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CAN THE FUNDAMENTAL (HOMOTOPY) GROUP OF A SPACE BE THE RATIONALS?

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ABSTRACT. We prove that for any topological space which is metric, compact (hence separable) path connected and locally path connected, its homotopy group is not the additive group of the rational, moreover if it is not finitely generated then it has the cardinality of the continuum.

We prove here

1. THEOREM. *Let X be a compact metric (topological) space which is path connected and locally path connected. If the homotopy group of X is not finitely generated then it has the power of the continuum (in fact there is a perfect set of nonhomotopic f's).*

REMARK. X is locally path connected if for every open u and $y \in u$ there is a path connected u', $y \in u' \subseteq u$. We shall really use only "weakly" locally path connected (see Definition 4).

2. INTRODUCTORY REMARKS. We have taken some trouble to make this accessible to both algebraic topologists and logicians (hence most other mathematicians), resulting in making the proof longer.

Nevertheless we assume e.g. that the reader understands what the theorem says. On the topological notions we use, see e.g. [**Sp**].

A conclusion of the theorem is that the additive group of the rationals cannot be the homotopy group of such a space.

The theorem answers a problem of Mycielski. I thank M. Foreman for asking me about it. On theorems related to Lemma 7 and their history see Harrington and Shelah [**HS** and **Sh1**] to which our exposition is closer.

The proof gives not only continuum many but a perfect set of paths (from say x_0 to x_0) nonhomotopic in pairs.

I thank H. Miller for improving the presentation.

3. NOTATION. $\mathbf{Z}$ is the set of integers, $\mathbf{Z}^+$ the set of strictly positive integers. $\mathbf{R}$ is the set of reals, $\mathbf{R}^+$ the set of strictly positive reals.

Let $I = [0,1] = \{t \in \mathbf{R}: 0 \leq t \leq 1\}$, endowed with the natural topology.

A *path* f in a topological space X is a continuous function from I to X. We say f is from x to y if $f(0) = x$, $f(1) = y$. We let f^{-1} be defined by $f^{-1}(t) = f(1-t)$ (so it goes from $f(1)$ to $f(0)$). For paths f, g, fg is defined only if $f(1) = g(0)$ and

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then

$$fg(t) = \begin{cases} f(2t), & 0 \le t \le 1/2, \\ g(2t-1), & 1/2 \le t \le 1. \end{cases}$$

Let $E(x_0, X) =$ the family of paths from x_0 to x_0.

Let $[f]$ be the equivalence class of f under the equivalence relation of being homotopic relative to the endpoints. Dividing by it, $E(x_0, X)$ becomes a group under the two operations defined above, which we call $\pi(X, x_0)$.

In a topological space a set is perfect if it is closed nonempty with no isolated point. For what we actually get, see 12 and the definition of f_A.

For a metric space, d is the metric and $B_r(x) = B(x, r) = \{y \in X : d(y, x) < r\}$.

4. DEFINITION. Say that a space X is *graphical* if it admits an open cover $\mathscr{U}$ such that for $u, v \in \mathscr{U}$, and all $x \in u$, $y \in v$, any two paths from x to y in $u \cup v$ are homotopic (relative to the end points) in X.

REMARK. For metric compact X, X is graphical iff

$(*)$ for every $x \in X$ for some open u, $x \in u$, and for every $y, z \in u$ any two paths from x to y in u are homotopic (relative to end points) in X.

5. DEFINITION. (1) Say that X is *weakly locally path connected* (WLPC) if for every $x \in X$ and every neighborhood u of x, there exists a neighborhood v of x in U such that any point in v can be joined to x by a path through u.

6. DEFINITION. (2) Say that X is *semilocally simply connected* (SLPC) if every point $x \in X$ has a neighborhood u such that $\pi_1(u; x) \to \pi_1(X; x)$ is the trivial homomorphism (i.e. any two paths in $E(x, u)$ are homotopic in X).

7. LEMMA. *If X is a compact graphical WLPC space then $\pi_1(X, x_0)$ is a finitely generated group for any $x_0 \in X$.*

PROOF. As X is graphical, there is an open cover $\mathscr{U}$ of X as in Definition 4.

For every $y \in X$ choose $u^0_y \in \mathscr{U}$ such that $y \in u^0_y$. As X is WLPC for some open u^1_y, $y \in u^1_y \subseteq u^0_y$ and any point in u^1_y is connected to y by a path through u^0_y. So $X = \bigcup\{u^1_y : y \in X\}$ but X is compact hence for some finite $Y \subseteq X$, $X = \bigcup\{u^1_y : y \in Y\}$, w.l.o.g. $x_0 \in Y$.

