}| = \gamma$", if $V^{P_{\alpha_0} * \dot{Q}'_{\alpha_0}} \models$ "δ is γ supercompact", then $V^{P_{\alpha_0} * \dot{Q}'_{\alpha_0} * \dot{T}_{\alpha_0} * \dot{R}} = V^P \models$ "δ is γ supercompact via any ultrafilter $\mathcal{U} \in V^{P_{\alpha_0} * \dot{Q}'_{\alpha_0}}$".

To this end, we first note we will actually show that for $G_{\alpha_0} * G'_{\alpha_0}$ the portion of G V-generic over $P_{\alpha_0} * \dot{Q}'_{\alpha_0}$, the embedding j extends to $k : V[G_{\alpha_0} * G'_{\alpha_0}] \to M[H]$ for some $H \subseteq j(P)$. As $\langle j(\alpha) : \alpha < \gamma \rangle \in M$, this will be enough to allow the definition of the ultrafilter $x \in \mathcal{U}$ iff $\langle j(\alpha) : \alpha < \gamma \rangle \in k(x)$ to be given in $V[G_{\alpha_0} * G'_{\alpha_0}]$.

We construct H in stages. In M, as $\delta = \delta_{\alpha_0}$ is the critical point of j,

$$j(P_{\alpha_0} * \dot{Q}'_{\alpha_0}) = P_{\alpha_0} * \dot{R}'_{\alpha_0} * \dot{R}''_{\alpha_0} * \dot{R}'''_{\alpha_0},$$

where $\dot{R}'_{\alpha_0}$ will be a term for the full support iteration of $\langle P^0_{\omega,\lambda} * (P^1_{\omega,\lambda}[\dot{S}_\lambda] \times P^2_{\omega,\lambda}[\dot{S}_\lambda]) : \delta^+ \le \lambda < \gamma$ and λ is regular$\rangle * \langle P^0_{\omega,\gamma} * P^1_{\omega,\gamma}[\dot{S}_\gamma]\rangle$ (note that as $M^\gamma \subseteq M$, GCH implies that $M \models$ "δ is λ supercompact" if $\lambda < \gamma$ is regular, so since $M \models$ "δ is not γ supercompact", $\dot{R}'_{\alpha_0}$ is indeed as just stated), $\dot{R}''_{\alpha_0}$ will be a term for the rest of the portion of $j(P_{\alpha_0})$ defined below $j(\delta)$, and $\dot{R}'''_{\alpha_0}$ will be a term for $j(\dot{Q}'_{\alpha_0})$. This will allow us to define H as $H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H'''_{\alpha_0}$. Factoring G'_{α_0} as $\langle G^0_{\omega,\lambda} * (G^1_{\omega,\lambda} \times G^2_{\omega,\lambda}) : \delta^+ \le \lambda \le \gamma$ and λ is regular$\rangle$, we let $H_{\alpha_0} = G_{\alpha_0}$ and

$$H'_{\alpha_0} = \langle G^0_{\omega,\lambda} * (G^1_{\omega,\lambda} \times G^2_{\omega,\lambda}) : \delta^+ \le \lambda < \gamma \text{ and } \lambda \text{ is regular } \rangle * \langle G^0_{\omega,\gamma} * G^1_{\omega,\gamma}\rangle.$$

Thus, H'_{α_0} is the same as G'_{α_0}, except, since $M \models$ "δ is not γ supercompact", we omit the generic object $G^2_{\omega,\gamma}$.

To construct H''_{α_0}, we first note that the definition of P ensures $|P_{\alpha_0}| = \delta$ and, since δ is necessarily Mahlo, P_{α_0} is δ-c.c. As $V[G_{\alpha_0}]$ and $M[G_{\alpha_0}]$ are both models of GCH, the definition of R'_{α_0} in $M[H_{\alpha_0}]$, Lemmas 4, 5, and 8, and the remark immediately following Lemma 5 then ensure that $M[H_{\alpha_0}] \models$ "The portion of R'_{α_0} below γ is γ^+-c.c. and the portion of R'_{α_0} at γ is a γ-strategically closed partial ordering followed by a γ^+-c.c. partial ordering". Since $M^\gamma \subseteq M$ implies $(\gamma^+)^V = (\gamma^+)^M$ and P_{α_0} is δ-c.c., Lemma 6.4 of [Ba] shows $V[G_{\alpha_0}]$ satisfies these facts as well. This means applying the argument of Lemma 6.4 of [Ba] twice, in concert with an application of the fact that a portion of R'_{α_0} at γ is γ-strategically closed, shows $M[H_{\alpha_0} * H'_{\alpha_0}] = M[G_{\alpha_0} * H'_{\alpha_0}]$ is closed under γ sequences with respect to $V[G_{\alpha_0} * H'_{\alpha_0}]$, i.e., if $f : \gamma \to M[H_{\alpha_0} * H'_{\alpha_0}]$, $f \in V[G_{\alpha_0} * H'_{\alpha_0}]$, then $f \in M[H_{\alpha_0} * H'_{\alpha_0}]$. Therefore, as $M[H_{\alpha_0} * H'_{\alpha_0}] \models$ "R''_{α_0} is both γ-strategically closed and $\prec \gamma^+$-strategically closed", these facts are true in $V[G_{\alpha_0} * H'_{\alpha_0}]$ as well.

Observe now that GCH allows us to assume $\gamma^+ < j(\delta) < j(\delta^+) < \gamma^{++}$. Since $M[H_{\alpha_0} * H'_{\alpha_0}] \models$ "$|R''_{\alpha_0}| = j(\delta)$ and $|\mathcal{P}(R''_{\alpha_0})| = j(\delta^+)$" (this last fact follows from GCH in $M[H_{\alpha_0} * H'_{\alpha_0}]$), in $V[G_{\alpha_0} * H'_{\alpha_0}]$, we can let $\langle D_\alpha : \alpha < \gamma^+\rangle$ be an enumeration of the dense open subsets of R''_{α_0} present in $M[H_{\alpha_0} * H'_{\alpha_0}]$. The $\prec \gamma^+$-strategic closure of R''_{α_0} in both $M[H_{\alpha_0} * H'_{\alpha_0}]$ and $V[G_{\alpha_0} * H'_{\alpha_0}]$ now allows us to meet all of these dense subsets as follows. Work in $V[G_{\alpha_0} * H'_{\alpha_0}]$. Player I picks $p_\alpha \in D_\alpha$ extending $\sup(\langle q_\beta : \beta < \alpha\rangle)$ (initially, q_{-1} is the trivial condition), and player II responds by picking $q_\alpha \geq p_\alpha$ (so $q_\alpha \in D_\alpha$). By the $\prec \gamma^+$-strategic closure of R''_{α_0} in $V[G_{\alpha_0} * H'_{\alpha_0}]$, player II has a winning strategy for this game, so $\langle q_\alpha : \alpha < \gamma^+\rangle$ can be taken as an increasing sequence of conditions with $q_\alpha \in D_\alpha$ for $\alpha < \gamma^+$. Clearly, $H''_{\alpha_0} = \{p \in R''_{\alpha_0} : \exists\alpha < \gamma^+[q_\alpha \geq p]\}$ is our $M[H_{\alpha_0} * H'_{\alpha_0}]$-generic object over R''_{α_0} which has been constructed in $V[G_{\alpha_0} * H'_{\alpha_0}] \subseteq V[G_{\alpha_0} * G'_{\alpha_0}]$, so $H''_{\alpha_0} \in V[G_{\alpha_0} * G'_{\alpha_0}]$.

To construct H'''_{α_0}, we note first that, as in our remarks in Lemma 8, since γ must be below the least inaccessible $> \delta$ and γ is regular, $\gamma = \sigma^+$ for some σ. This allows us to write in $V[G_{\alpha_0}]$ $Q'_{\alpha_0} = Q''_{\alpha_0} * \dot{Q}'''_{\alpha_0}$, where Q''_{α_0} is the full support iteration of $\langle P^0_{\omega,\lambda} * (P^1_{\omega,\lambda}[\dot{S}_\lambda] \times P^2_{\omega,\lambda}[\dot{S}_\lambda] : \delta^+ \leq \lambda \leq \sigma$ and λ is regular$\rangle$ and $\dot{Q}'''_{\alpha_0}$ is a term for $P^0_{\omega,\gamma} * (P^1_{\omega,\gamma}[\dot{S}_\gamma] \times P^2_{\omega,\gamma}[\dot{S}_\gamma])$. This factorization of Q'_{α_0} induces through j in $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$ a factorization of R'''_{α_0} into $R^4_{\alpha_0} * \dot{R}^5_{\alpha_0} = \langle$the full support iteration of $\langle P^0_{\omega,\lambda} * (P^1_{\omega,\lambda}[\dot{S}_\lambda] \times P^2_{\omega,\lambda}[\dot{S}_\lambda]) : j(\delta^+) \leq \lambda \leq j(\sigma)$ and λ is regular$\rangle * \langle \dot{P}^0_{\omega,j(\gamma)} * (P^1_{\omega,j(\gamma)}[\dot{S}_{j(\gamma)}] \times P^2_{\omega,j(\gamma)}[\dot{S}_{j(\gamma)}])\rangle$.

Work now in $V[G_{\alpha_0} * H'_{\alpha_0}]$. In $M[H_{\alpha_0} * H'_{\alpha_0}]$, as previously noted, R''_{α_0} is γ-strategically closed. Since $M[H_{\alpha_0} * H'_{\alpha_0}]$ has already been observed to be closed under γ sequences with respect to $V[G_{\alpha_0} * H'_{\alpha_0}]$, and since any γ sequence of elements of $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$ can be represented, in $M[H_{\alpha_0} * H'_{\alpha_0}]$, by a term which is actually a function $f : \gamma \to M[H_{\alpha_0} * H'_{\alpha_0}]$, $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$ is closed under γ sequences with respect to $V[G_{\alpha_0} * H'_{\alpha_0}]$, i.e., if $f : \gamma \to M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$, $f \in V[G_{\alpha_0} * H'_{\alpha_0}]$, then $f \in M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$.

