# Forcing a countable structure to belong to the ground model

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Suppose that $P$ is a forcing notion, $L$ is a language (in $\mathbb{V}$), $\dot{\tau}$ a $P$-name such that $P \Vdash$ "$\dot{\tau}$ is a countable $L$-structure". In the product $P \times P$, there are names $\dot{\tau}_1$, $\dot{\tau}_2$ such that for any generic filter $G = G_1 \times G_2$ over $P \times P$, $\dot{\tau}_1[G] = \dot{\tau}[G_1]$ and $\dot{\tau}_2[G] = \dot{\tau}[G_2]$. Zapletal asked whether or not $P \times P \Vdash \dot{\tau}_1 \cong \dot{\tau}_2$ implies that there is some $M \in \mathbb{V}$ such that $P \Vdash \dot{\tau} \cong \check{M}$. We answer this question negatively and discuss related issues.

## 1 The isomorphism property

Let us start with describing the motivating question (asked by Zapletal) for this paper:

**Question 1.1** Let $P_1$, $P_2$ be forcing notions, and let $L$ be a language (vocabulary) such that both $P_1$ and $P_2$ force $L$ to be countable. Suppose that $\dot{\tau}_1$, $\dot{\tau}_2$ are, respectively, $P_1$ and $P_2$ names for countable $L$ structures whose universe, we may assume, is $\omega$. We also fix our universe $\mathbb{V}$. Let $\dot{\tau}_1'$ be a name in the forcing notion $P_1 \times P_2$ such that for any generic filter $G = G_1 \times G_2$ for $P_1 \times P_2$, $\dot{\tau}_1'[G] = \dot{\tau}_1[G_1]$, and similarly define $\dot{\tau}_2'$.

> Suppose that $P_1 \times P_2 \Vdash \dot{\tau}_1' \cong \dot{\tau}_2'$. Does it follow that for some $L$-structure $M \in \mathbb{V}$, $P_1 \Vdash \dot{\tau}_1 \cong \check{M}$? $(\star)$

We start by giving some comments on the question: Note that even if $\dot{\tau}_1$ is forced to be a finite structure, it is not immediate that the answer is "yes". Here's an example where we can force a new structure with finite universe. Let $L = \{P_i \mid i < \omega\}$ where $P_i$ are unary predicates. Let $P$ be the Cohen forcing adding one new real $\varepsilon \in \mathbb{V}^P$. Then in $\mathbb{V}^P$, we can define the structure $N$ whose universe is $\{0\}$, and such that $P_i^N = \varnothing$ iff $\varepsilon(i) = 0$. However, it turns out that in this case the answer is "yes" in our situation by Remark 2.8 below, so for simplicity we shall focus on infinite structures.

Note also that if the answer to Question 1.1 is yes, then if $P_2 \Vdash |\check{M}| = \aleph_0$, then also $P_2 \Vdash \dot{\tau}_2 \cong \check{M}$. Indeed, suppose that $G_2$ is a $P_2$-generic filter over $\mathbb{V}$, and that $G_1$ is $P_1$-generic over $\mathbb{V}[G_2]$. Then $G_1 \times G_2$ is $P_1 \times P_2$-generic (cf. [3, Lemma 15.9]) over $\mathbb{V}$, so $\mathbb{V}[G_1 \times G_2] \models \dot{\tau}_2 \cong \check{M}$. But in $\mathbb{V}[G_2]$, the set of pairs $(x, y)$ of elements of $\omega^\omega$ which code isomorphic $L$-structure is analytic (i.e., $\boldsymbol{\Sigma}_1^1$), and analytic properties are absolute between transitive models of ZFC by Mostowski's absoluteness (cf. [3, Theorem 25.4]), so this is true in $\mathbb{V}[G_2]$ as well. Another way to see this is using Scott sentences, cf. Remark 2.7.

Finally let us note that we can reduce the question to the case where $P_1 = P_2$. Consider $P_1 \times P_2 \times P_1$, and, abusing notation, let $\dot{\tau}_1'$, $\dot{\tau}_2'$ be $P_1 \times P_2 \times P_1$-names as above, and let $\dot{\tau}_1''$ be a $P_1 \times P_2 \times P_1$-name such that for any generic filter $G = G_1 \times G_2 \times G_3$ for $P_1 \times P_2 \times P_1$, $\dot{\tau}_1''[G] = \dot{\tau}_1[G_3]$. Then $P_1 \times P_2 \times P_1 \Vdash \dot{\tau}_1' \cong \dot{\tau}_1''$ (because for any such generic filter $G$, $G_1 \times G_2$ is $P_1 \times P_2$-generic and $G_2 \times G_3$ is $P_2 \times P_1$-generic). So for any generic filter $G = G_1 \times G_2 \times G_3$ for $P_1 \times P_2 \times P_1$, in $\mathbb{V}[G]$, $\dot{\tau}_1[G_1] \cong \dot{\tau}_1[G_3]$, and by Mostowski's absoluteness (see above), the same is true in $\mathbb{V}[G_1 \times G_3]$.

In order to simplify the discussion, let us introduce the following definition.

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**Definition 1.2** Suppose $P$ is a forcing notion, $L$ a language such that $P \Vdash |\check{L}| \leq \aleph_0$, and $\dot{\tau}$ is a $P$-name for an $L$-structure with universe $\omega$ such that $P \times P \Vdash \dot{\tau}_1 \cong \dot{\tau}_2$ where $\dot{\tau}_1$ is a $P \times P$-name such that whenever $G = G_1 \times G_2$ is generic for $P \times P$, $\dot{\tau}[G_1] = \dot{\tau}_1[G]$, and where $\dot{\tau}_2$ is defined similarly. Then we say that $(P, L, \dot{\tau})$ has the *isomorphism property*.

In light of the remarks above, we shall focus on the following question:

**Question 1.3** Suppose that $(P, L, \dot{\tau})$ has the isomorphism property. Does it follow that for some $L$-structure $M \in \mathbb{V}$, $P \Vdash \dot{\tau} \cong \check{M}$?

Let us say that a forcing notion $P$ is *good* if the answer to Question 1.3 is "yes" for every $L$ and $\dot{\tau}$. In Corollary 4.5 we give a more natural forcing-theoretic description of such forcing notions: $P$ is good iff $P$ does not collapse $\aleph_2$ to $\aleph_0$.

§ 2 consists of three sub-sections, which mainly serve to recall classical results and to motivate the subsequent sections. In §2.1 we give the necessary facts about product of forcing notions. §2.2 discusses Scott sentences. §2.3 translates the Scott sentence of a structure $M$ to a first order theory $T$ such that $M$ is an atomic model of $T$.

§ 3 translates Question 1.3 to a question on the existence of atomic models. §4 finally answers Question 1.3 negatively in full generality, but positively in some cases (e.g., when $P$ does not collapse $\aleph_2$ to $\aleph_0$). In addition we investigate when atomic models exist under classification theory assumptions. In particular, we prove that if $T$ is a superstable theory and $A$ a subset of a model of $T$ such that the isolated types over $A$ are dense, then: if $\mathrm{MA}_\kappa$ holds, and $|A| \leq \kappa^+$, then there is an atomic model over $A$ (this is Theorem 4.24). Without Martin's Axiom, this is not true (Example 4.12).

§ 5 deals with linear orders, viz. we analyze the situation when $\dot{\tau}$ is forced to be a linear order. We do not reach a definite conclusion but we get some equivalent formulation of the problem.

## 2 Preliminaries

### 2.1 Some remarks on product forcing

The following lemma is easy, and probably well known.

**Lemma 2.1** *Suppose $\mathbb{U}_1$ and $\mathbb{U}_2$ are both transitive models of* ZFC *such that $\mathbb{U}_1 \subseteq \mathbb{U}_2$. Suppose that $P \in \mathbb{U}_1$ is a forcing notion, $\dot{\tau} \in \mathbb{U}_1$ is a $P$-name, and $x \in \mathbb{U}_2$. Then, if $p \Vdash \dot{\tau} = \check{x}$ (in $\mathbb{U}_2$) for some $p \in P$, then $x \in \mathbb{U}_1$.*

P r o o f . We prove the lemma for any $P$, $p$, $\dot{\tau}$ and $x$ by induction on $\beta$ — the rank of $x$ ($\beta$ is the the smallest ordinal $\alpha$ such that $x \in \mathbb{U}_{2,\alpha+1}$). For $\beta = 0$ it is obvious. For $\beta > 0$, since $p \Vdash \dot{\tau} = \check{x}$, then for every $y \in x$, for some (actually for densely many) $q \in P$ stronger than $p$, there is some name $\dot{\tau}' \in \mathbb{U}_1$ ($\dot{\tau}'$ is a member of some pair consisting of a condition and a name in $\dot{\tau}$), such that $q \Vdash \dot{\tau}' = \check{y}$. By the induction hypothesis, $y \in \mathbb{U}_1$. So $x$ is the set of $y$'s in $\mathbb{U}_1$ such that $p \Vdash \check{y} \in \dot{\tau}$, and hence by specification, $x \in \mathbb{U}_1$. $\square$

**Corollary 2.2** *Suppose $P_1$, $P_2$ are forcing notions and that $\dot{\tau}_1$, $\dot{\tau}_2$ are $P_1$ and $P_2$-names, respectively. As usual we let $\dot{\tau}_1'$ and $\dot{\tau}_2'$ be the corresponding names in the product. Then if $(p, q) \Vdash \dot{\tau}_1' = \dot{\tau}_2'$ for some $(p, q) \in P_1 \times P_2$, then for some $x \in \mathbb{V}$, $p \Vdash \dot{\tau}_1 = \check{x}$ and $q \Vdash \dot{\tau}_2 = \check{x}$.*

P r o o f . Let $G_1$ be a generic filter for $P_1$ over $\mathbb{V}$ containing $p$. Let $\mathbb{U}_1 = \mathbb{V}$, $\mathbb{U}_2 = \mathbb{V}[G_1]$, $P = P_2$, $\dot{\tau} = \dot{\tau}_2$ and $x = \dot{\tau}_1[G_1]$. So over $\mathbb{U}_2$, $q \Vdash \dot{\tau} = \check{x}$, and hence by Lemma 2.1, $x \in \mathbb{V}$. So $q \Vdash \dot{\tau}_2 = \check{x}$. Similarly, for some $x' \in \mathbb{V}$, $p \Vdash \dot{\tau}_1 = x'$. Finally, it must be that $x = x'$. $\square$

**Corollary 2.3** *Assume as above that $P_1$, $P_2$ are forcing notions and let $G_1 \times G_2$ be a $P_1 \times P_2$-generic filter. Then $\mathbb{V}[G_1] \cap \mathbb{V}[G_2] = \mathbb{V}$.*

P r o o f . Suppose that $x$ is in the intersection. Let $\dot{\tau}_1$ and $\dot{\tau}_2$ be $P_1$ and $P_2$-names such that $\dot{\tau}_1[G_1] = \dot{\tau}_2[G_2]$, and let $\dot{\tau}_1'$ and $\dot{\tau}_2'$ be the corresponding $P_1 \times P_2$-names. Then for some $(p, q) \in G_1 \times G_2$, $(p, q) \Vdash \dot{\tau}_1' = \dot{\tau}_2'$. By Corollary 2.2, for some $x \in \mathbb{V}$, $p \Vdash \dot{\tau}_1 = \check{x}$ and $q \Vdash \dot{\tau}_2 = \check{x}$, so we are done. $\square$

### 2.2    Scott sentences

Recall that for a countable structure $M$ for a countable language, the *Scott sentence* of $M$ is an $L_{\omega_1,\omega}$-sentence $\Psi$ such that whenever $N \models \Psi$ and $|N| = \aleph_0$, $N \cong M$.