For a path f we say that f is of type $\langle y_0, \dots, y_n\rangle$ if there are reals $0 = t_0 \le t_1 \le \dots \le t_n = 1$ exemplifying it, i.e. such that for $l = 0, \dots, n-1$ $\mathrm{Rang}(f \restriction [t_l, t_{l+1}]) \subseteq u^0_{y_l} \cup u^0_{y_{l+1}}$, and for $l = 0, \dots, n$ $f(t_l) \in u^1_{y_l}$.

Note that one f may have many types; however every f has at least one type. (For every $t \in [0, 1]$, for some $y \in Y$, $h(t) \in u^1_y$, hence for some open interval J of $[0, 1]$, $t \in J$ and f maps J into u^1_y, and apply the compactness of $[0, 1]$.)

$(*)$ If $f^1, f^2 \in E(x_0, X)$ both have type $\langle y_0, \dots, y_{n-1}\rangle$ *then* f^1, f^2 are homotopic.

We shall prove this later. Let

$$F = \{[f] : f \in E(x_0, X), \text{and for some } n \le 2|Y| + 8 \text{ and } y_0, \dots, y_{n-1} \in Y, f \text{ has type } \langle y_0, \dots, y_{n-1}\rangle\}.$$

By $(*)$ F (which is a subset of the fundamental group) has power $\le \sum_{i \le 2|Y|+8} |Y|^{2i}$ hence is finite. So it suffices to show that F generates the fundamental group.

So let $f \in E(x_0, X)$, and we shall prove $[f]$ is in the subgroup which F generates. We know that for some n and $\langle y_0, \dots, y_{n-1}\rangle$, f is of type $\langle y_0, \dots, y_{n-1}\rangle$ and let $\langle t_0, \dots, t_{n-1}\rangle$ exemplify it. We prove by induction on n. We prove that $[f]$ belongs to the subgroup generated by F. If $n \le 2|Y| + 8$ this is obvious. If not, let $i = |Y|+4$. Let $\langle z_0, \dots, z_k\rangle$ be a sequence of minimal length such that $\{z_0, \dots, z_k\} \subseteq Y$, $z_0 = x_0$, $z_k = y_i$ and for each $l \le k-1$, $u^1_{z_l} \cap u^1_{z_{l+1}} \ne \varnothing$. Hence there is a path h_l from z_l to z_{l+1}, $\mathrm{Rang}(h_l) \subseteq u^0_{z_l} \cup u^0_{z_{l+1}}$. (Use paths from z_l and z_{l+1} to some point in $u^1_{z_l} \cap u^1_{z_{l+1}}$.) Also there is a path h from $y_i = z_k$ to $f(t_i)$, $\mathrm{Rang}(h) \subseteq u^0_{y_i}$ (as $f(t_i) \in u^1_{y_i}$). So $h_0 h_1 \cdots h_{k-1}$ is a path from $z_0 = x_0$ to $z_k = y_i$ of type $\langle z_0, \dots, z_{k-1}\rangle$.

Now let $f_m = f \restriction [t_m, t_{m+1}]$ so

$$[f] = [f_0 f_1 \cdots f_i \cdots f_{n-1}] = [f_0][f_1]\cdots[f_i]\cdots[f_{n-1}] = [f_0][f_1]\cdots$$
$$[f_{i-1}][h]^{-1}[h_{k-1}]^{-1}\cdots[h_0]^{-1}([h_0]\cdots[h_{k-2}][h_{k-1}][h])[f_i][f_{i+1}]\cdots$$
$$[f_{n-1}] = [f_0 f_1 \cdots f_{i-1} h^{-1} h^{-1}_{k-1} h^{-1}_{k-2} \cdots h_0^{-1}][h_0 h_1 \cdots h_{k-1} h f_i f_{i+1} \cdots f_{n-1}].$$

So $[f]$ is the product of two elements (from $\{[f'] \colon f' \in E(x_0, X)\}$. The first has naturally type $\langle y_0, \dots, y_{i-1}, y_i, z_{k-1}, z_{k-2}, \dots, z_0\rangle$, but $i = |Y| + 4$, $k \le |Y|$, so it belongs to F.

The second has naturally type $\langle z_0, \dots, z_{k-2}, z_{k-1}, y_i, y_{i+1} \cdots y_{n-1}\rangle$ which has length $n - i + k$, but as $i = |Y| + 4$, $k \le |Y|$ this $\le n-1$, so it belongs to F.

