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Journal of Mathematical Logic
Vol. 17, No. 1 (2017) 1750001 (18 pages)

DOI: 10.1142/S0219061317500015

Exact saturation in simple and NIP theories

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Received 10 October 2015
Accepted 21 November 2016
Published 2 February 2017

A theory T is said to have exact saturation at a singular cardinal κ if it has a κ-saturated model which is not κ^+-saturated. We show, under some set-theoretic assumptions, that any simple theory has exact saturation. Also, an NIP theory has exact saturation if and only if it is not distal. This gives a new characterization of distality.

Keywords: Saturated models; classification theory; NIP theories; distal theories; simple theories.

Mathematics Subject Classification 2010: 03C45, 03C95, 03C55

1. Introduction

A first-order theory T has exact saturation at κ if it has a κ-saturated model which is not κ^+-saturated. When $\kappa > |T|$ is regular, then any theory has exact saturation at κ (see Fact 2.5), hence we are only interested in the case κ singular.

Possibly adding set-theoretic assumptions, we expect that for a given theory T, having exact saturation at a singular cardinal κ does not depend on κ, and that this property is an interesting dividing line within first-order theories. We indeed show this for stable, simple and NIP theories.

The second author has shown previously [4, IV, Lemma 2.18] that stable theories have exact saturation at any κ. Since this is not stated exactly in this form there, and also for completeness, we added a proof here (see Theorem 2.4). He also showed that an NIP theory with an infinite indiscernible set has exact saturation at any singular κ with $2^\kappa = \kappa^+$ [5, Claim 2.26].

We establish here the precise dividing line for NIP theories: with the same assumptions on κ, an NIP theory has exact saturation at κ if and only if it is not distal. This gives a new characterization of distality within NIP theories, and allows an answer to Question 2.30 from [5]. See Corollary 4.11.

We also generalize the result on stable theories to simple theories: let T be simple and assume that κ is singular of cofinality greater than $|T|$, $2^\kappa = \kappa^+$ and $\square_\kappa$ holds, then T has exact saturation at κ.

2. Definitions and First Results

Definition 2.1. Suppose T is a first-order theory and κ is a cardinal. We say that T *has exact saturation at* κ if T has a κ-saturated model M which is not κ^+-saturated.

We will use the following notion throughout the paper.

Definition 2.2. Let T be any complete theory. Suppose that D is a collection of finitary types over some set A. A set B is a D-*set* if for every finite tuple b from B, $\mathrm{tp}(b/A) \in D$. For a D-set $B \supseteq A$, a type $p \in S^{<\omega}(B)$ is called a D-*type* if Bd is a D-set for some (any) $d \models p$. A D-*model* is a model of T which is a D-set.

2.1. *Stable theories*

Suppose T is a stable theory. This part is not new, but it is short, and we keep it for completeness.

Fact 2.3 ([4, IV, Lemma 2.18]). Suppose $p(x)$ is a partial type over $B \subseteq A$. Then there is $A_0 \subseteq A$ of size $\leq |T|$, and an extension $q \supseteq p$ over BA_0 which isolates a complete type over A.

Proof. Enumerate the formulas $\langle \varphi_i(x, y_i) \mid i < |T| \rangle$. Construct an increasing continuous sequence of types $\langle p_i \mid i < |T| \rangle$, where $p_0 = p$ such that p_{i+1} isolates a complete φ_i-type over A and $|p_{i+1} \backslash p_i| = 1$. To find p_{i+1}, let $\psi(x)$ be a formula over A with minimal R_{2,φ_i}-rank consistent with p_i (which exists by stability), and let $p_{i+1} = p_i \cup \{\psi\}$. Finally, let $q = \bigcup_{i<|T|} p_i$. □

Theorem 2.4. *Assume T is stable. Then for all $\kappa > |T|$, T has exact saturation at κ.*

Proof. Let I be an indiscernible set of cardinality κ. Let D be the collection of finitary types p over I such that for some $I_0 \subseteq I$ of cardinality $< \kappa$, $p|_{I_0} \vdash p$.

Suppose that $p(x) = \mathrm{tp}(c/I) \in D$, as witnessed by I_0. By stability, there is some $I' \subseteq I$ of size $\leq |T|$ such that $I \backslash I'$ is indiscernible over cI' (let I' be the set of parameters appearing in the formulas defining $\mathrm{tp}(c/I)$ over I, or see Fact 4.2 below). Let $I_0' = I' \cup I''$, where $I'' \subseteq I$ is any infinite countable set disjoint from $I' \cup I_0$. Then an easy argument[a] gives us that $p|_{I_0'} \vdash p$. This shows that we can always assume that I_0 has size $\leq |T|$.

By Fact 2.3, we can construct a κ-saturated D-model M containing I of cardinality 2^κ. It is enough to show that given D-sets $A \subseteq B$, where $|A| < \kappa$, and some type $p \in S(A)$, there is some realization $a \models p$ such that aB is a D-set. We may assume that $I \subseteq B$. By Fact 2.3, there is some $A_0 \subseteq B$ such that $|A_0| \leq |T|$ and a type $p_0 \supseteq p$ over $A_0 A$ such that p_0 isolates a complete type over B. Let $a \models p_0$. Then aB is a D-set: for a finite tuple c from B, let I_0 be such that $\mathrm{tp}\,(A_0 Ac/I_0) \vdash \mathrm{tp}\,(A_0 Ac/I)$ and $|I_0| \leq |A_0| + |A| + |T| < \kappa$. Then $\mathrm{tp}(ac/I_0) \vdash \mathrm{tp}(ac/I)$.

Now note that $Av(I/I)$ which is a type of a new element in I over I is not a D-type, so M is not κ^+-saturated. □

2.2. *Unstable theories*

Recall that a type $p(x) \in S(M)$ is called *invariant* over $A \subseteq M$ if it does not split over A: if $a, b \in M$ are such that $a \equiv_A b$, then for any formula $\varphi(x, y)$, $\varphi(x, a) \in p \Leftrightarrow \varphi(x, b) \in p$.

Fact 2.5. If T is not stable then T has exact saturation at any regular $|T| < \kappa$.

Proof. Let $M_0 \models T$ be of size $|T|$. For $i \leq \kappa$, define a continuous increasing sequence of models M_i, where $|M_{i+1}| = 2^{|M_i|}$ and M_{i+1} is $|M_i|^+$-saturated. Hence M_κ is κ-saturated and $|M_\kappa| = \beth_\kappa(|T|)$.

As T is unstable, $|S(M_\kappa)| > \beth_\kappa(|T|)$. Why? first note that M_κ is $\beth_\kappa(|T|)^+$-universal in the sense that if $N \models T$, $|N| \leq \beth_\kappa(|T|)$, then N can be elementarily embedded into M_κ. To show this, given N, we may assume $|N| = \beth_\kappa(|T|)$, and write N as a continuous increasing sequence $\bigcup\{N_i \mid i < \kappa\}$, where $|N_0| = |T|$ and $|N_{i+1}| = 2^{|N_i|}$. By induction, find an increasing continuous sequence of elementary maps $f_i : N_i \to M_{i+1}$ for all $i < \kappa$, using the fact that $|N_i| = |M_i|$ and M_{i+1} is $|M_i|^+$-saturated. Taking the limit will give us an elementary map $f : N \to M_\kappa$. Hence, if $|S(M_\kappa)| \leq \beth_\kappa(|T|)$, it would follow that for any model N of size $|N| \leq \beth_\kappa(|T|)$, $S(N) \leq \beth_\kappa(|T|)$, which would mean that T is $\beth_\kappa(|T|)$-stable.

However, as the number of types over M_κ invariant over M_i is $\leq 2^{2^{|M_i|}} \leq \beth_\kappa(|T|)$, there is $p(x) \in S(M_\kappa)$ which splits over every M_i. Hence for each $i < \kappa$, there is some formula $\varphi_i(x, y)$ and some $a_i, b_i \in M_\kappa$ such that $a_i \equiv_{M_i} b_i$ and

[a]Suppose $\varphi(x, a, d) \in p$, where $d \in I'$ and $I' \cap a = \emptyset$. Let $a' \in I''$ be such that $a' \equiv a$, so that $a' \equiv_{I'} a$ and $\varphi(x, a', d) \in p$. Let $\psi(x, b, e) \in p$, $b \in I_0$, $e \in I'$ and $b \cap I' = \emptyset$, be such that $\psi(x, b, e) \vdash \varphi(x, a', d)$. Let $b' \in I''$ be such that $b' \equiv b$ and $b' \cap aa' = \emptyset$. Then $\psi(x, b', e) \vdash \varphi(x, a', d)$ and $\psi(x, b', e) \in p$. Note that $a \equiv_{db'e} a'$ so that $\psi(x, b', e) \vdash \varphi(x, a, d)$.

$\varphi_i(x, a_i) \wedge \neg\varphi_i(x, b_i) \in p$. Let $q(x)$ be $\{\varphi_i(x, a_i) \wedge \neg\varphi_i(x, b_i) \mid i < \kappa\}$. Then q is not realized in M_κ. $\square$

3. Simple Theories

In the following definition, for a set of ordinals A, $\mathrm{Lim}(A)$ is the set of ordinals $\delta \in A$ which are limits of ordinals in A.