We need a precise set-theoretic definition and coding of $L_{\lambda,\omega}$-formulas in order to continue. As usually done, we can code formulas and terms as objects in $\mathbb{V}$. In the following paragraph, we do not distinguish between a formula (or term) and its code.

For instance, a code for a term is a variable $x \in \mathrm{Var}$ (where Var is an infinite large enough set of variables) or a tuple $(f, t_1, \ldots, t_n)$ where $f$ is an $n$-place function from $L$, and $t_1, \ldots, t_n$ are terms. Similarly we define codes for atomic formulas as tuples $(R, t_1, \ldots, t_n)$ where $R$ is an $n$-place relation symbol and $t_1, \ldots, t_n$ are terms.

We also fix a constant set for the logical symbols $\neg, \bigwedge, \exists$. The code for the negation of an $L_{\lambda,\omega}$-formula $\neg\varphi$ is the pair $(\neg, \varphi)$, and similarly the code for $\exists x \varphi$ is $(\exists, x, \varphi)$. The code for $\bigwedge_{i \in I} \psi_i$ (where $\psi_i$ are formulas, and $|I| < \lambda$) is the pair $(\bigwedge, \{\psi_i \mid i \in I\})$. The connectors $\bigvee$ and $\rightarrow$, and the quantifier $\forall$ are treated as abbreviations.

As usual, $L_{\infty,\omega}$ is the union of $L_{\lambda,\omega}$ running over all $\lambda$.

**Remark 2.4** The property of being a (code for) $L_{\lambda,\omega}$-sentence for some $\lambda$ is absolute. I.e., if $\mathbb{U}_1 \subseteq \mathbb{U}_2$ are transitive models of ZFC having the same ordinals, $x \in \mathbb{U}_1$, then $\mathbb{U}_1 \models$"$x$ is an $L_{\infty,\omega}$-formula (sentence)" iff the same is true in $\mathbb{U}_2$. This can be proved by induction on the rank of $x$ in $\mathbb{U}_1$.

The choice for the coding of $\bigwedge_{i \in I} \psi_i$ to be a set rather than a sequence is important as we shall see now: we give a canonical construction for Scott sentences as in, e.g., [5].

**Proposition 2.5** *Suppose $L$ is a countable language. There is a (class) function Sc whose domain is the class of all countable $L$-structures, and whose range is the set of all (codes for) $L_{\omega_1,\omega}$ sentences such that for all countable $L$-structures $M$, $N$, $M \cong N$ iff $N \models \mathrm{Sc}(M)$ iff $\mathrm{Sc}(M) = \mathrm{Sc}(N)$.*

P r o o f .    We repeat the construction from [5]. Given a countable $L$-structure $M$, we define $\mathrm{Sc}(M)$. By induction on $\alpha < \omega_1$, for every finite tuple $\bar{a} = (a_0, \ldots, a_{n-1}) \in M^{<\omega}$, we define an $L_{\omega_1,\omega}$-formula $\varphi_{\alpha,\bar{a}}(\bar{x})$ with $\bar{x} = (x_0, \ldots, x_{n-1})$ as follows:

For $\alpha = 0$, $\varphi_{\alpha,\bar{a}}(\bar{x}) = \bigwedge\{\varphi(\bar{x}) \mid M \models \varphi(\bar{a}), \ \varphi$ atomic or a negation of an atomic formula$\}$.

For $\alpha$ limit, $\varphi_{\alpha,\bar{a}}(\bar{x}) = \bigwedge\{\varphi_{\beta,\bar{a}}(\bar{x}) \mid \beta < \alpha\}$.

For $\alpha = \beta + 1$,

$$\varphi_{\alpha,\bar{a}}(\bar{x}) = \varphi_{\beta,\bar{a}}(\bar{x}) \wedge \forall x_{n+1} \bigvee\{\varphi_{\beta,\bar{a}\frown b}(\bar{x}, x_{n+1}) \mid b \in M\} \wedge \bigwedge\{\exists x_{n+1}\varphi_{\beta,\bar{a}\frown b}(\bar{x}, x_{n+1}) \mid b \in M\}.$$

Now, one can prove by induction on $\alpha < \omega_1$ that $\bar{a} \models \varphi_{\alpha,\bar{a}}(\bar{x})$.

For some $\beta < \omega_1$, $M \models \forall \bar{x}(\varphi_{\beta,\bar{a}}(\bar{x}) \rightarrow \varphi_{\beta+1,\bar{a}}(\bar{x}))$ for all $\bar{a} \in M^{<\omega}$. Let $\mathrm{Sc}(M)$ be

$$\varphi_{\beta,\varnothing} \wedge \bigwedge\{\forall\bar{x}(\varphi_{\beta,\bar{a}}(\bar{x}) \rightarrow \varphi_{\beta+1,\bar{a}}(\bar{x})) \mid \bar{a} \in M^{<\omega}\}.$$

By a back and forth argument as in [5], if $N \models \mathrm{Sc}(M)$ then $M \cong N$. In fact, if $\bar{a} \in M^{<\omega}$ and $\bar{b} \in N^{<\omega}$, and $M \models \varphi_{\beta,\bar{c}}(\bar{a})$, $N \models \varphi_{\beta,\bar{c}}(\bar{b})$ for some $\bar{c} \in M^{<\omega}$ of the same length as $\bar{a}$ and $\bar{b}$, then there is an isomorphism between $M$ and $N$ taking $\bar{a}$ to $\bar{b}$.

For the other direction, suppose $f : M \rightarrow N$ is an isomorphism. It is easy to see by induction on $\alpha < \omega_1$ that for every $\bar{a} \in M^{<\omega}$, $\varphi_{\alpha,\bar{a}} = \varphi_{\alpha,f(\bar{a})}$ (in the induction step we rely on the choice of coding — as sets and not sequences). In particular, $\mathrm{Sc}(M) = \mathrm{Sc}(N)$. □

**Proposition 2.6** *Assume that $(P, L, \dot{\tau})$ has the isomorphism property. Then there is an $L_{\infty,\omega}$-sentence $\Psi$ in $\mathbb{V}$ such that $P \Vdash$"$\check{\Psi}$ is the Scott sentence of $\dot{\tau}$". Furthermore, there is a complete first order theory $T \in \mathbb{V}$ in the language $L$ such that $P \Vdash \check{T} = \mathrm{Th}(\dot{\tau})$.*

P r o o f .    Let $\dot{\Psi}$ be a name for $\mathrm{Sc}(\dot{\tau})$ in $P$. Then $P \Vdash$"$\dot{\Psi}$ is an $L_{\omega_1,\omega}$-sentence". Let $\dot{\Psi}_1$ and $\dot{\Psi}_2$ be $P \times P$-names such that for any generic $G = G_1 \times G_2$ for $P \times P$, $\dot{\Psi}_1[G] = \dot{\Psi}[G_1]$, and similarly for $\dot{\Psi}_2$. Then $P \times P \Vdash \dot{\Psi}_1 = \dot{\Psi}_2$ by Proposition 2.5. Hence by Corollary 2.2, for some $\Psi \in \mathbb{V}$, $P \Vdash \dot{\Psi} = \check{\Psi}$, so by Remark 2.4, $\Psi$ is an $L_{\infty,\omega}$-sentence and we are done.

The furthermore part, regarding the first order theory, is proved similarly. □

**Remark 2.7** In the slightly different context of §1, where we had two forcing notions $P_1$ and $P_2$ and two names $\dot{\tau}_1$ and $\dot{\tau}_2$, if $P_1 \times P_2 \Vdash \dot{\tau}_1' \cong \dot{\tau}_2'$ and for some $L$-structure $M \in \mathbb{V}$, $P_1 \Vdash \dot{\tau}_1 \cong \check{M}$, then Proposition 2.6 gives another reason why in this case, if $P_2 \Vdash |\check{M}| = \aleph_0$, then also $P_2 \Vdash \dot{\tau}_2 \cong \check{M}$ (without using Mostowski's absoluteness). Why? As in the proof of said proposition, we can show that the Scott sentence $\Psi$ of $\dot{\tau}_1$ (which is the same as the one of $\dot{\tau}_2$) is in $\mathbb{V}$. But then $M \models \Psi$ (in $\mathbb{V}$) so $P_2 \Vdash \check{M} \models \Psi$ and we are done by Proposition 2.5.

**Remark 2.8** Why did we restrict to the case where the structures are infinite? If $(P, L, \dot{\tau})$ has the isomorphism property but $P \Vdash |\dot{\tau}| < \omega$, then the first order theory $T$ in $L$ which $P$ forces to be $\text{Th}(\dot{\tau})$ is in $\mathbb{V}$ by Proposition 2.6. Hence $T$ is a consistent first order theory in $L$, and hence has a model $M \in \mathbb{V}$. But now $P \Vdash \check{M} \equiv \dot{\tau}$, and as $M$ is finite, they must be isomorphic.