PROOF OF (*). Let $\langle t^l_0, \dots, t^l_n\rangle$ exemplify that f^l has type $\langle y_0, \dots, y_n\rangle$ for $l = 1, 2$. So $0 = t^l_0 \le t^l_1 \le \cdots \le t^l_{n-1} \le t^l_n = 1$ and $\mathrm{Rang}(f^l \restriction [t^l_m, t^l_{m+1}]) \subseteq u^0_{y_m} \cup u^0_{y_{m+1}}$, $f^l(t^l_m) \in u^1_{y_m}$.

For each $m = 1, \dots, n-1$ the points $f^1(t^1_m)$ and $f^2(t^2_m)$ are in $u^1_{y_m}$ so there is a path g_m from $f^1(t^1_m)$ to $f^2(t^2_m)$ such that $\mathrm{Rang}(g_m) \subseteq u^0_{y_m}$. (Use paths from $f^l(t^l_m)$ to y_m for $l = 1, 2$.)

Let $f^l_m = f^l \restriction [t_m, t_{m+1}]$ for $m = 0, \dots, n-1$. Now

$$[f^1] = [f^1_0 f^1_1 \cdots f^1_{n-1}] = [f^1_0][f^1_1]\cdots[f^1_{n-1}]$$
$$= ([f^1_0][g_1])([g_1]^{-1}[f^1_1][g_2])([g_2]^{-1}[f^1_2][g_2])\cdots([g_{n-1}]^{-1}[f^1_{n-1}]),$$

and

$$[f^2] = [f^2_0 f^2_1 \cdots f^2_{n-1}] = [f^2_0][f^2_1]\cdots[f^2_{n-1}].$$

So for proving equality it is enough to show:

(a) $[f^1_0 g_1] = [f^2_0]$,
(b) $[g^{-1}_m f^1_m g_{m+1}] = [f^2_m]$,
(c) $[g^{-1}_{m-1} f^1_{n-1}] = [f^2_{n-1}]$.

Now (b) holds as $g^{-1}_m f^1_m g_{m+1}$, (f^2_m) are homotopic by the choice of $\mathscr{U}$ and as $u^0_{y_m}, u^0_{y_{m+1}} \in \mathscr{U}$. For (a), (c) the situation is similar.

8. LEMMA. *A metric space is graphical provided it is WLPC and SLPC.*

PROOF. For each $x \in X$, let $\varepsilon(x) > 0$ be such that $\pi_1(B_{\varepsilon(x)}(x), x) \to \pi_1(X, x)$ is trivial (exists as X is SLPC). Let $\delta(x) > 0$ be such that any point in $B_{\delta(x)}(x)$ is connected to x by a path in $B_{\varepsilon(x)/3}(x)$ (exists as X is WLPC). Then $\{B_{\delta(x)}(x) \colon x \in X\}$ is an open cover of X. Pick a pair of points, x_1 and x_2 and let $B_i = B_{\delta(x_i)}(x_i)$. Let $y_i \in B_i$, $i = 1, 2$, and let α and β be paths from y_1 to y_2 in $B_1 \cup B_2$. Let γ_i

join x_i to y_i in $B_{\varepsilon(x_i)/3}(x_i)$, $i=1,2$. Then $\gamma_1\alpha\gamma_2^{-1}$ and $\gamma_1\beta\gamma_2^{-1}$ are paths from x_1 to x_2 in $C \stackrel{\text{def}}{=} B_{\varepsilon(x_1)/3}(x_1) \cup B_{\varepsilon(x_2)/3}(x_2)$. We may assume $\varepsilon(x_1) \geq \varepsilon(x_2)$. The triangle inequality implies then that $C \subseteq B_{\varepsilon(x_1)}(x_1)$, so $(\gamma_1\alpha\gamma_2^{-1})(\gamma_1\beta\gamma_2^{-1})^{-1} \simeq \gamma_1\alpha\beta^{-1}\gamma_2^{-1}$ is null, and consequently α is homotopic (relative to endpoints) to β. Q.E.D

9. COROLLARY. *If X is a WLPC compact metric space and $\pi_1(X)$ is not finitely generated, then there exists a point $x \in X$ such that every neighborhood U of x contains a loop at x (i.e. an $f \in \pi(X,x)$) which is essential in X (i.e. $[f]$ is not the unit in the homotopy group).*

So let x_0, f_n $(n \in \mathbf{Z}^+)$ be such that x_0 is as in conclusion 8, $f_n \in E(x_0, X)$, Rang $f_n \subseteq B_{1/n}(x_0)$, $[f_n]$ is not the unit in the homotopy group.