Factor (in $V[G_{\alpha_0} * G'_{\alpha_0}]$) G'_{α_0} as $G''_{\alpha_0} * G'''_{\alpha_0}$, with $G''_{\alpha_0} = \langle G^0_{\omega,\lambda} * (G^1_{\omega,\lambda} \times G^2_{\omega,\lambda}) : \delta^+ \leq \lambda \leq \sigma$ and λ is regular$\rangle$ and $G'''_{\alpha_0} = G^0_{\omega,\gamma} * (G^1_{\omega,\gamma} \times G^2_{\omega,\gamma})$, where G''_{α_0} is the projection of G'_{α_0} onto Q''_{α_0} and G'''_{α_0} is the projection of G'_{α_0} onto Q'''_{α_0}. By our definitions, $Q''_{\alpha_0} \in V[G_{\alpha_0}]$ and $G''_{\alpha_0} \in V[G_{\alpha_0} * H'_{\alpha_0}]$. Also, our construction to this point guarantees that in $V[G_{\alpha_0} * H'_{\alpha_0}]$, the embedding j extends to j^*: $V[G_{\alpha_0}] \to M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$. Thus, as GCH in $V[G_{\alpha_0} * H'_{\alpha_0}]$ implies $V[G_{\alpha_0} * H'_{\alpha_0}] \models$ "$|Q''_{\alpha_0}| = |G''_{\alpha_0}| = \gamma$", the last paragraph implies $\{j^*(p) : p \in G''_{\alpha_0}\} \in M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$. Since $\{j^*(p) : p \in G''_{\alpha_0}\} \subseteq R^4_{\alpha_0}$, $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}] \models$ "$R^4_{\alpha_0}$

is equivalent to a $j^*(\delta) = j(\delta)$-directed closed partial ordering", and $j(\delta) > \gamma$, $q = \sup\{j^*(p) : p \in G''_{\alpha_0}\}$ can be taken as a condition in $R^4_{\alpha_0}$.

Note that GCH in $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$ implies $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}] \models$ "$|R^4_{\alpha_0}| = j(\gamma)$", and by choice of $j : V \to M$, $V[G_{\alpha_0} * H'_{\alpha_0}] \models$ "$|j(\gamma)| = \gamma^+$ and $|j(\gamma^+)| = \gamma^+$". Hence, as the number of dense open subsets of $R^4_{\alpha_0}$ in $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$ is $(2^{j(\gamma)})^{M[H_{a_0} * H'_{a_0} * H''_{\alpha_0}]} = (j(\gamma)^+)^{M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]}$ which has cardinality $(\gamma^+)^V = (\gamma^+)^{V[G_{\alpha_0} * H'_{\alpha_0}]}$, we can let $\langle D_\alpha : \alpha < \gamma^+ \rangle \in V[G_{\alpha_0} * H'_{\alpha_0}]$ enumerate all dense open subsets of $R^4_{\alpha_0}$ in $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$. The γ-closure of $R^4_{\alpha_0}$ in $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$ and hence in $V[G_{\alpha_0} * H'_{\alpha_0}]$ now allows an $M[H_{a_0} * H'_{\alpha_0} * H''_{\alpha_0}]$-generic object $H^4_{\alpha_0}$ over $R^4_{\alpha_0}$ containing q to be constructed in the standard way in $V[G_{\alpha_0} * H'_{\alpha_0}]$; namely let $q_0 \in D_0$ be so that $q_0 \geq q$, and at stage $\alpha < \gamma^+$, by the γ-closure of $R^4_{\alpha_0}$ in $V[G_{\alpha_0} * H'_{\alpha_0}]$, let $q_\alpha \in D_\alpha$ be so that $q_\alpha \geq \sup(\langle q_\beta : \beta < \alpha \rangle)$. As before, $H^4_{\alpha_0} = \{p \in R^4_{\alpha_0} : \exists \alpha < \gamma^+ [q_\alpha \geq p]\} \in V[G_{\alpha_0} * H'_{\alpha_0}] \subseteq V[G_{\alpha_0} * G'_{\alpha_0}]$ is clearly our desired generic object.

By the above construction, in $V[G_{\alpha_0} * G'_{\alpha_0}]$, the embedding $j^* : V[G_{\alpha_0}] \to M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$ extends to an embedding $j^{**} : V[G_{\alpha_0} * G''_{\alpha_0}] \to M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}]$. We will be done once we have constructed in $V[G_{\alpha_0} * G'_{\alpha_0}]$ the appropriate generic object for $R^5_{\alpha_0} = P^0_{\omega,j(\gamma)} * (P^1_{\omega,j(\gamma)}[\dot{S}_{j(\gamma)}] \times P^2_{\omega,j(\gamma)}[\dot{S}_{j(\gamma)}]) = (P^0_{\omega,j(\gamma)} * P^2_{\omega,j(\gamma)}[\dot{S}_{j(\gamma)}]) * P^1_{\omega,j(\gamma)}[\dot{S}_{j(\gamma)}]$. To do this, first rewrite G'''_{α_0} as $(G^0_{\omega,\gamma} * G^2_{\omega,\gamma}) * G^1_{\omega,\gamma}$. By the nature of the forcings, $G^0_{\omega,\gamma} * G^2_{\omega,\gamma}$ is $V[G_{\alpha_0} * G''_{\alpha_0}]$-generic over a partial ordering which is (γ, ∞)-distributive. Thus, by a general fact about transference of generics via elementary embeddings (folklore; see [C, Section 1.2, Fact 2, pp. 5-6]), since $j^{**} : V[G_{\alpha_0} * G''_{\alpha_0}] \to M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}]$ is so that every element of $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}]$ can be written $j^{**}(F)(a)$ with $\mathrm{dom}(F)$ having cardinality γ, $j^{**}{}'' G^0_{\omega,\gamma} * G^2_{\omega,\gamma}$ generates an $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}]$-generic set $H^5_{\alpha_0}$.

It remains to construct $H^6_{\alpha_0}$, our $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0} * H^5_{\alpha_0}]$-generic object over $P^1_{\omega,j(\gamma)}[S_{j(\gamma)}]$. To do this, first note that $H^4_{\alpha_0}$ (which was constructed in $V[G_{\alpha_0} * H'_{\alpha_0}]$) is $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$-generic over $R^4_{\alpha_0}$, a partial ordering which in $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$ is $j(\delta)$-closed. Since $j(\delta) > \gamma$ and $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$ is closed under γ sequences with respect to $V[G_{\alpha_0} * H'_{\alpha_0}]$, we can apply earlier reasoning to infer $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}]$ is closed under γ sequences with respect to $V[G_{\alpha_0} * H'_{\alpha_0}]$, i.e., if $f : \gamma \to M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}]$, $f \in V[G_{\alpha_0} * H'_{\alpha_0}]$, then $f \in M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}]$.

Choose in $V[G_{\alpha_0} * G'_{\alpha_0}]$ an enumeration $\langle p_\alpha : \alpha < \gamma^+ \rangle$ of $G^1_{\omega,\gamma}$. Working now in $V[G_{\alpha_0} * G'_{\alpha_0}]$, let f be an isomorphism between (a dense subset of) $P^1_{\omega,\gamma}[S_\gamma]$ and Q^1_γ. This gives us a sequence $\langle f(p_\alpha) : \alpha < \gamma^+ \rangle$ of γ^+ many compatible elements of Q^1_γ. Letting $p'_\alpha = f(p_\alpha)$, we may hence assume that $I = \langle p'_\alpha : \alpha < \gamma^+ \rangle$ is an appropriately generic object for Q^1_γ. By Lemma 6, $V[G_{\alpha_0} * G''_{\alpha_0} * G^0_{\omega,\gamma} * G^1_{\omega,\gamma} * G^2_{\omega,\gamma}] = V[G_{\alpha_0} * G'_{\alpha_0}]$ and $V[G_{\alpha_0} * G''_{\alpha_0} * G^0_{\omega,\gamma} * G^1_{\omega,\gamma}] = V[G_{\alpha_0} * H'_{\alpha_0}]$ have the same γ sequences of elements of $V[G_{\alpha_0} * G''_{\alpha_0}]$ and hence of $V[G_{\alpha_0} * H'_{\alpha_0}]$. Thus, any γ sequence of elements of $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}]$ present in $V[G_{\alpha_0} * G'_{\alpha_0}]$ is actually an element of $V[G_{\alpha_0} * H'_{\alpha_0}]$ (so $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}]$ is really closed under γ sequences with respect to $V[G_{\alpha_0} * G'_{\alpha_0}]$).

For $\alpha \in (\gamma, \gamma^+)$ and $p \in Q^1_\gamma$, let $p|\alpha = \{\langle\langle \rho, \sigma\rangle, \eta\rangle \in Q^1_\gamma : \sigma < \alpha\}$ and $I|\alpha = \{p|\alpha : p \in I\}$. Clearly $V[G_{\alpha_0} * G'_{\alpha_0}] \models$ "$|I|\alpha| = \gamma$ for all $\alpha \in (\gamma, \gamma^+)$". Thus, since

$Q^1_{j(\gamma)} \in M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}]$ and $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}] \models$ "$Q^1_{j(\gamma)}$ is $j(\gamma)$-directed closed", the facts $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}]$ is closed under γ sequences with respect to $V[G_{\alpha_0} * G'_{\alpha_0}]$ and I is compatible imply that $q_\alpha = \bigcup\{j^{**}(p) : p \in I|\alpha\}$ for $\alpha \in (\gamma, \gamma^+)$ is well-defined and is an element of $Q^1_{j(\gamma)}$. Further, if $\langle \rho, \sigma\rangle \in \mathrm{dom}(q_\alpha) - \mathrm{dom}(\bigcup\limits_{\beta<\alpha} q_\beta)$ ($\bigcup\limits_{\beta<\alpha} q_\beta \in Q^1_{j(\gamma)}$ as $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}]$ is closed under γ sequences with respect to $V[G_{\alpha_0} * G'_{\alpha_0}]$), then $\sigma \in [\bigcup\limits_{\beta<\alpha} j(\beta), j(\alpha))$. (If $\sigma < \bigcup\limits_{\beta<\alpha} j(\beta)$, then let β be minimal so that $\sigma < j(\beta)$, and let ρ and σ be so that $\langle \rho, \sigma\rangle \in \mathrm{dom}(q_\alpha)$. It must thus be the case that for some $p \in I|\alpha$, $\langle \rho, \sigma\rangle \in \mathrm{dom}(j^{**}(p))$. Since by elementarity and the definitions of $I|\beta$ and $I|\alpha$, for $p|\beta = q \in I|\beta$, $j^{**}(q) = j^{**}(p)|j(\beta) = j^{**}(p|\beta)$, it must be the case that $\langle \rho, \sigma\rangle \in \mathrm{dom}(j^{**}(q))$. This means $\langle \rho, \sigma\rangle \in \mathrm{dom}(q_\beta)$, a contradiction.)