### 2.3    From a Scott sentence to a theory

The outline for this subsection is as follows. First we show the well known result that an $L_{\lambda^+,\omega}$-sentence $\Psi$ is "the same thing as" a first order theory $T_\Psi$ and a collection of types to be omitted. We then show that when $\Psi$ is a Scott sentence of a structure $M$, then in the language of $T_\Psi$, $M$ is an atomic model of $T_\Psi^{\text{co}}$ — the natural completion of $T_\Psi$. In the end of this section we remark that $T_\Psi^{\text{co}}$ can be constructed directly from $T_\Psi$ in an absolute way, without passing through $M$ (hence, when $M$ is constructed in some generic extension but $\Psi$ is already in the ground model, then so is $T_\Psi^{\text{co}}$).

**Lemma 2.9** *Suppose that $\Psi$ is a consistent $L_{\lambda^+,\omega}$-sentence in the language $L$. Then there is a language $L_\Psi$, a consistent first order $L_\Psi$-theory $T_\Psi$ of size $\leq \lambda$, a collection $\Gamma_\Psi$ of partial $T_\Psi$-types and a canonical bijection $H_\Psi$ which respects isomorphisms between the class of $L$-structures $M \models \Psi$ and the class of $L_\Psi$ structures $N \models T_\Psi$ such that $N$ omits all the types in $\Gamma_\Psi$.*

P r o o f .  Recall that for an $L_{\infty,\omega}$ formula $\psi$, a sub-formula is an $L_{\infty,\omega}$ formula that appears in the construction of $\psi$. So, e.g., sub-formulas of $\psi = \bigwedge\{\varphi_i \mid i \in I\}$ are $\psi$ and sub-formulas of $\varphi_i$ for some $i$, but <u>not</u> any conjunction $\bigwedge\{\varphi_i \mid i \in I'\}$ for $I' \subseteq I$.

We may assume that $|L| \leq \lambda$, by restricting to symbols which appear in $\Psi$. We may also assume that $\Psi$ contains as sub-formulas all formulas of the form $P(x_0, \ldots, x_{n-1})$ and $F(x_0, \ldots, x_{n-1}) = x_n$ for every $n$-place relation symbol $P$ and $n$-place function symbol $F$. Otherwise, we replace $\Psi$ with $\Psi \wedge \varphi$ where $\varphi$ is a big conjunction of sentences of the form $\forall \bar{x}(P(\bar{x}) \vee \neg P(\bar{x}))$ and $\forall \bar{x} \exists y F(\bar{x}) = y$.

Let $L_\Psi$ be comprised of $n$-ary relation symbols $R_\varphi$ for every sub-formula $\varphi(x_0, \ldots, x_{n-1})$ of $\Psi$ (note that any sub-formula has a finite number of free variables as $\Psi$ is a sentence). So $|L_\Psi| \leq \lambda$. The theory $T_\Psi$ will have the axioms:

1. $\exists x_n R_\varphi(x_0, \ldots, x_n) \leftrightarrow R_{\exists x_n \varphi}(x_0, \ldots, x_{n-1})$ whenever $\exists x_n \varphi(x_0, \ldots, x_n)$ is a sub-formula of $\Psi$.
2. $\neg R_\varphi \leftrightarrow R_{\neg\varphi}$ whenever $\neg\varphi$ is a sub-formula of $\Psi$.
3. $R_{\bigwedge\{\varphi_i \mid i \in I\}} \rightarrow R_{\varphi_i}$ for every $i \in I$ whenever $\bigwedge\{\varphi_i \mid i \in I\}$ is a sub-formula of $\Psi$.
4. $R_\Psi$.

The set of types $\Gamma_\Psi$ consists of types of the form $\Sigma_\Phi = \{R_{\varphi_i}(\bar{x}) \mid i \in I\} \cup \{\neg R_{\bigwedge\{\varphi_i \mid i \in I\}}(\bar{x})\}$ whenever $\Phi = \bigwedge\{\varphi_i(\bar{x}) \mid i \in I\}$ is a sub-formula of $\Psi$ (and $\bar{x}$ a tuple of variables).

Finally, given an $L$-structure $M$, the induced $L_\Psi$-structure $M_* = H_\Psi(M)$ has the same universe, and for every $R_\varphi \in L_\Psi$, $R_\varphi^{M_*} = \varphi^M$. Note that $M_*$ omits all the types in $\Gamma_\Psi$, and since $\Psi$ is consistent, $T_\Psi$ is consistent. Also, if $M_1 \cong M_2$ then $H_\Psi(M_1) \cong H_\Psi(M_2)$.

Conversely, given a model $N$ of $T_\Psi$ which omits all types in $\Gamma_\Psi$, define an $L$-structure $N_*$, which in fact will be $H_\Psi^{-1}(N)$, by $P^{N_*} = (R_{P(x_0, \ldots, x_{n-1})})^N$ for every $n$-place predicate $P \in L$, and similarly for every $n$-place function symbol $F \in L$. Note that this map also respects isomorphisms. $\qquad\square$

**Remark 2.10** The map $\Psi \mapsto (L_\Psi, T_\Psi, \Gamma_\Psi, H_\Psi)$ is absolute in the sense that if $\Psi \in \mathbb{V}$ and $\mathbb{U}$ is a transitive model of ZFC extending $\mathbb{V}$, then $(L_\Psi, T_\Psi, \Gamma_\Psi)$ are the same in $\mathbb{U}$ as in $\mathbb{V}$ and $H_\Psi$ defines the same function when restricted to $\mathbb{V}$. This is easily seen by induction on the rank of $\Psi$.

Recall the following definitions.

**Definition 2.11** Let $T$ be a consistent first order theory (perhaps incomplete), and let $\Sigma(x_0, \ldots, x_{n-1})$ be a partial type. We say that a formula $\varphi(x_0, \ldots, x_{n-1})$ *isolates* $\Sigma$ in $T$ if $T \cup \{\exists \bar{x} \varphi(\bar{x})\}$ is consistent and $T \vdash \varphi \rightarrow \psi$ for every $\psi \in \Sigma$. The partial type $\Sigma$ is *isolated* in $T$ if some formula isolates it in $T$.

**Definition 2.12** A structure $M$ for a language $L$ is called *atomic* if for every finite tuple $\bar{a} \in M^{<\omega}$, $\mathrm{tp}(\bar{a}/\varnothing)$ is isolated in $\mathrm{Th}(M)$. Similarly, we say that a set $A \subseteq M$ is *atomic* if for every finite tuple $\bar{a} \in A^{<\omega}$, $\mathrm{tp}(\bar{a}/\varnothing)$ is isolated in $\mathrm{Th}(M)$. We add "over $B$" for some set $B \subseteq M$ to mean that we replace $\varnothing$ with $B$.

**Proposition 2.13** *Suppose that $M$ is a countable structure, and that $\Psi = \mathrm{Sc}(M)$ is its Scott sentence (cf. Proposition 2.5). Then the induced theory $T_\Psi$ is countable and has a natural completion, $T_\Psi^{\mathrm{co}} = \mathrm{Th}(H_\Psi(M))$. Furthermore, the induced $L_\Psi$-structure $H_\Psi(M)$ is atomic.*

P r o o f .   The theory $T_\Psi$ is countable since $\Psi$ is in $L_{\omega_1,\omega}$. This also implies that $\Gamma_\Psi$ is countable.

In order to show that $H_\Psi(M)$ is atomic, take any tuple $\bar{a} \in M^{<\omega}$. We shall show that $R_{\varphi_{\beta,\bar{a}}}$ isolates $\mathrm{tp}(\bar{a}/\varnothing)$ (recall the construction of $\Psi = \mathrm{Sc}(M)$ in Proposition 2.5). Suppose that $\vartheta(\bar{x}) \in \mathrm{tp}(\bar{a}/\varnothing)$, and $H_\Psi(M) \models \exists \bar{x}(R_{\varphi_{\beta,\bar{a}}} \wedge \neg \vartheta(\bar{x}))$. Suppose that $\bar{b} \models R_{\varphi_{\beta,\bar{a}}}(\bar{x}) \wedge \neg \vartheta(\bar{x})$ where $\bar{b}$ is from $M$. But then $M \models \varphi_{\beta,\bar{a}}(\bar{b})$, so there is an automorphism $\sigma$ of $M$ which takes $\bar{b}$ to $\bar{a}$ (cf. the proof of Proposition 2.5). But then $\sigma$ is also an automorphism of $H_\Psi(M)$—a contradiction.    $\square$

**Remark 2.14** In the same context of Proposition 2.13, the completion $T_\Psi^{\mathrm{co}}$ can also be constructed directly from $T_\Psi$ without passing through $M$. Viz. for an ordinal $\alpha$ define a consistent $L_\Psi$-theory $T_\Psi^\alpha$ by induction on $\alpha$ by: $T_\Psi^0 = T_\Psi$; for $\alpha$ limit, take $T_\Psi^\alpha = \bigcup_{\beta < \alpha} T_\Psi^\beta$; finally, for $\alpha = \beta + 1$, define

$$T_\Psi^\alpha := T_\Psi^\beta \cup \{\neg \exists \bar{x}(\varphi(\bar{x})) \mid \varphi(\bar{x}) \text{ isolates } \Sigma_\Phi \text{ in } T_\Psi^\beta \text{ for some sub-formula } \Phi \text{ of } \Psi\}.$$

(Recall $\Sigma_\Phi$ from the construction of $T_\Psi$ in Lemma 2.9 above.)

Since $L_\Psi$ is countable we must get stuck at some countable ordinal $\alpha$, and we let $T_\Psi^{\mathrm{co}} = T_\Psi^\alpha$. Since $H_\Psi(M)$ omits all the types in $\Gamma_\Psi$, $H_\Psi(M) \models T_\Psi^\beta$ for all $\beta$, and in particular $H_\Psi(M) \models T_\Psi^\alpha$. The types in $\Gamma_\Psi$ are not isolated in $T_\Psi^\alpha$ (if $\varphi(\bar{x})$ isolates some $\Sigma_\Phi$ in $T_\Psi^\alpha$, then $T_\Psi^{\alpha+1} = T_\Psi^\alpha$ includes $\neg \exists \bar{x}(\varphi(\bar{x}))$). Moreover, this is true for any consistent extension $T_\Psi^\alpha \cup \{\vartheta\}$ for a sentence $\vartheta$ (if $\varphi(\bar{x})$ isolates some type in $T_\Psi^\alpha \cup \{\vartheta\}$, then $\varphi(\bar{x}) \wedge \vartheta$ isolates it in $T_\Psi^\alpha$). By the omitting type theorem for countable languages (cf. [12, Theorem 4.2.4]), $T_\Psi^\alpha \cup \{\vartheta\}$ has a countable model $N$ which omits all the types in $\Gamma_\Psi$, and so by Lemma 2.9, $H_\Psi^{-1}(N) \models \Psi$, and so $H_\Psi^{-1}(N) \cong M$ and hence $N \cong H_\Psi(M)$. We conclude that for any such $\vartheta$, $T_\Psi^\alpha \vdash \vartheta$. Hence $T_\Psi^\alpha$ is complete. This construction of $T_\Psi^{\mathrm{co}}$ is absolute from $\Psi$.

