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POSITIVE RESULTS IN ABSTRACT MODEL THEORY: A THEORY OF COMPACT LOGICS

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We prove that compactness is equivalent to the amalgamation property, provided the occurrence number of the logic is smaller than the first uncountable measurable cardinal. We also relate compactness to the existence of certain regular ultrafilters related to the logic and develop a general theory of compactness and its consequences. We also prove some combinatorial results of independent interest.

0. Introduction

Abstract model theory, after its promising successes due to Lindstrom, Barwise and Feferman[1], is widely considered to be in a crisis. Indeed, the expectations were high. The theory should give us more information on why which model theoretic properties hold in what logics, should characterize most logics or logic-families as 'natural' or should explain to us why certain properties of logics are rare. But instead, counterexamples emerged refuting many reasonable conjectures.

Many more logics appeared on the scene like Shelah's various compact quantifiers [32, 26], stationary logic first introduced by Shelah [32] and extensively studied in [4, 22], Souslin-logic [10], infinitary propositional connectives [12, 15] and topological logics [36, 23]. Many researchers turned pessimistic: There seemed no hope for positive results. In particular one problem (P1) remained open, whether Craig's Interpolation theorem and compactness characterize first-order logic (cf. [13, 25]). It turned out that though this problem is still unsolved, it did contribute quite a bit to the positive development of abstract model theory. In

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[1] For a bibliography we refer to [25, 26] and especially to [3].

this paper we present a rather satisfying theory of compact logics. Let us trace its development.

In [25] we had proved that (under some strange set theoretic hypothesis) some version of Robinson's Consistency lemma (RCL 1) together with a Feferman–Vaught-type theorem for pairs of structures implies compactness. In the light of (P1) it is still possible that this is an 'empty' theorem, i.e. the hypothesis is only satisfied by first-order logic.

A preprint of D. Mundici [30] drew our attention to a different version of Robinson's Consistency lemma (RCL) and we realized, while proof-reading [25] in spring 1979, that, under the same set-theoretic hypothesis as before, RCL was equivalent to compactness together with Craig's Interpolation theorem.

Unsatisfied by the set-theoretic hypothesis we started to search for improvements, new properties were singled out to be crucial and finally a rather pleasant picture evolved.

Friedman [14] already had observed that compactness implies, that every formula depended essentially only on a finite number of non-logical symbols. This lead us to define the occurrence number $\mathrm{OC}(\boldsymbol{L})$ of a logic $\boldsymbol{L}$ to be smallest cardinal κ such that every $\boldsymbol{L}$-sentence depends on less than κ many non-logical symbols (if it exists). With the help of this one realizes that the amalgamation property (AP) for $\boldsymbol{L}$-embeddings is both a consequence of either compactness or RCL. But assuming the existence of extendible cardinals, there are logics which have AP and are not compact. However, here is a surprise: If there are no uncountable measurable cardinals, no such logic exists: Then compactness is equivalent to AP. In fact, it suffices to assume the logic $\boldsymbol{L}$ has occurrence number $\mathrm{OC}(L) < \mu_0$, the first uncountable measurable cardinal. Another surprise: This result solves an open problem of Malitz and Reinhart [29]: The logics $L_{\omega,\omega}(Q_\kappa)$ with the quantifier 'there exist at least κ many' (κ infinite) do not satisfy AP.

Already from [25] we knew that 'chopping up compactness in (λ, κ)-compactness' was not a good idea. We introduced there the notion of κ-r.c. (κ-relative compact) which we generalize here to $[\kappa, \lambda]$-compactness and denote it $[\kappa, \lambda]$-compactness following the literature in topology [1, 35]. It turns out that this is a very natural notion and we develop a general theory connecting it with certain regular ultrafilter (and ultrapowers over them) introduced by Keisler [18].

More consequences of compactness, such as the upward Lowenheim–Skolem theorem and non-existence of maximal models (in both relativized and non-relativized versions) are examined. Here $[\omega, \omega]$-compactness and again the existence of measurable cardinals play an important role. The latter comes via a theorem of Rabin and Keisler [19] on elementary extensions of complete structures.

The paper is organized as follows: In Section 1 we collect the preliminaries, define the central concepts and discuss their obvious interrelations.

In Section 2 we discuss consequences of compactness and their interrelations. The results are collected in Fig. 1, at the end of Section 2.4.

In Section 3 we develop the general theory of $[\kappa, \lambda]$-compactness and (κ, λ)-regular ultrafilters. We define the class of ultrafilters $\mathbf{UF}(\boldsymbol{L})$ related to a logic $\boldsymbol{L}$ and show that $\boldsymbol{L}$ is $[\kappa, \lambda]$-compact iff some (κ, λ)-regular ultrafilter $\mathbf{D}$ is in $\mathbf{UF}(\boldsymbol{L})$. An ultrafilter $\mathbf{D}$ is related to a logic $\boldsymbol{L}$ if, roughly speaking, some generalization of Los' lemma for ultrapowers over $\mathbf{D}$ holds for $\boldsymbol{L}$. This correspondence theorem is the key to results like $[\kappa^+, \lambda^+]$-compactness implies $[\kappa, \kappa]$-compactness for every κ or $[\mathrm{cf}(\kappa), \mathrm{cf}(\kappa)]$-compactness implies $[\kappa, \kappa]$-compactness.

In Section 4 we introduce the occurrence number $\mathrm{OC}(\boldsymbol{L})$ of a logic $\boldsymbol{L}$. Our main result here is (assume there are no uncountable measurable cardinals): If $\boldsymbol{L}$ is a logic which is $[\kappa, \kappa]$-compact, then $\mathrm{OC}(\boldsymbol{L}) = \omega$. We conclude this section with a combinatorial result of independent interest: We characterize uncountable cardinals bigger than the first measurable in terms of co-continuous functionals.

In Section 5 we prove finally our main theorem: If $\mathrm{OC}(\boldsymbol{L}) < \mu_0$ (the first uncountable measurable cardinal) and $\boldsymbol{L}$ satisfies AP, then $\boldsymbol{L}$ is compact. We also consider the Joint Embedding Property (JEP) and RCL. The main theorem relies on the results from Sections 2, 3 and 4. But behind it is a theorem which can be stated and proved independently of the previous chapter (Theorem 5.3) and for which the main tool in the proof is the construction of 'homogeneous' models of some non-elementary class K. In spirit, though not in detail, this is similar to the method Shelah deals with abstract elementaty classes in [33].

The order of the sections is dictated by their interdependence. Although proofs are rather detailed every section is independently readable (provided one believes the quoted results). In fact the casual reader may want to read Section 5 first and then backtrack for the needed material. For the case of finite occurrence Section 5 is independent of the rest of the paper.

We hope to convince the reader that abstract model theory is still a challenging branch of model theory and logic and that this paper positively supplements the work presented in [25–27].

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In the time between completion and revision of the present paper a book has been prepared under the untiring editorship of J. Barwise and S. Feferman [3] which puts much of what we tried to say in the introduction into a larger perspective. What should be mentioned here is that Chapter 18 of [3] is to some extent based on this paper.

1. Preliminaries

1.1. The framework

We work again the framework of abstract model theory as described in [2] and [28, 25, 26]. In many respects this paper is an expansion and continuation of [25, §6]. We follow closely [2] for notation with one exception: We denote by L vocabularies (called languages by Barwise), by $\boldsymbol{L}$ logics (hence omitting the star in L^* of Barwise) and we use $\boldsymbol{L}(L)$ to denote the class of L-formulas in the logic $\boldsymbol{L}$. Furthermore, we assume in most of the cases, unless otherwise mentioned, that $\boldsymbol{L}(L)$ is a set whenever L is a set. To avoid confusion we nevertheless write $L_{\omega,\omega}$ to denote predicate calculus and to denote the set of first-order L-formulas. But the context will always make clear if we are thinking of the L-formulas or the logic. We use:

$\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{M}$, $\mathbf{N}$ for L-structures,
A, B, C, M, N for their underlying sets,
$L_0, L_1, \ldots$ for vocabularies,
$\kappa, \lambda, \mu, \nu, \ldots$ for cardinals,
$\alpha, \beta, \gamma, \delta, \ldots$ for ordinals.

Other letters are used freely, but their usage should be clear from the context.

1.2. Compactness properties

A logic $\boldsymbol{L}$ is (κ, λ)-*compact* if, given a set Σ of $\boldsymbol{L}$-sentences of cardinality κ such that every $\Sigma_0 \subset \Sigma$ of cardinality $<\lambda$ has a model, then Σ has a model. A logic $\boldsymbol{L}$ is *compact* (*fully compact*) if $\boldsymbol{L}$ is (κ, ω)-compact for every κ. A logic $\boldsymbol{L}$ is $[\kappa, \lambda]$-*compact* if given any set Σ of $\boldsymbol{L}$-sentences and a set Σ_1 of $\boldsymbol{L}$-sentences of cardinality κ such that for every $\Sigma_0 \subset \Sigma_1$ with $\mathrm{card}(\Sigma_0) < \lambda$, $\Sigma_0 \cup \Sigma$ has a model, then $\Sigma_1 \cup \Sigma$ has a model. Here Σ plays the role of the diagram (or a fragment thereof) of a given structure. Note that $[\kappa, \kappa]$-compactness, was called κ-r.c. in [25]. Clearly, $[\kappa, \lambda]$-compactness implies (κ, λ)-compactness, compactness implies $[\kappa, \lambda]$-compactness for every $\kappa \geqslant \lambda$, and some trivial monotonicity properties hold as well. Less trivial consequences may be found in Sections 2 and 3, in particular Theorem 2.8, 3.11 and 3.12.

1.3. Elementary embeddings

Let $\boldsymbol{L}$ be a logic and $\mathbf{A}$, $\mathbf{B}$ be two L-structures. We say that $\mathbf{A}$ and $\mathbf{B}$ are $\boldsymbol{L}(L)$-*equivalent* and write $\mathbf{A} \equiv \mathbf{B}(\boldsymbol{L}(L))$ if for every $\boldsymbol{L}(L)$-sentence φ $\mathbf{A} \models_{\boldsymbol{L}} \varphi$ iff $\mathbf{B} \models_{\boldsymbol{L}} \varphi$. We denote by $\mathrm{Th}_{\boldsymbol{L}(L)}(\mathbf{A})$ the set $\{\varphi \in \boldsymbol{L}(L) : \mathbf{A} \models_{\boldsymbol{L}} \varphi\}$.

We say that $\mathbf{A}$ *is an* $\boldsymbol{L}(L)$-*substructure of* $\mathbf{B}$ and write $\mathbf{A} <_{\boldsymbol{L}(L)} \mathbf{B}$ if $\mathbf{A}$ is a substructure of $\mathbf{B}$ and $\langle \mathbf{A}, a \in A \rangle \equiv \langle \mathbf{B}, a \in A \rangle$ $(\boldsymbol{L}(L_A))$ where L_A is the vocabulary L augmented by names for each element of $\mathbf{A}$. Note that for $\boldsymbol{L} = L_{\infty,\omega}$ $\mathbf{A} <_{\boldsymbol{L}} \mathbf{B}$ implies that $\mathbf{A} = \mathbf{B}$.

We say that a logic $\boldsymbol{L}$ has the *Amalgamation Property* (AP), if whenever $\mathbf{A}$, $\mathbf{B}_1$, $\mathbf{B}_2$ are L-structures such that $\mathbf{A} <_{\boldsymbol{L}} \mathbf{B}_i$ $(i = 1, 2)$, then there is an L-structure $\mathbf{C}$ such that $\mathbf{B}_i <_{\boldsymbol{L}} \mathbf{C}$ amalgamating $\mathbf{A}$.

We say that $\boldsymbol{L}$ has the *Joint Embedding Property* (JEP) if whenever $\mathbf{A}_1$, $\mathbf{A}_2$ are L-structures such that $\mathbf{A}_1 \equiv \mathbf{A}_2(\boldsymbol{L}(L))$, then there is an L-structure B such that $\mathbf{A}_i <_{\boldsymbol{L}(L)} \mathbf{B}$ $(i = 1, 2)$.

Proposition 5.1 gives some easy consequence of these definitions.

1.4. *The Robinson consistency lemma*

Let Σ be a set of $\boldsymbol{L}(L)$-sentences. We say that Σ is *complete* if given two models $\mathbf{A}$, $\mathbf{B}$ of Σ we have $\mathbf{A} \equiv \mathbf{B}(\boldsymbol{L}(L))$. We also say $\boldsymbol{L}(L)$*-complete* if the context is not clear.

Let $\boldsymbol{L}$ be a logic. We say that $\boldsymbol{L}$ satisfies the *Robinson Consistency Lemma* (RCL) if the following holds: Let L_i $(i = 0, 1, 2)$ be vocabularies such that $L_0 = L_1 \cap L_2$, and let Σ_i $(i = 0, 1, 2)$ be sets of $\boldsymbol{L}(L_i)$-sentences. If Σ_0 is complete and $\Sigma_0 \cup \Sigma_1$ and $\Sigma_0 \cup \Sigma_2$ have models, then $\Sigma_0 \cup \Sigma_1 \cup \Sigma_2$ has a model.

We say that $\boldsymbol{L}$ satisfies RCL1 for the case where Σ_1 and Σ_2 are finite.

Proposition 5.1 shows that RCL implies AP. Trivially RCL1 is a consequence of RCL. If $\boldsymbol{L}$ is compact, then RCL is a consequence of Craig's Interpolation theorem (cf. [25]).

1.5. *Models of arbitrary large cardinality*

Let $\boldsymbol{L}$ be a logic. $\boldsymbol{L}$ is said to satisfy the *Upward Lowenheim Skolem theorem* (ULS) if every set of $\boldsymbol{L}$-sentences which has an infinite model, has arbitrary large models.

This is equivalent to the following: Given an infinite L-structure $\mathbf{A}$, there are L-structures $\mathbf{B}$ of arbitrary large cardinality, such that $\mathbf{A} <_{\boldsymbol{L}(L)} \mathbf{B}$.

Note that for many-sorted structures the cardinality is defined as the sum of the cardinalities of its sorts. The logic $\boldsymbol{L}$ is said to satisfy the *Relativized ULS* (RULS) if, given an L-structures $\mathbf{A}$ with distinguished unary predicate $P^{\mathbf{A}}$, $P^{\mathbf{A}}$ infinite, there are L-structures $\mathbf{B}$ with $P^{\mathbf{B}}$ of arbitrary cardinality such that $\mathbf{A} <_{\boldsymbol{L}(L)} \mathbf{B}$. Trivially RULS implies ULS. These notions are studied in Section 2.

1.6. *Maximal models*

Let $\boldsymbol{L}$ be a logic and $\mathbf{A}$ an L-structure. $\mathbf{A}$ is $\boldsymbol{L}(L)$*-maximal* if there is no proper L-extension $\mathbf{B}$ of $\mathbf{A}$.

If $P^{\mathbf{A}}$ is a unary predicate on $\mathbf{A}$, we say that $P^{\mathbf{A}}$ is $\boldsymbol{L}$*-maximal* if for every $\boldsymbol{L}$-extension $\mathbf{B}$ of $\mathbf{A}$ $P^{\mathbf{B}} = P^{\mathbf{A}}$.

We say that $\boldsymbol{L}$ satisfies MAX if there is no infinite $\boldsymbol{L}$-maximal models. $\boldsymbol{L}$ satisfies RMAX, if there are no infinite $\boldsymbol{L}$-maximal predicates in any L-structure.

Trivially ULS implies MAX, RULS implies RMAX and RMAX implies MAX, and all of them are consequences of compactness. Section 2 is devoted to the study of these properties.

Let $\mathbf{A}$ be an L-structure. The *complete expansion* $\mathbf{A}^{\#}$ of $\mathbf{A}$ is the structure with universe A and for every $X \subset A^n$ which is not the interpretation of some n-place relation symbol there is a *new* predicate symbol P_X whose interpretation is X.

Note that in the many-sorted case we do *not* add sorts to form the complete expansion.

We say a structure is *complete* if $\mathbf{A}=\mathbf{A}^{\#}$.

The following is a version of the Rabin–Keisler theorem (cf. [17]) which we present here with an outline of a proof:

Theorem 1.1. *Let* $\mathbf{A}$ *be a complete structure of cardinality* $\lambda<\mu_0$, *where* μ_0 *is the first (countable) measurable cardinal,* P^A *be a countable predicate of* $\mathbf{A}$ *and* $\mathbf{B}$ *a proper* $L_{\omega,\omega}$*-extension of* $\mathbf{A}$. *Then* $P^A \subsetneqq P^B$.

Proof. Let $c\in B-A$ and put $\mathbf{F}=\{X\subset A:\mathbf{B}\models P_X(c)\}$, where P_X is the predicate of $\mathbf{A}$ representing X. Clearly $\mathbf{F}$ is an ultrafilter. Now assume $P^A=P^B$. We want to show that $\mathbf{F}$ is countably complete to conclude that $\mathrm{card}(A)\geqslant\mu_0$. So let $\{X_n:n\in\omega\}$ be a family with $X_n\in\mathbf{F}$ for every $n\in\omega$. Let $P^A=\{a_n:n\in\omega\}$ be an enumeration of P^A. Since $\mathbf{A}$ is complete there is a predicate F^A (with predicate symbol F) such that $F^A\subset P^A\times A$ and $\{a\in A:\mathbf{A}\models F(a_n,a)\}=X_n$ for each $n\in\omega$. Now we use $P^A=P^B$ and conclude that $\mathbf{B}\models\forall x(P(x)\rightarrow F(x,c))$ and hence $\bigcap_{n\in\omega}X_n\in\mathbf{F}$. $\square$

Note that if $\mathrm{card}(\mathbf{A})\geqslant\mu_0$, $\mathrm{card}(P^A)<\mu_0$ and $\mathbf{F}$ is a μ_0-complete ultrafilter on μ_0, then $\prod\mathbf{A}/\mathbf{F}=\mathbf{B}$ is a proper $L_{\omega,\omega}$-elementary extension of $\mathbf{A}$, but $P^A=P^B$.

Corollary 1.2. *Let* L *be a logic and* P^A *a countable* L*-maximal predicate in some structure* $\mathbf{A}$ *of cardinality* $\lambda<\mu_0$. *Then there are arbitrarily large* L*-maximal structures* $\mathbf{B}$ *of cardinality* $<\mu_0$.

The following proposition is easy: (cf. [11]).

Proposition 1.3. *Let* $\mathbf{L}$ *be a logic. The following are equivalent:*

(i) $\mathbf{L}$ *is not* $[\omega,\omega]$*-compact, and*

(ii) *there is a countable* $\mathbf{L}$*-maximal predicate.*

Corollary 1.4. *If* $\mathbf{L}$ *is not* $[\omega,\omega]$*-compact, then there are arbitrary large* $\mathbf{L}$*-maximal structures of cardinality less than* $<\mu_0$.