Definition 3.1 (Jensen's Square principle, [2, p. 443]). Let κ be an uncountable cardinal; $\square_\kappa$ (square-κ) is the following condition:

There exists a sequence $\langle C_\alpha \mid \alpha \in \mathrm{Lim}(\kappa^+)\rangle$ such that:

(1) C_α is a closed unbounded subset of α.
(2) If $\beta \in \mathrm{Lim}(C_\alpha)$, then $C_\beta = C_\alpha \cap \beta$.
(3) If $\mathrm{cof}(\alpha) < \kappa$, then $|C_\alpha| < \kappa$.

Remark 3.2. Suppose that $\langle C_\alpha \mid \alpha \in \mathrm{Lim}(\kappa^+)\rangle$ witnesses $\square_\kappa$. Let $C'_\alpha = \mathrm{Lim}(C_\alpha)$. Then the following holds for $\alpha \in \mathrm{Lim}(\kappa^+)$.

(1) If $C'_\alpha \neq \emptyset$, then either $\sup(C'_\alpha) = \alpha$, or C'_α has a last element $< \alpha$ in which case $\mathrm{cof}(\alpha) = \omega$.
(2) $C'_\alpha \subseteq \mathrm{Lim}(\alpha)$ and for all $\beta \in C'_\alpha$, $C'_\alpha \cap \beta = C'_\beta$.
(3) If $\mathrm{cof}(\alpha) < \kappa$, then $|C'_\alpha| < \kappa$.

Theorem 3.3. *Suppose that T is simple, μ is singular with $|T| < \kappa = \mathrm{cof}(\mu)$, $\mu^+ = 2^\mu$ and $\square_\mu$ holds. Then T has exact saturation at μ.*

The square assumption will only be used in the end of the proof.

Towards the proof, let us first fix an increasing continuous sequence $\langle \lambda_i \mid i < \kappa\rangle$ of cardinals whose limit is μ such that λ_{i+1} is regular for all $i < \kappa$ and such that $\lambda_0 > \kappa$.

If T is stable, then we already know that T has exact saturation at μ by Theorem 2.4. So assume that T is not stable.

As it is simple, by e.g. [8, Exercises 8.2.5 and 8.2.6], it has the independence property. Let $\varphi(x, y)$ witness this.

Notation 3.4. For a sequence of linear orders $\langle (X_i <_i) \mid i \in I\rangle$, where I is linearly ordered by $<$, let $\sum_{i \in I} X_i$ be the linear order whose set of elements is $\bigcup\{X_i \times \{i\} \mid i \in I\}$ ordered by $(x, i) < (y, j)$ if and only if $i < j$ or $i = j$ and $x <_i y$.

Let $\mathrm{Succ}(\kappa) = \kappa \setminus \mathrm{Lim}(\kappa)$. For $i \in \mathrm{Succ}(\kappa)$, let I_i be the linear order λ_i, and let $I = \sum_{i \in \mathrm{Succ}(\kappa)} I_i$. Let $\langle a_i \mid i \in I\rangle$ be an indiscernible sequence witnessing that φ has the independence property i.e. for every subset $s \subseteq I$, there is some b_s such that $\varphi(b_s, a_i)$ holds if and only if $i \in s$. Abusing notation, we will write $I_i = \langle a_j \mid j \in I_i\rangle$ and similarly for I.

Definition 3.5. For $i \in \mathrm{Succ}(\kappa)$, let D_i be the collection of finitary types $p \in S^{<\omega}(I_i)$ such that for any finite $s \subseteq I_i$ there is some $\alpha < \lambda_i$ such that $I_i^{\geq \alpha}$ (i.e. $I_i \restriction [\alpha, \lambda_i)$) is indiscernible over $s \cup d$ for some (any) $d \models p$.

Remark 3.6. Note that since λ_i is regular when i is a successor, a set A is a D_i-set if and only if for any subset $C \subseteq A$, $|C| < \lambda_i$, there is some $\alpha < \lambda_i$ such that $I_i^{\geq \alpha}$ is indiscernible over $C \cup I_i^{<\alpha}$. Indeed, given C, for every finite set $t \subseteq C$, let α_t be as in Definition 3.5 (for $s = \emptyset$). Let $\alpha_0 = \sup\{\alpha_t \,|\, t \subseteq C \text{ finite}\}$. Let $\alpha_0 < \alpha_1$ be defined similarly for $C \cup I_i^{<\alpha_0}$. Continue and finally put $\alpha = \sup\{\alpha_n \,|\, n < \omega\}$.

Definition 3.7. Let $\mathcal{M}$ be the class of sequences $\bar{A} = \langle A_i \,|\, i < \kappa \rangle$ such that:

- For some $i_0 \in \mathrm{Succ}(\kappa)$, for all $i_0 \leq i \in \mathrm{Succ}(\kappa)$, $I_i \subseteq A_i$, and for all $i < i_0$, $A_i = \emptyset$; $\langle A_i \,|\, i < \kappa \rangle$ is increasing and continuous and $|A_i| \leq \lambda_i$ for all $i \in \mathrm{Succ}(\kappa)$.
- For all $i \in \mathrm{Succ}(\kappa)$, A_i is a D_i-set.

Definition 3.8. For $\bar{A}, \bar{B} \in \mathcal{M}$, write $\bar{A} \leq_i \bar{B}$ for: for all $i \leq j < \kappa$, $A_j \subseteq B_j$; $\bar{A} \leq \bar{B}$ for: $\bar{A} \leq_0 \bar{B}$; and $\bar{A} \leq_* \bar{B}$ for: there is some $i < \kappa$ such that $\bar{A} \leq_i \bar{B}$.

Proposition 3.9. *Given $\bar{A} \in \mathcal{M}$, there is $\bar{A} \leq \bar{B} \in \mathcal{M}$ such that for all $i \in \mathrm{Succ}(\kappa)$, B_i is either $\emptyset$ or a model of T.*

Proof. For simplicity assume that $i_0 = 0$ in Definition 3.7. It is enough to prove the following. □

Claim 3.10. *Let $i \in \mathrm{Succ}(\kappa)$. Suppose that $\psi(x)$ is a formula over A_i. Then there is some $\bar{A} \leq \bar{B} \in \mathcal{M}$ such that B_i realizes ψ.*

Proof. Suppose $\psi = \psi(x, c)$. Let $\alpha < \lambda_i$ be such that $I_i^{\geq \alpha}$ is indiscernible over $A_{i-1} c \cup I_i^{<\alpha}$ (see Remark 3.6). Let $d \models \psi$. by Ramsey and compactness, there is some sequence J with the same order type and EM-type as $I_i^{\geq \alpha}$ over $cdA_{i-1} I_i^{<\alpha}$ which is indiscernible over $cdA_{i-1} I_i^{<\alpha}$. Hence $J \equiv_{cA_{i-1} I_i^{<\alpha}} I_i^{\geq \alpha}$, so apply an automorphism of $\mathfrak{C}$ to move J to $I_i^{\geq \alpha}$ over $cA_{i-1} I_i^{<\alpha}$ and let d' be the image of d. We get that $I_i^{\geq \alpha}$ is indiscernible over $cd' A_{i-1} I_i^{<\alpha}$ and still $d' \models \psi$, and even $d' \equiv_{A_{i-1}} d$ (this is not important here, but will be later). Let $p = \mathrm{tp}(d'/cA_{i-1} I_i)$.

($\star$) Suppose now that we are in a general situation, where we have some type $p_1 \in S(A_j B I_{j+1})$, where $B \subseteq A_{j+1}$ is of cardinality $< \lambda_{j+1}$ and there is some $\alpha < \lambda_{j+1}$ such that for any $d \models p_1$, $I_{j+1}^{\geq \alpha}$ is indiscernible over $dA_j B I_{j+1}^{<\alpha}$, and suppose $e \in A_{j+1}$. Then there is some $\lambda_{j+1} > \beta > \alpha$ such that $I_{j+1}^{\geq \beta}$ is indiscernible over $A_j B I_{j+1}^{<\beta} e$. By Ramsey and compactness, there is some indiscernible sequence J with the same EM and order type as $I_{j+1}^{\geq \beta}$ over $A_j B I_{j+1}^{<\beta} ed$ which is indiscernible over $A_j B I_{j+1}^{<\beta} ed$ for some fixed $d \models p_1$. Then $J \equiv_{A_j B e I_{j+1}^{<\beta}} I_{j+1}^{\geq \beta}$ and $J \equiv_{A_j B d I_{j+1}^{<\beta}} I_{j+1}^{\geq \beta}$. Hence, applying an automorphism fixing $A_j B I_{j+1}^{<\beta} e$ which maps J to $I_{j+1}^{\geq \beta}$, we move

d to some d' which still realizes p_1 but now $I_{j+1}^{\geq\beta}$ is indiscernible over $A_j B I_{j+1}^{\geq\beta} e d'$. Let $p_2 = \mathrm{tp}(d'/A_j B I_{j+1} e) \supseteq p_1$.