# 3   Translating the question

Here we shall translate Question 1.3 from §1 to a question about the existence of atomic models.

Suppose that $(P, L, \dot{\tau})$ has the isomorphism property. By Proposition 2.6, there is an $L_{\infty,\omega}$ sentence $\Psi_{\dot{\tau}}$ in $\mathbb{V}$ which $P$ forces to be $\tau$'s Scott sentence. It makes sense then to let $L_{\Psi_{\dot{\tau}}}$, $T_{\Psi_{\dot{\tau}}}$, $T_{\Psi_{\dot{\tau}}}^{\mathrm{co}}$ and $\Gamma_{\Psi_{\dot{\tau}}}$ be the induced language, theories and collection of types as in Lemma 2.9. By Remarks 2.10 & 2.14, they are the same in $\mathbb{V}$ as in $\mathbb{V}[G]$ for any generic filter $G$.

Recall the following definition.

**Definition 3.1** Suppose $T$ is a first order theory. We say that the *isolated types are dense in $T$* if whenever $\varphi(\bar{x})$ is a consistent formula, i.e., $T \cup \{\exists \bar{x} \varphi\}$ is consistent (where $\bar{x}$ is a finite tuple of variables), there is a consistent formula $\vartheta(\bar{x})$ such that $T \vdash \vartheta \rightarrow \varphi$ and $\vartheta$ isolates a complete type: for every formula $\psi(\bar{x})$, either $T \models \vartheta \rightarrow \psi$ or $T \models \vartheta \rightarrow \neg \psi$.

**Fact 3.2** Suppose that $T$ is a countable complete theory, then the following are equivalent:

1. The theory $T$ has a countable atomic model.
2. The isolated types are dense in $T$.

P r o o f .   Cf. [12, Theorem 4.2.10].    $\square$

**Fact 3.3** Suppose that $M$ and $N$ are countable atomic models of a countable complete theory $T$. Then $M \cong N$ [12, Theorem 4.2.8, Corollary 4.2.16].

**Theorem 3.4** *Suppose that $P$ is a forcing notion, $L$ a language which $P$ forces to be countable, and $\dot{\tau}$ a P-name for an L-structure with universe $\omega$ such that $(P, L, \dot{\tau})$ has the isomorphism property. Then the following are equivalent:*

1. *For some $M \in \mathbb{V}$, $P \Vdash \dot{\tau} \cong \check{M}$.*
2. *The theory $T^{\mathrm{co}}_{\Psi_{\dot{\tau}}}$ has an atomic model.*

P r o o f .  (1) $\Rightarrow$ (2): By (1), for some $M \in \mathbb{V}$, $P \Vdash \dot{\tau} \cong \check{M}$. Then $M \models \Psi_{\dot{\tau}}$, and $H_{\Psi_{\dot{\tau}}}(M) \models T^{\mathrm{co}}_{\Psi_{\dot{\tau}}}$. Let $G$ be a generic filter for $P$. Then by Proposition 2.13, in $\mathbb{V}[G]$, $H_{\Psi_{\dot{\tau}}}(M)$ is an atomic model of $T^{\mathrm{co}}_{\Psi_{\dot{\tau}}}$, but being an atomic model is absolute, hence the same is true in $\mathbb{V}$.

(2) $\Rightarrow$ (1): Let $G$ be any generic filter. Then by Proposition 2.13, in $\mathbb{V}[G]$, $T^{\mathrm{co}}_{\Psi_{\dot{\tau}}}$ is countable and $H_{\Psi_{\dot{\tau}}}(\dot{\tau}[G])$ is an atomic model of $T^{\mathrm{co}}_{\Psi_{\dot{\tau}}}$. By (2), $T^{\mathrm{co}}_{\Psi_{\dot{\tau}}}$ has an atomic model $N$ which we can assume has cardinality $\leq |L_{\Psi_{\dot{\tau}}}| = \aleph_0^{\mathbb{V}[G]}$ by taking an elementary substructure. Let $M = H_{\Psi_{\dot{\tau}}}^{-1}(N)$. In $\mathbb{V}[G]$, $N$ is countable, so isomorphic to $H_{\Psi_{\dot{\tau}}}(\dot{\tau}[G])$ in $\mathbb{V}[G]$ by Fact 3.3, hence $M \cong \dot{\tau}[G]$. $\square$

Recall that a forcing notion $P$ is good when the answer to Question 1.3 is "yes" for every $L$ and $\dot{\tau}$: whenever $(P, L, \dot{\tau})$ has the isomorphism property, for some $L$-structure $M \in \mathbb{V}$, $P \Vdash \dot{\tau} \cong \check{M}$.

**Theorem 3.5** *Suppose that $P$ is a forcing notion. Then the following are equivalent:*

1. *$P$ is good.*
2. *For every complete first order theory $T$ such that the isolated types are dense in $T$ and $P \Vdash |\check{T}| = \aleph_0$, $T$ has an atomic model.*

P r o o f .  (1) $\Rightarrow$ (2): Suppose $T$ is a complete first order theory in a language $L$ in which the isolated types are dense, and $P \Vdash |T| = \aleph_0$. Let $\dot{\tau}$ be a name for a countable atomic model of $T$ (which exists by Fact 3.2). Then by Fact 3.3, $(P, L, \dot{\tau})$ has the isomorphism property. By (1), for some $M \in \mathbb{V}$, $P \Vdash \dot{\tau} \cong \check{M}$. Let $G$ be a generic filter for $P$. Then in $\mathbb{V}[G]$, $M$ is an atomic model of $T$, but being an atomic model of $T$ is absolute, hence the same is true in $\mathbb{V}$.

(2) $\Rightarrow$ (1): Suppose that $(P, L, \dot{\tau})$ has the isomorphism property. By Theorem 3.4, it is enough to show that $T^{\mathrm{co}}_{\Psi_{\dot{\tau}}}$ has an atomic model, so by (2) it is enough to show that the isolated types are dense. Let $G$ be a generic filter. Then by Proposition 2.13, in $\mathbb{V}[G]$, $T^{\mathrm{co}}_{\Psi_{\dot{\tau}}}$ is complete and countable and $H_{\Psi_{\dot{\tau}}}(\dot{\tau}[G])$ is an atomic model of $T^{\mathrm{co}}_{\Psi_{\dot{\tau}}}$. Hence by fact 3.2, the isolated types are dense in $T^{\mathrm{co}}_{\Psi_{\dot{\tau}}}$. But this is an absolute property, hence the same is true in $\mathbb{V}$. $\square$

## 4   On the existence of atomic models

In §4.1 we give a general criterion for when $P$ is good. In §§4.2 & 4.3 we investigate when an atomic model exists under classification theoretic assumption.

### 4.1   A criterion: collapsing $\aleph_2$ to $\aleph_0$

Having translated Question 1.3 to a question on the existence of atomic models in Theorem 3.5, we can now start to provide some answers. In Corollary 4.3, we provide a positive answer (i.e., $P$ is good), provided that $P$ does not collapse $\aleph_2$ to $\aleph_0$. Conversely, in Corollary 4.5 we prove that if $P$ does collapse $\aleph_2$ to $\aleph_0$ then $P$ is not good.

First we note that by Fact 3.2 the following is immediate.

**Corollary 4.1** *If $P$ is a forcing notion that does not collapse $\aleph_1$ then $P$ is good.*

For completeness we provide a proof of the following proposition, which is really an adaptation of [14, Chapter IV, Theorem 5.5]. This theorem was also proved independently by Knight [7, Theorem 1.3] and Kueker [8, p. 168].

**Proposition 4.2** *Suppose $|T| = \aleph_1$ and the isolated types are dense in $T$. Then $T$ has an atomic model.*

P r o o f .    Let $M \models T$ be $\aleph_1$-saturated. Construct an atomic set $N = \{b_i \mid i < \omega_1\} \subseteq M$ such that for any formula $\varphi(x, \bar{b})$, with $\bar{b}$ a finite tuple from $N$, if $M \models \exists x \varphi(x, \bar{b})$ then for some $c \in N$, $M \models \varphi(c, \bar{b})$. Then $N$ is an atomic model of $T$. By an easy book-keeping argument, it is enough to show that if $A \subseteq M$ is a countable atomic set, $\bar{b}$ is a finite tuple from $A$, and $M \models \exists x \varphi(x, \bar{b})$, then for some $c \in M$, $M \models \varphi(c, \bar{b})$ and $A \cup \{c\}$ is atomic.

For a consistent formula $\psi(\bar{x})$, choose a consistent formula $\vartheta_\psi(\bar{x})$ which isolates a complete type and implies $\psi$.

Enumerate $A = \{a_i \mid i < \omega\}$, and assume that $\bar{b} = (a_0, \ldots, a_{n-1})$. For $i < \omega$, let $\bar{a}_i = (a_0, \ldots, a_{i-1})$, and let $\vartheta_i(\bar{z}_i)$ isolate $\mathrm{tp}(\bar{a}_i)$. Construct a sequence of formulas $\psi_i(x, \bar{z}_i)$ such that $\psi_n(x, \bar{z}_n) \to \varphi(x, \bar{z}_n)$; for all $i \geq n$, $\psi_i$ is consistent; $T \models \psi_{i+1} \to \psi_i$; $\psi_i$ isolates a complete type and $\bar{a}_i \models \exists x \psi_i(x, \bar{z}_i)$. The construction: $\psi_n = \vartheta_{\varphi(x, \bar{z}_n) \wedge \vartheta_n(\bar{z}_n)}$ and $\psi_{i+1} = \vartheta_{\psi_{i(x, \bar{z}_i)} \wedge \vartheta_{i+1}(\bar{z}_{i+1})}$. Finally, $\{\psi_i(x, \bar{a}_i) \mid n \leq i < \omega\}$ is consistent, so let $c \in M$ satisfy this type.                                                       □

**Corollary 4.3** *If $P$ is a forcing notion that does not collapse $\aleph_2$ to $\aleph_0$ then $P$ is good.*

**Fact 4.4** (Laskowski & Shelah; [11]).  There is a complete first order theory $T$ with a sort $V$ whose elements form an indiscernible set in any model $M$ of $T$, such that for any set $A \subseteq V^M$, the isolated types in $T(A)$ are dense but if $|A| \geq \aleph_2$, $T(A)$ has no atomic model.