We now define an $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0} * H^5_{\alpha_0}]$-generic object $H^{6,0}_{\alpha_0}$ over $Q^1_{j(\gamma)}$ so that $p \in f''G^1_{\omega,\gamma}$ implies $j^{**}(p) \in H^{6,0}_{\alpha_0}$. First, for $\beta \in (j(\gamma), j(\gamma^+))$, let $Q^{1,\beta}_{j(\gamma)} \in M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}]$ be the forcing for adding β many Cohen subsets to $j(\gamma)$, i.e., $Q^{1,\beta}_{j(\gamma)} = \{g : j(\gamma) \times \beta \to \{0, 1\} : g$ is a function so that $|\mathrm{dom}(g)| < j(\gamma)\}$, ordered by inclusion. Next, note that since $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0} * H^5_{\alpha_0}] \models \mathrm{GCH}$, $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0} * H^5_{\alpha_0}] \models$ "$Q^1_{j(\gamma)}$ is $j(\gamma^+)$-c.c. and $Q^1_{j(\gamma)}$ has $j(\gamma^+)$ many maximal antichains". This means that if $\mathcal{A} \in M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0} * H^5_{\alpha_0}]$ is a maximal antichain of $Q^1_{j(\gamma)}$, then $\mathcal{A} \subseteq Q^{1,\beta}_{j(\gamma)}$ for some $\beta \in (j(\gamma), j(\gamma^+))$. Also, since $V \subseteq V[G_{\alpha_0} * G''_{\alpha_0}] \subseteq V[G_{\alpha_0} * H'_{\alpha_0}] \subseteq V[G_{\alpha_0} * G'_{\alpha_0}]$ are all models of GCH containing the same cardinals and cofinalities, $V[G_{\alpha_0} * G'_{\alpha_0}] \models$ "$|j(\gamma^+)| = \gamma^+$". The preceding thus means we can let $\langle \mathcal{A}_\alpha : \alpha < \gamma^+\rangle \in V[G_{\alpha_0} * G'_{\alpha_0}]$ be an enumeration of the maximal antichains of $Q^1_{j(\gamma)}$ present in $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0} * H^5_{\alpha_0}]$.

Working in $V[G_{\alpha_0} * G'_{\alpha_0}]$, we define now an increasing sequence $\langle r_\alpha : \alpha \in (\gamma, \gamma^+)\rangle$ of elements of $Q^1_{j(\gamma)}$ so that $\forall \alpha < \gamma^+[r_\alpha \geq q_\alpha$ and $r_\alpha \in Q^{1,j(\alpha)}_{j(\gamma)}]$ and so that $\forall \mathcal{A} \in \langle \mathcal{A}_\alpha : \alpha \in (\gamma, \gamma^+)\rangle \exists \beta \in (\gamma, \gamma^+) \exists r \in \mathcal{A}[r_\beta \geq r]$. Assuming we have such a sequence, $H^{6,0}_{\alpha_0} = \{p \in Q^1_{j(\gamma)} : \exists r \in \langle r_\alpha : \alpha \in (\gamma, \gamma^+)\rangle[r \geq p]\}$ is our desired generic object. To define $\langle r_\alpha : \alpha \in (\gamma, \gamma^+)\rangle$, if α is a limit, we let $r_\alpha = \bigcup\limits_{\beta<\alpha} r_\beta$. By the facts $\langle q_\beta : \beta \in (\gamma, \gamma^+)\rangle$ is (strictly) increasing and $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}]$ is closed under γ sequences with respect to $V[G_{\alpha_0} * G'_{\alpha_0}]$, this definition is valid. Assuming now r_α has been defined and we wish to define $r_{\alpha+1}$, let $\langle \mathcal{B}_\beta : \beta < \eta \leq \gamma\rangle$ be the subsequence of $\langle \mathcal{A}_\beta : \beta \leq \alpha + 1\rangle$ containing each antichain $\mathcal{A}$ so that $\mathcal{A} \subseteq Q^{1,j(\alpha+1)}_{j(\gamma)}$. Since $q_\alpha, r_\alpha \in Q^{1,j(\alpha)}_{j(\gamma)}$, $q_{\alpha+1} \in Q^{1,j(\alpha+1)}_{j(\gamma)}$, and $j(\alpha) < j(\alpha + 1)$, the condition $r'_{\alpha+1} = r_\alpha \cup q_{\alpha+1}$ is well-defined, as by our earlier observations, any new elements of $\mathrm{dom}(q_{\alpha+1})$ won't be present in either $\mathrm{dom}(q_\alpha)$ or $\mathrm{dom}(r_\alpha)$. We can thus, using the fact $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}]$ is closed under γ sequences with respect to $V[G_{\alpha_0} * G'_{\alpha_0}]$, define by induction an increasing sequence $\langle s_\beta : \beta < \eta\rangle$ so that $s_0 \geq r'_{\alpha+1}$, $s_\rho = \bigcup\limits_{\beta<\rho} s_\beta$ if ρ is a limit, and $s_{\beta+1} \geq s_\beta$ is so that $s_{\beta+1}$ extends some element of $\mathcal{B}_\beta$. The just mentioned closure fact implies $r_{\alpha+1} = \bigcup\limits_{\beta<\eta} s_\beta$ is a well-defined condition.

In order to show $H^{6,0}_{\alpha_0}$ is $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0} * H^5_{\alpha_0}]$-generic over $Q^1_{j(\gamma)}$, we must show that $\forall \mathcal{A} \in \langle \mathcal{A}_\alpha : \alpha \in (\gamma, \gamma^+)\rangle \exists \beta \in (\gamma, \gamma^+) \exists r \in \mathcal{A}[r_\beta \geq r]$. To do this, we first note that $\langle j(\alpha) : \alpha < \gamma^+\rangle$ is unbounded in $j(\gamma^+)$. To see this, if $\beta < j(\gamma^+)$ is an ordinal, then for some $g : \gamma \to M$ representing β, we can assume that for $\lambda < \gamma$, $g(\lambda) < \gamma^+$. Thus, by the regularity of γ^+ in V, $\beta_0 = \bigcup_{\lambda<\gamma} g(\lambda) < \gamma^+$, and $j(\beta_0) > \beta$. This means by our earlier remarks that if $\mathcal{A} \in \langle \mathcal{A}_\alpha : \alpha < \gamma^+\rangle$, $\mathcal{A} = \mathcal{A}_\rho$, then we can let $\beta \in (\gamma, \gamma^+)$ be so that $\mathcal{A} \subseteq Q^{1,j(\beta)}_{j(\gamma)}$. By construction, for $\eta > \max(\beta, \rho)$, there is some $r \in \mathcal{A}$ so that $r_\eta \geq r$. Finally, since any $p \in Q^1_\gamma$ is so that for some $\alpha \in (\gamma, \gamma^+)$, $p = p|\alpha$, $H^{6,0}_{\alpha_0}$ is so that if $p \in f''G^1_{\omega,\gamma}$, then $j^{**}(p) \in H^{6,0}_{\alpha_0}$.

Note now that our earlier work ensures j^{**} extends to

$$j^{***} : V[G_{\alpha_0} * G''_{\alpha_0} * G^0_{\omega,\gamma} * G^2_{\omega,\gamma}] \to M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0} * H^5_{\alpha_0}].$$

By Lemma 4, the isomorphism f is definable over $V[G_{\alpha_0} * G''_{\alpha_0} * G^0_{\omega,\gamma} * G^2_{\omega,\gamma}]$. This means the notions $j^{***}(f)$ and $j^{***}(f^{-1})$ make sense, so $j^{***}(f)$ is a definable isomorphism over $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0} * H^5_{\alpha_0}]$ between (a dense subset of) $P^1_{\omega,j(\gamma)}[S_{j(\gamma)}]$ and $Q^1_{j(\gamma)}$, and $j^{***}(f^{-1})$ is its inverse. If $H^6_{\alpha_0} = \{j^{***}(f^{-1})(p) : p \in H^{6,0}_{\alpha_0}\}$, then it is now easy to verify that $H^6_{\alpha_0}$ is an $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0} * H^5_{\alpha_0}]$-generic object over (a dense subset of) $P^1_{\omega,j(\gamma)}[S_{j(\gamma)}]$, so that $p \in$ (a dense subset of) $P^1_{\omega,\gamma}[S_\gamma]$ implies $j^{***}(p) \in H^6_{\alpha_0}$. Therefore, for $H''' = H^4_{\alpha_0} * H^5_{\alpha_0} * H^6_{\alpha_0}$ and $H = H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H'''_{\alpha_0}$, $j : V \to M$ extends to $k : V[G_{\alpha_0} * G'_{\alpha_0}] \to M[H]$, so $V[G] \models$ "δ is γ supercompact" if γ is regular. This proves Lemma 9. □ Lemma 9

Lemma 10. *For γ regular, $V[G] \models$ "δ is γ strongly compact iff δ is γ supercompact".*

Proof of Lemma 10. Assume towards a contradiction that the lemma is false, and let $\delta < \gamma$ be so that $V[G] \models$ "δ is γ strongly compact, δ isn't γ supercompact, γ is regular, and γ is the least such cardinal". As before, let $\delta = \delta_\alpha$, i.e., δ is the αth inaccessible cardinal. If $V \models$ "δ_α is γ supercompact", then Lemma 9 implies $V[G] \models$ "δ_α is γ supercompact", so it must be the case that $V \models$ "δ_α isn't γ supercompact". We therefore have $\lambda_\alpha \leq \gamma$ for λ_α the least regular cardinal so that $V \models$ "δ_α isn't λ_α supercompact".