Using $(\star)$ iteratively, taking unions at limit stages, starting with p, we can find some $\psi \in p_i \in S(A_i)$ such that for every $d \models p_i$, $A_i d$ is a D_i-set.

We construct an increasing continuous sequence of types, $\langle p_j \,|\, i \leq j < \kappa \rangle$, $p_j \in S(A_j)$ such that for $j \in \mathrm{Succ}(\kappa)$, if $e \models p_j$ then $A_j e$ is a D_j-set. We then let $p = \bigcup_{i \leq j < \kappa} p_j$, $d \models p$ and define $B_j = A_j \cup \{d\}$ for all $i \leq j < \kappa$.

The construction of p_{j+1} uses a weak version of $(\star)$ in the first step (keeping only the type over A_j, as we did in the beginning), and then $(\star)$ as in the construction of p_i. □

Main Lemma 3.11. *Suppose that $\langle A_i \,|\, i < \kappa \rangle \in \mathcal{M}$, and $C \subseteq \bigcup_{i<\kappa} A_i$ is such that $|C| < \mu$. Let $p \in S(C)$. Then there is some $\bar{A} \leq \bar{B} \in \mathcal{M}$ which contains a realization of p.*

Proof. Here, we use the simplicity of T.

First, by Proposition 3.9, we may assume that for $i \in \mathrm{Succ}(\kappa)$, $A_i = \emptyset$ or is a model.

We may assume that there is some $E \subseteq C$ of size $\leq |T|$ such that p does not fork over E and moreover if q is a type over $\bigcup_{i<\kappa} A_i$ extending p, then q does not fork over E (if p was already realized in $\bigcup_{i<\kappa} A_i$, we can extend p and take E to be that realization). We get this by trying to construct an increasing continuous sequence $\langle (p_\alpha, E_\alpha) \,|\, \alpha < |T|^+ \rangle$ of subsets $E_\alpha \subseteq \bigcup_{i<\kappa} A_i$ of cardinality $\leq |T|$, and complete types p_α over $E_\alpha \cup C$ extending p starting with $(p, \emptyset)$ such that $p_{\alpha+1}|_{E_{\alpha+1}}$ forks over E_α. By local character of nonforking in simple theories (see [8, Proposition 7.2.5]), it follows that we must get stuck at some point in the construction, say α, and let $E = E_\alpha$, $p = p_\alpha$ (note that in particular, p_α does not fork over E_α because otherwise we could increase E_α).

Let $i_0 < \kappa$ be a successor ordinal such that $A_{i_0} \neq \emptyset$, $E \subseteq A_{i_0}$ and $|C| < \lambda_{i_0}$. (Here, we use the assumption that $\mathrm{cof}(\mu) = \kappa > |T|$.)

Now we make things easier:

(1) Enlarge C, so that for all $i_0 \leq i \in \mathrm{Succ}(\kappa)$, $C \cap A_i$ is a model of T. We can do this by building $C_{i,l}$ for $l < \omega$, $i_0 \leq i < \kappa$, so that $\langle C_{i,l} \,|\, i_0 \leq i < \kappa \rangle$ is increasing continuous with union C_l, $C \cap A_i \subseteq C_l \cap A_i \subseteq C_{i,l'}$ for $l' > l$ and where $C_{i,l} \cap A_i$ is a model for $i \in \mathrm{Succ}(\kappa)$ and $|C_{i,l}| \leq \lambda_{i_0}$. Finally, let $C' = \bigcup \{C_l \,|\, l < \omega\}$.

(2) Enlarge C again, so that for all $i_0 \leq i \in \mathrm{Succ}(\kappa)$, $C \mathop{\smile\!\!\!\!\!\mid}_{C \cap A_i} A_i$. To achieve this, build again $C_{i,l}$ as above such that for $i \in \mathrm{Succ}(\kappa)$, $C_{i,l} \mathop{\smile\!\!\!\!\!\mid}_{C_{i,l} \cap A_i} A_i$. (We construct $C_{i+1,l+1}$ as follows. Start with $C_{i,l+1} \cup (C_l \cap A_{i+1})$ and by local character find some $B_0 \subseteq A_{i+1}$ of cardinality $\leq \lambda_{i_0}$ such that $C_{i,l+1} \cup (C_l \cap A_{i+1}) \mathop{\smile\!\!\!\!\!\mid}_{B_0} A_{i+1}$, then let $C^1_{i+1,l+1} = C_{i,l+1} \cup (C_l \cap A_{i+1}) \cup B_0$. Continue this ω steps and take the union). Finally, let $C' = \bigcup \{C_{i,l} \,|\, i < \kappa, l < \omega\}$.

(3) Enlarge C by alternating steps (1) and (2) ω times, so that both $C \cap A_i$ is a model of T and $C \downarrow_{C \cap A_i} A_i$ for all $i \in \mathrm{Succ}(\kappa)$ such that $i \geq i_0$.

Now we want to find some $e \models p$ such that $\mathrm{tp}(e/A_{i_0})$ is a D_{i_0}-type. Start with any $e \models p$. Note that by the choice of E above, $\mathrm{tp}(e/\bigcup_{i<\kappa} A_i)$ does not fork over E.

Let $\alpha < \lambda_{i_0}$ be such that $I_{i_0}^{\geq \alpha}$ is indiscernible over $(C \cap A_{i_0}) I_{i_0}^{<\alpha}$. By Ramsey, there is some J with the same order type and EM-type as $I_{i_0}^{\geq \alpha}$ over $(C \cap A_{i_0}) I_{i_0}^{<\alpha} e$ which is indiscernible over $(C \cap A_{i_0}) I_{i_0}^{<\alpha} e$. Since $e \downarrow_E \bigcup_{i<\kappa} A_i$, in particular we have that $e \downarrow_{(C \cap A_{i_0})} I_{i_0}$ by the choice of i_0 above, so also $e \downarrow_{(C \cap A_{i_0})} I_{i_0}^{<\alpha} J$ (here, we use the fact that $I_{i_0}^{\geq \alpha}$ is indiscernible over $(C \cap A_{i_0}) I_{i_0}^{<\alpha}$, see also below). By applying an automorphism over $(C \cap A_{i_0}) I_{i_0}^{<\alpha}$ taking J to $I_{i_0}^{\geq \alpha}$, we can find some $d \equiv_{I_{i_0}^{<\alpha}(A_{i_0} \cap C)} e$ such that $d \downarrow_{C \cap A_{i_0}} I_{i_0}$ and $I_{i_0}^{\geq \alpha}$ is indiscernible over $d I_{i_0}^{<\alpha} (C \cap A_{i_0})$. By the independence theorem over models in simple theories (see [8, Theorem 7.3.11]), as $e \downarrow_{A_{i_0} \cap C} C$, $d \downarrow_{A_{i_0} \cap C} I_{i_0}$, $e \equiv_{A_{i_0} \cap C} d$ and $C \downarrow_{A_{i_0} \cap C} I_{i_0}$, there is some d' such that $d' \models p$ and $d' \equiv_{(C \cap A_{i_0}) I_{i_0}} d$.

This gives us some $e \models p$ such that $\mathrm{tp}(e/I_{i_0}(C \cap A_{i_0}))$ is a D_{i_0}-type.

Now, we use basically the same idea as in the proof of Proposition 3.9, using the independence theorem: we start with a D_{i_0}-type $q_1 \in S(B I_{i_0})$, where $C \cap A_{i_0} \subseteq B$, $|B| < \lambda_{i_0}$, consistent with p, and we want to extend it to a D_{i_0}-type $q_2 \in S(B I_{i_0} f)$ where $f \in A_{i_0}$ which is also consistent with p. Let $d \models q_1 \cup p$. By the choice of E, $d \downarrow_E \bigcup_{i<\kappa} A_i$. Let $\alpha < \lambda_i$ be such that $I_{i_0}^{\geq \alpha}$ is indiscernible over $d B I_{i_0}^{<\alpha}$. Let $\beta > \alpha$ be such that $I_{i_0}^{\geq \beta}$ is indiscernible over $f B I_{i_0}^{<\beta}$. Find J with the same EM-type as $I_{i_0}^{\geq \beta}$ over $I_{i_0}^{<\beta} B d f$ such that J is indiscernible over $I_{i_0}^{<\beta} B d f$. Then $J \equiv_{I_{i_0}^{<\beta} B d} I_{i_0}^{\geq \beta}$, and $J \equiv_{I_{i_0}^{<\beta} B f} I_{i_0}^{\geq \beta}$. As $d \models p$, we know that $d \downarrow_{A_{i_0} \cap C} I_{i_0} B f$ (by choice of i_0), hence also $d \downarrow_{A_{i_0} \cap C} I_{i_0}^{<\beta} J B f$ (if $\psi(x, j, m)$ witnessed forking, where $j \in J$ is an increasing tuple and $m \in I_{i_0}^{<\beta} B f$, then for some increasing tuple $j' \in I_{i_0}^{\geq \beta}$, $\psi(d, j', m)$ holds. But $j'm \equiv_{A_{i_0} \cap C} jm$ as $I_{i_0}^{\geq \beta}$ is indiscernible over $f B I_{i_0}^{<\beta}$) . Move J to $I_{i_0}^{\geq \beta}$ over $I_{i_0}^{<\beta} B f$, to get some $d' \equiv_{I_{i_0} B} d$, but now $I_{i_0}^{\geq \beta}$ is indiscernible over $I_{i_0}^{<\beta} B d' f$ and $d' \downarrow_{A_{i_0} \cap C} I_{i_0} B f$. Now, we can use the independence theorem as above, and find q_2.