The reader should compare these results with Lemmas 2.5, 2.6 and Theorem 2.7.

2. Some consequences of compactness and their interrelations

2.1. Characterizing cardinals

Let $\mathbf{L}$ be a logic and λ an infinite cardinal.

Definition. A cardinal λ is *cofinally characterizable* in $\mathbf{L}$ if there is an expansion $\mathbf{A}$ of $\langle\lambda,\in\rangle$ of the form $\mathbf{A}=\langle A, U^A, E^A,\ldots\rangle$ of the vocabulary L such that:

(i) $\langle U^A, E^A\restriction U\rangle\cong\langle\lambda,\in\rangle$, and

(ii) whenever $\mathbf{B}$ is an L-structure and $\mathbf{A} <_{\mathbf{L}(L)} \mathbf{B}$ then $\langle U^A, E^A \restriction U\rangle$ is cofinal in $\langle U^B, E^B \restriction U\rangle$, i.e. for every $b \in U^B$ there is $a \in U^A$ with $\mathbf{B} \models E(b, a)$.

A cardinal λ is *cardinallike characterizable in* $\boldsymbol{L}$ if there is an expansion $\mathbf{A}$ of $\langle \lambda, \in\rangle$ of the form $\mathbf{A} = \langle A, U^A, E^A, \ldots\rangle$ in the vocabulary L such that:

(i) $\langle U^A, E^A \restriction U\rangle \cong \langle \lambda, \in\rangle$, and

(ii) whenever $\mathbf{B}$ is an L-structure and $\mathbf{A} <_{\mathbf{L}(L)} \mathbf{B}$, then $\langle U^B, E^B \restriction U\rangle$ is a λ-like ordering.

We shall say that λ *is cofinally* (*cardinallike*) *characterizable in* $\boldsymbol{L}$ *via* $\mathbf{A}$.

In [25] we have proved:

Proposition 2.1. (i) *$\boldsymbol{L}$ is $[\lambda, \lambda]$-compact, λ regular, iff λ is not cofinally characterizable in $\boldsymbol{L}$.*

(ii) *$\boldsymbol{L}$ is compact iff no infinite λ is cofinally characterizable in $\boldsymbol{L}$.*

Note that (ii) follows immediately from (i).

Proposition 2.2. *ω is cofinally characterizable in $\boldsymbol{L}$ iff ω is cardinallike characterizable in $\boldsymbol{L}$.*

Proof. If ω is cofinally characterizable in $\boldsymbol{L}$, $\boldsymbol{L}$ is not $[\omega, \omega]$-compact, hence, by Proposition 1.3, ω is cardinallike characterizable. The other direction follows from the fact that no proper elementary extension (in $L_{\omega,\omega}$) of $\langle \omega, \in\rangle$ is ω-like. □

Note that $[\lambda, \lambda]$-compactness is essential here. If we want similar theorems for (λ, λ)-compactness, we have to replace the complete theory of $\langle \mathbf{A}, A\rangle$ in $\boldsymbol{L}$ by a theory of cardinality λ.

Definition. λ is *strongly cofinally* (*cardinallike*) *characterizable in* $\boldsymbol{L}$ if λ is cofinally (cardinal-like) characterizable in $\boldsymbol{L}$ via $\mathbf{A}$ such that $A = U^A$.

We shall use strong characterizability in Section 2.3.

2.2. The existence of maximal models

We first characterize the properties RMAX and MAX for a logic $\boldsymbol{L}$:

Theorem 2.3. *$\boldsymbol{L}$ satisfies* RMAX *iff $\boldsymbol{L}$ is $[\omega, \omega]$-compact.*

Proof. Assume for contradiction $\boldsymbol{L}$ is not $[\omega, \omega]$-compact. By Propositions 2.1(ii) and 2.2, ω is cardinallike characterizable via $\mathbf{A}$, for some L-structure $\mathbf{A}$. Clearly then $\mathbf{A}$ has an expansion with a unary predicate $U = \omega$ which is $\boldsymbol{L}$-maximal. So $\boldsymbol{L}$ does not satisfy RMAX.

To prove the other direction let $\mathbf{B}$ be a structure with an infinite $\boldsymbol{L}$-maximal predicate R. Expand $\mathbf{B}$ to $\bar{\mathbf{B}}$ with a new unary predicate $P^B \subset R^B$ and P^B

countable and let $\{B_i : i \in \omega\}$ be an enumeration of the names of elements of P^B. Let Σ be the $\boldsymbol{L}$-diagram of $\bar{\mathbf{B}}$ and $\Sigma_1 = \{c \neq b_i : i \in \omega\} \cup \{P(c)\}$ where c is a new constant symbol not in the language of $\bar{\mathbf{B}}$. By $[\omega, \omega]$-compactness $\Sigma \cup \Sigma_1$ has a model $\mathbf{C}$ which extends $\bar{\mathbf{B}}$ since P^C properly extends P^B, hence it extends R^P. □

To characterize MAX we introduce a new definition:

Definition. Let $\boldsymbol{L}$ be a logic and λ a cardinal. $\boldsymbol{L}$ satisfies MAX(λ) if all $\boldsymbol{L}$-maximal models have cardinality $<\lambda$. Clearly MAX(ω) is the same as MAX.

Proposition 2.4. *If $\boldsymbol{L}$ is $[\lambda, \lambda]$-compact, then $\boldsymbol{L}$ satisfies* MAX(λ).

The proof is similar to the proof of Theorem 2.3 and is left as an exercise.

Lemma 2.5. *Let λ_0 be an infinite cardinal, and let $\boldsymbol{L}$ satisfy* MAX(λ_0). *Then $\boldsymbol{L}$ is $[\chi, \chi]$-compact for some infinite cardinal χ.*

Proof. We prove the contrapositive. Assume, by Proposition 2.1(i), all regular cardinals λ are cofinally characterizable in $\boldsymbol{L}$ via some structure $\mathbf{B}(\lambda)$, which we assume without loss of generality of minimal cardinality $g(\lambda)$. Let μ be the first cardinal such that:

(i) if $\nu < \mu$, ν a cardinal, then $g(\nu) \leqslant \mu$;
(ii) $\lambda_0 \leqslant \mu$;
(iii) $\mathrm{cf}(\mu) = \omega$.

We prove that such a μ exists: Define μ_n inductively by $\mu_0 = \omega$, $\mu_{n+1} = (\sum \{g(\lambda) : \lambda < \mu_n\}) + \mu_n^+$, $\mu = \sum \mu_n$. Note that here we use the replacement axiom. Let $\mathbf{B}$ be the complete expansion of the structure $\langle \mu, \in \rangle$. We claim that $\mathbf{B}$ is maximal. For otherwise, let $\mathbf{C}$ be an $\boldsymbol{L}$-extension of $\mathbf{B}$. If $\mathbf{C}$ is a proper extension, there is $c \in C - B$. Remember $\mathrm{cf}(\mu) = \omega$ and let $\{b_n : n \in \omega\}$ be a cofinal sequence in $\mathbf{B}$. Since ω is cofinally characterizable in $\boldsymbol{L}$, $g(\omega) \leqslant \mu$ and $\mathbf{B}$ is a complete structure $\{b_n : n \in \omega\}$ is also cofinal in $\mathbf{C}$. So $\mathbf{C} \models c \in b_k$ for some $k \in \omega$. Now let $d \in B$ be the smallest (with respect to $\in$) element in $\mathbf{B}$ such that $\mathbf{C} \models c \in d$. So d is an ordinal. Let $\delta = \mathrm{cf}(d)$ and $\{d_i : i < \delta\}$ be a sequence cofinal to d in $\mathbf{B}$. Again since $g(\delta) \leqslant \mu$ and δ is cofinally characterizable in $\boldsymbol{L}$, $\{d_i : i < \delta\}$ is cofinal to d in $\mathbf{C}$. So there is $j < \delta$ with $\mathbf{C} \models c \in d_j$, which contradicts the minimality of d. By our definition of μ, $\lambda_0 \leqslant \mu$, so we proved that $\boldsymbol{L}$ does not satisfy MAX(λ_0). □

Lemma 2.6. *Let $\boldsymbol{L}$ be a logic and let λ_0 be the first cardinal such that $\boldsymbol{L}$ is $[\lambda_0, \lambda_0]$-compact. Then λ_0 is measurable (or $=\omega$).*

Proof. By Proposition 2.1(i) each regular $\lambda < \lambda_0$ is cofinally characterizable in $\boldsymbol{L}$ via a structure $\mathbf{B}(\lambda)$ with γ_λ the cardinality of $\mathbf{B}(\lambda)$. Therefore all ordinals less than λ_0 are not extended. Let μ be defined by

$$\mu = \sup\{\gamma_\lambda : \lambda < \lambda_0\} + \lambda_0^+$$

and let $\mathbf{B}$ be the complete expansion of the structure $\langle \mu, \in \rangle$. By $[\lambda_0]$-compactness $\mathbf{B}$ has an L-elementary extension $\mathbf{C}$ with some $c \in C - B$ and such that $\mathbf{C} \models c \in \lambda_0^B$. Since λ_0 is minimal we have for no $\lambda < \lambda_0$ that $\mathbf{C} \models c \in \lambda^C$. We now define an ultrafilter $\mathbf{F}$ on λ_0 by

$$\mathbf{F} = \{X \subset \lambda_0 : \mathbf{C} \models c \in X\}$$

where X is the name of the set X in $\mathbf{B}$. Clearly $\mathbf{F}$ is an ultrafilter. We propose to show that $\mathbf{F}$ is λ_0-complete. Let $\{X_\alpha : \alpha < \mu < \lambda_0\}$ be any family in $\mathbf{F}$. The function f with $f(\alpha) = X_\alpha$ is a function in $\mathbf{B}$ with name, say, f. Put now $X = \bigcap_{\alpha<\mu} X_\alpha$. So $\mathbf{B} \models \forall x\, (\forall i < \alpha \ \ (x \in f(i) \Rightarrow x \in \bigcap f(i)))$ and $\alpha^B = \alpha$, and therefore $\mathbf{C} \models c \in X$ since f is a function of $\mathbf{C}$ with $f^C \restriction B = f^B$. So $X \in \mathbf{F}$ and therefore $\mathbf{F}$ is μ-complete. An easy induction now gives that λ_0 is measurable. □

We are now ready to characterize MAX.

Theorem 2.7. *Assume there are no uncountable measurable cardinals. Then a logic* $\boldsymbol{L}$ *satisfies* MAX *iff* $\boldsymbol{L}$ *satisfies* RMAX *iff* $\boldsymbol{L}$ *is* $[\omega, \omega]$*-compact.*

Proof. It suffices to show that $\boldsymbol{L}$ satisfies MAX if $\boldsymbol{L}$ is $[\omega, \omega]$-compact, by Theorem 2.3. So one direction is Proposition 2.4 and the other direction is Lemma 2.5 and 2.6. □

From the Lemma 2.6 we get in fact more:

Theorem 2.8. *Assume* ω *is the only measurable cardinal. If* $\boldsymbol{L}$ *is* $[\lambda, \lambda]$*-compact for some* $\lambda \geq \omega$*, then* $\boldsymbol{L}$ *is* $[\omega, \omega]$*-compact.*

Let us end this section with a counter-example.

Example 2.9. Let μ_0 be the first uncountable measurable cardinal and let $Qxy(\varphi(x), \psi(y))$ be a binary quantifier (of type $\langle 1, 1\rangle$) which is defined by $\mathbf{A} \models Qxy(\varphi(x), \psi(y))$ if $\{a \in A : \mathbf{A} \models \varphi(a)\}$ is finite and $\{a \in A : \mathbf{A} \models \psi(a)\}$ is of cardinality bigger than 2^{μ_0}.

Let $\boldsymbol{L}$ be the logic $L_{\omega,\omega}(Q)$. Clearly $\boldsymbol{L}$ does not satisfy RMAX, since there is an expansion of $\langle (2^{\mu_0})^+, \in \rangle$ in which $\langle \omega, \in \rangle$ is $\boldsymbol{L}$-characterized. Now if $\mathbf{A}$ is an L-structure of cardinality $\leq 2^{\mu_0}$, Q is trivially false, so every $L_{\omega,\omega}$-extension of $\mathbf{A}$ of cardinality less than 2^{μ_0} is an $\boldsymbol{L}$-extension. If $\mathrm{card}(\mathbf{A}) > 2^{\mu_0}$, let $\mathbf{F}$ be a μ_0-complete ultrafilter on μ_0 and form $\prod \mathbf{A}/\mathbf{F} = \mathbf{B}$. Since μ_0 is small for $(2^{\mu_0})^+$ $\mathbf{B}$ is a proper $\boldsymbol{L}$-extension of $\mathbf{A}$ where definable sets of cardinality $\leq 2^{\mu_0}$ are preserved. Since $\mathbf{F}$ is μ_0-complete, finiteness is preserved, so $\mathbf{B}$ is an $\boldsymbol{L}$-extension of $\mathbf{A}$, hence $\boldsymbol{L}$ satisfies MAX and is $[\mu_0, \mu_0]$-compact.

With this we get:

Theorem 2.10. *The following are equivalent:*
(i) *For every logic* $\boldsymbol{L}$ MAX *holds iff* RMAX *holds.*
(ii) *There exists an uncountable measurable cardinal.*

Proof. Theorem 2.7 and the example above. □

2.3. The upward Lowenheim–Skolem theorems

Let us first characterize the properties ULS and RULS for a logic $\boldsymbol{L}$.

Theorem 2.11. *Let* $\boldsymbol{L}$ *be a logic. Then*
(i) $\boldsymbol{L}$ *satisfies* RULS *iff no infinite cardinal is cardinallike characterizable in* $\boldsymbol{L}$.
(ii) $\boldsymbol{L}$ *satisfies* ULS *iff no infinite cardinal is strongly cardinallike characterizable in* $\boldsymbol{L}$.

Proof. (i) Clearly, if κ is cardinallike characterizable in $\boldsymbol{L}$ via $\mathbf{A}$, then U^A has cardinality κ and so has U^B for every $\mathbf{B}$ which is an $\boldsymbol{L}$-extension of $\mathbf{A}$. This contradicts RULS. In the other direction let $\mathbf{A}$ be an L-structure with U^A contradicting RULS, i.e. for every $\mathbf{B}$ with $\mathbf{A} <_{\boldsymbol{L}(L)} \mathbf{B}$ $\operatorname{card}(U^B) \leqslant \kappa$ for some cardinal κ. Put

$$S(\mathbf{A}) = \{\mu \leqslant \kappa : \text{there is } \mathbf{B}, \mathbf{A} <_{\boldsymbol{L}(L)} \mathbf{B} \wedge \operatorname{card}(U^B) = \mu\}.$$

If $S(\mathbf{A})$ has a last element κ_0, then κ_0^+ is cardinallike characterizable in $\boldsymbol{L}$ (in some structure $\mathbf{C}$ coding the situation). Otherwise let $\kappa_1 = S(\mathbf{A})$. Then κ_1 is cardinallike characterizable in $\boldsymbol{L}$. The details and the proof of (ii) are left to the reader. □

2.4. Some examples

Example 2.12. Let Q_κ be the quantifier 'there exist at least κ many'. Let $\boldsymbol{L}$ be $L_{\omega,\omega}[Q_\kappa]$ with $\kappa > 2^\omega$. Since ω is small for such a κ, Los's lemma holds for $\boldsymbol{L}$ for any ultra-filter over ω (cf. also Section 3, Example 3.5) so $\boldsymbol{L}$ is $[\omega, \omega]$-compact, hence satisfies, by Theorem 2.7, RMAX. Obviously ULS does not hold for $\boldsymbol{L}$.

Example 2.13. Let $Q^{\mathrm{cf} \geqslant \kappa} xy\varphi(x, y)$ be the quantifier which say that φ is a linear ordering of its domain of cofinality $\geqslant \kappa$, κ a regular cardinal. Let $Qx : yz(\varphi(x), \psi(yz))$ be the quantifier which holds in $\mathbf{A}$ iff $\mathbf{A} \models \neg Q_{\omega_1}\varphi(x)$ and $\mathbf{A} \models Q^{\mathrm{cf} \geqslant (2^{\mu_0})^+}\psi(yz)$ where μ_0 is the first uncountable measurable cardinal. Put $\boldsymbol{L} = L_{\omega,\omega}(Q)$. Obviously $\boldsymbol{L}$ does not satisfy RULS. We want to show that $\boldsymbol{L}$ satisfies ULS. Again, as in Example 2.9, if $\mathbf{A}$ is an L-structure of cardinality less than 2^{μ_0} the quantifier is trivially false on $\mathbf{A}$, hence every $\mathbf{B}$ with $\mathbf{A} <_{L_{\omega,\omega}} \mathbf{B}$, is an $\boldsymbol{L}$-extension of $\mathbf{A}$. So w.l.o.g. $\operatorname{card}(\mathbf{A}) > 2^{\mu_0}$. Let φ be a L-formula in two variables

x, y and parameters from $\mathbf{A}$, $\varphi=\varphi(x, y, \bar{a})$. We define $\mathrm{cf}(\varphi, \mathbf{A})$ to be the cofinality of the linear ordering defined by φ in $\mathbf{A}$. If φ does not define a linear ordering we put $\mathrm{cf}(\varphi, \mathbf{A})=0$. If $\mathrm{cf}(\varphi, \mathbf{A})=\delta$, let $\{c(\varphi, \mathbf{A}, \gamma): \gamma<\delta\}$ be a witnessing cofinal sequence in $\mathbf{A}$. Now let $\mathbf{F}$ be a μ_0-complete ultra-filter on μ_0 and form $\mathbf{B}=\prod \mathbf{A}/\mathbf{F}$.

Claim 1. If $\mathrm{cf}(\varphi, \mathbf{A})<\mu_0$ or $\mathrm{cf}(\varphi, \mathbf{A})>2^{\mu_0}$, then $\delta=\mathrm{cf}(\varphi, \mathbf{A})=\mathrm{cf}(\varphi, \mathbf{B})$ and $\{c(\varphi, \mathbf{A}, \gamma): \gamma<\delta\}$ is cofinal in φ in $\mathbf{B}$.

To prove the claim we use that μ_0 is small for each $\kappa>2^{\mu_0}$ and that $\mathbf{F}$ is μ_0-complete.

Claim 2. If $\mathrm{cf}(\varphi, \mathbf{A}) \leqslant 2^{\mu_0}$, then $\mathrm{cf}(\varphi, \mathbf{B}) \leqslant 2^{\mu_0}$.