In the manner of Lemma 9, write $P = P_\alpha * \dot{Q}_\alpha * \dot{Q}'_\alpha$, where P_α is the iteration through stage α, $\dot{Q}_\alpha$ is a term for the full support iteration of $\langle P^0_{\omega,\lambda} * (P^1_{\omega,\lambda}[\dot{S}_\lambda] \times P^2_{\omega,\lambda}[\dot{S}_\lambda]) : \delta^+ \leq \lambda < \lambda_\alpha$ and λ is regular$\rangle * \langle \dot{P}^0_{\omega,\lambda_\alpha} * P^1_{\omega,\lambda_\alpha}[\dot{S}_{\lambda_\alpha}]\rangle$, and $\dot{Q}'_\alpha$ is a term for the rest of P. By our previous results, $V^{P_\alpha * \dot{Q}_\alpha} \models$ "δ_α isn't λ_α strongly compact", and $\Vdash_{P_\alpha * \dot{Q}_\alpha}$ "$\dot{Q}'_\alpha$ is $\delta_{\alpha+1}$-strategically closed" (where $\delta_{\alpha+1}$ is the least inaccessible $> \delta_\alpha$). It must thus be the case that $V^{P_\alpha * \dot{Q}_\alpha * \dot{Q}'_\alpha} = V^P \models$ "δ_α isn't λ_α strongly compact", so of course, as $\lambda_\alpha \leq \gamma$, $V[G] \models$ "δ_a isn't γ strongly compact". This proves Lemma 10. □ Lemma 10

Lemma 11. *For γ regular, $V[G] \models$ "δ is γ supercompact" iff $V \models$ "δ is γ supercompact".*

Proof of Lemma 11. By Lemma 9, if $V \models$ "δ is γ supercompact and γ is regular", then $V[G] \models$ "δ is γ supercompact". If $V[G] \models$ "δ is γ supercompact and γ is regular" but $V \models$ "δ is not γ supercompact", then as in Lemma 10, for the α so that $\delta = \delta_\alpha$, $\lambda_\alpha \leq \gamma$ for λ_α the least regular cardinal so that $V \models$ "δ_α isn't λ_α supercompact". The proof of Lemma 10 then immediately yields that $V[G] \models$ "δ_α isn't $\lambda_\alpha \leq \gamma$ strongly compact". This proves Lemma 11. □ Lemma 11

The proof of Lemma 11 completes the proof of our Theorem in the case when κ is the unique supercompact cardinal in the universe and has no inaccessibles above it. This guarantees the Theorem to hold non-trivially. $\square$ Theorem

3. The General Case

We will now prove our Theorem under the assumption that there may be more than one supercompact cardinal in the universe (including a proper class of supercompact cardinals) and that the large cardinal structure above any given supercompact can be rather complicated, including possibly many inaccessibles, measurables, etc. Before defining the forcing conditions, a few intuitive remarks are in order. We will proceed using the same general paradigm as in the last section, namely iterating the forcings of Section 1 using Easton supports so as to destroy those "bad" instances of strong compactness which can be destroyed and so as to resurrect and preserve all instances of supercompactness. For each inaccessible δ_i, a certain coding ordinal $\theta_i < \delta_i$ will be chosen when possible which we will use to define $P^0_{\theta_i,\lambda}$, $P^1_{\theta_i,\lambda}[S_{\theta_i,\lambda}]$, and $P^2_{\theta_i,\lambda}[S_{\theta_i,\lambda}]$, where $S_{\theta_i,\lambda}$ is the non-reflecting stationary set of ordinals of cofinality θ_i added to λ^+ by $P^0_{\theta_i,\lambda}$. We will need to have different values of θ_i, instead of having $\theta_i = \omega$ as in the last section, so as to destroy the λ strong compactness of some δ and yet preserve the λ supercompactness of a $\delta' \neq \delta$ when necessary. When θ_i can't be defined, we won't necessarily be able to destroy the λ strong compactness of δ_i, although we will be able to preserve the λ supercompactness of δ_i if appropriate. This will happen when instances of the results of [Me] and [A] occur, i.e., when there are certain limits of supercompactness.

Getting specific, let $\langle \delta_i : i \in \mathrm{Ord}\rangle$ enumerate the inaccessibles of $V \models$ GCH, and let $\lambda_i > \delta_i$ be the least regular cardinal so that $V \models$ "δ_i isn't λ_i supercompact" if such a λ_i exists. If no such λ_i exists, i.e., if δ_i is supercompact, then let $\lambda_i = \Omega$, where we think of Ω as some giant "ordinal" larger than any $\alpha \in \mathrm{Ord}$. If possible, choose $\theta_i < \delta_i$ as the least regular cardinal so that $\theta_i < \delta_j < \delta_i$ implies $\lambda_j < \delta_i$ (whenever $j < i$). Note that θ_i is undefined for δ_i iff δ_i is a limit of cardinals which are $< \delta_i$ supercompact, because for $j < i$, if δ_j is $< \delta_i$ supercompact, then $\lambda_j \geq \delta_i$.

We define now a class Easton support iteration $P = \langle\langle P_\alpha, \dot{Q}_\alpha\rangle : \alpha \in \mathrm{Ord}\rangle$ as follows: 1. P_0 is trivial. 2. Assuming P_α has been defined, $P_{\alpha+1} = P_\alpha * \dot{Q}_\alpha$, where $\dot{Q}_\alpha$ is a term for the trivial partial ordering unless α is regular and for some inaccessible $\delta = \delta_i < \alpha$ with θ_i defined, either δ_i is α supercompact or $\alpha = \lambda_i$. Under these circumstances $\dot{Q}_\alpha$ is a term for $\Big(\prod_{\{i<\alpha:\delta_i \text{is } \alpha \text{ supercompact}\}} (P^0_{\theta_i,\alpha} * P^2_{\theta_i,\alpha}[\dot{S}_{\theta_i,\alpha}]) * \prod_{\{i<\alpha:\delta_i \text{ is } \alpha \text{ supercompact}\}} P^1_{\theta_i,\alpha}[\dot{S}_{\theta_i,\alpha}]\Big) \times \Big(\prod_{\{i<\alpha:\alpha=\lambda_i\}} P^0_{\theta_i,\alpha} * \prod_{\{i<\alpha:\alpha=\lambda_i\}} P^1_{\theta_i,\alpha}[\dot{S}_{\theta_i,\alpha}]\Big) = (\dot{P}^0_\alpha * \dot{P}^1_\alpha) \times (\dot{P}^2_\alpha * \dot{P}^3_\alpha)$, with the proviso that elements of $\dot{P}^0_\alpha$ and $\dot{P}^2_\alpha$ will have full support, and elements of $\dot{P}^1_\alpha$ and $\dot{P}^3_\alpha$ will have support $< \alpha$. Note that unless $|\{i < \alpha : \delta_i \text{ is } < \alpha \text{ supercompact}\}| = \alpha$, the elements of $\dot{P}^i_\alpha$ will have full support for $i = 0, 1, 2, 3$.

The following lemma is the natural analogue to Lemma 8.

Lemma 12. *For G a V-generic class over P, V and $V[G]$ have the same cardinals and cofinalities, and $V[G] \models$ ZFC + GCH.*

Proof of Lemma 12. We show inductively that for any α, V and V^{P_α} have the same cardinals and cofinalities, and $V^{P_\alpha} \models$ GCH. This will suffice to show $V[G] \models$

GCH and has the same cardinals and cofinalities as V, since if $\dot{R}$ is a term so that $P_\alpha * \dot{R} = P$, then $\Vdash_{P_\alpha}$ "The iteration $\dot{R}$ is $< \alpha$-strategically closed", meaning $V^{P_\alpha * \dot{R}}$ and V^{P_α} have the same cardinals and cofinalities $\leq \alpha$ and GCH holds in both of these models for cardinals $< \alpha$.