Using this technique (constructing an increasing continuous sequence of types over small subsets of A_{i_0} augmented with I_{i_0}) we can find some $e \models p$ such that $\mathrm{tp}(e/A_{i_0})$ is a D_{i_0}-type.

Now we may continue. More formally, we find an increasing continuous sequence of types p_i for $i_0 \leq i < \kappa$ such that:

- $p_{i_0} = \mathrm{tp}(e/A_{i_0})$; $p_i \in S(A_i)$ and for $i \in \mathrm{Succ}(\kappa)$, p_i is a D_i-type and $p_i \cup p$ is consistent for all i.

We can do this by using the same technique as in the construction of p_{i_0}. Finally, let $p_\kappa = \bigcup_{i<\kappa} p_i$, let $e \models p_\kappa$, and let $B_i = \emptyset$ for $i < i_0$ and $A_i e$ for $i \geq i_0$. □

Proof of Theorem 3.3. Let $\langle C_\alpha \mid \alpha \in \mathrm{Lim}(\mu^+)\rangle$ be a sequence as in Remark 3.2. Note that $|C_\alpha| < \mu$ for all $\alpha < \mu^+$ as μ is singular. Let $\{S_\alpha \mid \alpha < \mu^+\}$ be a partition of μ^+ to sets of size μ^+. We construct a sequence $\langle (\bar{A}_\alpha, \bar{p}_\alpha) \mid \alpha < \mu^+\rangle$ such that:

(1) $\bar{A}_\alpha = \langle A_{\alpha,i} \mid i < \kappa\rangle \in \mathcal{M}$;
(2) $\bar{p}_\alpha$ is an enumeration $\langle p_{\alpha,\beta} \mid \beta \in S_\alpha \backslash \alpha\rangle$ of all complete types over subsets of $\bigcup_{i<\kappa} A_{\alpha,i}$ of size $< \mu$ (this uses $\mu^+ = 2^\mu$);
(3) If $\beta < \alpha$ then $\bar{A}_\beta \leq_* \bar{A}_\alpha$ (see Definition 3.8);
(4) If $\alpha \in S_\gamma$ and $\gamma \leq \alpha$, then $\bar{A}_{\alpha+1}$ contains a realization of $p_{\gamma,\alpha}$;
(5) If α is a limit ordinal, then for all $i < \kappa$ such that $|C_\alpha| < \lambda_i$, $A_{\alpha,i} \neq \emptyset$ and for all $\beta \in C_\alpha$, $\bar{A}_\beta \leq_i \bar{A}_\alpha$.

Start with $A_{0,i} = I_i$ for $i \in \mathrm{Succ}(\kappa)$ and otherwise defined by continuity.

For $\alpha + 1$, use Main Lemma 3.11.

For α limit, there are two possibilities.

Case 1. $\sup(C_\alpha) = \alpha$. Suppose $i_0 < \kappa$ is minimal such that $|C_\alpha| < \lambda_{i_0}$ (so necessarily $i_0 \in \mathrm{Succ}(\kappa)$). For $i < i_0$, let $A_{\alpha,i} = \emptyset$. For $i \geq i_0$ successor, let $A_{\alpha,i} = \bigcup_{\beta \in C_\alpha} A_{\beta,i}$. Note that $|A_{\alpha,i}| \leq \lambda_i$. We have to show that $\bar{A}_\alpha$ satisfies (1), (3) and (5). The latter is by construction and the fact that for $\beta \in C_\alpha$, $|C_\beta| \leq |C_\alpha|$.

For (1), suppose $s \subseteq A_{\alpha,i}$ is a finite set where $i_0 \leq i \in \mathrm{Succ}(\kappa)$. For every element $e \in s$, there is some $\beta_e \in C_\alpha$ such that $e \in A_{\beta_e,i}$. Let $\beta = \max\{\beta_e \mid e \in s\}$. Then β is a limit ordinal and $C_\alpha \cap \beta = C_\beta$. As $|C_\beta| < \lambda_{i_0}$, it follows by the induction hypothesis that $s \subseteq A_{\beta,i}$. As $A_{\beta,i}$ is a D_i-set for all such β, it follows that $A_{\alpha,i}$ is a D_i-set as well.

Lastly, (3) is easy by assumption of the case and transitivity of $\leq_*$.

Case 2. $\sup(C_\alpha) < \alpha$. In this case, if $C_\alpha \neq \emptyset$, then it has a last element, and $\mathrm{cof}(\alpha) = \omega < \kappa$. If $C_\alpha = \emptyset$, choose $\gamma = 0$, otherwise, it is the last element of C_α. Let $i^* < \kappa$ be minimal such that $|C_\alpha| < \lambda_{i^*}$.

Choose a cofinal set $S \subseteq \alpha$ above γ of size $\aleph_0 < \kappa$. For all $\varepsilon < \zeta \in S$, $\bar{A}_\varepsilon \leq_* \bar{A}_\zeta$ as is witnessed by some $i_{\varepsilon,\zeta} < \kappa$. As κ is regular, there is some $i^* < i_0 \in \mathrm{Succ}(\kappa)$ such that $\bar{A}_\varepsilon \leq_{i_0} \bar{A}_\zeta$ for all $\varepsilon < \zeta \in S$. By the same reasoning there is some $i_0 < i_1 \in \mathrm{Succ}(\kappa)$ such that $\bar{A}_\gamma \leq_{i_1} \bar{A}_\varepsilon$ for all $\varepsilon \in S$. Set $A_{\alpha,i} = A_{\gamma,i}$ for $i < i_1$ and $A_{\alpha,i} = \bigcup_{\beta \in S} A_{\beta,i}$ for $i \geq i_1$.

Now: (1) follows by choice of i_0 (so that each $A_{\alpha,i}$ is a D_i-set) and i_1 (so that $A_{\alpha,i}$ is increasing with i), (3) follows by the transitivity of $\leq_*$, so we are left with (5). The first part is easy: if $C_\alpha = \emptyset$, then it follows by our choice of $\bar{A}_0$. Otherwise, use the fact that $C_\gamma \subseteq C_\alpha$.

Suppose $\beta \in C_\alpha$. Then, either $\beta = \gamma$, in which case this clause is obvious, or $\beta < \gamma$, in which case $\beta \in C_\gamma$. By the induction hypothesis, $\bar{A}_\beta \leq_{i^*} \bar{A}_\gamma \leq \bar{A}_\alpha$, so we are done by the choice of i^*.

Finally, let $M = \bigcup_{\alpha<\mu^+, i<\kappa} A_{\alpha,i}$. Then M is a μ-saturated model of T by (4). However, it is not μ^+-saturated because the type $\{\varphi(x, a_j) \mid j \in I \text{ even}\} \cup \{\neg\varphi(x, a_j) \mid j \in I \text{ odd}\}$ is not realized in M: suppose b realizes it. We have that b is a finite tuple, but since $\bar{A}_\alpha$ is an increasing continuous sequence for all $\alpha < \mu^+$, there must be some $\alpha < \mu^+$ and $i \in \mathrm{Succ}(\kappa)$ such that $b \in A_{\alpha,i}$. But clearly $\mathrm{tp}(b/I_i)$ is not a D_i-type — contradiction. □

4. Dependent Theories

Here, we characterize the NIP (dependent) theories which have exact saturation at κ, assuming the Continuum Hypothesis at κ. They happen to be precisely the nondistal theories.

Throughout this section, assume that T is NIP: for no formula $\varphi(x, y)$ is it the case that there are $\langle a_i \mid i < \omega\rangle$ and $\langle b_s \mid s \subseteq \omega\rangle$ such that $\mathfrak{C} \models \varphi(a_i, b_s)$ holds if and only if $i \in s$.

We use the notation X^{opp} for a linear order X to denote X with the order reversed.

4.1. *Preliminaries*

4.1.1. *NIP theories*

Suppose I is an indiscernible sequence. We will identify I and its underlying order. For instance, we will say that I is *dense* if its underlying order type is. All our sequences will be infinite.

Shrinking of indiscernibles. Recall the following definition, which, in NIP theories, gives a complete type.

Definition 4.1. The *average type (at ∞)* of an indiscernible sequence with no end, $\langle a_i \mid i \in I\rangle$, over A, denoted by $Av_\infty(I/A)$, consists of formulas of the form $\phi(b, x)$ with $b \in A$, such that for some $i \in I$, $\mathfrak{C} \models \phi(b, a_j)$ for every $j \geq i$.