**Corollary 4.5** *If $P$ collapses $\aleph_2$ to $\aleph_0$ then $P$ is not good. In conclusion, $P$ is good iff $P$ does not collapse $\aleph_2$ to $\aleph_0$.*

## 4.2   Totally transcendental theories

**Definition 4.6** Recall that a complete first order theory $T$ is called *totally transcendental* if there is no sequence of formulas $\langle \varphi_s(\bar{x}, \bar{y}_s) \mid s \in 2^{<\omega} \rangle$ such that in some model $M$ of $T$ there are tuples $\langle \bar{a}_s \mid s \in 2^{<\omega} \rangle$ such that for each $\eta \in 2^\omega$ the type $\{\varphi_{\eta \restriction n}(\bar{x}, \bar{a}_{\eta \restriction n})^{\eta(n)} \mid n < \omega\}$ is consistent (where $\varphi^0 = \neg \varphi$, $\varphi^1 = \varphi$).

**Fact 4.7** If $T$ is totally transcendental, then $T$ has an atomic model [15, Lemma 5.3.4, Corollary 5.3.7].

Note that for a complete first order theory $T$, being totally transcendental is an absolute property. This follows from the fact that having an infinite branch in a tree is an absolute property. Viz. define the tree $\Sigma$ consisting of finite families of sequences of formulas of the form $\langle \varphi_s(\bar{x}, \bar{y}_s) \mid s \in 2^{<n} \rangle$ such that it is consistent with $T$ that there are tuples $\langle \bar{a}_s \mid s \in 2^{<n} \rangle$ with the property that for each $t \in 2^n$ the type $\{\varphi_{t \restriction k}(\bar{x}, \bar{a}_{t \restriction k})^{t(k)} \mid k < n\}$ is consistent. The order between two such finite families is

$$\langle \varphi_s(\bar{x}, \bar{y}_s) \mid s \in 2^{<n} \rangle \leq \langle \psi_s(\bar{x}, \bar{y}_s) \mid s \in 2^{<m} \rangle$$

iff $n \leq m$ and for each $s \in 2^{<n}$, $\varphi_s = \psi_s$. This is easily seen to be a set-theoretic tree. Then $T$ is totally transcendental iff $\Sigma$ has no infinite branch. This is a $\Delta_1$ property, and hence absolute, cf. [3, Lemma 13.11].

**Definition 4.8** Recall that a complete first order theory $T$ in a countable language is called *$\omega$-stable* if for every countable model $M \models T$, the number of complete 1-types over $M$ is at most $\aleph_0$.

It is easy to see that if $T$ is $\omega$-stable then it is also totally transcendental. The converse is also true when $T$ is countable. Cf. [15, Theorem 5.2.6].

**Corollary 4.9** *Suppose that $(P, L, \dot{\tau})$ has the isomorphism property, and that $\Psi = \Psi_{\dot{\tau}} \in L_{\infty, \omega}$ is the Scott sentence of $\dot{\tau}$. Let $T_\Psi^{\mathrm{co}}$ be the induced theory. Then if $P \Vdash$ "$T_\Psi^{\mathrm{co}}$ is $\omega$-stable", then for some $M \in \mathbb{V}$, $P \Vdash \dot{\tau} \cong \check{M}$.*

P r o o f .    Since $P$ forces that $T_\Psi^{\mathrm{co}}$ is $\omega$-stable and countable, it follows that $P \Vdash$ "$T_\Psi^{\mathrm{co}}$ is totally transcendental". But then $T_\Psi^{\mathrm{co}}$ is totally transcendental by absoluteness. So $T_\Psi^{\mathrm{co}}$ has an atomic model by Fact 4.7, and hence by Theorem 3.4 we are done.                                                       □

Warning: it is tempting to think that if $(P, L, \dot{\tau})$ has the isomorphism property and $P \Vdash$ "Th$(\dot{\tau})$ is $\omega$-stable" then $T_{\Psi_{\dot{\tau}}}^{co}$ is $\omega$-stable. However this is not the case.

**Example 4.10** It is well known that there is a graph on $\omega$ with language $\{S\}$ (so $S$ is a 2-place relation) such that the full theory $(\mathbb{N}, +, \cdot, 0, 1)$ can be interpreted in $(\omega, S)$; cf., e.g., [2, Theorem 5.5.1]. Let $L = \{P, Q, \pi_1, \pi_2\} \cup \{Q_k \mid k < \omega\}$ where $P$, $Q$ are unary predicates, $\pi_1, \pi_2$ are unary function symbols, and for $k < \omega$, $Q_k$ is a unary predicate. Let $M$ be the following $L$-structure: its universe is the union of $P^M \cup Q^M$ where $P^M = \omega$ and

$$Q^M = \{(n, m, \alpha) \mid n, m < \omega, \alpha \le \omega, (n \, S \, m \to \alpha < \omega)\}.$$

The function $\pi_1^M : Q^M \to P^M$ is the projection to the first coordinate ($\pi_1^M(n, m, \alpha) = n$) and $\pi_2^M : Q^M \to P^M$ is the projection to the second coordinate (meaning that on $P^M$, $\pi_1$ and $\pi_2$ are the identity). Finally, $Q_k^M = \{(n, m, k) \mid n, m < \omega\}$. Let $T = \text{Th}(M)$.

It is easy to see that $T$ has quantifier elimination. If $N \equiv M$ is a countable model, then the number of 1-types over $N$ is countable: given $c \notin N$ in some elementary extension, the type of $c$ over $N$ is determined as follows. If $c \in P$ its type is the unique (non-algebraic) type. Otherwise the type of $c$ is determined by the unique $k < \omega$ so that $c \in Q_k$ (if there is any) and by the type of the pair $(\pi_1(c), \pi_2(c))$ over $N$. Hence $T$ is $\omega$-stable.

Let $\Psi = \Psi_M$ be the Scott sentence of $M$. For any $a \in Q^M \setminus \bigcup_{k<\omega} Q_k^M$, $\bigwedge_{k<\omega} \neg Q_k(x)$ holds. Even though formally $\varphi(x) = \bigwedge_{k<\omega} \neg Q_k(x)$ is not a sub-formula of $\Psi$ (because $\varphi_{0,a}$ contains the full atomic type of $a$), $\varphi^M$ is a definable set in $H_\Psi(M)$. It follows that in $H_\Psi(M)$, $\omega$ is definable (by $R_P$) and also $S$: $n \, S \, m$ iff $R_P(n)$, $R_P(m)$ and $\forall z \in R_Q((\pi_1(z) = n \land \pi_2(z) = m) \to \neg(\varphi^M(z)))$. So a model of $T_\Psi$ can interpret the full theory of arithmetic. In particular, $T_\Psi$ is not $\omega$-stable.

### 4.3 Superstable theories

Recall the following definition.

**Definition 4.11** A complete first order theory $T$ is superstable if there exists some cardinal $\lambda$ such that for all model $M \models T$, $|S_1(M)| \le |M| + \lambda$.

We start by giving an example, similar in spirit to the one in Fact 4.4, but here we replace $\aleph_2$ with $(2^{\aleph_0})^+$, and the theory involved is superstable.

**Example 4.12** (Laskowski) Let $N \in \omega$. Let $L_N = \{U, V, \pi\} \cup \{E_n \mid n < N + 1\}$ where $U$, $V$ are unary predicates, $\pi$ is a unary function symbol and for $n < N + 1$, $E_n$ is a binary relation. Let $T_N^\forall$ be the following theory:

1. The sets $U$, $V$ are disjoint.
2. $\pi : U \to V$.
3. For each $n < N$: $E_n$ is an equivalence relation on $U$; $E_{n+1} \subseteq E_n$; $E_{n+1}$ has at most two classes in any $E_n$ class; $E_0$ has just one class.

Now, $T_N^\forall$ is a universal theory with the amalgamation and disjoint embedding properties. Hence by, e.g., [2, Theorem 7.4.1], it has a model completion, $T_N$, which eliminates quantifiers and is $\omega$-categorical. One can also check that $T_{N+1} \supseteq T_N$. Let $T = \bigcup_{N<\omega} T_N$.

**Claim 4.13** The theory $T$ is superstable but not $\omega$-stable.

P r o o f . If $M$ is a countable model of $T$, then for each $n < \omega$, $M$ contains representatives for all the $2^n$ classes of $E_n$. Hence there are $2^\omega$ many types over $M$, so $T$ is not $\omega$-stable. However, it is superstable by quantifier elimination, as there are at most $2^{\aleph_0} + \lambda$ 1-types over a model of size $\lambda$. $\square$

**Claim 4.14** The isolated types are dense in $T$.

P r o o f . Suppose that $\psi(x_0, \ldots, x_{n-1})$ is a consistent formula in $T$. Suppose $\psi$ is in $L_{N+1}$ for some $N$. Let $\bar{a} = a_0, \ldots, a_{n-1}$ realize $\psi$ in some model $M \models T$ and assume that $\bar{a} \subseteq U^M$. There are $2^N$ many $E_N$-classes. Let us partition $\bar{a}$ into these classes, and suppose the largest such class has $l$ elements. Let $K \ge N$ be such that

$2^{K-N} \geq l$, so each $E_N$ class contains at least $l$ distinct $E_K$-classes. Let $\bar{b} = b_0, \ldots, b_{n-1}$ in $M$ have the same type as $\bar{a}$ in $L_{N+1}$ but such that $b_i$ and $b_j$ are not $E_K$-equivalent for $i \neq j$ (such a $\bar{b}$ exists since $M{\restriction}L_{K+1}$ is an existentially closed model of $T^{\forall}_{K+1}$, and by our choice of $K$). Let $p(\bar{x})$ be the quantifier free type of $\bar{b}$ in $L_{K+1}$, so it is a finite set, and let $\vartheta(\bar{x}) = \bigwedge p$. Then $T \models \vartheta \to \psi$ and $\vartheta$ isolates a complete type (since if $\bar{c} \models \vartheta$, then the partition of $\bar{c}$ by any equivalence relation $E_i$ is determined by $\vartheta$). If $\bar{a}$ contains also some elements from $V$, then a simple adjustment of the above argument will work.    $\square$

Hence $T$ has an atomic model $M_0$ by Fact 3.2. Note that, by quantifier elimination, in any model $N \models T$, $V^N$ is an infinite indiscernible set: all sets of elements of size $n$ have the same type. Let $N \models T$ be of size $\geq (2^{\aleph_0})^+$ and assume that $N$ contains some $A \subseteq V^N$ of size $\geq (2^{\aleph_0})^+$. Let $T(A)$ be the expansion of $T$ by the full theory of $N$ in the language $L(A)$ (where we add constants for elements of $A$).