Again we use that μ_0 is small for $\kappa>2^{\mu_0}$.

Claim 3. If $\mathrm{cf}(\varphi, \mathbf{A})=\mathrm{cf}(\varphi, \mathbf{B})=\delta$ and $\{c(\varphi, \mathbf{A}, \gamma): \gamma<\delta\}$ is cofinal for φ in $\mathbf{B}$ and $\mathbf{C}=\prod \mathbf{B}/\mathbf{F}$, then $\{c(\varphi, \mathbf{A}, \gamma): \gamma<\delta\}$ is cofinal for φ in $\mathbf{C}$ and hence $\mathrm{cf}(\varphi, \mathbf{C})=\delta$.

Proof. Let $f \in B_{\mu_0}$ and put $X_\gamma=\{i \in \mu_0: \mathbf{B} \vDash \varphi(c(\varphi, \mathbf{A}, \gamma), f(i))\}$ and assume for contradiction that $X_\gamma \in \mathbf{F}$ for each $\gamma<\delta$. Now $\mathbf{B} \vDash \varphi(c(\varphi, \mathbf{A}, \alpha), f(i))$ for some $\alpha<\delta$ since $\{c(\varphi, \mathbf{A}, \gamma): \gamma<\delta\}$ was cofinal for φ in $\mathbf{B}$. Put now $g(i)=c(\varphi, \mathbf{A}, \alpha_i)$ where α_i is the smallest index such that $f(i) \leqslant c(\varphi, \mathbf{A}, \alpha_i)$. This defines a function $g \in A^{\mu_0}$ with $\{i \in \mu_0: \mathbf{A} \vDash \varphi(c(\varphi, \mathbf{A}, \gamma), g(i))\}=X_\gamma$. Since each $X_\gamma \in \mathbf{F}$ we conclude that $\{c(\varphi, \mathbf{A}, \gamma): \gamma<\delta\}$ is not cofinal for φ in $\mathbf{B}$.

Claim 4. $\boldsymbol{L}$ satisfies ULS.

Proof. Clearly we get from Claims 1, 2 that $\mathbf{A} <_{\boldsymbol{L}} \mathbf{B}$. Now let α be any ordinal and put $\mathbf{A}_0=\mathbf{A}$, $\mathbf{A}_{i+1}=\prod \mathbf{A}_i/\mathbf{F}$ and $\mathbf{A}_\delta=\bigcup_{\beta<\delta} \mathbf{A}_\beta$ for $i<\alpha$ and δ limit, $\delta<\alpha$. From Claims 1–3 we get that $\{\mathbf{A}_\beta: \beta<\delta\}$ is an $\boldsymbol{L}$-chain. Choosing α big enough we prove that $\boldsymbol{L}$ satisfies ULS.

Eample 2.14. Now we put $\boldsymbol{L}=L_{\omega,\omega}[Q^{\mathrm{cf} \geqslant \lambda}]$ for each $\lambda \leqslant (\beth_{\omega_1})^+=\kappa$ and ω small for λ, i.e. we add all these quantifiers simultaneously to $L_{\omega,\omega}$. Clearly $\boldsymbol{L}$ is not $[\omega_1, \omega_1]$-compact. We put

$$\Sigma=\{\varphi_1, \varphi_2\} \cup \{\psi_\lambda: \lambda<\beth_{\omega_1}\}$$

with φ_1 says that "φ is a linear ordering", φ_2 is $-Q^{\mathrm{cf} \geqslant \kappa} x y \varphi(x, y)$ and ψ_λ is $Q^{\mathrm{cf} \geqslant \lambda} x y \varphi(x, y)$.

Using similar arguments as in Example 2.13 we prove RULS for $\boldsymbol{L}$. The details are left to the reader.

2.5. *The equivalence of* RULS *and* ULS

Let us discuss here if RULS and ULS could possibly be equivalent, as this is the case for MAX and RMAX if there are no uncountable measurable cardinals (Theorem 2.7). For this we digress a bit to the theory of ultrafilters.

Look at the following assumption, where λ is an infinite cardinal.

$A(\lambda)$: If $\mathbf{F}$ is a uniform ultrafilter on λ, then $\mathbf{F}$ is μ-descendingly incomplete for every $\mu \leqslant \lambda$.

We denote by $A(\infty)$ the statement "for every λ $A(\lambda)$".

Jensen and Koppelberg [16], Magidor [20] and Donder [9] have studied this assumption. The following theorem resumes their results:

Theorem 2.15. (i) (Jensen–Koppelberg). *Assume* $\neg 0^{\#}$. *Then for every regular cardinal* λ *we have* $A(\lambda)$.

(ii) (Donder). *Assume there is no inner model of* ZFC *with an uncountable measurable cardinal. Then* $A(\infty)$.

(iii) *If* $A(\infty)$ *holds, then there are no uncountable measurable cardinals.*

(iv) *Assume there are supercompact cardinals. Then it is consistent with* ZFC *that* $A(\omega_1)$ *fails.*

The assumption $A(\infty)$ is intimately connected with compactness properties, as we shall see in Section 3.

Theorem 2.16. *Assume* $A(\infty)$ *and* $\boldsymbol{L}$ *is a logic. Then* $\boldsymbol{L}$ *satisfies* ULS *iff* $\boldsymbol{L}$ *satisfies* RULS.

To prove Theorem 2.16 we first state two lemmas.

Lemma 2.17. *Let* $\boldsymbol{L}$ *be a logic which does not satisfy* RULS. *Then there is an* L*-structure* $\mathbf{M}=\langle M, P^M, \ldots\rangle$, P^M *infinite, and a unary predicate symbol* P *such that for each* L*-structure* $\mathbf{N}$ *with* $\mathbf{M}<_{\boldsymbol{L}}\mathbf{N}$ *we have* $\mathrm{card}(P^M)=\mathrm{card}(P^N)$.

Proof. Let $\mathbf{M}$ be an L-structure which is a counter example to RULS, i.e. $\mathbf{M}=\langle M, P^M, \ldots\rangle$ and

$$\{\kappa\in\mathrm{Card}: \exists\mathbf{N}(\mathbf{M}<_{\boldsymbol{L}}\mathbf{N})\wedge\mathrm{card}(P^N)=\kappa\}=S_M.$$

is bounded. If S_M has a maximal element there is nothing to prove. So we can assume that $S_M=\{\kappa\in\mathrm{Card}:\kappa<\alpha\}$ for some limit α. We then construct a structure $\mathbf{N}=\langle N, <, P, \ldots\rangle$ such that $S_N=\{\alpha\}$ by coding all S_0 in it. $\square$

Lemma 2.18. *Let* $\boldsymbol{L}$ *be a logic which satisfies* ULS, *and* $\mathbf{M}$ *be an* L*-structure. Then there is an* L*-structure* $\mathbf{N}$, $\mathbf{M}<_{\boldsymbol{L}}\mathbf{N}$ *and in* $\mathbf{N}$ *there is a sequence* $\{a_i : i<\mathrm{card}(\mathbf{M})^+\}$ *of elements such that for each first order-formula* $\varphi(x, y)$ *in two variables and each* $i<j$, $k<l$ *we have that* $\mathbf{N}\models\varphi(a_i, a_j)$ *iff* $\mathbf{N}\models\varphi(a_k, a_l)$.

Sketch of proof. Take $\mathbf{N}$ to be large enough so one can apply a Ramsey-type argument, as they are now standard in model theory, cf. [7].

Proof of Theorem 2.16. Let $\boldsymbol{L}$ a logic which satisfies ULS but not RULS. By the hypothesis we can apply both lemmas.

Fix $\mathbf{M}$ as in Lemma 2.17 and let $\mathbf{M}^{\#}$ be the complete expansion of $\mathbf{M}$ and $\mu=\mathrm{card}(\mathbf{M})^+$. Now let $\mathbf{N}$ and $\{a_i : i<\mu\}$ be as in Lemma 2.18 applied to $\mathbf{M}^{\#}$. We define an ultrafilter $\mathbf{F}_0$ on M. If $X\subset M$ let R_X be the unary predicate symbol of

$\mathbf{M}^{\#}$ whose interpretation is X. Then we put $\mathbf{F}_0 = \{X \subset M : \mathbf{N} \models R_X(a_0)\}$. Clearly $\mathbf{F}_0$ is an ultrafilter. Let $A \in \mathbf{F}_0$ be a set of smallest cardinality and define an ultrafilter $\mathbf{F}$ on A by $\mathbf{F} = \mathbf{F}_0/A$. $\mathbf{F}$ is a uniform ultrafilter on A.

Claim 1. $\text{card}(A) > \text{card}(P^M)$.

Assume $\text{card}(A) \leq \text{card}(P^M)$. Then there is a 1–1-function $f: A \to P^M$. Let F be the function symbol of $\mathbf{M}^{\#}$ representing f and R_A the predicate symbol representing A. Since $A \in \mathbf{F}$ the domain of F in $\mathbf{N}$ contains a_0 and hence all the a_i for $i < \mu$. But since $\mathbf{M}^{\#} <_{\boldsymbol{L}} \mathbf{N}$, F is 1–1 in $\mathbf{N}$, so we conclude that $\text{card}(P^N) \geq \mu$. But by our assumptions (Lemma 2.16) $\text{card}(P^N) = \text{card}(P^M) < \mu$, a contradiction. Now let κ be the smallest cardinal such that $\text{card}(P^N)^\kappa > \text{card}(P^N)$. Put $\nu = \text{card}(P^N)^\kappa$.

Claim 2. $\kappa < \text{card}(A)$.

Trivially $\kappa \leq \text{card}(P^N) = \text{card}(P^M) < \text{card}(A)$ using Claim 1.

Now we apply our hypothesis $A(\lambda)$ for $\lambda = \text{card}(A)$ and conclude with Claim 2 that

Claim 3. $\mathbf{F}$ is not κ-descendingly complete.

Let $\{A_i : i < \kappa\}$ be a decreasing sequence of elements of $\mathbf{F}$ with $\bigcap_{i<\kappa} A_i = \emptyset$. Now, since $\text{card}(P^M)^\kappa > \text{card}(P^M)$, we can find $\bar{c}_i = \{c_{ij} : j < \kappa\}$ for each $i < \nu$ with $c_{ij} \in P^M$ and $c_{ij} \neq c_{lj}$ for each j, $i \neq l$. We now define functions $f_i : A \to P^M$ for $i < \nu$ by $f_i(a) = c_{ij}$ where j is the least index such that $a \notin A_j$. This is well defined, since $\bigcap_{i<\kappa} A_i = \emptyset$. Let F_i be the function symbol representing f_i in $\mathbf{M}^{\#}$. By the choice of the f_i $\mathbf{M}^{\#} \models \forall a (F_i(a) \neq F_j(a))$ for $i \neq j$. Since the domain of F_i is A we have $\mathbf{N} \models P(F_i(a_0))$ and $\mathbf{N} \models F_i(a_0) \neq F_j(a)$ for $i \neq j$. So $\nu \leq \text{card}(P^N)$, a contradiction. $\square$

Fig. 1 gives a synopsis of the situation.

3. Compactness and ultrafilters

3.1. Introduction

In this section we present a general theory of compactness of logics $\boldsymbol{L}$, centering around the notion of $[\lambda, \mu]$-compactness and relating it to (λ, μ)-regular ultrafilters. This section shows clearly that $[\lambda, \mu]$-compactness is a natural notion, in fact much more natural than (λ, μ)-compactness. We reprove, with different methods, most of the results of [25, §6] and improve some of them. For non-defined notions on ultrafilters we refer to the reader to the monograph [8].

In Section 3.2 we collect some facts about ultrafilters. In Section 3.3 we prove our main theorem of this chapter. And in Section 3.4 we apply this to study $[\lambda, \mu]$-compactness.

3.2. (λ, μ)-regular ultrafilters

The following notion was introduced in [18], but his notation is reversed. (The reader should think of the brackets as unordered pairs.)

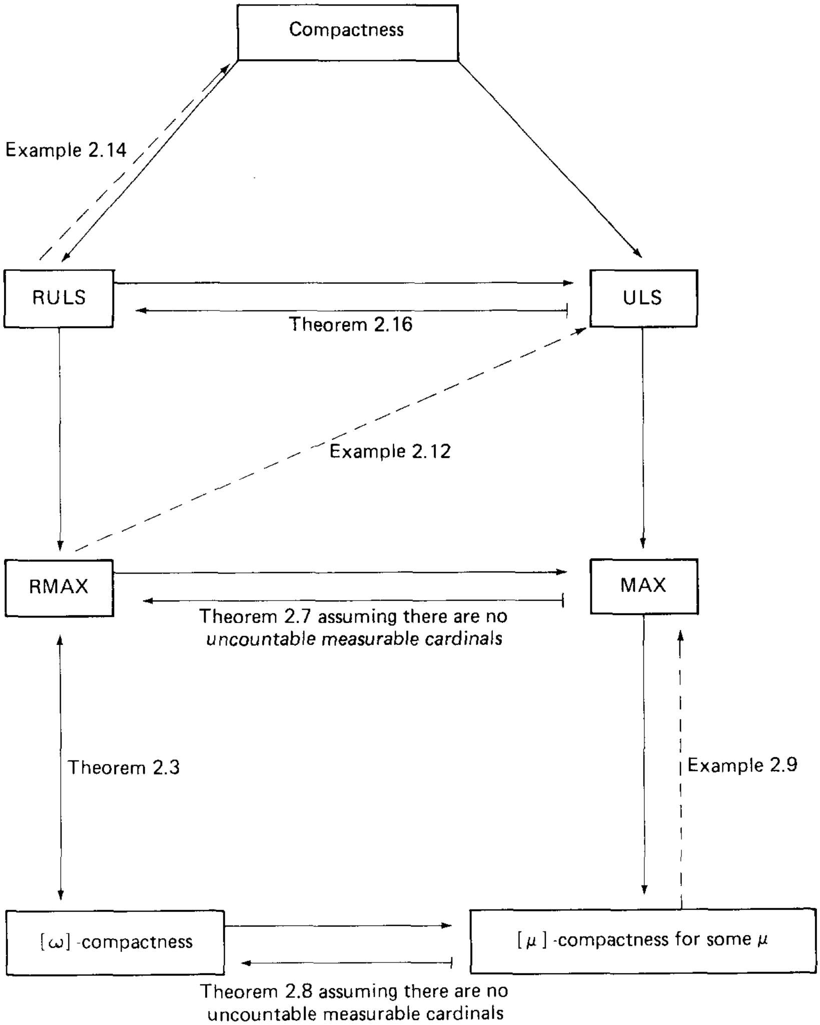

Fig. 1. Drawn lines show implications which hold, broken lines such which do not hold.

Definition. Let **F** be an ultrafilter on I, and λ, μ be regular cardinals with $\lambda \geqslant \mu$.

(i) **F** is said to be (λ, μ)-*regular* if there is a family $\{X_\alpha, \alpha < \lambda\}$, $X_\alpha \in \mathbf{F}$ such that if S is a subset of λ of cardinality μ, then $\bigcap_{\alpha \in S} X_\alpha = \emptyset$. The family $\{X_\alpha, \alpha < S\}$ is called a (λ, μ)-regular family.

(ii) A (λ, ω)-regular ultrafilter on λ is called *regular.*

(iii) **F** is λ-descendingly incomplete if there exists a family $\{X_\alpha : \alpha < \lambda\}$, $X_\alpha \in \mathbf{F}$ with $X_\alpha \subset X_\beta$ for $\alpha < \beta < \lambda$ such that $\bigcap_{\alpha<\lambda} X_\alpha = \emptyset$.

(iv) **F** is *uniform on* λ if every $X \in \mathbf{F}$ has cardinality λ.

Lemma 3.1. (i) *If* **F** *is* (λ, μ)-*regular and* $\mu \leqslant \mu_1 \leqslant \lambda_1 \leqslant \lambda$, *then* **F** *is* (λ_1, μ_1)-*regular.*

(ii) *If* λ *is a regular cardinal and* $\mathbf{F}$ *is* λ*-descendingly incomplete, then* $\mathbf{F}$ *is* (λ, λ)*-regular.*

(iii) *If* $\mathbf{F}$ *is uniform on* λ, *then* $\mathbf{F}$ *is* (λ, λ)*-regular.*

(iv) *If* $\mathbf{F}$ *is* $(\mathrm{cf}(\lambda), \mathrm{cf}(\lambda))$*-regular, then* $\mathbf{F}$ *is* (λ, λ)*-regular.*

Proof. (i) is obvious.

(ii) Let $\{X_\alpha : \alpha < \lambda\}$ be a decreasing family with empty intersection. Since λ is regular $\{X_\alpha : \alpha < \lambda\}$ is a (λ, λ)-regular family.

(iii) Follows from the fact that $\mathbf{F}$ is uniform on λ iff for every $X \subset \lambda$ such that $\mathrm{card}(\lambda - X) < \lambda$ we have that $X \in \mathbf{F}$, cf. [7, Exercise 4.3.2].

(iv) Let $\{X_\alpha : \alpha < \mathrm{cf}(\lambda)\}$ be a $(\mathrm{cf}(\lambda), \mathrm{cf}(\lambda))$-regular family and $\{\beta_i : i < \mathrm{cf}(\lambda)\}$ cofinal in λ. Put now $Y_\xi = X_{\beta_i}$ for every ξ with $\beta_i \leq \xi < \beta_{i+1}$. Then $\{Y_\xi : \xi < \lambda\}$ is a (λ, λ)-regular family. $\square$

This is essentially Lemma 7.10 in [8] and due to [18]. The next lemma is from [17, Corollary 2.4] and [8, Corollary 8.36].

Lemma 3.2. (i) [17]. *If* $\mathbf{F}$ *is uniform on* λ^+ *and* λ *is singular, then* $\mathbf{F}$ *is* (λ^+, λ)*-regular.*

(ii) [31, 6]. *If* $\mathbf{F}$ *is uniform on* λ^+ *and* λ *is regular, then* $\mathbf{F}$ *is* λ*-descendingly incomplete, and hence* (λ, λ)*-regular.*

We now proceed to give a model theoretic characterization of (λ, μ)-regular ultrafilters. Let $H(\lambda)$ denote the set of sets hereditarily of cardinality $< \lambda$, and $\in$ the natural membership relation on $H(\lambda)$.