We will use shrinking of indiscernibles (which is a stronger version of the existence of averages), as formulated in the following fact. Given a linear order $(X, <)$, a *finite convex equivalence relation* on X is an equivalence relation with finitely many classes which are convex.

Fact 4.2 (Shrinking, see e.g. [7, Theorem 3.33]). Suppose I is an indiscernible sequence over some set A, and suppose that b is some finite tuple and Δ a finite set of formulas. Then there is a finite convex equivalence relation $\sim$ on I such that each $\sim$-class C is Δ-indiscernible over $Ab \cup I\backslash C$.

Moreover, given a formula $\varphi(x_0, \ldots, x_{n-1}, y, z)$ there is such an equivalence relation $\sim$ such that for any two finite increasing sequences $\bar{i}, \bar{j}$ of length n from I, if $\bar{i} \sim \bar{j}$ (i.e. $i_0 \sim j_0, \ldots, i_{n-1} \sim j_{n-1}$) and $a \in A^z$ then $\varphi(a_{i_0}, \ldots, a_{i_{n-1}}, b, a)$ holds if and only if $\varphi(a_{j_0}, \ldots, a_{j_{n-1}}, b, a)$.

A *cut* in an indiscernible sequence I has the form $\mathfrak{c} = (I_1, I_2)$ for I_1 an initial segment of I and I_2 its corresponding end segment. Our cuts will *always* be internal (both I_1, I_2 are not empty) and have infinite cofinality from both sides, unless we specifically say otherwise.

Here is a useful and easy corollary of Shrinking of indiscernibles (Fact 4.2). We leave its proof as an exercise.

Corollary 4.3. *Suppose that I is indiscernible, $|T| \leq \theta$, and that $\mathfrak{c}_i$ for $i < \theta^+$ are distinct cuts, each of cofinality at least θ^+ from both sides. Then for any set A with $|A| \leq \theta$, there is some $i < \theta^+$ such that there is an interval I_0 around $\mathfrak{c}_i$ which is indiscernible over $A \cup I \backslash I_0$.*

Invariant types and Morley sequences. Recall that for a global A-invariant type $p \in S(\mathfrak{C})$, and for $B \supseteq A$, the sequence $\langle a_i \,|\, i < \omega \rangle$ generated by realizing $a_i \models p|_{Ba_{<i}}$ is always indiscernible over B. This sequence is a *Morley sequence generated by p over B*. In general, Morley sequences need not be of order type ω. A sequence $\langle a_i \,|\, i \in I \rangle$ of any order type $(I, <)$ is a *Morley sequence of p over B* if for any $i \in I$, $a_i \models p|_{Ba_{<i}}$. Let $p^{(I)}|_B = \mathrm{tp}(\langle a_i \,|\, i \in I \rangle / B)$ and $p^{(I)}$ be the global A-invariant type $\bigcup_{A \subseteq B} p^{(I)}|_B$. See e.g. [8, Example 7.2.10; 7, Sec. 2.2.1] for more.

In NIP, any endless indiscernible sequence over A is a Morley sequence of some invariant type over a set containing A (extend I to $I + I^{\mathrm{opp}}$, and let p be the average type of I^{opp} at $-\infty$, so p is I^{opp}-invariant and I is a Morley sequence of p over AI^{opp}).

4.1.2. *Distal theories*

Suppose that I is indiscernible and that $\mathfrak{c} = (I_1, I_2)$ is a cut in I. For a set A, denote by $\lim(\mathfrak{c}^- / A) = Av_\infty(I_1 / A)$, and similarly $\lim(\mathfrak{c}^+ / A)$ is the average type of I_2 at $-\infty$ over A. This is the *limit type of $\mathfrak{c}^-$ (or $\mathfrak{c}^+$) over A*. Note that if $A = \mathfrak{C}$, this is an I_1 (or I_2)-invariant type.

Note that $\lim(\mathfrak{c}^+ / I) = \lim(\mathfrak{c}^- / I)$ (so we just write $\lim(\mathfrak{c}/I)$), and that if $b \models \lim(\mathfrak{c}/I)$ then b *fills* I: when b is put in $\mathfrak{c}$, the augmented sequence $I \cup b$ is indiscernible. In fact this is equivalent to satisfying $\lim(\mathfrak{c}/I)$.

More generally, if $\bar{c}$ is some (possibly infinite) ordered tuple of tuples in same length as the tuples in I, we will say that $\bar{c}$ *fills* $\mathfrak{c}$ if when we put $\bar{c}$ in $\mathfrak{c}$, in the right order, the augmented sequence $I \cup \bar{c}$ is indiscernible.

For instance, if $\bar{c}$ is of order type ω, then, using the notation from above, we have that if $\bar{c} \models \lim(\mathfrak{c}^+ / \mathfrak{C})^{(\omega)}|_I$, then $\bar{c}$ fills $\mathfrak{c}$.

When I is indiscernible over A, we can add "over A" everywhere, meaning that we name the elements of A.

Definition 4.4. We say that two types $p(x), q(y) \in S(A)$ are *orthogonal* if their union implies a complete type in x, y over A (usually this notion is called "weakly orthogonal", but full orthogonality will not be used in this paper).

If $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are two distinct cuts in a dense indiscernible sequence I, and $b_i \models \lim(\mathfrak{c}_i/I)$ for $i = 1, 2$, we will say that b_1, b_2 are *I-independent*, if, when placed in their appropriate cuts, $I \cup \{b_1, b_2\}$ is indiscernible. The two limit types are orthogonal if and only if this happens to every such b_1, b_2.

Definition 4.5. A dense indiscernible sequence I is called *distal* if whenever $\mathfrak{c}_1, \mathfrak{c}_2$ are two distinct cuts in I, then $\lim(\mathfrak{c}_1/I)$ and $\lim(\mathfrak{c}_2/I)$ are orthogonal.

Remark 4.6. (1) If I is a dense indiscernible sequence which is not distal, then there are two distinct cuts $\mathfrak{c}_i$ for $i = 1, 2$ and $b_i \models \lim(\mathfrak{c}_i/I)$ such that $I \cup \{b_1, b_2\}$ (i.e. I augmented with b_1, b_2 placed in their corresponding cuts) is not indiscernible. By compactness and indiscernibility, it is easy to see that this is true for any dense indiscernible sequence with the same EM-type as I and any distinct cuts $\mathfrak{d}_1, \mathfrak{d}_2$ there.

(2) Distal indiscernible sequences and distal theories (see below) were defined and discussed at length in [6]. There, they are defined a bit differently, namely: an infinite indiscernible sequence (not necessarily dense) is distal if it has the same EM-type as a dense indiscernible sequence which is distal. On the face of it, this defines a larger class even inside dense sequences, but this is not the case by [6, Lemma 2.3].

Definition 4.7. An NIP theory T is called *distal* if all infinite dense indiscernible sequences in it are distal.

Fact 4.8 ([7; 1, Theorems 9.21 and 9.22]). A theory T is distal if and only if for any formula $\varphi(x, y)$ there is a formula $\theta(x, z)$ such that: for any $M \models T$, $A \subseteq M$ of size at least 2, $a \in M^x$ and a finite $C \subseteq A^y$ there is $b \in A^z$ such that $M \models \theta(a, b)$ and $\theta(x, b) \vdash \mathrm{tp}_\varphi(a/C)$. Equivalently, in some elementary extension $(M, A) \prec (M', A')$ of the pair, there is $b \in (A')^z$ such that $M' \models \theta(a, b)$ and $\theta(x, b) \vdash \mathrm{tp}_\varphi(a/A)$.

Example 4.9 ([6, Corollary 2.30]). Examples of distal theories include o-minimal theories (e.g. RCF, DLO), and the theory of the p-adics.

4.2. *Results*

Now we are ready to state our main theorem for this section.

Theorem 4.10. *Suppose that κ is a singular cardinal such that $\kappa^+ = 2^\kappa$. An* NIP *theory T with $|T| < \kappa$ is distal if and only if it does not have exact saturation at κ.*

In [5, Question 2.30], the following question appears. Is there a dependent theory T with exact saturation at some singular cardinal κ of cofinality $> |T|$ such that even in T^{eq} there is no infinite indiscernible set?

In [7, Sec. 9.3.4], there is an example of an NIP theory which is not distal and yet has no nontrivial generically stable type, even in T^{eq}. Having an infinite indiscernible set is equivalent to having a generically stable type (see [7, Sec. 2.2.2, Remark 2.32]). Together, we get the following.

Corollary 4.11. *The answer to* [5, Question 2.30] *is "yes", provided that for some singular κ with* $\mathrm{cof}(\kappa) > |T|$, $\kappa^+ = 2^\kappa$.

Left to Right.