Assume now V and V^{P_α} have the same cardinals and cofinalities, and $V^{P_\alpha} \models$ GCH. We show V and $V^{P_{\alpha+1}} = V^{P_\alpha * \dot{Q}_\alpha}$ have the same cardinals and cofinalities, and $V^{P_{\alpha+1}} \models$ GCH. If $\dot{Q}_\alpha$ is a term for the trivial partial ordering, this is clearly the case, so we assume $\dot{Q}_\alpha$ is not a term for the trivial partial ordering. Let $\dot{Q}'_\alpha$ be a term for $(\dot{P}^0_\alpha * \dot{P}^1_\alpha) \times (\prod_{\{i<\alpha:\alpha=\lambda_i\}} (\dot{P}^0_{\theta_i,\alpha} * P^2_{\theta_i,\alpha}[\dot{S}_{\theta_i,\alpha}]) * \dot{P}^3_\alpha) = (\dot{P}^0_\alpha * \dot{P}^1_\alpha) \times (\dot{P}^4_\alpha * \dot{P}^3_\alpha)$, where as earlier, the elements of $\dot{P}^0_\alpha$ and $\dot{P}^4_\alpha$ will have full support, and the elements of $\dot{P}^1_\alpha$ and $\dot{P}^3_\alpha$ will have support $< \alpha$. We are now able to rewrite $\dot{Q}'_\alpha$ as $(\prod_{\{i<\alpha:\delta_i \text{ is } \alpha \text{ supercompact or } \alpha=\lambda_i\}} (P^0_{\theta_i,\alpha} * P^2_{\theta_i,\alpha}[\dot{S}_{\theta_i,\alpha}])) * (\prod_{\{i<\alpha:\delta_i \text{ is } \alpha \text{ supercompact or } \alpha=\lambda_i\}} P^1_{\theta_i,\alpha}[\dot{S}_{\theta_i,\alpha}]) = \dot{P}^5_\alpha * \dot{P}^6_\alpha$, where the elements of $\dot{P}^5_\alpha$ will have full support, and the elements of $\dot{P}^6_\alpha$ will have support $< \alpha$. By Lemma 4, in V^{P_α}, each $P^0_{\theta_i,\alpha} * (P^1_{\theta_i,\alpha}[\dot{S}_{\theta_i,\alpha}] \times P^2_{\theta_i,\alpha}[\dot{S}_{\theta_i,\alpha}])$ is equivalent to $Q^0_\alpha * \dot{Q}^1_\alpha$. We therefore have that in V^{P_α}, Q'_α is equivalent to $(\prod_{\beta<\gamma} Q^0_\alpha) * (\prod_{\beta<\gamma} \dot{Q}^1_\alpha)$, where $\gamma = |\{i < \alpha : \delta_i$ is α supercompact or $\alpha = \lambda_i\}|$ (γ is a cardinal in both V and V^{P_α} by induction), i.e., the full support product of γ copies of Q^0_α followed by the $< \alpha$ support product of γ copies of Q^1_α. Since $\gamma \leq \alpha$, $\prod_{\beta<\gamma} Q^0_\alpha$ is isomorphic to the usual ordering for adding γ many Cohen subsets to α^+ using conditions of support $< \alpha^+$, and since $\prod_{\beta<\gamma} Q^1_\alpha$ is composed of elements having support $< \alpha$, $\prod_{\beta<\gamma} Q^1_\alpha$ is isomorphic to a single partial ordering for adding α^+ many Cohen subsets to α using conditions of support $< \alpha$. Hence, $V^{P_\alpha * \dot{Q}'_\alpha}$ and V^{P_α} have the same cardinals and cofinalities, and $V^{P_\alpha * \dot{Q}'_\alpha} \models$ GCH, so $V^{P_\alpha * \dot{Q}'_\alpha}$ and V have the same cardinals and cofinalities. And, for G_α the projection of G onto P_α, if H is $V[G_\alpha]$-generic over Q'_α, for any $i < \alpha$ so that $\alpha = \lambda_i$, we can omit the portion of H generic over $P^2_{\theta_i,\alpha}[\dot{S}_{\theta_i,\alpha}]$ and thus obtain a $V[G_\alpha]$-generic object H' for Q_α. Since $V \subseteq V[G_\alpha][H'] \subseteq V[G_\alpha][H]$, as in Lemma 5, it must therefore be the case that V, $V^{P_\alpha * \dot{Q}_\alpha} = V^{P_{\alpha+1}}$, and $V^{P_\alpha * \dot{Q}'_\alpha}$ all have the same cardinals and cofinalities and satisfy GCH.

To complete the proof of Lemma 12, if now α is a limit ordinal, the proof that V and V^{P_α} have the same cardinals and cofinalities and $V^{P_\alpha} \models$ GCH is the same as the proof given in the last paragraph of Lemma 8, since the iteration still has enough strategic closure and can easily be seen by GCH to be so that for any $\beta < \alpha$, $|P_\beta| < \alpha$. And, since for any α, $\Vdash_{P_\alpha}$ "$\dot{Q}_\alpha$ is $< \alpha$-strategically closed", all functions $f : \gamma \to \beta$ for $\gamma < \alpha$ and β any ordinal in $V[G]$ are so that $f \in V^{P_\alpha}$. Thus, since P is an Easton support iteration, as in Lemma 8, $V[G]$ satisfies Power Set and Replacement. This proves Lemma 12. $\square$ Lemma 12

We remark that if we rewrite $\dot{Q}_\alpha$ as $(\dot{P}^0_\alpha \times \dot{P}^2_\alpha) * (\dot{P}^1_\alpha \times \dot{P}^3_\alpha)$, then the ideas used in the proof of Lemma 12 combined with an argument analogous to the one in the remark following the proof of Lemma 5 show $\Vdash_{P_\alpha * (\dot{P}^0_\alpha \times \dot{P}^2_\alpha)}$ "$\dot{P}^1_\alpha \times \dot{P}^3_\alpha$ is α^+-c.c." Also, by their definitions, $\Vdash_{P_\alpha}$ "$\dot{P}^0_\alpha \times \dot{P}^2_\alpha$ is α-strategically closed". These observations will be used in the proof of the following lemma, which is the natural analogue to Lemma 9.

Lemma 13. *If $\delta < \gamma$ and $V \models$ "δ is γ supercompact and γ is regular", then for G V-generic over P, $V[G] \models$ "δ is γ supercompact".*

Proof of Lemma 13. We mimic the proof of Lemma 9. Let $j : V \to M$ be an elementary embedding witnessing the γ supercompactness of δ so that $M \models$ "δ is not γ supercompact", and let α_0 be so that $\delta = \delta_{\alpha_0}$.

Let $P = P_\delta * \dot{Q}'_\delta * \dot{R}$, where P_δ is the iteration through stage δ, $\dot{Q}'_\delta$ is a term for the iteration $\langle\langle P_\alpha/P_\delta, \dot{Q}_\alpha\rangle : \delta \le \alpha \le \gamma\rangle$, and $\dot{R}$ is a term for the rest of P. As before, since $\Vdash_{P_\delta * \dot{Q}'_\delta}$ "$\dot{R}$ is γ-strategically closed", the regularity of γ and GCH in $V^{P_\delta * \dot{Q}'_\delta}$ mean it suffices to show $V^{P_\delta * \dot{Q}'_\delta} \models$ "δ is γ supercompact".

We will again show that $j : V \to M$ extends to $k : V[G_\delta * G'_\delta] \to M[H]$ for some $H \subseteq j(P)$. In M, $j(P_\delta * \dot{Q}'_\delta) = P_\delta * \dot{R}'_\delta * \dot{R}''_\delta * \dot{R}'''_\delta$, where $\dot{R}'_\delta$ will be a term for the iteration (as defined in M^{P_δ}) $\langle\langle P_\alpha/P_\delta, \dot{Q}_\alpha\rangle : \delta \le \alpha \le \gamma\rangle$, $\dot{R}''_\delta$ will be a term for the iteration (as defined in $M^{P_\delta * \dot{R}'_\delta}$) $\langle\langle P_\alpha/P_{\gamma+1}, \dot{Q}_\alpha\rangle : \gamma + 1 \le \alpha < j(\delta)\rangle$, and $\dot{R}'''_\delta$ will be a term for the iteration (as defined in $M^{P_\delta * \dot{R}'_\delta * \dot{R}''_\delta}$) $\langle\langle P_\alpha/P_{j(\delta)}, \dot{Q}_\alpha\rangle : j(\delta) \le \alpha \le j(\gamma)\rangle$. By the facts that GCH holds in both V and M, $M^\gamma \subseteq M$, and M $\models$ "δ is $< \gamma$ supercompact but δ is not γ supercompact", $\dot{R}'_\delta$ will actually be a term for the iteration $\langle\langle P_\alpha/P_\delta, \dot{Q}_\alpha\rangle$: $\delta \le \alpha < \gamma\rangle * \langle(\dot{P}^0_\gamma * \dot{P}^1_\gamma) \times (\dot{P}^2_\gamma * \dot{P}^3_\gamma)\rangle$, where the term for the iteration $\langle\langle P_\alpha/P_\delta, \dot{Q}_\alpha\rangle : \delta \le \alpha < \gamma\rangle$ is the same as in V, any term of the form $(\dot{P}^0_{\theta_i,\gamma} * P^2_{\theta_i,\gamma}[\dot{S}_{\theta_i,\gamma}]) * P^1_{\theta_i,\gamma}[\dot{S}_{\theta_i,\gamma}]$ appearing in $\dot{R}'_\delta$ (more specifically, in $\dot{P}^0_\gamma * \dot{P}^1_\gamma$) is identical to one appearing in $\dot{Q}'_\delta$, and if $\dot{P}^0_{\theta_i,\gamma} * P^1_{\theta_i,\gamma}[\dot{S}_{\theta_i,\gamma}]$ appears in $\dot{R}'_\delta$ (more specifically, in $\dot{P}^2_\gamma * \dot{P}^3_\gamma$), then either it appears as an identical term in $\dot{Q}'_\delta$, or (as is the case, e.g., when $i = \alpha_0$ and θ_i is defined) it appears as the term $(\dot{P}^0_{\theta_i,\gamma} * P^2_{\theta_i,\gamma}[\dot{S}_{\theta_i,\gamma}]) * P^1_{\theta_i,\gamma}[\dot{S}_{\theta_i,\gamma}]$ in $\dot{Q}'_\delta$. This allows us to define $H_\delta = G_\delta$, where G_δ is the portion of G V-generic over P_δ, and $H'_\delta = K * K'$, where K is the projection of G onto $\langle\langle P_\alpha/P_\delta, \dot{Q}_\alpha\rangle$: $\delta \le \alpha < \gamma\rangle$ and K' is the projection of G onto $(P^0_\gamma * \dot{P}^1_\gamma) \times (P^2_\gamma * \dot{P}^3_\gamma)$ as defined in M.

To construct the next portion of the generic object H''_δ, note that as in Lemma 9, the definition of P_δ ensures $|P_\delta| = \delta$ and P_δ is δ-c.c. Thus, as before, GCH in $V[G_\delta]$ and $M[G_\delta]$, the definition of $\dot{R}'_\delta$, the fact $M^\gamma \subseteq M$, and some applications of Lemma 6.4 of [Ba] allow us to conclude that $M[H_\delta * H'_\delta] = M[G_\delta * H'_\delta]$ is closed under γ sequences with respect to $V[G_\delta * H'_\delta]$. Thus, any partial ordering which is $\prec \gamma^+$-strategically closed in $M[H_\delta * H'_\delta]$ is actually $\prec \gamma^+$-strategically closed in $V[G_\delta * H'_\delta]$.