Lemma 3.3. *For an ultrafilter* $\mathbf{F}$ *on a set* I *the following are equivalent:*

(i) $\mathbf{F}$ *is* (λ, μ)*-regular.*

(ii) *In the structure*

$$\mathbf{N} = \prod_I \langle H(\lambda^+), \in \rangle / \mathbf{F}$$

there is an element $\mathbf{a} = a/\mathbf{F}$ *where* $\mathbf{a} : I \to H(\lambda^+)$ *such that*

$$\mathbf{N} \models \mathbf{a} \subset \lambda^N \;\&\; \mathrm{card}(\mathbf{a}) < \mu^N$$

but for every $\alpha < \lambda$

$$\mathbf{N} \models \alpha^N \in \mathbf{a}^N.$$

Proof. (i) $\to$ (ii). Define $\mathbf{a} : I \to H(\lambda^+)$ by $a(t) = \{\alpha \in \lambda : t \in X_\alpha\}$ for $t \in I$ and $\{X_\alpha : \alpha < \lambda\}$ a (λ, μ)-regular family. Now $X_\alpha = \{t \in I : \alpha \in a(t)\}$, so $\mathbf{N} \models \bar{\alpha} \in \bar{a}$, since for each $\alpha < \lambda$, $X_\alpha \in \mathbf{F}$. But $a(t)$ has cardinality less than μ for each $t \in I$, since $\{X_\alpha : \alpha \in \lambda\}$ is (λ, μ)-regular, so $\mathbf{N} \models$ '$\mathrm{card}(\bar{a}) < \mu$'. Trivially, $\mathbf{N} \models \bar{a} \subset \bar{\lambda}$.

(ii) $\to$ (i). Let $\bar{\mathbf{a}} = a/\mathbf{F}$ be the required element in $\mathbf{N}$. Define a' by: $a'(t) = a(t)$ if $a(t) \in P_{<\mu}(\lambda)$ and $\emptyset$ otherwise.

Obviously $a/\mathbf{F}=a'/\mathbf{F}$ since $\mathbf{N}\models \bar{\mathbf{a}}\in\bar{\lambda}$ and $\mathbf{N}\models\text{card}(\bar{a})<\mu$. We want to construct a (λ,μ)-regular family. We put $X_\alpha=\{t\in I:\alpha\in a'(t)\}$, for each $\alpha<\lambda$. Now suppose that for some $\{\alpha_i:i<\mu\}$, $\bigcap_{i<\mu}X_{\alpha_i}\neq\emptyset$. So there is $t\in I$ such that for each $i<\mu$ $\alpha_i\in a'(t)$, which contradicts the fact that $a'(t)\in P_{<\mu}(\lambda)$. $\square$

Definition. Let $\mathbf{F}_i$ be ultrafilters on I_i $(i=1,2)$. $\mathbf{F}_2$ is a *projection* of $\mathbf{F}_1$ if there is map $f:I_1\to I_2$ which is onto and such that $\mathbf{F}_2=\{X\subset I_2:f^{-1}[X]\in\mathbf{F}_1\}$.

This is also known as the Rudin–Keisler order on ultrafilters.

Lemma 3.4. *If* λ *is regular and* $\mathbf{F}$ *is an* (λ,λ)*-regular ultrafilter on I, then there is a uniform ultrafilter* $\mathbf{F}$ *on* λ *which is a projection of* $\mathbf{F}$.

Proof. We use Lemma 3.3 for $\lambda=\mu$. Let $\mathbf{N}^{\#}$ be the complete expansion of $\mathbf{N}=\prod_I\langle H(\lambda^+),\in\rangle/\mathbf{F}$ and $a:I\to H(\lambda^+)$ as in Lemma 3.3. Put now $b(t)$ to be the supremum of the set $a(t)$, so $b(t)\in\lambda$ since λ is regular and card $(a(t))<\lambda$; moreover $\mathbf{N}\models\bar{\mathbf{a}}\leqslant\bar{\mathbf{b}}$. Thus $b:I\to\lambda$ is well defined. We now define $\mathbf{F}=\{S\subset\lambda:\mathbf{N}^{\#}\models\bar{b}\in\bar{S}\}$ where $\bar{S}$ is the name of S in $\mathbf{N}^{\#}$. Clearly $\mathbf{F}$ is an ultrafilter on λ.

Claim 1. $\mathbf{F}$ is uniform.

Assume for contradiction $S\in\mathbf{F}$, but card$(S)<\lambda$. So S is bounded by some $\alpha_S\in\lambda$ and for every β in $\mathbf{N}$ such that $\mathbf{N}^{\#}\models\bar{\beta}>\bar{\alpha}_S$ we have $\mathbf{N}^{\#}\models\bar{\beta}\notin\bar{S}$. But $\mathbf{N}^{\#}\models\bar{\alpha}_S\in\bar{a}\leqslant\bar{b}$, hence $\mathbf{N}\models\bar{\beta}\notin\bar{S}$ a contradiction.

Claim 2. $\mathbf{F}$ is projection of $\mathbf{F}$ by $b:I\to\lambda$.

Clearly $S\in\mathbf{F}$ iff $\mathbf{N}\models\bar{b}\in S$ iff $\{t\in I:b(t)\in S\}\in\mathbf{F}$ iff $b^{-1}(S)\in\mathbf{F}$. $\square$

Remark. In Lemma 3.3 we can find $\mathbf{F}$ on λ also if $\mu\leqslant\text{cf}(\lambda)$, like in the proof of 3.4, but $\mathbf{F}$ need not be uniform. The function $\bar{\mathbf{a}}$ in Lemma 3.3 is from [18].

3.3. *Ultrafilters and compactness*

The key definition of this section is motivated by Lemma 3.3.

Definition. (i) Let $\boldsymbol{L}$ be a logic and $\mathbf{F}$ be an ultrafilter over I. We say that $\mathbf{F}$ *is related to* $\boldsymbol{L}$ if for every τ and for every τ-structure $\mathbf{A}$ there exists a τ-structure $\mathbf{B}$ extending $\prod_I\mathbf{A}/\mathbf{F}$ such that for every formula $\varphi\in L[\tau]$, $\varphi=\varphi(x_1,x_2,\ldots,x_i,\ldots)_{i<\alpha}$ and every $f_i\in A^I$, $i<\alpha$ we have:

$$\mathbf{B}\models\varphi(f_1/\mathbf{F},f_2/\mathbf{F},\ldots,f_i/\mathbf{F},\ldots)$$

iff

$$\{j\in I:A\models\varphi(f_1(j),f_2(j),\ldots,f_i(j),\ldots)\}\in\mathbf{F}.$$

(ii) We define $\mathbf{UF}(\boldsymbol{L})$ to be the class of ultrafilters $\mathbf{F}$ which are related to $\boldsymbol{L}$.

Remark. Note that $\mathbf{B}$ is always an elementary extension of $\prod_I\mathbf{A}/\mathbf{F}$.

Before we continue let us look at some examples:

Examples 3.5. (i) Every ultrafilter is in $\mathbf{UF}(L_{\omega,\omega})$.

(ii) Let $\boldsymbol{L}$ be $L_{\omega,\omega}(Q_k)$, i.e. first-order logic with the additional quantifier 'there exist at least κ many'. Then every ultrafilter on ω is related to $\mathbf{L}$, provided ω is small for κ. This follows from the ultrapower theorem for $\boldsymbol{L}$ as stated in [5, Chapter 13, Theorem 2.2].

In the above examples $\mathbf{N}=\prod_I M^I/\mathbf{F}$. In general, as we see in the proof of the theorem below, this is not the case.

Proposition 3.6. *$\boldsymbol{L}$ is compact iff every ultrafilter is related to $\boldsymbol{L}$.*

Proof. Let $\mathbf{M}$ be an L-structure and $\mathbf{F}$ an ultrafilter on a set I. For every $f\in M^I$ let c_f be a new constant symbol not in L. Put

$$T=\{\varphi(c_{f_1},c_{f_2},\ldots):\varphi\in\boldsymbol{L}(L)\wedge\{t\in I:M\models\varphi(f_1(t),f_2(t),\ldots)\}\in\mathbf{F}\}$$

Obviously every finite subset of T has a model: We just expand $\mathbf{M}$ appropriately. So let $\mathbf{N}$ be a model of T. Clearly

$$\prod_I \mathbf{M}^I/\mathbf{F}\subset\mathbf{N}$$

and by the definition of T, $\mathbf{N}$ satisfies the requirements for $\mathbf{F}\in\mathbf{UF}(\boldsymbol{L})$. The converse is trivial if we code ultraproducts in ultrapowers of modified structures. $\square$

Example 3.7. There are many compact logics, the simplest being the logics with the cofinality quantifiers, cf. [26] and [32].

Remark. If all ultrafilters are in $\mathbf{UF}(\boldsymbol{L})$ and $\mathbf{N}=\prod\mathbf{A}/\mathbf{F}$ for every $\mathbf{F}\in\mathbf{UF}(\boldsymbol{L})$, then $\boldsymbol{L}\equiv L_{\omega,\omega}$, since, by the Keisler–Shelah ultrapower theorem (cf. [7]) elementarily equivalent structures (in $L_{\omega,\omega}$) have isomorphic ultrapowers, and hence by the isomorphism axiom ($\boldsymbol{L}$ does not distinguish between isomorphic structures) and Proposition 3.6 the claim follows easily.

Lemma 3.8. $\mathbf{UF}(\boldsymbol{L})$ *is closed under projections.*

Proof. Let $f:I\to I'$ be the projection of $\mathbf{F}$ onto $\mathbf{F}'$. Choose g be such that fg is the identity on I'. g induces a map g^* from $\mathbf{A}^{I'}$ into $\mathbf{A}^I$ such that $g^*(h')=h$ for $hg=h'$ and g^* is compatible with filters, since f is a projection. $\square$

We are now in position to state and prove our main theorem, the Abstract Compactness Theorem.

Theorem 3.9 (Abstract Compactness Theorem). *Let $\boldsymbol{L}$ be a logic, λ, μ be cardinals and $\lambda \geqslant \mu$. The following are equivalent:*

(i) *There is (λ, μ)-regular ultrafilter $\mathbf{F}$ on $I = P_{<\mu}(\lambda)$ which is in $\mathbf{UF}(\boldsymbol{L})$.*

(ii) *For every expansion $\mathbf{A}$ of $\mathbf{H}(\lambda^+)$ there is an $\boldsymbol{L}$-extension $\mathbf{B}$ and an element $b \in B$ such that $\mathbf{B} \models \mathrm{card}(b) < \mu^B$ but for every $\alpha < \lambda$ we have $\mathbf{B} \models \alpha^B \in b$.*

(iii) *$\boldsymbol{L}$ is $[\lambda, \mu]$-compact.*

If λ is regular, then the following are equivalent:

(iv) *There is a uniform ultrafilter $\mathbf{F}$ on λ which is in $\mathbf{UF}(\boldsymbol{L})$.*

(v) *$\boldsymbol{L}$ is $[\lambda, \lambda]$-compact.*

In particular we have:

(vi) *If there is a (λ, μ)-regular ultrafilter $\mathbf{F}$ on any set I which is in $\mathbf{UF}(\boldsymbol{L})$, then $\boldsymbol{L}$ is $[\lambda, \mu]$-compact.*

Proof. (i)$\rightarrow$(ii). Let $\mathbf{F}$ be a (λ, μ)-regular ultrafilter in $\mathbf{UF}(\boldsymbol{L})$ and let $\mathbf{M}$ be any expansion of $\langle H(\lambda^+), \in\rangle$. Put $\mathbf{N}_0$ to be the ultrapower $\prod_I \mathbf{M}/\mathbf{F}$ and $\mathbf{N}_1$ the extension of $\mathbf{N}_0$ as required for $\mathbf{F} \in \mathbf{UF}(\boldsymbol{L})$. First we observe that $\mathbf{N}_0 < \mathbf{N}_1(L_{\omega,\omega})$ and, by Lemma 3.3, there is an element a in $\mathbf{N}_0$ with the required properties. But then the same element a has the same properties also in $\mathbf{N}_1$ since $\mathbf{N}_0 < \mathbf{N}_1(L_{\omega,\omega})$. But by the definition of $\mathbf{N}_1, \mathbf{M} < \mathbf{N}_1(\boldsymbol{L})$, so we are done.

(ii)$\rightarrow$(iii). Let Σ, Σ_1 be $\boldsymbol{L}$-sentences satisfying the hypothesis of $[\lambda, \mu]$-compactness. We define an expansion $\mathbf{M}(\Sigma, \Sigma_1)$ of $\langle H(\lambda^+), \in\rangle$ to apply (ii). For this purpose let $\{S_\alpha : \alpha < P_{<\mu}(\lambda)\}$ be an enumeration of all the subsets of Σ_1 of cardinality less then μ, $\mathbf{A}_\alpha$ be a model of $\Sigma \cup S_\alpha$ and $\{c_\alpha : \alpha < \lambda^{<\mu}\}$ an enumeration of all the subsets of λ of cardinality less than μ. Finally we put $\nu = (\sup_\alpha(\mathrm{card}(\mathbf{A}_\alpha))) + \lambda^+)$, and define $\lambda_\alpha = \mathrm{card}(\mathbf{A}_\alpha)$.

We now define $\mathbf{M}(\Sigma, \Sigma_\alpha)$ to be $\langle H(\nu), d_\alpha, E, R, P\rangle_{\alpha<\lambda^+, P\in L}$ such that d_α is the name of $\alpha < \lambda^+$ and E is membership, R is a binary predicate not in L and the range of R is λ. We arrange it such that for each $\alpha < \lambda$ the set $R_\alpha = \{x \in H(\nu) : (\alpha, x) \in R\}$ has cardinality λ_α and such that $\langle R_\alpha, P\rangle_{P\in L} \cong \mathbf{A}_\alpha$. In other words put all the models $\mathbf{A}_\alpha$ into $\mathbf{M}(\Sigma, \Sigma_1)$ in way, that when we now apply (ii) we shall get a model for $\Sigma \cup \Sigma_1$. More precisely, we observe that for each formula $\varphi \in \Sigma$:

(a) $\mathbf{M}(\Sigma, \Sigma_1) \models \mathrm{card}(c) < d_\mu \rightarrow \varphi^{R_c}$

and for each $\beta < \lambda$ and for $\Sigma_1 = \{\varphi_i : i < \lambda\}$ an enumeration of Σ_1 we have

(b) $\mathbf{M}(\Sigma, \Sigma_1) \models (d_\beta E c \wedge \mathrm{card}(c) < d_\mu) \rightarrow \varphi^{R_{c\beta}}$.

Now let $\mathbf{N}$, $a \in N$ be as in the conclusion of (ii) $\prod \mathbf{M}(\Sigma, \Sigma_1)/\mathbf{F}$.

Claim. $\langle R_a, P\rangle_{P\in\boldsymbol{L}} \models \Sigma \cup \Sigma_1$.

This follows from the definition of (a) and (b).

(ii)$\rightarrow$(i). So assume $\boldsymbol{L}$ is $[\lambda, \mu]$-compact but no (λ, μ)-regular ultrafilter on $\lambda^{<\mu}$ $\mathbf{F}$ is related to $\boldsymbol{L}$. So for every such $\mathbf{F}$ there is a $L_{\mathbf{F}}$-structure $\mathbf{A}_{\mathbf{F}}$ exemplifying this.

We now proceed to construct an ultrafilter $\mathbf{F}_0$ on $\lambda^{<\mu}$ which contradicts the choice of the $\mathbf{A_F}$'s. For this we construct first a rich enough structure $\mathbf{M}$ such that

(1) For each $\mathbf{A_F}$ there is a unary predicate $P_{\mathbf{F}}$ in $\mathbf{M}$ with $\langle P_{\mathbf{F}}, P\rangle_{P\in L} \cong \mathbf{A_F}$.

(2) $\mathbf{M}$ is a model of enough set theory to carry out the argument.

(3) $\mathbf{M}$ is an expansion of $\langle H(\lambda^+), \in\rangle$.

Let $\mathbf{M}^{\#}$ be the complete expansion of $\mathbf{M}$ and put $\Sigma = \mathrm{Th}_{L^{\#}}(\mathbf{M}^{\#})$, the first-order theory of $\mathbf{M}^{\#}$ where $L^{\#}$ is the vocabulary of $\mathbf{M}^{\#}$. Furthermore put

$$\Sigma_1 = \{c \subset d_\lambda \wedge \mathrm{card}(c) < d_\mu \wedge d_\alpha \in c : \alpha < \lambda\}.$$

Clearly Σ and Σ_1 satisfy the hypothesis of $[\lambda, \mu]$-compactness using the model $\mathbf{M}^{\#}$. So $\Sigma \cup \Sigma_1$ has a model $\mathbf{N}$. We want to use $\mathbf{A}$ to construct our filter $\mathbf{F}_0$. First we observe that $\mathbf{M}^{\#} <_{\mathbf{L}} \mathbf{N}$. Let a_c be the interpretation of c in $\mathbf{N}$. We define $\mathbf{F}_0$ on $P_{<\mu}(\lambda)$ by $\mathbf{F}_0 = \{R \in P_{<\mu}(\lambda) : \mathbf{N} \models a_c \in R^{\mathbf{N}}\}$. This makes sense, since $\mathbf{M}^{\#}$ is a complete expansion and hence every subset of λ of cardinality $<\mu$ corresponds to a predicate in $\mathbf{M}^{\#}$ (remember $\langle H(\lambda^+), \in\rangle$ is present in $\mathbf{M}^{\#}$).

To complete the proof we have to verify several claims:

Claim 1. $\mathbf{F}_0$ is ultrafilter.

Obvious.

Claim 2. $\mathbf{F}_0$ is (λ, μ)-regular.

Let $X_\alpha = \{t \in P_{<\mu}(\lambda) : \alpha \in t\}$ for $\alpha < \lambda$. Now $X_\alpha \in \mathbf{F}_0$, for if X_α corresponds to R_α, then $\mathbf{N} \models a_c \in R_\alpha$ iff $\mathbf{N} \models d_\alpha \in a_c$, which is true for all $\alpha < \lambda$ by the definition of a_c. Now let $\{X_{\alpha_i} : i < \mu\}$ be a subfamily of the A_α's. Clearly, $\bigcap_{i<\mu} X_{\alpha_i} = \emptyset$, since each t in some X_a has cardinality $<\mu$.

Now consider the product $\prod \mathbf{M}^{\#}/\mathbf{F}_0 = \mathbf{N}_0$. If g is an element of $\mathbf{N}_0$, then g is an $\mathbf{F}_0$-equivalence class of functions $g : P_{<\mu}(\lambda) \to \mathbf{M}^{\#}$. So g corresponds to a function $\mathbf{g}^{\mathbf{M}}$ in $\mathbf{M}^{\#}$ with name $\mathbf{g}$, (since $\mathbf{M}^{\#}$ is the complete expansion) and $a_c \in \mathrm{Dom}(\mathbf{g}^{\mathbf{M}})$. So we define an embedding $f : \mathbf{N}_0 \to \mathbf{N}$ by $g/\mathbf{F}_0 \to_f \mathbf{g}^{\mathbf{N}}(a_c)$.