Proposition 4.12. *If T is distal and $|T| < \kappa$ is singular, then every κ-saturated model M is also κ^+-saturated.*

Proof. Suppose $A \subseteq M$ and $|A| = \kappa$. Suppose $\mu = \mathrm{cof}(\kappa) < \kappa$. Let $p \in S(A)$, and we want to show that p is realized in M. Write $A = \bigcup_{i<\mu} A_i$ with $|A_i| < \kappa$. For each $i < \mu$, let $b_i \models p \restriction A_i$, $b_i \in M$ (exists as M is κ-saturated). Let $\varphi(x, y)$ be some formula. By Fact 4.8, there is some $(M'_i, A'_i) \succ (M, A_i)$, $d_i^\varphi \in A'_i$ and θ^φ such that $M'_i \models \theta^\varphi(b_i, d_i^\varphi)$ and $\theta^\varphi(x, d_i^\varphi) \vdash \mathrm{tp}_\varphi(b_i/A_i)$ (as usual, we assume that everything happens in the monster model $\mathfrak{C}$ of T). Let $q_i = \{\theta^\varphi(x, d_i^\varphi) \,|\, \varphi(x, y) \in L\}$. □

Claim 4.13. *If $\mu > j \geq i$ then $b_j \models q_i$ (in M'_i).*

Proof. For $j = i$, this is by choice of θ^φ, so suppose $j > i$ and that $\theta^\varphi(b_i, d_i^\varphi) \wedge \neg\theta^\varphi(b_j, d_i^\varphi)$ for some φ. Hence $(M'_i, A'_i) \models \exists z \in P(\theta^\varphi(b_i, z) \wedge \neg\theta^\varphi(b_j, z))$, where P is a predicate symbol interpreted as A'_i. Hence the same is true in (M, A_i). But $b_i \equiv_{A_i} b_j$ so this cannot happen. □

Let $d_i = \langle d_i^\varphi \,|\, \varphi \in L\rangle$ for $i < \mu$, and find $e_i \models \mathrm{tp}(d_i/A_i \cup \{b_i \,|\, i < \mu\})$ in M, which exists by κ-saturation. Enumerate it as $e_i = \langle e_i^\varphi \,|\, \varphi \in L\rangle$. Let $r_i(x) = \{\theta^\varphi(x, e_i^\varphi) \,|\, \varphi \in L\}$. Note that for each φ and $i < \mu$, $\theta^\varphi(x, e_i^\varphi) \vdash \mathrm{tp}_\varphi(b_i/A_i)$ and that $b_j \models r_i$ for $j \geq i$ by Claim 4.13.

Let $r = \bigcup_{i<\mu} r_i$. By the previous paragraph, r is a consistent type in M, and it is a type over a set of size $\leq \mu \cdot |T| < \kappa$, so it is realized, say by $c \in M$. Then $c \models p$. □

Right to left — Technical lemmas.

Definition 4.14. Suppose $s \subseteq \mathfrak{C}$ is a finite set, and I is an indiscernible sequence. Let us say that a cut $\mathfrak{c}$ in I is *generic for s* if there is a neighborhood $I_0 = (i_1, i_2)$ of $\mathfrak{c}$ in I such that I_0 is indiscernible over $s \cup I \backslash I_0$.

Similarly, for a small set A, $\mathfrak{c}$ is A-generic if it is generic for every finite subset from A. We can similarly say that $\mathfrak{c}$ is q-generic for a complete finitary type q over I, meaning that $\mathfrak{c}$ is c-generic for some (any) $c \models q$ (formally, for $\bigcup c$).

Definition 4.15. For an indiscernible sequence I, let D_I be the collection of finitary types $q \in S^{<\omega}(I)$ such that q is orthogonal (see Definition 4.4) to $\lim(\mathfrak{c}/I)$ for some q-generic cut $\mathfrak{c}$.

Remark 4.16. Suppose that $\mathfrak{c}$ is generic for c in some sequence I as witnessed by (i_1, i_2). Then $\lim(\mathfrak{c}^-/Ic) = \lim(\mathfrak{c}^+/Ic)$ and if $b \models \lim(\mathfrak{c}/Ic)$ then b fills $\mathfrak{c}$ and moreover, the interval $(i_1, i_2) \cup b$ is indiscernible over $c \cup I \backslash (i_1, i_2)$.

Hence, to say that q is orthogonal to $\mathfrak{c}$ in Definition 4.15 means that whenever $c \models q$ and $a \models \lim(\mathfrak{c}/I)$, we have that $a \models \lim(\mathfrak{c}/Ic)$.

Remark 4.17. Note that if I is dense indiscernible, and $\mathfrak{c}$ is c-generic then $\mathfrak{c}$ is c-generic also in I^{opp} (I with reverse order). Similarly, $q \in D_I$ if and only if $q \in D_{I^{\mathrm{opp}}}$ with the same cut witnessing this. Moreover, note that I is a D_I-set (for any finite $s \subseteq I$, any cut $\mathfrak{c}$ will witness this), and that if A is a D_I-set then so is AI.

We want to show that this definition behaves well.

Main Lemma 4.18. *Suppose I is dense indiscernible. If $q \in D_I$ and $\mathfrak{c}$ is q-generic, then q is orthogonal to* $\lim(\mathfrak{c}/I)$.

The proof uses two ingredients, both from [6]. One is the finite co-finite theorem, and the other is the external characterization of domination.

First, a technical claim.

Claim 4.19. *Suppose that I is a dense indiscernible sequence, and that* $\mathrm{tp}(c/I) \in D_I$ *as witnessed by $\mathfrak{c} = (I_1, I_2)$. Let $I_1 \ni i_1 < i_2 \in I_2$ be such that (i_1, i_2) is indiscernible over $I \backslash (i_1, i_2) \cup c$. Suppose that J is some dense sequence with no minimum such that $I' = I_1 + J + I_2$ is indiscernible and, in I', we still have that (i_1, i_2) is indiscernible over $I \backslash (i_1, i_2) \cup c$. Then* $\mathrm{tp}(c/I') \in D_{I'}$ *as witnessed by $(I_1, J + I_2)$.*

Similarly, if J has no maximum, the same is true for $(I_1 + J, I_2)$.

Proof. What this claim says is that, letting $\mathfrak{d} = (I_1, J + I_2)$, $\mathfrak{d}$ is c-generic in I' (by assumption) and if $a \models \lim(\mathfrak{d}/I')$, then $a \models \lim(\mathfrak{d}/I'c)$. Suppose not. Then for some formula $\varphi(x, x_1, \ldots, x_{k-1})$ over $c \cup I_1 \cup I \backslash (i_1, i_2)$ there are $\mathfrak{d} < b_1 < \cdots < b_{k-1} < i_2$ from $J + I_2$ such that $\neg\varphi(a, b_1, \ldots, b_{k-1})$ holds even though for all $c_0 < \cdots < c_{k-1} < i_2$ from I_2, $\varphi(c_0, \ldots, c_{k-1})$ holds.

Choose $b'_1 < \cdots < b'_{k-1} < i_2$ from I_2. Let $\Gamma_0(x)$ be the type over $c \cup I$ saying that x fills $\mathfrak{c}$ and let $\Gamma = \Gamma_0 \cup \{\neg\varphi(x, b'_1, \ldots, b'_{k-1})\}$. Then Γ is consistent: a finite part Γ'_0 of Γ_0 says that $\langle a'_j \,|\, j < m\rangle \frown \langle x\rangle \frown \langle c'_j \,|\, j < l\rangle$ is Δ-indiscernible where $\langle a'_j \,|\, j < m\rangle$, $\langle c'_j \,|\, j < l\rangle$ are increasing sequences from I_1 and I_2, respectively and Δ is a finite set of L-formulas. For $0 < i < k$, let $s'_i = \{j < l \,|\, b'_{i-1} < c'_j \leq b'_i\}$, where $b'_0 = -\infty$ in I_2. As $J + I_2$ is dense, we can find an increasing sequence $\langle c_j \,|\, j < l\rangle$ from $J + I_2$ such that, letting $s_i = \{j < l \,|\, b_{i-1} < c_j \leq b_i\}$ (where $b_0 = -\infty$ in J), $s'_i = s_i$. As (i_1, i_2) is indiscernible over $c \cup I \backslash (i_1, i_2)$ in I', there is an automorphism σ fixing $c \cup I_1 \cup I \backslash (i_1, i_2)$ taking b'_i to b_i and c'_j to c_j. But $\sigma^{-1}(a)$, then satisfies Γ'_0 and also $\neg\varphi(x, b'_1, \ldots, b'_{k-1})$ as we wanted.

However, Γ cannot be satisfied by the assumption that $\mathrm{tp}(c/I) \in D_I$.

The analogous claim on $(I_1 + J, I_2)$ is proved similarly. $\square$

We continue with the finite-co-finite theorem [6, Theorem 3.30]. This theorem states that if $I = I_1 + I_2 + I_3$ is indiscernible, both I_1, I_3 are infinite and $I_1 + I_3$ is indiscernible over A, then for any $a \in A$, and any $\varphi(x, y)$, the set $\{b \in I_2 \mid \mathfrak{C} \models \varphi(a, b)\}$ is either finite or co-finite.