**Claim 4.15** The theory $T(A)$ is superstable of size $|A|$. The isolated types are dense in $T(A)$ but $T(A)$ has no atomic model.

P r o o f .    The first sentence is clear. To see that the isolated types are dense in $T(A)$, one can either repeat the proof of Claim 4.14, or note that since $A$ is an indiscernible set, given a consistent formula $\varphi(\bar{x}, \bar{a})$ ($\bar{a}$ is a tuple from $A$), there is some tuple $\bar{b}$ in $V^{M_0}$ with the same type as $\bar{a}$. Suppose that $\psi(\bar{y})$ isolates the type of $\bar{b}$. There is a formula $\vartheta(\bar{x}, \bar{y})$ which isolates a type and implies $\varphi(\bar{x}, \bar{y}) \wedge \psi(\bar{y})$, so $\vartheta(\bar{x}, \bar{b})$ is consistent, implies $\varphi(\bar{x}, \bar{b})$ and isolates a complete type and hence so is $\vartheta(\bar{x}, \bar{a})$.

Suppose $N \models T(A)$ is atomic. Since $\pi$ is onto (which follows from our choice of $T$), $|U^N| > 2^{\aleph_0}$. Hence there are $a, b \in U^N$ such that $a \, E_n \, b$ for all $n < \omega$. But easily, the type $\mathrm{tp}(a, b)$ is not isolated by quantifier elimination.    $\square$

This example gives a weaker version of Corollary 4.5.

**Corollary 4.16** *Example* 4.12 *shows that if $P$ collapses $(2^{\aleph_0})^+$ to $\aleph_0$ then $P$ is not good, and (2) in Theorem* 3.5 *can be witnessed with a superstable theory.*

Now, we shall show that under Martin's Axiom, if $T$ is superstable, then such an example as in 4.4 cannot exist. Let $(P, <)$ be a partial order. We use the standard interpretation of "strength" in $P$ when we think of it as a forcing notion, i.e., $a$ is stronger than $b$ if $a < b$.

**Definition 4.17** For an infinite cardinal $\kappa$, let $\mathsf{MA}_\kappa$ (Martin's Axiom for $\kappa$) be the following statement:

> If $(P, <)$ is a partially ordered set that satisfies the countable chain condition and if $D$ is a collection of at most $\kappa$ dense subsets of $P$, then there exists a $D$-generic filter $G$ on $P$ (i.e., $G$ meets every element of $D$, $G$ is downward directed: if $p, q \in G$ then there is a condition stronger than both $p, q$, and if $r$ is weaker than $p$, then $r$ is in $G$).

**Remark 4.18** We shall in fact need a weakening of Martin's Axiom, viz., $\mathsf{MA}(\mathrm{Cohen})_\kappa$, which states that whenever $D$ is a collection of at most $\kappa$ dense subsets of the Cohen forcing (adding one real), then there is a $D$-generic filter $G$ on $P$. It is well known that any non-trivial countable forcing notion is forcing-equivalent to Cohen forcing (i.e., they generate the same generic extension). Hence $\mathsf{MA}(\mathrm{Cohen})_\kappa$ is equivalent to $\mathsf{MA}(\mathrm{countable})_\kappa$ (the restriction of $\mathsf{MA}_\kappa$ to countable forcing notions). Note that $\mathsf{MA}(\mathrm{Cohen})_\kappa$ implies that $\kappa < 2^{\aleph_0}$.

To prove the next theorem we shall also have to recall the following definition [14, II, Definition 1.1]:

**Definition 4.19** Let $T$ be a complete first order theory with monster model $\mathfrak{C}$. Let $p(\bar{x})$ be a partial type. We define by induction on $\alpha$ when is $R_\infty(p) \geq \alpha$ as follows:

1. $R_\infty(p) \geq 0$ iff $p$ is consistent.
2. When $\delta$ is a limit ordinal, then $R_\infty(p) \geq \delta$ iff $R_\infty(p) \geq \alpha$ for all $\alpha < \delta$.
3. $R_\infty(p) \geq \alpha + 1$ iff for every finite $q \subseteq p$, and for every cardinal $\mu$, there are partial types $\{q_i(\bar{x}) \mid i < \mu\}$ such that $R_\infty(q_i \cup q) \geq \alpha$, and $q_i \cup q_j$ are explicitly inconsistent for $i \neq j$ (i.e., there is a formula $\varphi(\bar{x}, \bar{y})$ such that for some $\bar{a} \in \mathfrak{C}$, $\varphi(\bar{x}, \bar{a}) \in q_i$, $\neg\varphi(\bar{x}, \bar{a}) \in q_j$).

We shall need some facts about forking in order to continue. Let $T$ be a complete first order theory, and suppose that $\mathfrak{C}$ is its monster model. Given a set $A \subseteq \mathfrak{C}$ (whose size is, as usual, smaller than $|\mathfrak{C}|$) there is a class of formulas

with parameters in $\mathfrak{C}$ which are called the *forking formulas* over $A$. The precise definition can be found in, e.g., [15, Definition 7.1.7]. Given a tuple $\bar{a}$ and sets $A \subseteq B \subseteq \mathfrak{C}$, we write $\bar{a} \underset{A}{\downarrow} B$ when $p = \mathrm{tp}(\bar{a}/B)$ does not fork over $A$, meaning that no formula in $p$ forks over $A$. For our purposes we shall need the following facts:

**Fact 4.20** Let $A \subseteq B \subseteq \mathfrak{C}$. A formula forks over $A$ iff it forks over $\mathrm{acl}(A)$ iff any equivalent formula forks over $A$, so forking is a property of definable sets and not of formulas. The set of formulas over $B$ which fork over $A$ form an ideal (a finite disjunction of forking formulas over $A$ forks over $A$, if $\psi$ forks over $A$ and $\varphi \vdash \psi$ then $\varphi$ forks over $A$, and $\bar{x} \neq \bar{x}$ forks over $A$). As a result, if $q(\bar{x})$ is a partial type over $B$ which does not fork over $A$, then there is a complete type $q \subseteq p$ over $B$ which does not fork over $A$ (this is a non-forking extension of $q$).

If $T$ is stable, then any type over $A$ does not fork over $A$. If moreover $A = \mathrm{acl}^{\mathrm{eq}}(A)$ (here we assume $\mathfrak{C} = \mathfrak{C}^{\mathrm{eq}}$), then any complete type over $A$ has a unique non-forking extension to $B$ [15, §8.5].

The connection between forking and ranks is given in:

**Fact 4.21** Suppose that $p(\bar{x})$ is a partial type over $A$, $\varphi(\bar{x}, \bar{a})$ a formula and $R_\infty(p) = R_\infty(p \cup \{\varphi\}) < \infty$. Then $\varphi$ does not fork over $A$ [14, III, Lemma 1.2].

**Fact 4.22** A complete theory $T$ is superstable iff for any formula $\varphi(x_0, \ldots, x_{n-1})$ (perhaps with parameters), $R_\infty(\varphi) < \infty$. Superstable theories are stable [14, II, Theorem 3.14].

Before stating the main theorem, let us recall another fact:

**Lemma 4.23** *If $M$ is a structure and $A \subseteq M$ is an atomic set, then so is $\mathrm{acl}(A)$.*

P r o o f . The proof is an easy exercise in the definitions. $\square$

**Theorem 4.24** *Assume* $\mathrm{MA(Cohen)}_\kappa$. *Suppose $T$ is superstable and countable, and $A \subseteq M \models T$ has size $\leq \kappa^+$. If the isolated types are dense in $T(A)$, then $T(A)$ has an atomic model.*

P r o o f . Let $\mathfrak{C} \models T$ be a monster model of $T$ containing $A$ (so $\mathfrak{C}$ is a reduct of the monster model of $T(A)$ to the language $L$ of $T$). We may assume that $\mathfrak{C} = \mathfrak{C}^{\mathrm{eq}}$. As in the proof of Proposition 4.2, it is enough to show that if $B \subseteq \mathfrak{C}$ is an atomic set over $A$ of size $\leq \kappa$, $\bar{b}$ is a finite tuple from $B$, $\bar{a}$ a finite tuple from $A$, and $\mathfrak{C} \models \exists x \varphi(x, \bar{b}, \bar{a})$, then for some $c \in \mathfrak{C}$, $\mathfrak{C} \models \varphi(c, \bar{b}, \bar{a})$ and $B \cup \{c\}$ is atomic over $A$.

Since $T$ is superstable, there is some consistent formula $\psi(x, \bar{c})$ over $D = \mathrm{acl}^{\mathrm{eq}}(A \cup B)$ such that $\psi \vdash \varphi$, and $\alpha = R_\infty(\psi)$ is minimal among all consistent formulas over $D$ which imply $\varphi$. Let $C = \mathrm{acl}^{\mathrm{eq}}(\bar{c}) \subseteq D$, so $C$ is a countable set (the algebraic closure is taken in $T$ and not in $T(A)$). Let $\pi(x)$ be the partial type over $D$ consisting of all formulas of the form $\neg\chi(x, \bar{e})$ where $\bar{e}$ is from $D$ and $\chi$ forks over $C$ (equivalently, $\chi$ forks over $\bar{c}$).