Claim 3. f is well defined and 1–1.

Let $g/\mathbf{F}_0 = g'/\mathbf{F}_0$. We want to show that this is equivalent to $\mathbf{N} \models \mathbf{g}(a_c) = \mathbf{g}'(a_c)$, iff $Y = \{t \in P_{<\mu}(\lambda) : g(t) = g'(t)\} \in \mathbf{F}_0$. But the latter is true iff $a_c \in Y^{\mathbf{N}}$ which is equivalent to $\mathbf{g}(a_c) = \mathbf{g}'(a_c)$.

So we have shown that f is an embedding of $\mathbf{N}_0$ into $\mathbf{N}$.

Now let $\bar{g} = \{g_i/\mathbf{F}_0 : i < \alpha\}$ be in $\mathbf{N}_0$.

Claim 4. For every L-formula φ we have $\mathbf{N} \models \varphi(\bar{g})$ iff

$$Y = \{t \in P_{<\mu}(\lambda) : \mathbf{M} \models \varphi(g_1(t), g_2(t), \ldots)\} \in \mathbf{F}_0.$$

Now $Y \in \mathbf{F}_0$ iff $Y^{\mathbf{N}}$ contains a_c iff $\mathbf{N} \models \varphi(g_1(a_c), g_2(a_c), \ldots)$.

Now look at $\mathbf{A}_{\mathbf{F}_0}$. By assumption there is no $\mathbf{N}'$ extending $\prod \mathbf{A}_{\mathbf{F}_0}/\mathbf{F}_0$ satisfying Claim 4. But $\langle P_{\mathbf{F}_0}^{\mathbf{N}}, P\rangle_{P\in \mathbf{L}_{\mathbf{F}_0}}$ is such an $\mathbf{N}'$ by construction.

(iv) $\to$ (v). This follows from the previous together with Lemma 3.1(iii).

(v) $\to$ (iv). The proof is similar to (iii) $\to$ (i), but instead of $\mathbf{F}_0$ we construct $\mathbf{F}_1$ on λ by

$$\mathbf{F}_0 = \{R \subset \lambda : R \text{ is an initial segment of } \lambda \text{ and } \mathbf{N} \models a_c \in R^{\mathbf{N}}\}.$$

To get $\mathbf{F}_1$ uniform we use Lemmas 3.4 and 3.8.

(vi) For this we just observe that in the proof of (i) $\to$ (ii) we did not use the special form of the set I, as Lemma 3.3 does not depend on it.

This proves Theorem 3.9. □

Note that, in this proof, expansions may include extensions of the universe.

3.4. The compactness spectrum

Let $\mathrm{INC}(\boldsymbol{L}) = \{\lambda \in \mathrm{card} : \boldsymbol{L}$ is not $[\lambda, \lambda]$-*compact*$\}$ and $\mathrm{RCOMP}(\boldsymbol{L}) = \mathrm{reg} - \mathrm{INC}(\boldsymbol{L})$, where reg is the class of regular cardinals.

Theorem 3.10. $\mathrm{INC}(\boldsymbol{L})$ *is closed under successor* $(\lambda \in \mathrm{INC}(\boldsymbol{L}) \to \lambda^+ \in \mathrm{INC}(\boldsymbol{L}))$ *and cofinality* $(\lambda \in \mathrm{INC}(\boldsymbol{L}) \to \mathrm{cf}(\lambda) \in \mathrm{INC}(\boldsymbol{L}))$.

Proof. *Successor:* Assume for contradiction $\boldsymbol{L}$ is $[\lambda^+, \lambda^+]$-compact. So by 3.9(v) there is uniform ultrafilter $\mathbf{F}$ on λ^+ with $\mathbf{F} \in \mathbf{UF}(\boldsymbol{L})$. Now we have two cases:

λ *regular*: By 3.2(ii) $\mathbf{F}$ is (λ, λ)-regular, hence by 3.9 $\boldsymbol{L}$ is $[\lambda, \lambda]$-compact.

λ *singular*: By Lemma 3.2(i) $\mathbf{F}$ is (λ^+, λ)-regular and by Lemma 3.1(i) (λ, λ)-regular. So we can apply 3.9(vi).

Cofinality: This is Lemma 3.1(iv). □

Theorem 3.11. *Let* λ, μ, ν *be cardinals and* $\boldsymbol{L}$ *a logic. The following are equivalent:*

(i) $\boldsymbol{L}$ *is* $[\mu, \mu]$*-compact for every regular cardinal* $\nu > \mu \geqslant \lambda$.

(ii) $\boldsymbol{L}$ *is* $[\mu, \mu]$*-compact for every cardinal* $\nu > \mu \geqslant \lambda$.

(iii) $\boldsymbol{L}$ *is* $[\mu, \lambda]$*-compact for every cardinal* $\nu > \mu \geqslant \lambda$.

Proof. (i) $\to$ (ii). Assume μ is singular and $\lambda \leqslant \mu < \nu$. By (i) $\boldsymbol{L}$ is $[\mu^+, \mu^+]$-compact. Now by Theorem 3.9(iv) and (v) there is a uniform ultrafilter $\mathbf{F}$ on μ^+ which is in $\mathbf{UF}(\boldsymbol{L})$. So by Lemma 3.2(i) $\mathbf{F}$ is (μ^+, μ)-regular and by 3.9(i) and (ii) $\boldsymbol{L}$ is $[\mu^+, \mu]$-compact and therefore, using the monotonicity properties, $[\mu, \mu]$-compact. The other implications are similar using Lemma 3.1 and Theorem 3.9. □

The last result which we would like to state here concerns the structure of $\mathrm{RCOMP}(\boldsymbol{L})$. The following was proven in [25, Lemma 6.4]:

Lemma 3.12. *Let* $\lambda > \mu$ *be two regular cardinals and* $\boldsymbol{L}$ *be a logic such that* $\lambda \in \mathrm{RCOMP}(\boldsymbol{L})$ *but* $\mu \notin \mathrm{RCOMP}(\boldsymbol{L})$. *Then there is a* μ*-descendingly complete ultrafilter on* λ.

From this we obtain immediately:

Theorem 3.13. *Assume* $A(\infty)$ *holds (cf. Section* 2.5). *Then* $\mathrm{RCOMP}(\boldsymbol{L})$ *is an initial segment of the regular cardinals, i.e.* $\lambda \in \mathrm{RCOMP}(\boldsymbol{L})$ *and* $\mu < \lambda$ *implies that* $\mu \in \mathrm{RCOMP}(\boldsymbol{L})$.

Note that the hypothesis $A(\infty)$ implies that there are no uncountable measurable cardinals (cf. Theorem 2.15(iii)).

4. The occurrence number

4.1. Introduction

This section is entirely devoted to the notion of the occurrence number of a logic $\mathbf{L}$. In Section 4.2 we give the precise definitions and review previous results. We also state the main theeorem (4.3), the Finite Occurrence Theorem. In Section 4.3 we prove a sequence of lemmas (4.5–4.9) to establish a connection between the existence of certain ultrafilters and the occurrence number. Section 4.4 is purely set theoretic, establishing some new results about certain ultrafilters and measurable cardinals. Finally, in Section 4.5 we show that our results are best possible. We study there an example in detail.

It should be noticed that there is a side theme evolving here: The role of the existence of uncountable measurable cardinals in abstract model theory. We have seen this already in Sections 2.2 and 3.4 and we shall see it again in Section 5.2.

4.2. The occurrence number: definition and main theorem

In his unpublished notes Friedman [14] proved the following easy, but fundamental, theorem (cf. [11]):

Theorem 4.1. *Let $\mathbf{L}$ be a logic which is compact. Then each $\mathbf{L}(L)$-sentence φ depends only on finitely many non-logical symbols of L.*

Let us define the notions involved in the theorem precisely: Let $\mathbf{L}$ be a logic, L a vocabulary and φ an $\mathbf{L}(L)$-sentence. Let $L_0 \subset L$. We say that φ *depends on L_0 only*, if for any two L-structures $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{A} \upharpoonright L_0 \cong \mathbf{B} \upharpoonright L_0$ we have $\mathbf{A} \models \varphi$ iff $\mathbf{B} \models \varphi$.

We say that φ *depends on L_0* (*properly*) iff there are L-structures $\mathbf{A}$, $\mathbf{B}$ such that $\mathbf{A} \upharpoonright L - L_0 \cong \mathbf{B} \upharpoonright L - L_0$ but $\mathbf{A} \models \varphi$ and $\mathbf{B} \models \neg\varphi$ and φ *does not depend* on L_0 if it depends on $L - L_0$ only.

We define the *occurrence number* $\mathrm{OC}(\mathbf{L})$ *of a logic* $\mathbf{L}$ to be the smallest cardinal μ such that for every vocabulary L (which is a set) and every $\mathbf{L}(L)$-sentence φ there is a subvocabulary $L_0 \subset L$ of cardinality less than μ such that φ depends on L_0 only.

The occurrence number $\mathrm{OC}(\mathbf{L})$ need not exist. In this case we stipulate $\mathrm{OC}(\mathbf{L}) = \infty$, e.g. $L_{\infty,\omega}$ has no occurrence number. The next example illustrates an other point:

Example 4.1. Let κ be a cardinal and $\mathbf{F}$ an ultrafilter on κ. We define a logic $L_{\mathbf{F},\omega}$ by adding to first order logic $L_{\omega,\omega}$ the following formation rule: If $\{\varphi_i : i < \kappa\}$ are sentences of $L_{\mathbf{F},\omega}$ so is $\bigwedge_{\mathbf{F}}\{\varphi_i : i < \kappa\}$ and for a structure $\mathbf{A}$ we define $\mathbf{A} \models_{L_{\mathbf{F},\omega}} \bigwedge_{\mathbf{F}}\{\varphi_i : i < \kappa\}$ to hold iff $\{i \in \kappa : \mathbf{A} \models \varphi_i\} \in \mathbf{F}$.

Obviously, $\mathrm{OC}(L_{\mathbf{F},\omega}) \leq \kappa^+$, but if $L_0 \subset L$, $\mathrm{card}(L) = \kappa$, $\mathrm{card}(L_0) < \kappa$ and we change in an L-structure $\mathbf{A}$ all the interpretation of symbols in L_0, then validity in

$\mathbf{A}$ for $L_{\mathbf{F},\omega}$ is not changed. In other words a $L_{\mathbf{F},\omega}(L)$-sentence φ depends on $L-L_0$ only, but φ may be chosen such that whenever φ depends on $L_1 \subset L$ only, then $\operatorname{card}(L_1)=\kappa$. Hence the occurrence number is κ^+. We conclude that in $L_{\mathbf{F},\omega}$ there is no semantical version of the occurrence axiom, i.e. given a $L_{\mathbf{F},\omega}(L)$-sentence φ, it is not necessarily true that there is a smallest $L_0 \subset L$ such that φ depends on L_0 only. Nevertheless, if we are concerned with 'physical' occurrence such an L_0 does exist.

Here is a generalization of Theorem 4.1. In Section 4.5 we shall see that Theorem 4.2 is best possible.

Theorem 4.2. *If* $\mathbf{L}$ *is* (λ, κ)*-compact and* $\mathrm{OC}(\mathbf{L}) \leqslant \lambda^+$, *then* $\mathrm{OC}(\mathbf{L}) < \kappa$.

Proof. Let L be a vocabulary, $\kappa \leqslant \operatorname{card}(L) \leqslant \lambda$ and φ be a $\mathbf{L}(L)$-sentence. Let L_1 and L_2 be two disjoint copies of L and denote for each constant symbol c, function symbol f and relation symbol R in L the corresponding symbols by c_i, f_i, R_i $(i=1,2)$ respectively. Put $L_3 = L_1 \cup L_2$. Assume for contradiction that φ depends on at least κ many symbols of L. then for every $L_0 \subset L$ with $\operatorname{card}(L_0) < \kappa$ there are L-structures $\mathbf{A}$, $\mathbf{B}$ such that

$$(1) \qquad \mathbf{A} \models \varphi, \qquad \mathbf{B} \models \neg\varphi \quad \text{and} \quad \mathbf{A} \restriction L_0 \cong \mathbf{B} \restriction L_0.$$

Look at the following set of sentences:

$$\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \{\varphi_1, \neg\varphi_2\}$$

with

$$\Sigma_1 = \{c_1 = c_2 : c \text{ a constant symbol of } L\},$$
$$\Sigma_2 = \{\forall \bar{x}(f_1(\bar{x}) = f_2(\bar{x})) : f \text{ a function symbol of } L\},$$
$$\Sigma_3 = \{\forall \bar{x}(R_1(\bar{x}) \leftrightarrow R_2(\bar{x}) : R \text{ a relation symbol of } L\},$$

and φ_i is the $\mathbf{L}(L_i)$-sentence obtained from φ by substituting L_i for L. Clearly $\operatorname{card}(\Sigma) \leqslant \lambda$. Now let $\Sigma_0 \subset \Sigma$, $\operatorname{card}(\Sigma_0) < \kappa$. By (1) Σ_0 has a model. But any model of Σ would violate the isomorphism axiom. $\square$

Our main theorem in this section is the Finite Occurrence Theorem:

Theorem 4.3 (Finite Occurrence Theorem). *Assume a logic* $\mathbf{L}$ *is* $[\omega, \omega]$*-compact and* $\mathrm{OC}(\mathbf{L}) \leqslant \mu_0$, *where* μ_0 *is the first uncountable measurable cardinal (or* $<\infty$ *if no such cardinal exists). Then* $\mathrm{OC}(\mathbf{L}) = \omega$.

As an immediate corollary from Theorem 2.8 we get

Corollary 4.4. *Assume a logic* $\mathbf{L}$ *is* $[\lambda, \lambda]$*-compact for* $\lambda < \mu_0$ *and* $\mathrm{OC}(\mathbf{L}) \leqslant \mu_0$. *Then* $\mathrm{OC}(\mathbf{L}) = \omega$.

The proof of Theorem 4.3 is rather involved. We need a series of lemmas manipulating the occurrence number and a main lemma on the existence of measurable cardinals.

4.3. The lemmas

Let us fix a $[\lambda, \lambda]$-compact logic $\boldsymbol{L}$, a vocabulary L and a sentence $\varphi \in \boldsymbol{L}(L)$. We want to study subsets of L on which φ does not depend.

Lemma 4.5. (i) *For every* $L_1 \subset L$ *with* $\mathrm{card}(L_1) \leqslant \lambda$ *there is a* $L_0 \subset L$ *with* $\mathrm{card}(L_0) < \lambda$ *such that* φ *does not depend on* $L_1 - L_0$.

(ii) *There is a* $\mu < \lambda$ *such that for every* $L_1 \subset L$ *with* $\mathrm{card}\ (L_1) \leqslant \lambda$ *there is a* $L_0 \subset L_1$ *with* $\mathrm{card}(L_0) \leqslant \mu$ *such that* φ *does not depend on* $L_1 - L_0$.

Proof. (i) For $\mathrm{card}(L) = \lambda$ this follows from Theorem 4.2, otherwise put Σ_{L_1} to be Σ of the proof of Theorem 4.2 and

$$\Sigma_{L-L_1} = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$$

for $L - L_1$ instead of L. Now use $[\lambda, \lambda]$-compactness.

(ii) If λ is a successor this is equivalent to (i). So let λ be a limit cardinal. Assume for contradiction that for every $\mu < \lambda$ there is $L_1^\mu \subset L$ with $\mathrm{card}(L_1^\mu) \leqslant \lambda$ making the lemma false. Put $L_1 = \bigcup_{\mu<\lambda} L_1^\mu$, so $\mathrm{card}(L_1) \leqslant \lambda$. Now by the previous lemma there is $L_0 \subset L_1$ with $\mathrm{card}(L_0) < \lambda$ such that φ does not depend on $L_1 - L_0$. But then for every $\mu < \lambda$ φ does not depend on $L_1^\mu - (L_0 \cap L_1^\mu)$, by the definition of L_1. But this contradicts the definition of L_1^μ. $\square$

The next lemma is one of the three lemmas used in the proof of Theorem 4.3.

Lemma 4.6. *There is a* $L_1 \subset L$ *with* $\mathrm{card}(L_1) < \lambda$ *such that for every* $L_0 \subset L - L_1$ *with* $\mathrm{card}(L_0) \leqslant \lambda$, φ *does not depend on* L_0.

Proof. Let μ be as in Lemma 4.5(ii). We define by induction on $\alpha \leqslant \mu^+$ vocabularies $L^\alpha \subset L$ with $\mathrm{card}(L^\alpha) \leqslant \lambda$:

$$L^0 = \emptyset \quad \text{and} \quad L^\delta = \bigcup_{\beta<\delta} L^\beta \quad \text{if } \delta \text{ is limit.}$$

For $\beta = \alpha + 1$ we first apply Lemma 4.5(ii) to L^α. So there is $L_1^\alpha \subset L^\alpha$ with $\mathrm{card}(L_1^\alpha) \leqslant \mu$ and φ does not depend on $L^\alpha - L_1^\alpha$.

Assume for contradiction that the lemma is false, so in particular it is false for $L_1 = L_1^\alpha$. Hence there is $L_0^\alpha \subset L - L_1^\alpha$ with $\mathrm{card}(L_0^\alpha) \leqslant \lambda$ such that φ depends properly on L_0^α. Clearly $L_1^\alpha \cap L_0^\alpha = \emptyset$. As φ does not depend on $L^\alpha - L_1^\alpha$ w.l.o.g. $L^\alpha - L_1^\alpha = \emptyset$. So we put $L^\beta = L^\alpha \cup L_0^\alpha$. Since $\mu^+ < \lambda$ and $\mathrm{card}(L^{\mu^+}) \leqslant \lambda$, we can apply Lemma 4.5(ii) to L^{μ^+}. Hence there is $L_1^{\mu^+} \subset L^{\mu^+}$ with $\mathrm{card}(L_1^{\mu^+}) \leqslant \mu$ and φ does not depend on $L^{\mu^+} - L_1^{\mu^+}$. But $\mathrm{cf}(\mu^+) > \mu \geqslant \mathrm{card}(L_1^{\mu^+})$ so for some $\alpha < \mu^+$ we have $L_1^{\mu^+} \subset L^\alpha$ and

$$L_0^\alpha \subset L^{\alpha+1} - L^\alpha \subset L^{\mu^+} - L_1^{\mu^+}$$

so φ does not depend on L_0^α, a contradiction. $\square$

The second lemma used in the proof of Theorem 4.3 gives us the connection to ultrafilters. Here we use some material from Section 3.3, in particular the definition of $\mathbf{UF}(\boldsymbol{L})$.