Fact 4.20 ([6, Corollary 3.32]). Let $I_1 + I_2 + I_3$ be an indiscernible sequence, such that I_1 and I_3 are without endpoints. Suppose that $I_1 + I_3$ is indiscernible over A. Then there is some $I_2' \subseteq I_2$ such that $|I_2 \backslash I_2'| \leq |T| + |A|$ and $I_1 + I_2' + I_3$ is A-indiscernible.

Proposition 4.21. *Let I be dense indiscernible. Suppose that $q \in D_I$ and that $\mathfrak{c}$ is a cut in I that witnesses it. Then q is orthogonal to $\lim(\mathfrak{c}^+/\mathfrak{C})^{(X)}|_I$ for any linear order $(X, <)$.*

Proof. Suppose (i_1, i_2) is an interval around $\mathfrak{c}$ which witnesses that $q \in D_I$. What we have to show is that if $c \models q$ and $\bar{a} \models \lim(\mathfrak{c}^+/\mathfrak{C})^{(X)}|_I$ (i.e. $\bar{a}$ is an ordered sequence of order type $(X, <)$ which fills $\mathfrak{c}$), then $(i_1, i_2) \cup \bar{a}$ is indiscernible over $c \cup I \backslash (i_1, i_2)$.

We may assume that X is finite. We prove by induction on n that the proposition holds for all I with $X = n$ as an order. For $n = 1$ this is just the assumption that $q \in D_I$. Suppose this is true for n and prove it for $n + 1$.

Let $\mathfrak{c} = (I_1, I_2)$ inside (i_1, i_2). Let $a_0, \ldots, a_n$ fill $\mathfrak{c}$.

Let J be a dense indiscernible sequence of cofinality $(|I| + |T|)^+$ from both sides (and such that between any two elements there are $(|I| + |T|)^+$ elements), such that $I_1 + a_0 + \cdots + a_{n-1} + J + a_n + I_2$ is indiscernible over $I \backslash (i_1, i_2)$. By Fact 4.20, we may assume that $I_1 + J + I_2$ is indiscernible over $c \cup I \backslash (i_1, i_2)$.

Let $\mathfrak{d} = (I_1 + J, I_2)$, which we identify with the corresponding cut in the extended sequence $I' = I \cup J$. By Claim 4.19, $\mathfrak{d}$ witnesses that $\operatorname{tp}(c/I') \in D_{I'}$, and hence $I_1 + J + a_n + I_2$ is indiscernible over $c \cup I \backslash (i_1, i_2)$. Note that $J + a_n$ is dense with no minimum, so by applying Claim 4.19 again on the sequence $I'' = I' \cup a_n$, we get that $\operatorname{tp}(c/I'') \in D_{I''}$ as witnessed by the cut corresponding to $(I_1, J + a_n + I_2)$. By the induction hypothesis, $I_1 + a_0 + \cdots + a_{n-1} + J + a_n + I_2$ is indiscernible over $c \cup I \backslash (i_1, i_2)$. In particular, we get what we wanted. □

Now, we need to discuss domination in indiscernible sequences. Although we will not use it directly, we give the definition.

Definition 4.22. Suppose that I is a dense Morley sequence of an A-invariant type p over A. Suppose $\mathfrak{c}$ is a cut in I, $\bar{a}$ is an ordered tuple which fills $\mathfrak{c}$ over A, and that $a \models p|_{IA}$. We will say that $\bar{a}$ dominates a over (I, A) if whenever $\mathfrak{d} \neq \mathfrak{c}$ is a cut in I, and $\bar{b}$ is any ordered tuple which fills $\mathfrak{d}$ over A, if $\bar{a} \downarrow_I \bar{b}$ (i.e. when put in their appropriate cuts in the right order, $I \cup \{\bar{a}, \bar{b}\}$ is A-indiscernible) then $a \downarrow_I \bar{b}$ (i.e. $a \models p|_{IA\bar{b}}$).

Say that $\bar{a}$ strongly dominates a over (I, A) if for any dense A-indiscernible sequence J containing I such that $\bar{a}$ still fills some cut in J and $a \models p|_{AJ}$, $\bar{a}$ dominates a over (J, A).

Remark 4.23. In the definition in [6, Definition 3.2], the tuple $\bar{b}$ is dense. However, by compactness, if $\bar{a}$ dominates a over (I, A) for dense tuples, then there is domination for any ordered tuple.

The following fact says that we can always find strong domination.

Fact 4.24 ([6, Proposition 3.6]). Let I be a dense Morley sequence (of finite tuples) of a global A-invariant type p over A and $a \models p|_{AI}$. Suppose $\mathfrak{c}$ is a cut in I. Then there is an ordered tuple $\bar{a}$ of length $\leq |T|$ that fills $\mathfrak{c}$ which strongly dominates a over (I, A).

Fact 4.25 ([6, Proposition 3.7]). Let I be a dense Morley sequence of an A-invariant type p over A, $a \models p|_{AI}$. Suppose $\bar{a}$ fills a cut $\mathfrak{c}$ and that $\bar{a}$ strongly dominates a over (I, A). Let $d \in \mathfrak{C}$. Assume that:

There is a partition $I = J_1 + J_2 + J_3 + J_4$ such that J_2 and J_4 are infinite, $\mathfrak{c}$ is interior to J_2, $J_2 \cup \{\bar{a}\}$ is indiscernible over $J_{\neq 2}Ad$ and J_4 is a Morley sequence of p over $J_{\neq 4}Ad$.

Then $a \models p|_{AId}$.

Proof of Main Lemma 4.18. So assume that I is dense indiscernible and $\mathrm{tp}(c/I) \in D_I$. Suppose that $\mathfrak{c}$ is a cut which witnesses this.

Let $\mathfrak{d}$ be another cut which is generic for c. We want to show that $\mathrm{tp}(c/I)$ is orthogonal to $\lim(\mathfrak{d}/I)$. Suppose without loss of generality that $\mathfrak{c} < \mathfrak{d}$ (otherwise, reverse the order of I, see Remark 4.17). Suppose that $a \models \lim(\mathfrak{d}/I)$. We want to show that $a \models \lim(\mathfrak{d}/Ic)$. Let $i_1 < i_2 < j_1 < j_2$ witness that $\mathfrak{c}$ and $\mathfrak{d}$ are c-generic respectively. Let $\mathfrak{d} = (I_1, I_2)$. Let $p = \lim(\mathfrak{d}^+/\mathfrak{C})$, so that p is I_2-invariant. We want to show that $a \models p|_{Ic}$.

By Fact 4.24, we may find some $\bar{a}$ filling $\mathfrak{c}$ over I_2 such that $\bar{a}$ strongly dominates a over (I_1, I_2). As $\mathfrak{c}$ witnesses that $\mathrm{tp}(c/I) \in D_I$, and as $\bar{a}$ fills $\mathfrak{c}$ in I, Proposition 4.21 implies that $(i_1, i_2) \cup \bar{a}$ is indiscernible over $cI \backslash (i_1, i_2)$.

Let $J_2 = (i_1, i_2)$, $J_4 = (j_1, +\infty)$ in I_1 and let J_1, J_3 fill the other parts of I_1 so that $I_1 = J_1 + J_2 + J_3 + J_4$. Let us check that the assumptions of Fact 4.25 are satisfied, with I there being I_1, $d = c$ and $A = I_2$. Note that I_1 is a Morley sequence of p over I_2. Also $\mathfrak{c}$ is interior to J_2, and J_4 is a Morley sequence of p over $J_{\neq 4}I_2c$ by the choice of (j_1, j_2). We already mentioned that $J_2 \cup \{\bar{a}\}$ is indiscernible over $J_{\neq 2}I_2c$. Finally, the conclusion of Fact 4.25 is exactly what we want. □

Right to Left. Assume that T is NIP but not distal, and that $|T| < \kappa$ is singular such that $\kappa^+ = 2^\kappa$.

For an ordinal $\alpha < \kappa$, let $(Y_\alpha, <)$ be a dense linear order of cofinality $|\alpha|^+$, and of power $|\alpha|^+$. Let $(X_\alpha, <)$ be the linear order $Y_\alpha + Y_\alpha^{\mathrm{opp}}$. Let $\mathfrak{c}_\alpha$ be the cut $(Y_\alpha, Y_\alpha^{\mathrm{opp}})$.

Since T is not distal, there is a dense indiscernible sequence $\mathcal{I}$ which is not distal. By Remark 4.6, we may assume that $\mathcal{I}$ is of order type $\sum_{\alpha<\kappa} X_\alpha$ (see Notation 3.4). Abusing notation, we let $\mathfrak{c}_\alpha$ denote the appropriate cuts in $\mathcal{I}$. Note that $|\mathcal{I}| = \kappa$.

Definition 4.26. Let $D = D_{\mathcal{I}}$.

Note that by Shrinking (Fact 4.2), given some set A of size $< \kappa$, any cut $\mathfrak{c}$ in $\mathcal{I}$ which is not one of the cuts[b] induced by the finite equivalence relations induced by A (there are at most $|T| + |A|$ such) which has cofinality $(|T| + |A|)^+$ from both sides is A-generic. In particular, given such an A, for some $\alpha < \kappa$, $\mathfrak{c}_\alpha$ is generic for A.