We shall say that two formulas $\xi(x), \zeta(x)$ (with parameters from $\mathfrak{C}$) *are equivalent modulo $\pi$* if $\pi \vdash \zeta \leftrightarrow \xi$.

**Claim 4.25** If $\xi(x, \bar{d})$ is a consistent formula over $D$ such that $\xi \vdash \psi$ then $\xi$ is equivalent modulo $\pi$ to a consistent formula over $C$.

P r o o f . By choice of $\psi$, $R_\infty(\xi) = R_\infty(\psi)$. By Fact 4.21, $\xi(x)$ does not fork over $C$. Suppose $\mathfrak{C} \models \xi(e)$ for some $e \models \pi$. Let $p = \mathrm{tp}(e/C)$. Then $q = \mathrm{tp}(e/D)$ is a non-forking extension of $p$, but by Fact 4.20, $q$ is the unique non-forking extension of $p$, so $\pi \cup p \vdash \xi$, and by compactness, for some $\zeta_e \in p$, $\pi \cup \{\zeta_e\} \vdash \xi$. Hence by compactness, $\xi$ is equivalent modulo $\pi$ to a finite disjunction of such formulas $\zeta_i(x)$ over $C$. $\square$

Let $P$ be the set of consistent formulas $\xi(x)$ over $C$ which imply $\psi$. We define an order $<$ on $P$ by: $\xi < \zeta$ iff $\mathfrak{C} \models \xi \to \zeta$. Equivalently, $\pi \vdash \xi \to \zeta$ (as $\xi$ and $\zeta$ are formulas over $C$ — if $a \models \xi$ and $\pi \vdash \xi \to \zeta$, then $p = \mathrm{tp}(a/C)$ does not fork over $C$, so we can find some $a' \models p$ such that $a' \models \pi$, so $a' \models \zeta$, and hence $\zeta \in p$, so $a \models \zeta$). Note that $P$ is countable.

Fix some finite tuple $\bar{d}$ from $B$. Let $X_{\bar{d}}$ be the set of formulas $\xi$ from $P$ such that $\pi \vdash \xi(x) \to \beta(x, \bar{d}', \bar{a})$ for some formula $\beta$, where $\bar{a}$ is a finite tuple from $A$ and $\bar{d}'$ is a finite tuple from $D$ containing $\bar{d}$, such that $\beta(x, \bar{y}, \bar{a})$ isolates a complete type over $A$.

We claim that $X_{\bar{d}}$ is dense in $P$. Indeed, let $\zeta(x, \bar{e}) \in P$ (so $\bar{e}$ is a finite tuple from $C$). Since $\bar{e}$ is algebraic over $\bar{c}$, by Lemma 4.23, $\mathrm{tp}(\bar{e}\bar{d}/A)$ is isolated, say by $\vartheta_{\bar{e}\bar{d}}(\bar{y}, \bar{z}, \bar{a}')$. Let $\beta'(x, \bar{y}, \bar{z}, \bar{a}') = \zeta(x, \bar{y}) \wedge \vartheta_{\bar{e}\bar{d}}(\bar{y}, \bar{z}, \bar{a}')$. This is a consistent formula, so by assumption there is some consistent $\beta(x, \bar{y}, \bar{z}, \bar{a}) \vdash \beta'(x, \bar{y}, \bar{z}, \bar{a}')$ which isolates a complete type over $A$, where $\bar{a}$ is a finite tuple from $A$. Note that $\beta(x, \bar{e}, \bar{d}, \bar{a})$ is consistent and implies $\zeta(x, \bar{e})$. By

Claim 4.25, $\beta(x, \bar{e}, \bar{d}, \bar{a})$ is equivalent modulo $\pi$ to a consistent formula $\xi(x, \bar{e}')$ over $C$ from $P$. Then $\pi \vdash \xi \rightarrow \zeta$ and $\xi \in X_{\bar{d}}$ so we are done.

By $\mathsf{MA(Cohen)}_\kappa$, there is some $\{X_{\bar{d}} \mid \bar{d} \in B^{<\omega}\}$-generic filter $G \subseteq P$. Note that $G$ is a type (i.e., consistent). This is because all the elements of $P$ are consistent and because $G$ is downward directed.

Being a partial type over $C$, $G$ does not fork over $C$, so $G \cup \pi$ is consistent. Let $c \models G \cup \pi$.

First of all, $c \models G$ and any formula in $G$ implies $\psi(x, \bar{c})$, which implies $\varphi(x, \bar{b}, \bar{a})$, so $\mathfrak{C} \models \varphi(c, \bar{b}, \bar{a})$. Now, suppose that $\bar{d} \frown \langle c \rangle$ is some finite tuple from $B \cup \{c\}$. By choice of $G$, there is some $\xi \in G \cap X_{\bar{d}}$, so for some $\beta(x, \bar{d}', \bar{a})$ as above, $\beta(c, \bar{d}', \bar{a})$ holds (since $c \models \pi$), $\bar{d}' = \bar{d} \frown \bar{d}''$ for some $\bar{d}''$, and $\beta(x, \bar{y}, \bar{y}'', \bar{a})$ isolates a complete type over $A$. Then $\exists \bar{y}'' \beta(x, \bar{y}, \bar{y}'', \bar{a}) \in \mathrm{tp}(c\bar{d}/A)$ isolates a complete type, so $B \cup \{c\}$ is atomic. $\qquad\square$

**Remark 4.26** In the stable case, our methods allow us to construct only a locally atomic model (without Martin's Axiom, but still assuming that the underlying language is countable). This is a classical result by Lachlan, cf. [9], later improved upon by Newelski [13], where he replaces the assumption that the theory is countable by a weaker one.

By Fact 4.7, the remark after Definition 4.8, Example 4.12 and Theorem 4.24 we conclude:

**Corollary 4.27** *Suppose $T$ is a countable first order theory, $A \subseteq M \models T$ and $\kappa$ is a cardinal such that the isolated types in $T(A)$ are dense and $|A| \le \kappa^+$. Then:*

1. *If $T$ is $\omega$-stable then $T(A)$ has an atomic model.*
2. *If $T$ is superstable and $\mathsf{MA(Cohen)}_\kappa$ holds then $T(A)$ has an atomic model.*
3. *If $T$ is superstable but $\mathsf{MA(Cohen)}_\kappa$ does not hold then $T(A)$ may not have an atomic model.*

**Problem 4.28** Does Theorem 4.24 hold when $T$ is stable?

## 5   On linear orders

In this section we shall try to focus on the particular case when the structures involved are linear orders. Assume that $L = \{<\}$ (where $<$ is a binary relation symbol). Let $P$ be a forcing notion, and $\dot{\tau}$ a $P$-name for an infinite linear order with universe $\omega$. Again we ask: suppose that $(P, L, \dot{\tau})$ has the isomorphism property (cf. Definition 1.2). Does it follow that for some linear order $I = (X, <) \in \mathbb{V}$, $P \Vdash \check{I} \cong \dot{\tau}$?

In private communications with Zapletal, we were told that he solved the problem positively for set-theoretic trees (i.e., structures of the form $(X, <)$ where $<$ is a partial order and the set below every element is well-ordered).

In §5.1 we translate the problem to that of finding an atomic model in certain classes of theories, contained in a bigger class $\mathbf{K}_\mu$. In §5.2 we shall further analyze these classes and give an equivalent definition. In §5.3, we give an example of a theory in $\mathbf{K}_\mu$, for $\mu = (2^{\aleph_0})^+$, with no atomic model.

### 5.1   The class $\mathbf{K}_\mu$

Let $\mathrm{Coll}(\mu, \aleph_0)$ be the Levy collapse of $\mu$ to $\aleph_0$. Note that as this forcing notion is weakly homogeneous, for every first order sentence $\psi$ in the language of set theory with parameters from $\mathbb{V}$, $\mathbb{V}[G_1] \models \psi$ iff $\mathbb{V}[G_2] \models \psi$ for any two generic filters $G_1$ and $G_2$. Hence we may write $\mathbb{V}^{\mathrm{Coll}(\mu, \aleph_0)} \models \psi$ meaning $\mathbb{V}[G] \models \psi$ for any generic $G$. Cf. [4, Proposition 10.19].

**Definition 5.1** For a cardinal $\mu$, let $L_\mu = \{<\} \cup \{P_i \mid i \in \mu\} \cup \{Q_j \mid j \in \mu\}$ where $P_i$ are unary predicates and $Q_j$ binary relation symbols. let $\mathbf{K}_\mu$ be the class of complete first order theories $T$ in a language $L_T = \{<\} \cup \{P_i \mid i \in u_T\} \cup \{Q_j \mid j \in v_T\} \subseteq L_\mu$ equipped with a function $f_T : v^2 \to v$ such that

> the theory $T$ says that $<$ is a linear order; the sets $P_i$ are nonempty and disjoint; the sets $Q_j$ are nonempty and disjoint; $Q_j(x, y) \vdash x < y$; *additivity* of $T$: for all $x < y < z$, $Q_{j_1}(x, y) \wedge Q_{j_2}(y, z) \vdash Q_{f_T(j_1, j_2)}(x, z)$; the formula $P_i(x)$ isolates a complete type for $i \in u_T$; the formula $Q_j(x, y)$ isolates a complete type for $j \in v_T$; the isolated types are dense in $T$, and for any (equivalently, some) countable atomic model $M$ of $T$ in $\mathbb{V}^{\mathrm{Coll}(\mu, \aleph_0)}$, the sets $P_i^M$ and $Q_j^M$ form a partition of $M$, $<^M$ respectively.

**Remark 5.2** Suppose that $T \in \mathbf{K}_\mu$. Then for every $n < \omega$, if a formula of the form $\bigwedge_{i<n} Q_{j_i}(x_i, x_{i+1})$ is consistent, then it isolates a complete type. Indeed, it is enough to show that this is true in a countable atomic model $M$ of $T$ in $\mathbb{V}^{\mathrm{Coll}(\mu, \aleph_0)}$. So let $\bar{x} = (x_0, \ldots, x_n)$ and $\bar{y} = (y_0, \ldots, y_n)$ be finite increasing tuples from $M$ such that $Q_{j_i}^M(x_i, x_{i+1})$ and $Q_{j_i}^M(y_i, y_{i+1})$ hold for all $i < n$. Then, as $Q_{j_i}$ isolate complete types, and as atomic models are homogeneous, there are automorphisms $f_i$ of $M$ which take $x_i, x_{i+1}$ to $y_i, y_{i+1}$. Let

$$f = f_0 \upharpoonright (-\infty, x_1] \cup f_1 \upharpoonright (x_1, x_2] \cup \ldots \cup f_{n-1} \upharpoonright (x_{n-1}, \infty).$$

Then, as the $Q_j$'s form a partition of $<^M$, it follows that $f$ is an automorphism of $M$ by the additivity of $T$. It follows that these types are dense in $T$.