Lemma 4.7. *Let* $\boldsymbol{L}$ *be a logic and* φ *a* $\boldsymbol{L}(L)$*-sentence which depends on* $L_2 \subset L$ *but for each* $L_1 \subset L_2$ *with* $\operatorname{card}(L_1) \leqslant \lambda$, φ *does not depend on* L_1. *Then there is a function* $f : P(L_2) \to \{0, 1\}$ *such that*:

(i) f *is non-constant.*

(ii) *For every* $K_1, K_2 \subset L_1$ *with* $\operatorname{card}(\kappa_1 \Delta K_2) \leqslant \lambda$ *we have* $f(K_1) = f(K_2)$.

(iii) *For every ultrafilter* $\mathbf{F} \in \mathbf{UF}(\boldsymbol{L})$ *(on some* μ*)* f *is* $\mathbf{F}$*-continuous.*

Recall that $\lim_{\mathbf{F}}$ is defined as follows: If $\{i_\alpha : \alpha < \mu\}$ is a sequence, then $\lim_{\mathbf{F}} i_\alpha = i$ iff $\{\alpha < \mu : i_\alpha = i\} \in \mathbf{F}$; if $\{f_\alpha : \alpha < \mu\}$ is a family of functions, then $\lim_{\mathbf{F}} f_\alpha$ is defined pointwise and if $\{K_\alpha : \alpha < \mu\}$ is a family of sets, then $\lim_{\mathbf{F}} K_\alpha$ is defined via the limit of its characteristic functions.

Proof of Lemma 4.7. Let $L_2 = \{P_i : i < \kappa\}$ where P_i are predicate symbols only for notational simplicity. Since φ depends on L_2 there are L_2-structures, $\mathbf{A}$, $\mathbf{B}$ on the same universe A, $\mathbf{A} = \langle A, R_i \rangle_{i<\kappa}$, $\mathbf{B} = \langle A, S_i \rangle_{i<\kappa}$ with $\mathbf{A} \models \varphi$ and $\mathbf{B} \models \neg\varphi$. Let $\mathbf{H}$ be a structure rich enough to contain $H(\kappa^+)$, $\mathbf{A}$ *and* $\mathbf{B}$. Let Q_i $(i < \kappa)$ be new relation symbols with one place more than P_i. We interpret Q_i on $\mathbf{H}$ such that the new variables range over the power set $P(\kappa)$ and put for each $X \subset \kappa$:

$\mathbf{H} \models Q_i(X, \bar{a})$ iff $i \in X$ and $\bar{a} \in R_i$ or i not in X and $\bar{a} \in S_i$. X serves as a new parameter. We put φ^X to be the result of substituting $Q_i(X, -)$ for $P_i(-)$ in φ. We are now ready to define f. For $X \subset \kappa$ we put $f(X)$ to be the truth value of φ^X in $\mathbf{H}$.

Claim 1. f is not constant.

Clear, since φ depends on L_2 and therefore $f(\emptyset) = 0$ and $f(\kappa) = 1$.

Claim 2. If X_1, X_2 and $\operatorname{card}(X_1 \Delta X_2) \leqslant \lambda$, then $f(X_1) = f(X_2)$.

This is true, since φ does not depend on any $L_0 \subset L_1$ with $\operatorname{card}(L_0) \leqslant \lambda$.

Claim 3. f is $\mathbf{F}$-continuous for any $\mathbf{F} \in \mathbf{UF}(\boldsymbol{L})$.

Let $X = \lim_{\mathbf{F}} X_i$ for some ultrafilter $\mathbf{F}$ on μ in $\mathbf{UF}(\boldsymbol{L})$. We have to show that $f(X) = \lim_{\mathbf{F}} f(X_i)$. For this we use the definition of $\mathbf{UF}(\boldsymbol{L})$. Let $\mathbf{N}$ be the extension of $\prod \mathbf{G}/\mathbf{F}$ as required for $\mathbf{F} \in \mathbf{UF}(\boldsymbol{L})$. Since X is an element of $\mathbf{H}$, so it is of $\mathbf{N}$, and $\mathbf{H} \models \varphi^X$ iff $\mathbf{N} \models \varphi^X$, and similarly for each X_i $(i < \mu)$. Put $\bar{X} = (X_0, X_1, \ldots)_{i<\lambda}/\mathbf{F}$. Now we have for each $\alpha \in \kappa$ that

$$(*) \qquad \mathbf{N} \models \alpha \in X \quad \text{iff} \quad \mathbf{N} \models \alpha \in \bar{X},$$

since $X = \lim_{\mathbf{F}} X_i$. Next we observe that

$$(**) \qquad \mathbf{N} \models \varphi^X \quad \text{iff} \quad \mathbf{N} \models \varphi^{\bar{X}}$$

because φ^X is obtained from φ by substituting $Q_i(X,-)$ for $P_i(-)$ and similarly for $\varphi^{\bar{X}}$ and for every $\alpha \in \kappa$ we have $\mathbf{N} \models Q_i(X,-)$ *iff* $\mathbf{N} \models Q_i(\bar{X},-)$ by (*).

Finally we have that

$$(***) \qquad \mathbf{N} \models \varphi^{\bar{X}} \quad \text{iff} \quad \{i \in \mu : \mathbf{H} \models \varphi^{X_i}\} \in \mathbf{F}$$

by the property $\mathbf{F} \in \mathbf{UF}(\boldsymbol{L})$.

Now clearly $\lim_{\mathbf{F}} f(X_i) = 1$ iff $\mathbf{N} \models \varphi^{\bar{X}}$ (by (***)) and $f(X) = 1$ iff $\mathbf{N} \models \varphi^X$ by the definition of f, so (**) gives the claim. □

The third lemma, used in the proof of Theorem 4.3, gives us the connection to measurable cardinals:

Lemma 4.8. *If* $\mathbf{F}$ *is a uniform ultrafilter on* ω *and* $f : P(\kappa) \to \{0, 1\}$ *satisfies* (i)–(iii) *of the previous lemma, then there is an measurable cardinal* μ_0 *such that* $\omega < \mu_0 \leq \kappa$.

Lemma 4.8 is a special case of Proposition 4.9 in the next section. But we are now ready to prove Theorem 4.3.

Proof of Theorem 4.3. Assume $\boldsymbol{L}$ is $[\omega, \omega]$-compact and $\mathrm{OC}(\boldsymbol{L}) > \omega$. Then there is an $\boldsymbol{L}(L)$-sentence φ which does not depend only on a finite subset of L. So $\mathrm{card}(L) \geq \omega$, and if $\mathrm{card}(L) = \omega$ we are done by Theorem 4.2. So $\mathrm{card}(L) > \omega$. By Lemma 4.6 (for $\lambda = \omega$) we can assume that φ does not depend on any countable subset of L. Now we apply Lemma 4.7 to construct the function f and by 3.9 and 4.6 we know that f is $\mathbf{F}$-continuous for some uniform ultrafilter on ω. So by Lemma 4.8 we know that $\mathrm{card}(L) \geq \mu_0$, the first uncountable measurable cardinal. But this shows that $\mathrm{OC}(\boldsymbol{L}) \geq \mu_0$. □

4.4. *Some set theory*

Let us now prove the needed facts about measurable cardinals:

Proposition 4.9. *Let* I *be a set and* $S \cup T = P(I)$ *a partition of the power of set* I *such that*

(i) $\phi \in S$, $I \in T$.

(ii) *If* $A \in S$ *and* $\mathrm{card}(A \,\Delta\, A') \leq \omega$, *then* $A' \in S$.

(iii) S *and* T *are closed under* ω-*limits* (*monotone unions and intersections*).

Then $\mathrm{card}(I)$ *is bigger than the first uncountable measurable cardinal.*

Proof of Lemma 4.8. f gives rise to a partition by $f(A) = 0$ iff $A \in S$ and $f(A) = 1$ iff $A \in T$. Now (i) and (ii) of the hypothesis of Proposition 4.9 are trivially satisfied by the hypothesis on f, and (iii) is weaker then $\mathbf{F}$-continuity. So the result follows. □

Proof of Proposition 4.9. *Claim* 1. There is $A^* \in T$ such that for each $B \subset C \subset A^*$ $C \in S$ implies that $B \in S$.

For assume we have $\{A_n : n \in \omega\}$ such that $A_{2n} \in T$ and $A_{2n+1} \in S$ and $A_{n+1} \subset A_n$. If the sequence is proper, we get a contradiction since by (iii) $\bigcap_{n\in\omega} A_{2n} \in T$ and $\bigcap_{n\in\omega} A_{2n+1} \in S$ but $\bigcap_{n\in\omega} A_{2n} = \bigcap_{n\in\omega} A_{2n+1}$. So for some $n \in \omega$ the sequence stops. Let A^* be the last $A_{2n} \in T$. Obviously A^* has the required property.

Now, w.l.o.g. we can assume that $A^* = I$, for if we put

$$S^* = \{A \subseteq A^*; A \in S\}, \qquad T^* = P(A^*) - S^*,$$

properties (i)–(iii) are preserved.

Now let $W \subset S$ be a maximal ideal. Clearly W is closed under countable unions, by (iii) and the maximality of W. We now need a sublemma:

Sublemma 4.10. *If there is no measurable cardinal* μ, $\omega < \mu \leqslant \operatorname{card}(I)$, *then there are* $\{B_n \subset I : n \in \omega\}$, $\{C_n \subset I : n \in \omega\}$ *such that*

(i) $B_n \in P(I) - W$, $C_n \in W$,
(ii) $B_{n+1} \subset B_n$, $C_n \subset C_{n+1}$,
(iii) $B_n \cup C_n \in T$, $B_{n+1} \cup C_n \in S$.

Proposition 4.9 now follows immediately, for if there is no such measurable cardinal, then $A = \bigcap (B_n \cup C_n) = \bigcap (B_{n+1} \cup C_n)$ so $A \in S \cap T = \emptyset$, a contradiction. □

Proof of Sublemma 4.10. We define B_n, C_n $(n \in \omega)$ by induction on $n \in \omega$. For $n = 0$ we put $B_0 = I$, $C_0 = \emptyset$.

For $n+1$ we first define B_{n+1}. Assume no B_{n+1} exists satisfying conditions (i)–(iii). Then for all $X \subset B_n$ we have

$$(*) \qquad \begin{cases} \text{if } X \notin W, \text{ then } X \cup C_n \in T, \\ \text{if } X \in W, \text{ then } X \cup C_n \in S. \end{cases}$$

(Here we use that W is an ideal and $W \subseteq S$.)

Claim. 1. The quotient space $P(B_n)/[P(B_n) \cap W]$ is an infinite boolean algebra.

For otherwise we get a countably complete ultrafilter on a subset of I, contrary to our assumptions.

So by Claim 1, there are $D_k \subset B_n$ $(k \in \omega)$, pairwise disjoint, $D_k \notin W$. Now put $E_k = \bigcup_{k \leqslant l < \omega} D_l$, so $C_n = \bigcap_{k<\omega} (E_k \cup C_n)$, hence, as $C_n \in S$ and T is closed under ω-limits, for some $k_0 \in \omega$ $E_{k_0} \cup C_n \in S$. But $E_{k_0} \notin W$, so by $(*)$ $E_{k_0} \cup C_n \in T$, a contradiction. We conclude that there is B_{n+1} satisfying (i)–(iii), in fact, we put $B_{n+1} = E_{k_0}$.

Now we define C_{n+1}. Since W is a maximal ideal in S, there is $E \in W$ and $D \subset B_{n+1}$ such that $D \cup E \in T$. So we put $C_{n+1} = C_n \cup E$. Clearly the conditions (i)–(iii) are satisfied. □

We finish this section with three propositions which improve some of our lemmas and may be of independent interest.

Proposition 4.11. *Let I be a set,* $\mathrm{card}(I) < \mu_0$, *the first uncountable measurable cardinal, $f: P(I) \to \{0, 1\}$ a map which is nontrivial and commutes with ω-limits. Then there is a finite $J \subset I$ such that for all $A \subset I$ $f(A) = f(A \cap J)$.*

Proposition 4.12. *If f is as in Proposition* 4.11, *then f is* **F***-continuous for any ultrafilter* **F** *on $\alpha < \mu_0$.*

Proposition 4.13. *If $f: P(I) \to \{0, 1\}$ is* **F***-continuous for an ultrafilter* **F** *on κ,* **F** *κ-complete (hence κ is measurable). Then there is $\alpha < \kappa$ and a partition I_i $(i < \alpha)$ of I and there are ultrafilters $\mathbf{F}_i$ on I_i $(i < \alpha)$ such that*

(i) *$f(A)$ depends only on the truth value of $A \cap I_i \in \mathbf{F}_i$ $(i < \alpha)$.*

(ii) *The characteristic function of $\mathbf{F}_i$ on $P(I_i)$ is* **F***-continuous for all $i < \alpha$.*

Remark. Proposition 4.11 is proved in a similar way as the lemmas in Section 4.3. Proposition 4.12 follows easily from 4.11: Let **F** be an ultrafilter on $\alpha < \mu_0$ and A_i $(i < \alpha)$ be a monotone sequence of subsets of I. We define an equivalence relation E on I by $(x, y) \in E$ iff for all $i < \alpha$ $x \in A_i \Leftrightarrow y \in A_i$. Now apply 4.11 to I/E.

Proposition 4.13 is similar to 4.9, but uses more machinery from combinatorics (Ramsey's theorem).

4.5. *An example*

In this section we return to Example 4.1 in Section 4.2. But we assume that each $L_{\mathbf{F},\omega}$-formula has less then μ free variables.

Theorem 4.14. *Let* **F** *be an μ-complete ultrafilter on μ. Then $L_{\mathbf{F},\omega}$ is $[\lambda, \lambda]$-compact for every $\lambda < \mu$.*

To prove 4.14 we first prove a lemma:

Lemma 4.15. *Let* **D** *be any ultrafilter on $\lambda < \mu$ and $\{\mathbf{A}_i : i < \lambda\}$ a family of L-structures. Let φ be a $L_{\mathbf{F},\omega}(L)$-formula and f_j $(j \leqslant \nu < \mu)$ be functions in $\prod A_i$. Then the following are equivalent:*

(i) $\prod A_i/\mathbf{D} \models \varphi[f_1, f_2, \ldots, f_j, \ldots]_{j \leqslant \nu}$,

(ii) $\{i \in \lambda : \mathbf{A}_i \models \varphi[f_1(i), f_2(i), \ldots, f_j(i), \ldots]_{j \leqslant \nu}\} \in \mathbf{D}$.

Proof. This is like the proof for first-order logic by induction over the complexity of φ. Here we only show the case for $\varphi = \bigwedge_{\mathbf{F}} \psi_j$. We use $[\bar{f}]$ and $[\bar{f}(i)]$ as the obvious abbreviations.

Assume (ii), so

$$X_\varphi = \{i<\lambda : \mathbf{A}_i \models \varphi[\bar{f}(i)]\} \in \mathbf{D}$$

Put

$$F_i = \{j<\mu : \mathbf{A}_i \models \psi_j[\bar{f}(i)]\}$$

So, for each $i \in X_\varphi$ we have $F_i \in \mathbf{F}$ by the definition of φ, hence $F = \bigcap_{i\in X_\varphi} F_i \in \mathbf{F}$ since $\mathbf{F}$ is μ-complete and $\lambda<\mu$. So for each $j \in F, \prod \mathbf{A}_i/\mathbf{D} \models \psi_j(\bar{f})$ by the induction hypothesis and since $F \in \mathbf{F}$ we have $\prod \mathbf{A}_i/\mathbf{D} \models \varphi_j[\bar{f}]$, which proves (i). Now assume (ii) is false, hence $X_\varphi \notin \mathbf{D}$ and, since $\mathbf{D}$ is an ultrafilter, $\lambda - X_\varphi \in \mathbf{D}$. But

$$\lambda - X_\varphi = X_{\neg\varphi} = \{i<\lambda : \mathbf{A}_i \models \neg\varphi[\bar{f}]\}.$$

Put

$$\bar{F}_i = \{j<\mu : \mathbf{A}_i \models \neg\psi[\bar{f}(i)]\}.$$

Since $\mathbf{F}$ is an ultrafilter, for each $i \in X_\varphi$ $\bar{F}_i \in \mathbf{F}$ and therefore $\bar{F} = \bigcap_{i\in X_{\neg\varphi}} \bar{F}_i \in \mathbf{F}$. So for each $j \in \bar{F}$, $\prod \mathbf{A}_i/\mathbf{D} \models \psi_i[\bar{f}]$ by the induction hypothesis, hence $\prod \mathbf{A}_i/\mathbf{D} \models \varphi[\bar{f}]$ since $\bar{F} \in \mathbf{F}$ and $\mathbf{F}$ is an ultrafilter. This proves that (i) is false. □

To prove Theorem 4.14 we now apply Theorem 3.9. □

Theorem 4.14 shows that neither Theorem 4.2 nor Theorem 4.3 can be improved.

5. Amalgamation and compactness

5.1. An easy theorem

In this section we prove a few simple propositions and discuss some examples. The definitions were stated in Sections 1 and 2.

Proposition 5.1. (i) *Let* $\mathbf{L}$ *be a logic. Then* (A)$\to$(B), (B)$\to$(C) *and* (D)$\to$(B), *where*

(A) $\mathbf{L}$ *is compact*;
(B) $\mathbf{L}$ *has* JEP;
(C) $\mathbf{L}$ *has* AP;
(D) $\mathbf{L}$ *has* RCL.

(ii) (Mundici [30]) (B) *implies that the class of cardinals* λ *such that* $\mathbf{L}$ *is not* $[\lambda,\lambda]$-*compact is a set. More precisely, if for every countable vocabulary* L card$(\mathbf{L}(L)) \leqslant \lambda_0$, *then there are at most* 2^{λ_0} *such cardinals.*

(iii) *If we assume* $A(\infty)$, *as defined in Section* 2.5, *then* (B) *implies compactness.*

Proof. (A)$\to$(B). Let $\mathbf{L}$ be compact and $\mathbf{B}$ and $\mathbf{C}$ be two L-structures with $\mathbf{B} \equiv \mathbf{C}(\mathbf{L}(L))$. Let us assume that $B \cap C = \emptyset$ and that $L_B \cap L_C = L$. Put $T = \mathrm{Th}_{\mathbf{L}}(\mathbf{B}, B) \cup \mathrm{Th}_{\mathbf{L}}(\mathbf{C}, C)$. It suffices to show that T has a model, so by compactness that every finite subset $T_0 \subset T$ has a model. By Theorem 4.2 every $\boldsymbol{L}$-formula depends only on some finite subset of $L_B \cup L_C$, in particular only on some finitely many constants $b_1, \ldots, b_n, c_1, \ldots, c_m$ from $B \cup C$. W.l.o.g. T_0 is $\varphi(\bar{b}) \wedge \psi(\bar{c})$. But $\mathbf{B} \models \varphi(\bar{b})$ and $\mathbf{C} \models \psi(\bar{c})$, so, since $\mathbf{B} \equiv \mathbf{C}(\boldsymbol{L})$, $\mathbf{B} \models \exists \bar{x}\bar{y}(\varphi(\bar{x}) \wedge \psi(\bar{y}))$, hence $\mathbf{B}$ can be expanded to a model of T_0. Note that we used closure under finite conjunction and existential quantification of $\boldsymbol{L}$.