Observe now that if $\langle T_\alpha : \alpha < \eta\rangle$ is so that each T_α is $\prec \rho^+$-strategically closed for some cardinal ρ, then $\prod_{\alpha<\eta} T_\alpha$ is also $\prec \rho^+$-strategically closed, for if $\langle f_\alpha : \alpha < \eta\rangle$ is so that each f_α is a winning strategy for player II for T_α, then $\prod_{\alpha<\eta} f_\alpha$, i.e., pick the αth coordinate according to f_α, is a winning strategy for player II for $\prod_{\alpha<\eta} T_\alpha$. This observation easily implies $\Vdash_{P_\delta * \dot{R}'_\delta}$ "$\dot{R}''_\delta$ is $\prec \gamma^+$-strategically closed" in either $V[G_\delta * H'_\delta]$ or $M[H_\delta * H'_\delta]$. The definition of the iteration R''_δ then allows us, as in Lemma 9, to construct in $V[G_\delta * H'_\delta] \subseteq V[G_\delta * G'_\delta]$ an $M[H_\delta * H'_\delta]$-generic object

H''_δ over R''_δ. As in Lemma 9, $M[H_\delta * H'_\delta * H''_\delta]$ is closed under γ sequences with respect to $V[G_\delta * H'_\delta]$.

Write $\dot{R}'''_\delta$ as $\dot{R}^4_\delta * \dot{R}^5_\delta$, where $\dot{R}^4_\delta$ is a term for the iteration $\langle\langle P_\alpha/P_{j(\delta)}, \dot{Q}_\alpha\rangle : j(\delta) \le \alpha < j(\gamma)\rangle$ and $\dot{R}^5_\delta$ is a term for $\dot{Q}_{j(\gamma)}$. Also, write in V $\dot{Q}'_\delta = \dot{Q}''_\delta * \dot{Q}'''_\delta$, where $\dot{Q}''_\delta$ is a term for the iteration $\langle\langle P_\alpha/P_\delta, \dot{Q}_\alpha\rangle : \delta \le \alpha < \gamma\rangle$ and $\dot{Q}'''_\delta$ is a term for $\dot{Q}_\gamma$, and let $G'_\delta = G''_\delta * G'''_\delta$ be the corresponding factorization of G'_δ. For any non-trivial term $\dot{Q}_\alpha = (\dot{P}^0_\alpha * \dot{P}^1_\alpha) \times (\dot{P}^2_\alpha * \dot{P}^3_\alpha)$ appearing in $\dot{R}^4_\delta$, Lemma 4 and the fact that elements of $\dot{P}^0_\alpha$ will have full support and elements of $\dot{P}^1_\alpha$ will have support $< \alpha$ imply that in M, for $T = P_\delta * \dot{R}'_\delta * \dot{R}''_\delta * \langle\langle P_\beta/P_{j(\delta)}, \dot{Q}_\beta\rangle : j(\delta) \le \beta < \alpha\rangle$, $\Vdash_T$ "(a dense subset of) $\dot{P}^0_\alpha * \dot{P}^1_\alpha$ is γ^+-directed closed". Further, if $\alpha \in [j(\delta), j(\gamma)]$ is so that for some i, $\alpha = \lambda_i$, then it must be the case that $j(\delta) < \delta_i$, for if $\delta_i \le j(\delta)$, then by a theorem of Magidor [Ma2], since $M \models$ "δ_i is $< j(\delta)$ supercompact and $j(\delta)$ is $j(\gamma)$ supercompact", $M \models$ "δ_i is $j(\gamma)$ supercompact", a contradiction to the fact $M \models$ "$\alpha = \lambda_i < j(\gamma)$". Hence, by the definition of θ_i, it must be the case that $j(\delta) \le \theta_i$, i.e., since $j(\delta) > \gamma$, $\theta_i > \gamma$. This means $\Vdash_T$"$\dot{P}^0_{\theta_i,\alpha}$ and $P^1_{\theta_i,\alpha}[\dot{S}_{\theta_i,\alpha}]$ are γ^+-directed closed", so as elements of $\dot{P}^2_\alpha$ will have full support and elements of $\dot{P}^3_\alpha$ will have support $< \alpha$, $\Vdash_T$"$\dot{P}^2_\alpha * \dot{P}^3_\alpha$ is γ^+-directed closed", i.e., $\Vdash_T$ "(A dense subset of) $(\dot{P}^0_\alpha * \dot{P}^1_\alpha) \times (\dot{P}^2_\alpha * \dot{P}^3_\alpha)$ is γ^+-directed closed". Thus, in M, $\Vdash_{P_\delta * \dot{R}'_\delta * \dot{R}''_\delta}$ "(A dense subset of) $\dot{R}^4_\delta$ is γ^+-directed closed". Therefore, using the extension of j, $j^* : V[G_\delta] \to M[H_\delta * H'_\delta * H''_\delta]$ which we have produced in $V[G_\delta * H'_\delta]$, the fact that GCH in $M[H_\delta * H'_\delta * H''_\delta]$ implies $M[H_\delta * H'_\delta * H''_\delta] \models$ "$|R^4_\delta| = j(\gamma)$ and $2^{j(\gamma)} = j(\gamma^+)$", $V[G_\delta * H'_\delta] \models$ "$|j(\gamma^+)| = (\gamma^+)^V = \gamma^+$", and the closure properties of $M[H_\delta * H'_\delta * H''_\delta]$, we can produce in $V[G_\delta * H'_\delta]$ as in Lemma 9 an upper bound q for $\{j^*(p) : p \in G''_\delta\}$ and an $M[H_\delta * H'_\delta * H''_\delta]$-generic object H^4_δ for R^4_δ so that $q \in H^4_\delta$. Again, as in Lemma 9, $M[H_\delta * H'_\delta * H''_\delta * H^4_\delta]$ is closed under γ-sequences with respect to $V[G_\delta * H'_\delta]$. Therefore, by the remarks after the proof of Lemma 12 and the proof of Lemma 6, $M[H_\delta * H'_\delta * H''_\delta * H^4_\delta]$ is closed under γ-sequences with respect to $V[G_\delta * G'_\delta]$.

Rewrite $\dot{R}^5_\delta$ as

$$\begin{aligned}
&\Big(\prod_{\{i<j(\gamma):\delta_i \text{ is } j(\gamma) \text{ supercompact}\}} (\dot{P}^0_{\theta_i,j(\gamma)} * P^2_{\theta_i,j(\gamma)}[\dot{S}_{\theta_i,j(\gamma)}]) \\
&\qquad\qquad \times \prod_{\{i<j(\gamma):j(\gamma)=\lambda_i\}} \dot{P}^0_{\theta_i,j(\gamma)}\Big) \\
&\quad * \Big(\prod_{\{i<j(\gamma):\delta_i \text{ is } j(\gamma) \text{ supercompact or } j(\gamma)=\lambda_i\}} \dot{P}^1_{\theta_i,j(\gamma)}[\dot{S}_{\theta_i,j(\gamma)}]\Big) \\
&\quad = \dot{R}^6_\delta * \dot{R}^7_\delta,
\end{aligned}$$

where all elements of $\dot{R}^6_\delta$ will have full support, and all elements of $\dot{R}^7_\delta$ will have support $< j(\gamma)$. By our earlier observation that products of (appropriately) strategically closed partial orderings retain the same amount of strategic closure, it is clearly the case that Q^*_γ, the portion of Q_γ corresponding to R^6_δ, i.e., $Q^*_\gamma = \prod_{\{i<\gamma:\delta_i \text{ is } \gamma \text{ supercompact}\}} (P^0_{\theta_i,\gamma} * P^2_{\theta_i,\gamma}[\dot{S}_{\theta_i,\gamma}]) \times \prod_{\{i<\gamma:\gamma=\lambda_i\}} P^0_{\theta_i,\gamma}$, is γ-strategically closed and therefore is (γ,∞)-distributive. Hence, as we again have that in $V[G_\delta * H'_\delta]$, j^* extends to $j^{**} : V[G_\delta * G''_\delta] \to M[H_\delta * H'_\delta * H''_\delta * H^4_\delta]$, we can use j^{**} as in the proof of Lemma 9 to transfer G^4_δ, the projection of G'''_δ onto Q^*_γ, via the general transference principle given in [C], Section 1.2, Fact 2, pp. 5-6, to an $M[H_\delta * H'_\delta * H''_\delta * H^4_\delta]$-generic object H^5_δ over R^6_δ.

By its construction, since $p \in G^4_\delta$ implies $j^{**}(p) \in H^5_\delta$, j^{**} extends in $V[G_\delta * G'_\delta]$ to $j^{***} : V[G_\delta * G''_\delta * G^4_\delta] \to M[H_\delta * H'_\delta * H''_\delta * H^4_\delta * H^5_\delta]$. And, since R^6_δ is γ-strategically closed, $M[H_\delta * H'_\delta * H''_\delta * H^4_\delta * H^5_\delta]$ and $M[H_\delta * H'_\delta * H''_\delta * H^4_\delta]$ contain the same γ sequences of elements of $M[H_\delta * H'_\delta * H''_\delta * H^4_\delta]$ with respect to $V[G_\delta * G'_\delta]$. As any γ sequence of elements of $M[H_\delta * H'_\delta * H''_\delta * H^4_\delta * H^5_\delta]$ can be represented, in $M[H_\delta * H'_\delta * H''_\delta * H^4_\delta]$, by a term which is actually a function $f : \gamma \to M[H_\delta * H'_\delta * H''_\delta * H^4_\delta]$, and as $M[H_\delta * H'_\delta * H''_\delta * H^4_\delta]$ is closed under γ sequences with respect to $V[G_\delta * G'_\delta]$, $M[H_\delta * H'_\delta * H''_\delta * H^4_\delta * H^5_\delta]$ is closed under γ sequences with respect to $V[G_\delta * G'_\delta]$.