10. PROOF OF THEOREM 1. Fix $0 = t_0 < t_1 < t_2 < \cdots < 1$ s.t. $1 = \lim_{n\to\infty} t_n$ (so $0 < t_n < t_{n+1} < 1$ for $n \in \mathbf{Z}^+$) and for $A \subset \mathbf{Z}^+$ put

$$f_A(t) = \begin{cases} f_n((t-t_{n-1})/(t_n - t_{n-1})) & \text{if } n \in A,\ t \in [t_{n-1}, t_n], \\ x_0 & \text{if } n \notin A,\ t \in [t_{n-1}, t_n], \end{cases}$$

So f_A is in a sense the infinitary product $\prod_{n\in A} f_n$. By the choice of the f_n's, f_A is continuous. Hence it represents a member of the homotopy group of X. Also $f_{\{n\}}$ represents the same element as f_n and

(∗) *if* n is the first member of A, $B = A - \{n\}$ then f_A and $f_n f_B$ are homotopic.

11. FACT. If A, $B \subseteq \mathbf{Z}^+$, $n \notin B$, $A = B \cup \{n\}$ then f_A, f_B are not homotopic.

PROOF OF 11. We prove it by induction on the number of $m < n$ which are in B. If this number is zero but f_A, f_B are homotopic then by (∗) above "f_A and $f_n f_B$ are homotopic" we get that f_n is homotopic to zero, a contradiction to its choice.

Now Theorem 1 will follow immediately by Claim 12 and Lemma 13 below.

We define a relation $\mathscr{E}$ on $\Gamma = \{A \colon A \text{ a subset of } \mathbf{Z}^+\}$ by $A\mathscr{E}B$ *iff* the mappings f_A, f_B are homotopic.

12. CLAIM. $\mathscr{E}$ is an analytic (see below) equivalence relation.

12A. REMARK. (1) $\mathscr{E}$ is analytic means that the set $\{(A,B) \colon A, B \in \Gamma, A \in B\}$ is an analytic subset of the product space $\Gamma \times \Gamma$.

(2) $\Delta \subset \Gamma$ is *analytic* if for some complete separable metric space Y and Borel subset P of $Y \times \Gamma$

$$\Delta = \{A \in \Gamma \colon \text{for some } y \in Y, (y, A) \in P\}.$$

We can replace "Borel" by "closed".

(3) Of course Γ is endowed with the Tychonov topology (e.g. use the metric $d(A,B) = \operatorname{Inf}\{1/2^n \colon A \cap \{1,\ldots,n\} = B \cap \{1,\ldots,n\}\}$.

PROOF OF 12. By the basic properties of homotopy (see e.g. [**Sp**]) $\mathscr{E}$ is an equivalence relation. It is also true that $\mathscr{E}$ is an analytic relation. As X is a compact and metric it is necessarily separable. So let $\{z_n \colon n \in \mathbf{Z}^+\} \subseteq X$ be a dense subset of X. Suppose f_A, f_B are homotopic, then there is a continuous function g from $[0,1] \times [0,1]$ into X, such that for every real r $g(r,0) = f_A(r)$, $g(r,1) = f_B(r)$. [More formally we should have written $f((r,0))$, $f((r,1))$.] We say in such cases that "g exemplifies the homotopy of f_A and f_B". As $[0,1] \times [0,1]$ is compact, g is uniformly continuous, so there is a function $h\colon \mathbf{Z}^+ \to \mathbf{Z}^+$ such that

$$m \in \mathbf{Z}^+, (x_1,y_1), (x_2,y_2) \in [0,1] \times [0,1], |x_1 - x_2| + |y_i - y_i| \\ < 1/h(m) \text{ implies } d(g(x_1,y_1),\ g(x_2,y_2)) < 1/m.$$

We can code g by a set C_g of octuples of positive natural number (i.e. members of $\mathbf{Z}^+$),

$$C_g = \{\langle l_1,k_1,l_2,k_2,n,m,i,j\rangle : l_1,k_1,l_2,k_2 : n,m,i,j \text{ belongs to } \mathbf{Z}^+, \\ d(g(l_1/k_1, l_2/k_2), z_n) < n/m \text{ and } h(i) = j\}.$$

Let $P = \{\langle A,B,C_g\rangle : g$ exemplifies the homotopy of $f_A, f_B\}$ and $\Gamma_8 = \{A\colon A \subseteq (\mathbf{Z}^+)^8\}$ again with the Tichonov topology. It is clear that

(1) C_g determine g (i.e. for every $C \subseteq (\mathbf{Z}^+)^8$ for at most one g as above, $C_g = C$).
(2) Γ_8 is a separable metrizable space.
(3) P is a Borel subset of $\Gamma_8 \times \Gamma \times \Gamma$.
(4) f_A, f_B are homotopic iff for some C, $\langle A,B,C\rangle \in P$.