(B) $\rightarrow$ (C). Let $\mathbf{A} < \mathbf{B}(\boldsymbol{L})$, $\mathbf{A} < \mathbf{C}(\boldsymbol{L})$ be three L-structures. By our hypothesis $\langle \mathbf{A}, A\rangle \equiv \langle \mathbf{B}, A\rangle \equiv \langle \mathbf{C}, A\rangle(\boldsymbol{L}(L_A))$. So we can apply JEP to $\langle \mathbf{B}, A\rangle$ and $\langle \mathbf{C}, A\rangle$.

(D) $\rightarrow$ (B). For $\mathbf{A} \equiv \mathbf{B}(\boldsymbol{L}(L))$ put $T = \mathrm{Th}_{\boldsymbol{L}(L)}(\mathbf{A}) = \mathrm{Th}_{\boldsymbol{L}(L)}(\mathbf{B})$ and $T_1 = \mathrm{Th}_{\boldsymbol{L}(L_A)}(\mathbf{A}, A)$, $T_2 = \mathrm{Th}_{\boldsymbol{L}(L_B)}(\mathbf{B}, B)$ with $L_A \cap L_B = L$. By RCL $T_1 \cup T_2$ has a model which proves JEP. This ends the proof of (i).

(ii) We use Proposition 2.1. First we observe that if $\mathbf{M}(\lambda)$ and $\mathbf{M}(\mu)$ are two L-structures characterizing cofinally λ and μ, then there is no joint embedding. Next we observe that, w.l.o.g. we can assume that there is a countable vocabulary L such that, whenever $\boldsymbol{L}$ is not $[\lambda, \lambda]$-compact, λ regular, then λ is cofinally characterizable in some L-structure $\mathbf{M}(\lambda)$. We simply use constants to code many n-ary relation symbols by one $(n+1)$-ary relation symbol. To finish the proof we observe that there are only 2^{λ_0} many L-theories in $\boldsymbol{L}(L)$.

(iii) This now follows from (ii) and Theorem 3.13. $\square$

Note that (ii) and, since $A(\infty)$ implies that there are no uncountable measurable cardinals, (iii) follow immediately from our Theorem 5.2 below.

5.2. *The main theorem*

Our main theorem is:

Theorem 5.2. *Let* $\boldsymbol{L}$ *be a logic with* $\mathrm{OC}(\boldsymbol{L}) = \lambda_0 < \mu_0$ *(where* μ_0 *is the first uncountable measurable cardinal), satisfying* AP. *Then* $\boldsymbol{L}$ *is compact. In particular, if there are no uncountable measurable cardinals, and* $\boldsymbol{L}$ *is a logic with some occurrence number* $\mathrm{OC}(\boldsymbol{L}) = \lambda_0$, *satisfying* AP, *then* $\boldsymbol{L}$ *is compact.*

The theorem will follow from the apparently weaker Theorem 5.3, which we prove in the next section. For a logic $\boldsymbol{L}$ with finite occurrence, i.e. $\mathrm{OC}(\boldsymbol{L}) = \omega$, compactness follows without Theorems 2.8 and 4.3.

Theorem 5.3. *Let* $\boldsymbol{L}$ *be a logic with* $\mathrm{OC}(\boldsymbol{L}) \leq \lambda$, λ *regular, satisfying* AP. *Then* $\boldsymbol{L}$ *is* $[\lambda, \lambda]$*-compact.*

Proof of Theorem 5.2. By Theorem 5.3, $\boldsymbol{L}$ is $[\lambda_0, \lambda_0]$-compact. Since there are no uncountable measurable cardinals, by Theorem 2.8, $\boldsymbol{L}$ is $[\omega, \omega]$-compact, so by Theorem 4.3 $\mathrm{OC}(\boldsymbol{L}) = \omega$. Now we apply Theorem 5.3 again to get that $\boldsymbol{L}$ is $[\lambda, \lambda]$-compact for every regular $\lambda \geq \omega$, hence, by Theorem 3.12, $\boldsymbol{L}$ is compact. $\square$

Corollary 5.4. *Assume there are no uncountable measurable cardinals and* $\mathbf{L}$ *satisfies* RCL *and* OC($\mathbf{L}$) *exists. Then* $\mathbf{L}$ *is compact.*

This improves Theorem 6.4 of [25] and Theorem 5.1(iii).

Corollary 5.5. *The logics* $L^\kappa = L_{\omega,\omega}(Q_\kappa)$ *with the quantifier 'there exists at least* κ *many',* κ *infinite, do not satisfy* AP.

Proof. L^κ is not $[\kappa, \kappa]$-compact. □

This solves Problem 1 in [29, p. 155].

Incidentally, Problem 3 [ibid] was already solved in [28], which gives another example of how abstract model theory can be used to solve concrete problems: There are logics $\mathbf{L}$ which are (ω, ω)-compact, not (ω_1, ω)-compact and such that L^{ω_1} is not contained in $\mathbf{L}$. Let Qxy any binary quantifier definable in $\Delta(L^{\omega_1})$ but not in L^{ω_1}, which is (ω_1, ω)-incompact, e.g. $Qxy\varphi(xy)$ which says: 'φ is a linear ω-like ordering'. $\mathbf{L} = L_{\omega,\omega}(Q)$ is (ω, ω)-compact, since $\Delta(L^{\omega_1})$ is (ω, ω)-compact.

Problem 2 [ibid] asks whether in the counterexample to AP for L^κ all the structures can be chosen to be isomorphic. As will follow from the proof in the next section, the answer is yes.

Let us conclude this section with three examples which show that some assumptions on the occurrency number OC($\mathbf{L}$) are necessary for Theorem 5.2.

Example 5.6. Look at the logic $L_{\infty,\infty}$, which has no occurrence number. Since in $L_{\infty,\infty}$ every complete theory is categorical, one easily verifies that $L_{\infty,\infty}$ satisfies RCL, hence has AP. But $L_{\infty,\infty}$ is not $[\lambda, \lambda]$-compact for any λ.

Example 5.7. Let $L^2_{\kappa,\lambda}$ be the infinitary logic with conjunctions of length $<\kappa$ and both existential quantification of elements and relations of length $<\lambda$. Magidor [21] has shown that $L^2_{\kappa,\lambda}$ is (∞, κ)-compact iff κ is extendible. Obviously the occurrence number $\mathrm{OC}(L^2_{\kappa,\lambda}) = \max(\kappa, \lambda)$.

Proposition 5.8. *Assume* $L^2_{\kappa,\kappa}$ *is* (∞, κ)*-compact. Then* $L^2_{\kappa,\kappa}$ *satisfies* RCL.

Proof. Let T_i $(i = 0, 1, 2)$ be theories satisfying the hypothesis of RCL. We have to show that every subset $S \subset T = \bigcup_i T_i$ of cardinality less than κ has a model. But such and S is equivalent to a single formula and the predicates and constants not in T_0 can be quantified away. So any model of T_0 can be expanded to satisfy S since T_0 is complete. □

Proposition 5.9. *Let* κ *be extendible. Then* $L^2_{\kappa,\kappa}$ *is* (∞, κ)*-compact.*

Proof. By [21] $L^2_{\kappa,\omega}$ is (∞, κ)-compact. We propose to show that every $L^2_{\kappa,\kappa}$-formula is equivalent to an $L^2_{\kappa,\omega}$-formula using additional predicates. The idea is to replace an existential quantification over less than κ variables (or relation symbols) by a single quantification over finitely many relation symbols coding the κ many. For this we need that in $L^2_{\kappa,\omega}$ the power set operation is absolute and that we have conjunctions over less than κ many formulas. Also, every ordinal $<\kappa$ is describable in $L^2_{\kappa,\omega}$ by a single sentence, which will give us the parameters. The details are left to the reader. From this (∞, ω)-compactness follows immediately. □

Example 5.7 shows that some large cardinals assumption on the occurrence number of L is needed in Theorem 5.2.

Unfortunately extendible cardinals are even bigger than the first supercompact cardinal (cf. [21]) but Vopenka's principle implies that extendible cardinals are stationary. For more results related to compactness of infinitary logics the reader may consult Stavi [34] and [24]. In the latter it is shown that Vopenka's principle is equivalent to the assumption that every logic which is finitely generated has some $[\kappa, \kappa]$-compactness.

Example 5.8. The logic $L_{\infty,\omega}$ satisfies AP trivially, since $\mathbf{A} <_{L_{\infty,\omega}} \mathbf{B}$ implies that $\mathbf{A} = \mathbf{B}$, but does not satisfy RCL (cf. [25]). $L_{\infty,\omega}$ has no occurrence number and $L_{\infty,\omega}$ is not compact.

5.3. *Proof of the main theorem*

We give first an outline of the proof, to help the reader. We assume for contradiction that λ is regular and $\boldsymbol{L}$ is not $[\lambda, \lambda]$-compact. Using Theorem 2.1 we construct a class K of linear orderings with additional predicates in which points of cofinality λ are absolute. Inside K we show the existence of some sufficiently homogeneous structure $\mathbf{N}$. In $\mathbf{N}$ we shall find $\mathbf{M}_i$ $(i = 0, 1, 2)$ being a counterexample to AP for $\boldsymbol{L}$. The occurrence number and the isomorphism axiom will be needed to show that $\mathbf{M}_0 <_{\boldsymbol{L}} \mathbf{M}_i$ $(i = 1, 2)$ and the absoluteness of 'cofinality λ' to show that there is no amalgamating structure.

The counterexample to amalgamation is patterned after the following example: Let K be the class of dense linear orderings with an additional unary predicate *Red* such that both *Red* and its complement are dense. Let $\mathbf{A} <_K \mathbf{B}$ hold if $\mathbf{A}$ is an elementary substructure of $\mathbf{B}$ and the universe of $\mathbf{A}$ is a dense subset of the universe of $\mathbf{B}$. We shall show that K with this notion of substructure $<_K$ does not allow amalgamation: For this let $\mathbf{A}_0$ be the rationals properly coloured, and let $\mathbf{A}_i$ $(i = 1, 2)$ the rationals augmented by one element (say π) coloured *Red* in $\mathbf{A}_1$ and not coloured in $\mathbf{A}_2$. Clearly, $\mathbf{A}_0 <_K \mathbf{A}_i$ $(i = 1, 2)$, but no amalgamating structure exists, since otherwise π is simultaneously coloured and not coloured.

Now, let $\lambda \geqslant \mathrm{OC}(\boldsymbol{L})$ be regular and $\boldsymbol{L}$ not $[\lambda, \lambda]$-compact. By Theorem 2.1, λ is cofinally characterizable in $\boldsymbol{L}$ in a structure $\mathbf{M}$. We need some more information on $\mathbf{M}$: Let $\Sigma, \Sigma_1=\{\varphi_\alpha:\alpha<\lambda\}$ be the counter-example to $[\lambda,\lambda]$-compactness. Put $\Sigma^{\alpha_1}=\{\varphi_\beta:\beta<\alpha\}$ and $\mathbf{M}_\alpha\models\Sigma\cup\Sigma^{\alpha_1}$. W.l.o.g. the $\mathbf{M}_\alpha$'s are structures of some countable language L (coding more predicates with parameters), and have the same power $\mu\geqslant\lambda$, $\mathbf{M}_\alpha=\langle M_\alpha, Q_n(n\in\omega)\rangle$.

We want to code all the $\mathbf{M}_\alpha$'s into one structure. So we let $\mathbf{M}$ to be such that

(1) $\mathbf{M}=\langle M, <, \bar{Q}_n, c_j(n\in\omega, j\in\lambda)\rangle$.

(2) $\langle M, <\rangle$ is a linear order of cofinality λ such that every initial segment has power μ (of order type $\mu^*+\lambda$, e.g.).

(3) $\{c_j:j<\lambda\}\subset M$ is increasing and unbounded.

(4) If $x\leqslant c_j$ but $x>c_i$ for every $i<j$, then $\langle y\in M:y<x\}$, $\bar{Q}_n(x,-,-,\ldots,-)\rangle\cong\mathbf{M}_\alpha$. Let $T=\mathrm{Th}_{\boldsymbol{L}}(\mathbf{M})$ for some fixed $\mathbf{M}$ as described above.

Claim. *Then T cofinally characterizes λ.*

This is proved like Theorem 2.1.

For the rest of this section $\mathbf{M}$ is fixed. We now define a class of structures $\mathbf{K}(\mathbf{M})$: The vocabulary of $\mathbf{K}(\mathbf{M})$ is that of $\mathbf{M}$ without the constant symbols for c_j but with two additional unary predicate symbols P and R and one additional binary predicate symbol E. Actually our main focus is on the order together with P, R, and E is used to code copies of $\mathbf{M}$, which we need to guarantee the absoluteness of cofinality λ.

A model in $\mathbf{K}(\mathbf{M})$ is of the form $\mathbf{A}=\langle A, <, \bar{Q}_i, P, R, E\rangle$ with the requirements:

(K1) If $x\in P$, then the cofinality of x in $\langle A, <\rangle$ is λ with a witnessing sequence $\{c_j(x):j<\lambda\}$.

(K2) $(a, x)\in E$ implies that $a<x$.

(K3) $(a, x)\in E$ implies that $x\in P$ and $a\notin P$.

(K4) $P(x)$ implies that $E(c_j(x), x)$ for every $j\in\lambda$.

Put $J^x_{\mathrm{A}}=\{a\in A:(a, x)\in E\}$, and J^x_{A} be the substructure of $\langle A, <, \bar{Q}_i\rangle$ induced by J^x_{A}.

(K5) The structure $\langle \mathbf{J}^x_{\mathrm{A}}, c_j(x)\rangle$ is isomorphic to $\mathbf{M}$.

(K6) $R\subset P$.

We call a structure in $\mathbf{K}(\mathbf{M})$ *pure* if additionally

(K7) $\bar{Q}_i$ is false where not defined by the previous requirements.

Comments. Note that if $\mathbf{A}\in\mathbf{K}(\mathbf{M})$ is pure and P in $\mathbf{A}$ is empty, then $\mathbf{A}$ is just a linear ordering, i.e. all the other relations are empty, too, by (K7).

If we add to $\mathbf{M}$ one point at the end, say x and let $P=\{x\}$, we get a structure in $\mathbf{K}(\mathbf{M})$. We denote this structure by $\mathbf{M}^{+1}$.

In general the structures in $\mathbf{K}(\mathbf{M})$ are linearly ordered structures where every point in P has a copy of $\mathbf{M}$ attached to it in way that different points have almost disjoint copies of $\mathbf{M}$, and $\mathbf{M}$ cofinally reaches its point in P. The choice of R can be any subset of P. More precisely:

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Fact 1. *For every* $\mathbf{A} \in \mathbf{K}(\mathbf{M})$ *and every* $a, a' \in A$ $J^a_{\mathrm{A}} \cap J^{a'}_{\mathrm{A}}$ *is bounded below both* a, a'.

This is proved using the fact that $\mathbf{M}$ is of order type $\mu^* + \lambda$. Note that this is first order expressible and could have been stated also as an axiom among (K1–K7).

Fact 2. *If* $\mathbf{A} \in \mathbf{K}(\mathbf{M})$ *and* $a \in P^{\mathrm{A}}$ *and we form* $\mathbf{A}'$ *by changing the truth value of* $a \in R^{\mathrm{A}}$, *but leaving everything else fixed, then* $\mathbf{A}' \in \mathbf{K}(\mathbf{M})$.

Next we define the notion of K-substructure, $\mathbf{A} \subset_K \mathbf{B}$, for $\mathbf{A}, \mathbf{B} \in \mathbf{K}(\mathbf{M})$ by:

(K8) $\mathbf{A} \subset \mathbf{B}$.

(K9) If $x \in P^{\mathrm{A}}$, then $J^x_{\mathrm{B}} \subset A$.

(K10) If $x \in P^B - P^{\mathrm{A}}$, then $\{a \in A : a < x\}$ is bounded below x in $\mathbf{B}$, i.e. there is $b_x \in B$ such that $b_x < x$ and for each $a \in A$ with $a < x$ we have $a < b_x$.

The idea behind this is that in $\mathbf{B}$ points are added to P^B in a way that they are not limits of points from $\mathbf{A}$, and that points in $\mathbf{A}$ are of cofinally λ also in $\mathbf{B}$ with the same copy of $\mathbf{M}$ ensuring this as in $\mathbf{A}$.

This ends the definition of $\mathbf{K}(\mathbf{M})$ and of K-substructures.

Before we proceed with the proof of Theorem 5.3 we collect some more facts:

Definition. If $\mathbf{A}_1, \mathbf{A}_2 \in \mathbf{K}(\mathbf{M})$ we define $\mathbf{A}_1 + \mathbf{A}_2$ to be the disjoint union of $\mathbf{A}_1$, $\mathbf{A}_2$ with the linear ordering of $\mathbf{A}_1$ and $\mathbf{A}_2$ for their elements and $a_1 < a_2$ for every $a_1 \in A_1$, $a_2 \in A_2$. For the other relations we just take their unions.

Fact 3. *If* $\mathbf{A}_1, \mathbf{A}_2 \in \mathbf{K}(\mathbf{M})$ *so* $\mathbf{A}_1 + \mathbf{A}_2 \in \mathbf{K}(\mathbf{M})$ *and* $\mathbf{A}_i \subset_K \mathbf{A}_1 + \mathbf{A}_2$ $(i = 1, 2)$.

This is clear from the definitions.

Definition. Denote by $I^x_{\mathrm{A}} = \{a \in A : a < x\}$ and by $\mathbf{I}^x_{\mathrm{A}}$ the structure $\mathbf{A} \restriction I^x_{\mathrm{A}}$.