It remains to construct the $M[H_\delta * H'_\delta * H''_\delta * H^4_\delta * H^5_\delta]$-generic object H^6_δ over R^7_δ. To do this, take Q^{**}_γ to be the portion of Q_γ corresponding to R^7_δ, i.e., Q^{**}_γ is the $< \gamma$ support product $\prod_{\{i<\gamma : \delta_i \text{ is } \gamma \text{ supercompact or } \gamma = \lambda_i\}} P^1_{\theta_i,\gamma}[S_{\theta_i,\gamma}]$, with G^5_δ the projection of G'''_δ onto Q^{**}_γ. Next, for the purpose of the remainder of the proof of this lemma, if $p \in R^6_\delta$ and $i < j(\gamma)$ is an ordinal, say that $i \in \text{support}(p)$ iff for some non-trivial component $\bar p$ of p, $\bar p \in P^0_{\theta_i,j(\gamma)}$. Analogously, it is clear what $i \in \text{support}(p)$ for $p \in R^7_\delta$ means. Now, let $A = \{i < j(\gamma) : \text{For some } p \in j^{**\prime\prime}G^4_\delta,\ i \in \text{support}(p)\}$, and let $B = \{i < j(\gamma) : \text{For some } q \in R^7_\delta,\ i \in \text{support}(q) \text{ but } i \notin \text{support}(p)$ for any $p \in j^{**\prime\prime}G^4_\delta\}$. Write $A = A_0 \cup A_1$, where $A_0 = \{i \in A : j(\gamma) = \lambda_i\}$ and $A_1 = \{i \in A : j(\gamma) \neq \lambda_i\}$. Note that since $H^5_\delta = \{q \in R^6_\delta : \exists p \in j^{**\prime\prime}G^4_\delta[q \le p]\}$, $A, A_0, A_1, B \in M[H_\delta * H'_\delta * H''_\delta * H^4_\delta * H^5_\delta]$.

If $i \in A_1$, then by the genericity of H^5_δ, $P^1_{\theta_i,j(\gamma)}[S_{\theta_i,j(\gamma)}]$ contains a dense subordering P^*_i given by Lemma 4 which is isomorphic to $Q^1_{j(\gamma)}$. Hence, we can infer that the ($< j(\gamma)$ support) product $\prod_{i \in A_1} P^*_i$ is dense in the ($< j(\gamma)$ support) product $\prod_{i \in A_1} P^1_{\theta_i,j(\gamma)}[S_{\theta_i,j(\gamma)}]$. We thus without loss of generality consider $\prod_{i \in A_1} P^*_i$ instead of $\prod_{i \in A_1} P^1_{\theta_i,j(\gamma)}[S_{\theta_i,j(\gamma)}]$. Further, if $i \in A_0$, then since $j(\gamma) = \lambda_i$, by our earlier remarks, $\theta_i > \gamma$. This means $P^1_{\theta_i,j(\gamma)}[S_{\theta_i,j(\gamma)}]$ is γ^+-directed closed.

As we observed in the proof of Lemma 4, for any $i \in A$ and any $\langle w^i, \alpha^i, \bar r^i, Z^i\rangle \in P^1_{\theta_i,j(\gamma)}[S_{\theta_i,j(\gamma)}]$, the first three coordinates $\langle w^i, \alpha^i, \bar r^i\rangle$ are a re-representation of an element of $Q^1_{j(\gamma)}$. Since the $< j(\gamma)$ support product of $j(\gamma)$ many copies of $Q^1_{j(\gamma)}$ is isomorphic to $Q^1_{j(\gamma)}$, for any condition

$$\begin{aligned} p &= \langle \langle w^i, \alpha^i, \bar r^i, Z^i\rangle_{i<\ell_0<j(\gamma)}, \langle w^i, \alpha^i, \bar r^i, Z^i\rangle_{i<\ell_1<j(\gamma)}\rangle \\ &\in \prod_{i\in A_0} P^1_{\theta_i,j(\gamma)}[S_{\theta_i,j(\gamma)}] \times \prod_{i \in A_1} P^*_i, \end{aligned}$$

we can in a unique and canonical way write p as $\langle \bar p, \bar Z\rangle$, where $\bar p \in Q^1_{j(\gamma)}$ and $\bar Z = \langle \langle Z^i : i < \ell_0 < j(\gamma)\rangle, \langle Z^i : i < \ell_1 < j(\gamma)\rangle\rangle$. Further, this rearrangement can be taken so as to preserve the order relation on $\prod_{i\in A_0} P^1_{\theta_i,j(\gamma)}[S_{\theta_i,j(\gamma)}] \times \prod_{i \in A_1} P^*_i$. Therefore, since our remarks in the last paragraph imply $\prod_{i\in A_0} P^1_{\theta_i,j(\gamma)}[S_{\theta_i,j(\gamma)}] \times \prod_{i \in A_1} P^*_i$ is γ^+-directed closed, the fact that $M[H_\delta * H'_\delta * H''_\delta * H^4_\delta * H^5_\delta]$ is closed under γ sequences with respect to $V[G_\delta * G'_\delta]$ means that we can in essence ignore each sequence $\bar{Z}$ as above and apply the arguments used in Lemma 9 to construct the generic object for $Q^1_{j(\gamma)}$ to construct an $M[H_\delta * H'_\delta * H''_\delta * H^4_\delta * H^5_\delta]$-generic object $H^{6,0}_\delta$ for $\prod_{i\in A_0} P^1_{\theta_i,j(\gamma)}[S_{\theta_i,j(\gamma)}] \times \prod_{i\in A_1} P^*_i$. As before, since $\prod_{i\in A_0} P^1_{\theta_i,j(\gamma)}[S_{\theta_i,j(\gamma)}] \times \prod_{i\in A_1} P^*_i$ is γ^+-directed closed, $M[H_\delta * H'_\delta * H''_\delta * H^4_\delta * H^5_\delta * H^{6,0}_\delta]$ is closed under γ sequences with respect to $V[G_\delta * G'_\delta]$.

By our remarks following the proof of Lemma 12 and the ideas used in the remark following the proof of Lemma 5, $\prod_{i\in B} P^1_{\theta_i,j(\gamma)}[S_{\theta_i,j(\gamma)}]$ is $j(\gamma^+)$-c.c. in $M[H_\delta * H'_\delta * H''_\delta * H^4_\delta * H^5_\delta]$ and $M[H_\delta * H'_\delta * H''_\delta * H^4_\delta * H^5_\delta * H^{6,0}_\delta]$. Since $\prod_{i\in B} P^1_{\theta_i,j(\gamma)}[S_{\theta_i,j(\gamma)}]$ is a $< j(\gamma)$ support product and $P^1_{\theta_i,j(\gamma)}[S_{\theta_i,j(\gamma)}]$ has cardinality $j(\gamma^+)$ in $M[H_\delta * H'_\delta * H''_\delta * H^4_\delta * H^5_\delta * H^{6,0}_\delta]$ for any $i < j(\gamma)$, $\prod_{i\in B} P^1_{\theta_i,j(\gamma)}[S_{\theta_i,j(\gamma)}]$ has cardinality $j(\gamma^+)$ in $M[H_\delta * H'_\delta * H''_\delta * H^4_\delta * H^5_\delta * H^{6,0}_\delta]$. We can thus as in Lemma 9 let $\langle \mathcal{A}_\alpha : \alpha < \gamma^+\rangle$ enumerate in $V[G_\delta * G'_\delta]$ the maximal antichains of $\prod_{i\in B} P^1_{\theta_i,j(\gamma)}[S_{\theta_i,j(\gamma)}]$ with respect to $M[H_\delta * H'_\delta * H''_\delta * H^4_\delta * H^5_\delta * H^{6,0}_\delta]$, and we can once more mimic the construction in Lemma 9 of H''_{α_0} to produce in $V[G_\delta * G'_\delta]$ an $M[H_\delta * H'_\delta * H''_\delta * H^4_\delta * H^5_\delta * H^{6,0}_\delta]$-generic object $H^{6,1}_\delta$ over $\prod_{i\in B} P^1_{\theta_i,j(\gamma)}[S_{\theta_i,j(\gamma)}]$. If we now let $H^6_\delta = H^{6,0}_\delta * H^{6,1}_\delta$ and $H = H_\delta * H'_\delta * H''_\delta * H^4_\delta * H^5_\delta * H^6_\delta$, then our construction guarantees $j : V \to M$ extends to $k : V[G_\delta * G'_\delta] \to M[H]$, so $V[G] \models$ "δ is γ supercompact". This proves Lemma 13. $\square$ Lemma 13

We remark that the proof of Lemma 13 will work whether or not θ_{α_0} is defined.
We prove now the natural analogue of Lemma 10.

Lemma 14. *For γ regular, $V[G] \models$ "δ is γ strongly compact iff δ is γ supercompact, except possibly if for the i so that $\delta = \delta_i$, θ_i is undefined".*

Proof of Lemma 14. As in Lemma 10, we assume towards a contradiction that the lemma is false, and let $\delta = \delta_{i_0} < \gamma$ be so that $V[G] \models$ "δ is γ strongly compact, δ isn't γ supercompact, θ_{i_0} is defined, γ is regular, and γ is the least such cardinal". Since Lemma 13 implies that if $V \models$ "δ is γ supercompact", then $V[G] \models$ "δ is γ supercompact", as in Lemma 10, it must be the case that $\lambda_{i_0} \le \gamma$.

Write $P = P_{\lambda_{i_0}} * \dot{Q}_{\lambda_{i_0}} * \dot{R}$, where $P_{\lambda_{i_0}}$ is the forcing through stage λ_{i_0}, $\dot{Q}_{\lambda_{i_0}}$ is a term for the forcing at stage λ_{i_0}, and $\dot{R}$ is a term for the rest of the forcing. In $V^{P_{\lambda_{i_0}}}$, since $V \models$ "$\delta = \delta_{i_0}$ isn't λ_{i_0} supercompact", we can write $Q_{\lambda_{i_0}}$ as $T_0 \times T_1$, where T_1 is $P^0_{\theta_{i_0},\lambda_{i_0}} * P^1_{\theta_{i_0},\lambda_{i_0}}[\dot{S}_{\theta_{i_0},\lambda_{i_0}}]$, and T_0 is the rest of $Q_{\lambda_{i_0}}$. Since $V^{P_{\lambda_{i_0}}} \models$ "$T_0 \times P^0_{\theta_{i_0},\lambda_{i_0}}$ is $< \lambda_{i_0}$-strategically closed" (and hence adds no new bounded subsets of λ_{i_0} when forcing over $V^{P_{\lambda_{i_0}}}$), the arguments of Lemma 3 apply in $V^{P_{\lambda_{i_0}} * (\dot{T}_0 \times \dot{P}^0_{\theta_{i_0},\lambda_{i_0}})}$ to show $V^{(P_{\lambda_{i_0}} * (\dot{T}_0 \times \dot{P}^0_{\theta_{i_0},\lambda_{i_0}})) * P^1_{\theta_{i_0},\lambda_{i_0}}[\dot{S}_{\theta_{i_0},\lambda_{i_0}}]} = V^{P_{\lambda_{i_0}} * \dot{Q}_{\lambda_{i_0}}} \models$ "δ_{i_0} isn't λ_{i_0} strongly compact since λ_{i_0} doesn't carry a δ_{i_0}-additive uniform ultrafilter".