Lemma 4.27. *Suppose A is a D-set, $|A| < \kappa$. Suppose that $p(x) \in S(A)$. Then there is some $q \supseteq p$, $q \in S(AI_0)$ where $I_0 \subseteq \mathcal{I}$, $|I_0| \leq |T|$ such that if $b \models q$ then bA is a D-set.*

Proof. Try to construct an increasing continuous sequence of partial types $\langle p_\varepsilon \mid \varepsilon < |T|^+\rangle$ and intervals $I_\varepsilon = (i_1^\varepsilon, i_2^\varepsilon) \subseteq X_{\alpha_\varepsilon}$ around distinct A-generic cuts $\mathfrak{c}_{\alpha_\varepsilon}$ in $\mathcal{I}$ such that $p_0 = p$, $p_{\varepsilon+1} \backslash p_\varepsilon$ contains one formula over AI_ε. Also, we ask that each I_ε is indiscernible over $A\mathcal{I}\backslash I_\varepsilon$ (in other words, it witnesses that $\mathfrak{c}_{\alpha_\varepsilon}$ is generic for A).

Suppose p_ε (or any completion of it) is not as we wanted. This means that there is some $b \models p_\varepsilon$ such that bA is not a D-set. Let $\alpha_\varepsilon < \kappa$ be such that $\mathfrak{c}_{\alpha_\varepsilon}$ is a generic cut for bA and such that $\alpha_\varepsilon \notin \{\alpha_\zeta \mid \zeta < \varepsilon\}$, and suppose this is witnessed by $I_\varepsilon = (i_1^\varepsilon, i_2^\varepsilon) \subseteq X_{\alpha_\varepsilon}$. By definition, some finite tuple ba from bA is not a D-set, which means that $\mathrm{tp}(ba/\mathcal{I})$ is not orthogonal to $\lim(\mathfrak{c}_{\alpha_\varepsilon}/\mathcal{I})$. This means that there is some c filling $\mathfrak{c}_{\alpha_\varepsilon}$ such that c does not fill $\mathfrak{c}_{\alpha_\varepsilon}$ in I_ε over $ba \cup \mathcal{I}\backslash I_\varepsilon$.

Hence, there is a formula $\varphi(y_0, \ldots, y_{i-1}, y, y_{i+1}, \ldots, y_k, x, w)$ over $\mathcal{I}\backslash I_\varepsilon$ and some $c_0 < \cdots < c_{i-1} < \mathfrak{c}_{\alpha_\varepsilon} < c_{i+1} < \cdots < c_k$ from I_ε such that $\neg\varphi(c_0, \ldots, c_{i-1}, c, c_{i+1}, \ldots, c_k, b, a)$ holds while for all $d_0 < \cdots < d_k < \mathfrak{c}_{\alpha_\epsilon}$ from I_ε, $\varphi(d_0, \ldots, d_k, b, a)$ holds.

As A is a D-set, by Main Lemma 4.18, c fills $\mathfrak{c}_{\alpha_\varepsilon}$ in I_ε over $A \cup \mathcal{I}\backslash I_\varepsilon$.

Let $c' < c'' \in I_\varepsilon$ be such that $c_{i-1} < c' < c'' < \mathfrak{c}_{\alpha_\varepsilon}$. Then for any formula $\theta \in p_\varepsilon$,

$$\begin{aligned}\mathfrak{C} \models \exists x \theta(x) \wedge \neg\varphi(c_0, \ldots, c_{i-1}, c, c_{i+1}, \ldots, c_k, x, a) \\ \wedge\, \varphi(c_0, \ldots, c_{i-1}, c', c_{i+1}, \ldots, c_k, x, a),\end{aligned}$$

and hence

$$\begin{aligned}\mathfrak{C} \models \exists x \theta(x) \wedge \neg\varphi(c_0, \ldots, c_{i-1}, c'', c_{i+1}, \ldots, c_k, x, a) \\ \wedge\, \varphi(c_0, \ldots, c_{i-1}, c', c_{i+1}, \ldots, c_k, x, a).\end{aligned}$$

[b]Here, we use the term "cuts" in the most general sense, as opposed to our convention so far where cuts had infinite cofinality from both sides.

Put

$$p_{\varepsilon+1} = p_\varepsilon \cup \{\neg\varphi(c_0, \ldots, c_{i-1}, c'', c_{i+1}, \ldots, c_k, x, a)$$
$$\wedge \varphi(c_0, \ldots, c_{i-1}, c', c_{i+1}, \ldots, c_k, x, a)\}.$$

Then $p_{\varepsilon+1}$ is consistent.

Finally, let $p_\infty = \bigcup_{\varepsilon<|T|^+} p_\varepsilon$, and let $b \models p_\infty$. Then we have a sequence of mutually indiscernible sequences $\langle I_\varepsilon \mid \varepsilon < |T|^+\rangle$ over A and some b such that for each $\varepsilon < |T|^+$, I_ε is not indiscernible over b. This means that the dp-rank of $\mathrm{tp}(b/A)$ is $\geq |T|^+$, which implies the independence property. See for example [3, Corollary 2.3].

Hence, we must get stuck somewhere, and we are done. □

Corollary 4.28. *Suppose $A \subseteq B$ are D-sets and $|A| < \kappa$, $|B| \leq \kappa$. Assume that $p(x) \in S(A)$, then there is some $b \models p$ such that Bb is a D-set.*

Proof. Write B as an increasing continuous sequence $\bigcup_{\alpha<\kappa} B_\alpha$, where $|B_\alpha| < \kappa$ and $A = B_0$. Construct an increasing continuous sequence of types $\langle q_\alpha \mid \alpha < \kappa\rangle$ and subsets $\langle I_\alpha \mid \alpha < \kappa\rangle$ of $\mathcal{I}$ such that $p = q_0$, $I_0 = \emptyset$, $q_{\alpha+1} \in S(B_{\alpha+1} \cup I_{\alpha+1})$, $|I_\alpha| \leq |\alpha||T| < \kappa$ for all $\alpha < \kappa$ and if $b \models q_{\alpha+1}$ then $bB_{\alpha+1}$ is a D-set.

For $\alpha = 0$ and limit there is nothing to do. For $\alpha + 1$, first choose $q'_{\alpha+1} \in S(B_{\alpha+1} \cup I_\alpha)$ extending q_α. Apply Lemma 4.27 to get some $J \subseteq \mathcal{I}$ of size $\leq |T|$ and a type $q_{\alpha+1} \in S(B_{\alpha+1} \cup I_\alpha J)$ such that if $b \models q_{\alpha+1}$, then $bB_{\alpha+1}$ is a D-set (which is the same as saying that $bB_{\alpha+1}\mathcal{I}$ is a D-set, see Remark 4.17). Finally, let $I_{\alpha+1} = I_\alpha J$.

When the construction is done, let $q = \bigcup_{\alpha<\kappa} q_\alpha$ and $b \models q$. □

We can finally prove the right to left direction of Theorem 4.10. We wish to construct a κ-saturated model M which is not κ^+-saturated.

Proposition 4.29. *Any D-model $M \supseteq \mathcal{I}$ is not κ^+-saturated.*

Proof. As $\mathcal{I}$ is not distal, the limit type of any cut $\mathfrak{c}$ in it is not orthogonal to any limit type of a another cut. See Remark 4.6. This means that the type $\lim(\mathfrak{c}/\mathcal{I})$ is not a D-type, and hence not realized in M. □

Proof of Theorem 4.10. By Proposition 4.29, it is enough to construct a κ-saturated D-model containing $\mathcal{I}$. We do this in similar way to the one in the proof of Theorem 3.3.

Let $\langle S_\alpha \mid \alpha < \kappa^+\rangle$ be a partition of κ^+ to sets of size κ^+. Construct an increasing continuous sequence of D-sets $\langle A_\alpha \mid \alpha < \kappa^+\rangle$ and sequences of types $\langle \bar{p}_\alpha \mid \alpha < \kappa^+\rangle$ such that:

(1) $|A_\alpha| \leq \kappa$;
(2) $\bar{p}_\alpha$ is an enumeration $\langle p_{\alpha,\beta} \mid \beta \in S_\alpha \backslash \alpha\rangle$ of all complete types over subsets of A_α of size $< \kappa$ (this uses $\kappa^+ = 2^\kappa$);
(3) If $\alpha \in S_\gamma$ and $\gamma \leq \alpha$, then $A_{\alpha+1}$ contains a realization of $p_{\gamma,\alpha}$.

Start with $A_0 = \mathcal{I}$, see Remark 4.17. Step (3) is done by Corollary 4.28. Finally, let $M = \bigcup_{\alpha<\kappa^+} A_\alpha$ and we are done. □

Acknowledgment

The first author would like to thank the Israel Science Foundation for partial support of this research (Grant No. 1533/14). The research leading to these results has received funding from the European Research Council, ERC Grant Agreement n. 338821. No. 1074 on the second author's list of publications. Partially supported by ValCoMo (ANR-13-BS01-0006).

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