If $\mathbf{B} \in \mathbf{K}(\mathbf{M})$ and $A \subset B$ we define a substructure $\mathbf{C}(A)$ of $\mathbf{B}$ by

$$\mathbf{C}(A) = \mathbf{B} \restriction \bigcup_{a \in A} J^a_{\mathrm{B}} \cup A.$$

This makes sense by Fact 1 and ensures that:

Fact 4. *For every* $\mathbf{B} \in \mathbf{K}(\mathbf{M})$, $A \subset B$, $\mathbf{C}(A) \subset_K \mathbf{B}$, *but in general* $\mathbf{C}(A)$ *is not pure. Furthermore, if* A *is bounded in* $\mathbf{B}$ *by* b, *i.e. there is* $b \subset B$ *with* $A \subset I^b_{\mathrm{B}}$, *so* $\mathbf{C}(A) \subset I^b_{\mathrm{B}}$ *and* $\mathbf{C}(I^b_{\mathrm{B}}) = \mathbf{I}^b_{\mathrm{B}}$.

Fact 5. *If* $\mathbf{A} \in \mathbf{K}(\mathbf{M})$ *and* $d \in P^{\mathrm{A}}$, *then* $\mathbf{A} \restriction I^d_{\mathrm{A}} \subset_K \mathbf{A}$.

Fact 6. *If* $\{\mathbf{A}_i : i < \alpha\}$ *is a sequence of structures in* $\mathbf{K}(\mathbf{M})$ *such that* $\mathbf{A}_i \subset_K \mathbf{A}_{i+1}$, *then* $\mathbf{A} = \bigcup_{i<\alpha} \mathbf{A}_i \in \mathbf{K}(\mathbf{M})$ *and* $\mathbf{A}_i \subset_K \mathbf{A}$ *for each* $i < \alpha$.

Definition. If $\mathbf{A}_1, \mathbf{A}_2 \in \mathbf{K}(\mathbf{M})$, $\mathbf{B}_i \subset_K \mathbf{A}_i$ $(i=1,2)$ and $f:\mathbf{B}_1 \cong \mathbf{B}_2$ is an isomorphism, we define $\mathbf{A}_1 +_f \mathbf{A}_2$ in the following way: Form the disjoint union of $\mathbf{A}_1$ and $\mathbf{A}_2$ modulo f (i.e. identify elements only via f). This makes it into a partially ordered structure where $a_i \in A_i$ $(i=1,2)$ are comparable only if one of them is in the range or domain of f, or there is b between a_1, a_2 which has been identified. For incomparable a_1, a_2 we extend the order on $\mathbf{A}_1 +_f \mathbf{A}_2$ setting $a_1 < a_2$.

Fact 7. *If* $\mathbf{A}_1, \mathbf{A}_2 \in \mathbf{K}(\mathbf{M})$ *and* $f:\mathbf{B}_i \cong \mathbf{B}_2$, $\mathbf{B}_i \subset_K \mathbf{A}_i$ $(i=1,2)$, *then* $\mathbf{A}_1 +_f \mathbf{A}_2 \in \mathbf{K}(\mathbf{M})$ *and* $\mathbf{A}_i \subset_K \mathbf{A}_1 +_f \mathbf{A}_2$.

The proofs of the facts are left to the reader.

The next lemma is crucial for our construction:

Lemma 5.9. *If* $\mathbf{A} \in \mathbf{K}(\mathbf{M})$ *and* $\mathbf{B}$ *is a* $\boldsymbol{L}$*-extension of* $\mathbf{A}$ *and* $\{d_j : j < \lambda\}$ *is cofinal in* J^a_{A} *for* $a \in P^{\mathrm{A}}$, *then* $\{d_j : j < \lambda\}$ *is cofinal in* J^a_{B}.

Proof. Let $a \in P^{\mathrm{A}}$, so $\mathbf{J}^a_{\mathrm{A}} \cong \mathbf{M}$ by (K5) and by our assumption on $\boldsymbol{L}$ and $\mathbf{M}$, $\boldsymbol{L}$ cofinally characterizes λ in $\mathbf{M}$. Using relativization of $\boldsymbol{L}$ the structure $\mathbf{J}^a_{\mathrm{B}}$ is an L-extension of $\mathbf{M}$ so $\mathbf{M}$ is cofinal in $\mathbf{J}^a_{\mathrm{B}}$, hence $\{d_j : j < \lambda\}$ is cofinal in $\mathbf{J}^a_{\mathrm{B}}$ which proves the lemma. □

The next lemma is proved in the following section. But from it we can now complete the proof of Theorem 5.3.

Lemma 5.10. *There is a structure* $\mathbf{N}$ *in* $\mathbf{K}(\mathbf{M})$ *and* $d_1 < d_2 < d_3$ *in* $\mathbf{N}$ *with* $d_i \in P^N$ $(i=1,2,3)$, $d_1 \in R^N$, $d_2 \notin R^N$ *such that*

(i) $\mathbf{N} \restriction I^{d_1}_N \cong \mathbf{N} \restriction I^{d_2}_N \cong \mathbf{N} \restriction I^{d_3}_N$.

(ii) *If* $\mathbf{A} \subset_K \mathbf{N} \restriction I^{d_i}_N$ $(i=1,2)$ *is bounded in* $\mathbf{N} \restriction I^{d_i}_N$, *then* $\mathbf{N} \restriction I^{d_i}_N \cong \mathbf{N} \restriction I^{d_3}_N$ *over* $\mathbf{A}$ $(i=1,2)$.

Proof of Theorem 5.3. Put $\mathbf{M}_i = \mathbf{N} \restriction I^{d_i}_N$ $(i=1,2,3)$. We have to verify some claims:

Claim 1. $\mathbf{M}_i <_{\boldsymbol{L}} \mathbf{M}_3$ $(i=1,2)$.

Proof. Let φ be an $\boldsymbol{L}(M_L)$-sentence. Since the occurrence number $\mathrm{OC}(\boldsymbol{L}) \leq \lambda$, φ depends on $<\lambda$ many constants, hence there is $a \in M_i$ and all the constants of φ are in $I^a_{M_i}$. So by Fact 4$\mathbf{M}_i \restriction I^a_{M_i}$ is a bounded K-substructure of both $\mathbf{M}_i$ and $\mathbf{M}_3$. So by Lemma 5.10(ii) $\langle \mathbf{M}_i, I^a_{M_i}\rangle$ is isomorphic to $\langle \mathbf{M}_3, I^a_{M_3}\rangle$ hence by the basic isomorphism axiom, $\langle \mathbf{M}_i, I^a_{M_i}\rangle \models \varphi$ iff $\langle \mathbf{M}_3, I^a_{M_3}\rangle \models \varphi$.

Now let $f:\mathbf{M}_1 \cong \mathbf{M}_2$ be the isomorphism from Lemma 5.10(i) and $g_i:\mathbf{M}_i \to \mathbf{M}_3$ $(i=1,2)$ the $\boldsymbol{L}$-embeddings from Claim 1.

Since $\boldsymbol{L}$ has AP, let $\mathbf{A}$ be the amalgamation for $g_1:\mathbf{M}_1 \to \mathbf{M}_3$, $g_2 f:\mathbf{M}_1 \to \mathbf{M}_3$.

Claim 2. $\mathbf{A} \models d_1 = d_2$.

Proof. $d_i \in P^{M_3}$ $(i=1,2)$ are both of cofinality λ and $g_i(M_1)$ is cofinal in $\mathbf{M}_3 \restriction I^{d_1}_{M_3}$, and $g_2 f:(M_1)$ is cofinal in $\mathbf{M}_3 \restriction I^{d_2}_{M_3}$, so by Lemma 5.9 also in $\mathbf{A} \restriction I^{d_1}_{\mathrm{A}}$ and $\mathbf{A} \restriction I^{d_2}_{\mathrm{A}}$, hence $\mathbf{A} \models d_1 = d_2$.

But Claim 2 contradicts our assumption of Lemma 5.10 that $d_1 \in R^{\mathbf{A}}$ and $d_2 \notin R^{\mathbf{A}}$. This completes the proof of 5.3. □

5.4. Proof of Lemma 5.10

The proof of 5.10 is similar to the classical proof of the existence of homogeneous structures. Compare this also with [33] or its expository version as chapter 20 of [3]. The first step is the amalgamation lemma for K-extensions:

Lemma 5.11. *If* $\mathbf{A} \in \mathbf{K}(\mathbf{M})$, $d_1, d_2 \in P^{\mathbf{A}}$, $\mathbf{B}_i \subset_K \mathbf{A} \restriction I^{d_i}_{\mathbf{A}}$ $(i=1,2)$ *and* $f : \mathbf{B}_1 \cong \mathbf{B}_2$ *and both* $\mathbf{B}_i$ *are either bounded or unbounded in* $\mathbf{A} \restriction I^{d_i}_{\mathbf{A}}$, *then there are* $\mathbf{A}' \in \mathbf{K}(\mathbf{M})$ *and* $\mathbf{B}'_1$, $\mathbf{B}'_2$ *and* f' *such that*

(i) $\mathbf{A} \subset_K \mathbf{A}'$,
(ii) $\mathbf{B}_1 \subset_K \mathbf{I}^{d_1}_{\mathbf{A}'}$ *unbounded*,
(iii) $\mathbf{B}_2 = \mathbf{I}^{d_2}_{\mathbf{A}} \subset_K \mathbf{I}^{d_2}_{\mathbf{A}'}$ *unbounded*,
(iv) $f \subset f'$ *with* $f' : \mathbf{B}'_1 \cong \mathbf{B}'_2$.

Proof. There are two cases:

Case 1: $d_1 < d_2$. Put $\mathbf{A}_i = \mathbf{I}^{d_i}_{\mathbf{A}}$ $(i=1,2)$ and form $\mathbf{A}_3 + \mathbf{A}_2 +_f \mathbf{A}_2$. There are natural embeddings $\pi_i : \mathbf{A}_2 \to \mathbf{A}_3$ depending if $\mathbf{A}_2$ is identified with the first or second copy of $\mathbf{A}_2$ in $\mathbf{A}_3$. Let $g : \pi_1(\mathbf{A}_2) \cong \mathbf{A}_2 \subset_K \mathbf{A}$ and form now $\mathbf{A}_3 +_g \mathbf{A} = \mathbf{A}'$. There is a natural embedding $h : \mathbf{A} \subset_K \mathbf{A}'$ which determines the value of d_1, d_2 in $\mathbf{A}'$. We have to define f'. The range of f' is $h(\mathbf{A}_2)$ and on $h(\mathbf{B}_2)$ f' is f, otherwise it is g^{-1}. So we have proved (i) and (iv). To prove (ii) and (iii) we have only to show that $\mathbf{B}'_2 = h(\mathbf{A}_2)$ is unbounded in $\mathbf{I}^{h(d_2)}_{\mathbf{A}'}$ and similarly for $\mathbf{B}'_1 = (f')^{-1}(\mathbf{B}'_2)$. In the first case this is trivial and in the second case this comes from f if $\mathbf{B}_1$ is unbounded and from g^{-1} if $\mathbf{B}_1$ is bounded.

Case 2: $d_2 < d_1$. the proof is similar, but put $\mathbf{A}_3 = \mathbf{A}_2 +_f \mathbf{A}_1$. In both cases we use heavily Facts 4 and 6. □

Lemma 5.12. *Under the same hypothesis as in Lemma* 5.11 *we can even have* $\mathbf{B}'_1 = \mathbf{I}^{d_1}_{\mathbf{A}'}$ *and* $\mathbf{B}'_2 = \mathbf{I}^{d_2}_{\mathbf{A}'}$.

Proof. Use Lemma 5.11 countably many times with alternating roles of d_1, d_2 and take unions of chains using Fact 5. □

Proof of Lemma 5.10. Let $\mathbf{A}$ be in $K(\mathbf{M})$ such that $P^{\mathbf{A}}$ is cofinal in $\mathbf{A}$ and both $(P-R)^{\mathbf{A}}$ and $R^{\mathbf{A}}$ are cofinal in $P^{\mathbf{A}}$. For instance take $\mathbf{M}$ and put $\mathbf{M}^{+1}$ and for x the new element set $P(x)$ in the obvious way. We have freedom to choose both $R(x)$ and $\neg R(x)$. So put $\mathbf{M}_i = \mathbf{M}^{+1}$ with $x \in (P-R)$ if i is even and $x \in R$ if i is odd. Then form $\mathbf{A}_{n+1} = \mathbf{A}_n + \mathbf{M}_{n+1}$ and $\mathbf{A}_0 = \mathbf{M}_0$. Now put $\mathbf{A} = \bigcup_{n \in \omega} \mathbf{A}_n$.

Now choose any d_1, d_2, d_3 with $d_1 < d_2 < d_3$ in $P^{\mathbf{A}}$ with $d_1 \in R^{\mathbf{A}}$, $d_2 \in (P-R)^{\mathbf{A}}$. Let $\{c_j(d_i) : j < \lambda\}$ $(i=1,2,3)$ be cofinal witnessing sequences for d_i respectively and assume w.l.o.g. $c_0(d_1) = c_0(d_2) = c_0(d_3)$. By Lemma 5.12 we can assume that there is $\mathbf{B}_0$, $\mathbf{A} \subset_K \mathbf{B}_0$ with $\mathbf{I}^{d_i}_{\mathbf{B}_0}$ all isomorphic over $\mathbf{I}^{c_0(d_i)}_{\mathbf{B}_0}$. We can achieve this using an iteration, alternating the roles of the d_i's in pairs. Iterating this we construct $\mathbf{B}_{\alpha+1}$ extending $\mathbf{B}_\alpha$ such that $\mathbf{I}^{c_{\alpha+1}(d_i)}_{\mathbf{B}_{\alpha+1}}$ are all isomorphic over $\mathbf{I}^{c_\alpha(d_i)}_{\mathbf{B}_\alpha}$ for $i = 1, 2, 3$. For α limit we take unions. Clearly $\mathbf{B}_\lambda$ is the required model. □

References

[1] P. Alexandroff and P. Urysohn, Memoire sur les espaces topologiques, Verhandlingen der Koninklijke Akademie van Wetenschappen te Amsterdam, Afdeling Natuurkunde (Eerste Sectie), XIV (1929) 1–96.

[2] K.J. Barwise, Axioms for abstract model theory, Ann. Math. Logic 7 (1974) 221–265.

[3] K.J. Barwise and S. Feferman, ed., Model Theoretic Logics. (Springer-Verlag, Berlin, to appear).

[4] K.J. Barwise, M. Kaufmann and M. Makkai, Stationary Logic, Ann. Math. Logic 13 (1978) 171–224.

[5] J.L. Bell and A.B. Slomson, Models and Ultraproducts (North-Holland, Amsterdam, 1969).

[6] G.V. Cudnovskii and D.V. Cudnovskii, Regular and descending complete ultrafilters, Soviet Math. Dokl. 12 (1971) 901–905.

[7] C.C. Chang and J. Keisler, Model Theory (North-Holland, Amsterdam, 1977) 2nd ed.

[8] W. Comfort and S. Negrepontis, The Theory of Ultrafilter, Grundlehren der math. Wissenschaften, Vol 211 (Springer-Verlag, Berlin, 1974).

[9] D. Donder, R.B. Jensen and B. Koppelberg, Some applications of the core model, Lecture Notes in Math. Vol. 872 (1981).

[10] E. Ellentuck, Foundations of Souslin logic, J. Symbolic Logic 41(1) (1976) 59–72.

[11] J. Flum, First order logic and its extensions, Logic Conference, Kiel 1974, Proceedings, Lecture Notes in Math, Vol. 499 (Springer-Verlag, Berlin, 1976) 248–310.

[12] H. Friedman, Adding prepositional connectives to countably infinitary logic, mimeographed.

[13] H. Friedman, Hundred four problems in mathematical logic, J. Symbolic Logic 40(2) (1974) 22–26.

[14] H. Friedman, Characterizing first order logic, mimeographed.

[15] L. Harrington, Extensions of countable infinitary logic which preserve most of its nice properties, Arch. f. Math. Logic 20 (1980) 95–102.

[16] R.B. Jensen and B.J. Koppelberg, A note on ultrafilters, Notices AMS 78T-E21.

[17] A. Kanamori, Weakly normal ultrafilters and irregular ultrafilters, Trans. AMS 220 (1976) 393–399.

[18] J. Keisler, On cardinalities of ultraproducts, Bull. AMS 70 (1964) 644–647.

[19] J. Keisler, Limit ultrapowers, Trans. AMS 107 (1963) 383–408.

[20] M. Magidor, On the existence of descendingly complete ultrafilters, in preparation.

[21] M. Magidor, On the role of supercompact and extendible cardinals in logic, Israel J. Math. 10(2) (1971) 147–157.

[22] J.A. Makowsky, Quantifying over countable sets: positive vs. stationary log, Proceedings of the Logic Colloquium 1977 (North-Holland, Amsterdam, 1978) 183–194.

[23] J.A. Makowsky, Topological model theory: a survey, Model Theory and Applications (Rome, 1975).

[24] J.A. Makowsky, Vopenka's principle and compact logics, to appear.

[25] J.A. Makowsky and S. Shelah, The theorems of Beth and Craig in abstract model theory I: The abstract setting, Trans. AMS 256 (1979) 215–239.

[26] J.A. Makowsky and S. Shelah, The theorems of Beth and Craig in abstract model theory II: Compact logics, Arch. f. Math. Logic., 21 (1981) 13–35.

[27] J.A. Makowsky and S. Shelah, The theorems of Beth and Craig in abstract model theory III: Infinite logic, in preparation.

[28] J.A. Makowsky, S. Shelah and J. Stavi, Δ-logics and generalized quantifiers, Ann. Math. Logic 10 (1976) 155–192.

[29] J. Malitz and W. Reinhart, Maximal models in the language with the quantifier "there exist uncountably many", Pacific J. Math. 40(1) (1972) 139–155.
[30] D. Mundici, Interpolation, compactness and JEP in soft model theory, Arch. f. Math. Logik 22 (1982) 61–67.
[31] K. Prikry, On descendingly complete ultrafilters, in: 1971 Cambridge Summer School in Mathematical Logic, Lecture Notes in Math. Vol. 337 (Springer-Verlag, Berlin, 1973) 459–488.
[32] S. Shelah, Generalized quantifiers and compact logic, Trans. AMS 204 (1975) 342–364.
[33] S. Shelah, Classification theory for non-elementary classes II, to appear.
[34] J. Stavi, Compactness properties of infinitary and abstract languages, Logic Colloquium 77 (North-Holland, Amsterdam, 1978) 267–274.
[35] J.E. Vaughan, Some properties related to $[a, b]$-compactness, Fund. Math. 87 (1975) 251–260.
[36] M. Ziegler, A language for topological structures which satifies a Lindstrom theorem, Bull. AMS 82 (1976) 586–570.