By (3) and (4) and definition of analytic, we finish.

13. LEMMA. *If $\mathscr{E}$ is an analytic equivalence relation on $\Gamma = \{A\colon A \subseteq \mathbf{Z}^+\}$ which satisfies*

(*) *if A, $B \subset \mathbf{Z}^+$, $n \notin B$, $A = B \cup \{n\}$ then A, B are not $\mathscr{E}$-equivalent,*
then there is a perfect subset of Γ of pairwise nonequivalent $A \subseteq \mathbf{Z}^+$.

REMARK. The proof uses some knowledge of set theory.

PROOF. Let N be a countable elementary submodel of $(H((2^{\aleph_0})^+), \mathscr{E})$ to which the real parameter in the definition of $\mathscr{E}$ belongs. Now

(**) *if* $\langle A_1, A_2\rangle$ be a pair of subsets of $\mathbf{Z}^+$ which is Cohen generic over N [this means that it belongs to no first category subset of $\Gamma \times \Gamma$ which belongs to N] then

(1) A_1, A_2 are $\mathscr{E}$-equivalent in $N[A_1, A_2]$ if they are $\mathscr{E}$-equivalent.
(2) A_1, A_2 are non-$\mathscr{E}$-equivalent in $N[A_1, A_2]$.

PROOF OF (**). (1) By the absoluteness criterions.

(2) If not then some finite information forces this, hence for some n

(a) If $\langle A'_1, A'_2\rangle$ is Cohen generic over N and $A'_1 \cap \{1,\dots,n\} = A_2 \cap \{1,\dots,n\}$ and $A'_2 \cap \{1,\dots,n\} = A_2 \cap \{1,\dots,n\}$ then A'_1, A'_2 are $\mathscr{E}$-equivalent in $N[A'_1, A'_2]$.

Let A''_1 be $A_1 \cup \{n+1\}$ if $(n+1) \notin A_1$ and $A_1 - \{n+1\}$ if $(n+1) \in A_1$.

Trivially also $\langle A''_1, A_2\rangle$ is Cohen generic over N, hence by (α) above A''_1, A_2 are $\mathscr{E}$-equivalent in $N[A''_1, A_2]$. By (**) (1) we know that really A''_1, A_2 are $\mathscr{E}$-equivalent. As equivalence is a transitive relation clearly A_1, A''_1 are $\mathscr{E}$-equivalent. But this contradicts the hypothesis (*).

END OF THE PROOF OF THEOREM 1. We can easily find a perfect (nonempty) subset P of $\{A\colon A \subseteq \mathbf{Z}^+\}$ such that for any distinct A, $B \in P$, (A,B) is Cohen generic over N. So for A, $B \in P$, $N[A,B] \models$ "A,B are not $\mathscr{E}$-equivalent" and by (**) (1) A,B are not $\mathscr{E}$-equivalent. This finishes the proof of 13 hence of Theorem 1. We can similarly [**HSh**, **Sh1**] prove

14. LEMMA. *Suppose $\mathscr{E}$ is a co-κ-Souslin equivalence relation on some κ-Souslin set $\Gamma_1 \subseteq P = \{A\colon A \subseteq \mathbf{Z}^+\}$, or $\Gamma_1 \subseteq (\mathbf{Z}^+)^{(\mathbf{Z}^+)} = \{h\colon h$ a function from $\mathbf{Z}^+$ to $\mathbf{Z}^+\}$ (equivalently ${}^\omega\omega$) and Γ_1 has cardinality $> \kappa$. Suppose further*

(*) *for every $n \in \mathbf{Z}^+$ and function h_0 from $\mathbf{Z}^+$ into $\mathbf{Z}^+$, there are functions $h_1, h_2\colon \mathbf{Z}^+ \to \mathbf{Z}^+$, $\bigwedge_{i=1}^{n} h_0(i) = h_1(i) = h_2(i)$, not $\mathscr{E}$-equivalent but $\{m \in \mathbf{Z}^+\colon h_0(m) \neq h_l(m)\}$ is finite for $l = 1, 2$.*

(**) *If we add a generic Cohen real r to our universe V, in $V[r]$ $\mathscr{E}$ is still an equivalence relation (i.e. its definition defines one).*

Then there is a perfect family of sets $A \subseteq \mathbf{Z}^+$, pairwise non-$\mathscr{E}$-equivalent.

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