It remains to show that $V^{P_{\lambda_{i_0}} * \dot{Q}_{\lambda_{i_0}} * \dot{R}} = V^P \models$ "δ_{i_0} isn't λ_{i_0} strongly compact". If this weren't the case, then let $\dot{\mathcal{U}}$ be a term in $V^{P_{\lambda_{i_0}} * \dot{Q}_{\lambda_{i_0}}}$ so that $\Vdash_R$"$\dot{\mathcal{U}}$ is a δ_{i_0}-additive uniform ultrafilter over λ_{i_0}". Since $\Vdash_{P_{\lambda_{i_0}} * \dot{Q}_{\lambda_{i_0}}}$ "$\dot{R}$ is $\prec \lambda^+_{i_0}$-strategically closed" and $V^{P_{\lambda_{i_0}} * \dot{Q}_{\lambda_{i_0}}} \models$ GCH, if we let $\langle x_\alpha : \alpha < \lambda^+_{i_0}\rangle$ be in $V^{P_{\lambda_{i_0}} * \dot{Q}_{\lambda_{i_0}}}$ a listing of all of the subsets of λ_{i_0}, as in the construction of H''_{α_0} in Lemma 9, we can let

$\langle r_\alpha : \alpha < \lambda_{i_0}^+ \rangle$ be an increasing sequence of elements of R so that $r_\alpha \|$"$x_\alpha \in \dot{\mathcal{U}}$". If now in $V^{P_{\lambda_{i_0}} * \dot{Q}_{\lambda_{i_0}}}$ we define $\mathcal{U}'$ by $x_\alpha \in \mathcal{U}'$ iff $r_\alpha \Vdash$ "$x_\alpha \in \dot{\mathcal{U}}$", then it is routine to check that $\mathcal{U}'$ is a δ_{i_0}-additive uniform ultrafilter over λ_{i_0} in $V^{P_{\lambda_{i_0}} * \dot{Q}_{\lambda_{i_0}}}$, which contradicts that there is no such ultrafilter in $V^{P_{\lambda_{i_0}} * \dot{Q}_{\lambda_{i_0}}}$. Thus, $V^P \models$ "δ_{i_0} isn't λ_{i_0} strongly compact", a contradiction to $V[G] \models$ "δ is γ strongly compact". This proves Lemma 14. $\square$ Lemma 14

Note that the analogue to Lemma 11 holds if $\delta = \delta_i$ and θ_i is defined, i.e., for γ regular, $V[G] \models$ "δ is γ supercompact" iff $V \models$ "δ is γ supercompact" if $\delta = \delta_i$ and θ_i is defined. The proof uses Lemmas 13 and 14 and is exactly the same as the proof of Lemma 11.

Lemmas 12–14 complete the proof of our Theorem in the general case.

$\square$ Theorem

4. Concluding Remarks

In conclusion, we would like to mention that it is possible to use generalizations of the methods of this paper to answer some further questions concerning the possible relationships amongst strongly compact, supercompact, and measurable cardinals. In particular, it is possible to show, using generalizations of the methods of this paper, that the result of [Me] which states that the least measurable cardinal κ which is the limit of strongly compact or supercompact cardinals is not 2^κ supercompact is best possible. Specifically, if $V \models$ "ZFC + GCH + κ is the least supercompact limit of supercompact cardinals + $\lambda > \kappa^+$ is a regular cardinal which either is inaccessible or is the successor of a cardinal of cofinality $> \kappa$ + $h : \kappa \to \kappa$ is a function so that for some elementary embedding $j : V \to M$ witnessing the $< \lambda$ supercompactness of κ, $j(h)(\kappa) = \lambda$", then there is some generic extension $V[G] \models$ "ZFC + For every cardinal $\delta < \kappa$ which is an inaccessible limit of supercompact cardinals and every cardinal $\gamma \in [\delta, h(\delta))$, $2^\gamma = h(\delta)$ + For every cardinal $\gamma \in [\kappa, \lambda)$, $2^\gamma = \lambda$ + κ is $< \lambda$ supercompact + κ is the least measurable limit of supercompact cardinals".

It is also possible to show using generalizations of the methods of this paper that if $V \models$ "ZFC + GCH + $\kappa < \lambda$ are such that κ is $< \lambda$ supercompact, $\lambda > \kappa^+$ is a regular cardinal which either is inaccessible or is the successor of a cardinal of cofinality $> \kappa$ + $h : \kappa \to \kappa$ is a function so that for some elementary embedding $j : V \to M$ witnessing the $< \lambda$ supercompactness of κ, $j(h)(\kappa) = \lambda$", then there is some cardinal and cofinality preserving generic extension $V[G] \models$ "ZFC + For every inaccessible $\delta < \kappa$ and every cardinal $\gamma \in [\delta, h(\delta))$, $2^\gamma = h(\delta)$ + For every cardinal $\gamma \in [\kappa, \lambda)$, $2^\gamma = \lambda$ + κ is $< \lambda$ supercompact + κ is the least measurable cardinal". This generalizes a result of Woodin (see [CW]), who showed, in response to a question posed to him by the first author, that it was possible to start from a model for "ZFC + GCH + $\kappa < \lambda$ are such that κ is λ^+ supercompact and λ is regular" and use Radin forcing to produce a model for "ZFC + $2^\kappa = \lambda$ + κ is δ supercompact for all regular $\delta < \lambda$ + κ is the least measurable cardinal". In addition, it is possible to iterate the forcing used in the construction of the above model to show, for instance, that if $V \models$ "ZFC + GCH + There is a proper class of cardinals κ so that κ is κ^+ supercompact", then there is some cardinal and cofinality preserving generic extension $V[G] \models$ "ZFC + $2^\kappa = \kappa^{++}$ iff κ is inaccessible + There is a proper class of measurable cardinals + $\forall\kappa[\kappa$ is measurable iff κ is κ^+ strongly compact iff κ is κ^+ supercompact] + No cardinal κ is κ^{++} strongly compact". In this result, there is nothing special about κ^+, and each κ can be λ supercompact for $\lambda = \kappa^{++}$, $\lambda = \kappa^{+++}$, or λ essentially any "reasonable" value below 2^κ. The proof of these results will appear in [AS].

Acknowledgement

The authors wish to thank Menachem Magidor for several helpful conversations on the subject matter of this paper. In addition, the authors would like to express their gratitude to the referee for his thorough and careful reading of the manuscript for this paper. The referee's many corrections and helpful suggestions considerably improved the presentation of the material contained herein and have been incorporated into this version of the paper.

References

[A] A. Apter, *"On the Least Strongly Compact Cardinal"*, Israel J. Math. **35**, 1980, 225–233. MR **81h:**03099

[AS] A. Apter, S. Shelah, *"Menas' Result is Best Possible"*, Trans. Amer. Math. Soc. (to appear).

[Ba] J. Baumgartner, *"Iterated Forcing"*, in: A. Mathias, ed., Surveys in Set Theory, Cambridge University Press, Cambridge, England, 1983, 1–59. MR **87c:**03099

[Bu] J. Burgess, *"Forcing"*, in: J. Barwise, ed., Handbook of Mathematical Logic, North-Holland, Amsterdam, 1977, 403–452. MR **58:**27475

[C] J. Cummings, *"A Model in which GCH Holds at Successors but Fails at Limits"*, Transactions AMS **329**, 1992, 1–39. MR **92h:**03076

[CW] J. Cummings, H. Woodin, *Generalised Prikry Forcings*, circulated manuscript of a forthcoming book.

[J] T. Jech, *Set Theory*, Academic Press, New York, 1978. MR **80a:**03062

[KaM] A. Kanamori, M. Magidor, *"The Evolution of Large Cardinal Axioms in Set Theory"*, in: Lecture Notes in Mathematics **669**, Springer-Verlag, Berlin, 1978, 99–275. MR **80b:**03083

[KiM] Y. Kimchi, M. Magidor, *"The Independence between the Concepts of Compactness and Supercompactness"*, circulated manuscript.

[Ma1] M. Magidor, *"How Large is the First Strongly Compact Cardinal?"*, Annals Math. Logic **10**, 1976, 33–57. MR **55:**2578

[Ma2] M. Magidor, *"On the Role of Supercompact and Extendible Cardinals in Logic"*, Israel J. Math. **10**, 1971, 147–157. MR **45:**4966

[Ma3] M. Magidor, *"There are Many Normal Ultrafilters Corresponding to a Supercompact Cardinal"*, Israel J. Math. **9**, 1971, 186–192. MR **50:**110

[Ma4] M. Magidor, unpublished; personal communication.

[Me] T. Menas, *"On Strong Compactness and Supercompactness"*, Annals Math. Logic **7**, 1975, 327–359. MR **50:**9589

[MS] A. Mekler, S. Shelah, *"When κ-Free Implies Strongly κ-Free"*, in: Proceedings of the Third Conference on Abelian Group Theory, Gordon and Breach, Salzburg, 1987, 137–148. MR **90f:**20082

[SRK] R. Solovay, W. Reinhardt, A. Kanamori, *"Strong Axioms of Infinity and Elementary Embeddings"*, Annals Math. Logic **13**, 1978, 73–116. MR **80h:**03072

Department of Mathematics, Baruch College of CUNY, New York, New York 10010
E-mail address: awabb@cunyvm.cuny.edu

Department of Mathematics, The Hebrew University, Jerusalem, Israel
E-mail address: shelah@sunrise.huji.ac.il

Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08904
E-mail address: shelah@math.rutgers.edu