Now note that an equivalent definition to Definition 5.1 can be obtained by replacing the additivity requirement, the requirement that the isolated types are dense, and the requirement that the $P_i$'s and $Q_j$'s form a partition of a countable atomic model (after the collapse) by asking that formulas of the form $\bigwedge_{i<n} Q_{j_i}(x_i, x_{i+1})$, if consistent, isolate a complete type, and that types of such forms, as well as $P_i(x)$ are dense in $T$ (we should also allow formulas of the form $P_i(x) \wedge x = y$, etc., to be completely formal).

For notational simplicity it is useful to write $Q(a, b) = j$ for $a \neq b \in M \models T$ when $j \in v_T$ is the unique element such that $(a, b) \in Q_j^M$ or $(b, a) \in Q_j^M$, and similarly $P(a) = i$ if $a \in P_i^M$.

We define two special subclasses of $\mathbf{K}_\mu$.

**Definition 5.3** Let $\mathbf{K}_\mu^+$ be the class of theories $T \in \mathbf{K}_\mu$ such that for any $j \in v_T$ there is an $L_{\mu^+, \omega}$-formula $\psi_j(x, y)$ in the language $\{P_i \mid i \in u_T\} \cup \{<\}$ such that for any countable atomic model $M$ of $T$ in $\mathbb{V}^{\mathrm{Coll}(\mu, \aleph_0)}$, $\psi_j^M = Q_j^M$. (After the collapse, $\psi_j$ is an $L_{\omega_1, \omega}$-formula.)

Let $\mathbf{K}_\mu^*$ be the class of theories $T \in \mathbf{K}_\mu$ such that for any $j \in v_T$ there is an $L_{\mu^+, \omega}$-formula $\psi_j(x, y)$ in the language $\{<\}$ such that for any countable atomic model $M$ of $T$ in $\mathbb{V}^{\mathrm{Coll}(\mu, \aleph_0)}$, $\psi_j^M = Q_j^M$.

As usual, this definition does not depend on the choice of atomic model or the generic extension. Note that $\mathbf{K}_\mu \supseteq \mathbf{K}_\mu^+ \supseteq \mathbf{K}_\mu^*$.

**Remark 5.4** If $T \in \mathbf{K}_\mu^*$, then for every $i \in u_T$, there is a formula $\varphi_i$ in $L_{\mu^+, \omega}$ such that in any countable atomic model $M$ of $T$ in $\mathbb{V}^{\mathrm{Coll}(\mu, \aleph_0)}$, $\varphi_i^M = P_i^M$. Why? Suppose that $M$ is a countable atomic model of $T$ in $\mathbb{V}^{\mathrm{Coll}(\mu, \aleph_0)}$. Let $a \in P_i^M$, and let $\Psi_a(x)$ be the Scott formula which isolates the complete $L_{\omega_1, \omega}$-type of $a$ in $\{<\}$ (in the context of Proposition 2.5, this is $\varphi_{\beta, a}$). As $P_i$ isolates a complete type, this formula does not depend on $a$, so we define $\varphi_i = \Psi_a$. In addition, $\varphi_i$ does not depend on $M$, and hence by 2.2, $\varphi_i$ is in $\mathbb{V}$. By definition, $P_i^M \subseteq \varphi_i^M$ for any such $M$. It remains to show that if $a \models \varphi_i$ in some such $M$, and $a \in P_{i'}^M$, then $i' = i$. If not, by choice of $\varphi_i$, there is some $b \models \varphi_i$ in $P_i^M$, and suppose that $a < b$. Let $j \in v_T$ be such that $(a, b) \in Q_j^M$. Since $a$ and $b$ satisfy $\varphi_i$, there is an automorphism $\sigma$ of $M \upharpoonright \{<\}$ taking $a$ to $b$. Then by the definition of $\mathbf{K}_\mu^*$, $(\sigma(a), \sigma(b)) \in Q_j^M$. However $Q_j$ isolates a complete type, so it must be that $Q_j(x, y) \vdash P_{i'}(x)$ and hence $b = \sigma(a) \in P_{i'}^M$ so $i = i'$.

The following theorem translates the question of finding a linear order with the isomorphism property which is not realized in $\mathbb{V}$ to the question of finding a theory in $\mathbf{K}_\mu^+$ without an atomic model, as in §3. It seems that in order to produce a counterexample, $\mathbf{K}_\mu^+$ allows more freedom than $\mathbf{K}_\mu^*$ (as it contains more theories), but in the end they are equivalent in this sense.

**Theorem 5.5** *Let $\mu$ be some cardinal. The following are equivalent:*

1. *For any forcing notion $P$ such that $P \not\Vdash |(\check{\mu}^+)| < \omega_1$ and any $P$-name $\dot{\tau}$ such that $P$ forces $\dot{\tau}$ to be a linear order on $\omega$, if $(P, \{<\}, \dot{\tau})$ has the isomorphism property, then for some linear order $I = (X, <) \in \mathbb{V}$, $P \Vdash \check{I} \cong \dot{\tau}$.*
2. *Every theory $T \in \mathbf{K}_\mu^+$ has an atomic model of size $\mu$.*
3. *Every theory $T \in \mathbf{K}_\mu^*$ has an atomic model of size $\mu$.*

P r o o f . $(1) \Rightarrow (2)$: Suppose that $T \in \mathbf{K}_\mu^+$. Let $P = \mathrm{Coll}(\mu, \aleph_0)$ be the Levy collapse of $\mu$ to $\aleph_0$. As the isolated types are dense in $T$, $P$ forces that $T$ has a countable atomic model, so let $\dot{M}$ be a name for it. Now let us work in $\mathbb{V}^P$, and let $M \in \mathbb{V}[G]$ be a countable atomic model of $T$ with universe $\omega$.

Let us introduce the following notation. For a family $\{(X_i, <_i) \mid i \in I\}$ of linear orders, where $I$ is linearly ordered by $\prec$, we let $\sum_{i \in I} X_i$ be the linear order whose universe is the disjoint union of the sets $X_i$ and the order $<$ is such that $< \restriction X_i = <_i$ and for $i \neq j$, $a \in X_i$ and $b \in X_j$, $a < b$ iff $i \prec j$ (in other words, it is the lexicographical sum of $X_i$ for $i \in I$).

Denote by $\mathbb{Q}$ the usual dense linear order on the rational numbers. Define the linear order $X_M = (X, <)$ as the sum of linear orders $\sum_{a \in M} X_a$ where for each $a \in M$, $X_a = \mathbb{Q} + (P(a) + 2)$ (recall that $P(a)$ is the unique $i \in u_T$ — now a countable ordinal — such that $a \in P_i$. It is well defined because in an atomic model, the $P_i$'s form a partition). For instance, if $a \in P_0$, then $X_a = \mathbb{Q} + \{0, 1\}$.

Let $\dot{\tau}$ be a $P$-name for $X_{\dot{M}}$. First we claim that $(P, \{<\}, \dot{\tau})$ has the isomorphism property. This follows easily from the fact that after the collapse, if $M \cong N$ are two countable atomic models of $T$, then $X_M \cong X_N$ (as linear orders). Now, $P \times P \Vdash \dot{M}_1 \cong \dot{M}_2$ (where, as usual, $\dot{M}_1$ is the $P \times P$-name for $\dot{M}[G_1]$ where $G_1 \times G_2$ is a generic for $P \times P$, etc.), so in $\mathbb{V}^{P \times P}$, $\dot{\tau}_1 \cong \dot{\tau}_2$ as desired.

By (1), there is some linear order $I = (X, <) \in \mathbb{V}$ such that $P \Vdash \check{I} \cong \dot{\tau}$. Now we want to recover an atomic model of $T$ from $I$.

Let $X_0 \subseteq X$ be the set of all $x \in X$ with a densely ordered open neighborhood without endpoints, and let $X_1 = X \setminus X_0$. For each $x \in X_1$, there are unique $x_0 = x_0(x) \leq x \leq x_1(x) = x_1$ such that $x_0 < x_1$; $x_0, x_1 \in X_1$; every point in $[x_0, x_1)$ has a successor; for every $z < x_0$ there is some $y \in (z, x_0) \cap X_0$, and similarly, for every $x > x_1$ there is some $y \in (x_1, x) \cap X_0$.

Why? This sentence is absolute and since it is true after the collapse, it is also true in $\mathbb{V}$. We now know that after the collapse, the closed interval $[x_0, x_1]$ is well-ordered and has the same order type as $i + 2$ for some countable ordinal $i \in u_T$, so in $\mathbb{V}$ the same is true (except that now $i < \mu^+$).

There is a natural convex equivalence relation $\sim$ on $X_1$: two points are equivalent if they define the same $x_0$ and $x_1$. Let $Y = X_1 / \sim$, and define an $L_T$-structure $N$ with universe $Y$ such that

$$P_i^N = \{[x]_\sim \in Y \mid \text{otp}\,[x_0(x), x_1(x)] = i + 2\},$$

and such that $Q_j^N$ is defined using the $L_{\mu^+, \omega}$-formula promised in Definition 5.3.

Now, by absoluteness of the construction, $N$ is also the result of this construction in $\mathbb{V}^P$, and there, as $I \cong X_M$, we get that $M \restriction \{<\} \cup \{P_i \mid i \in u_T\}$ is isomorphic to $N \restriction \{<\} \cup \{P_i \mid i \in u_T\}$, and hence $N \cong M$ and we are done.

$(2) \Rightarrow (3)$: Immediate as $\mathbf{K}_\mu^* \subseteq \mathbf{K}_\mu^+$.

$(3) \Rightarrow (1)$: Suppose that $(P, \{<\}, \dot{\tau})$ has the isomorphism property. Let $G$ be a generic filter for $P$, and let $M = \dot{\tau}[G]$. For each $a \in M$, let $\Psi_a(x)$ be as in Remark 5.4 and similarly define $\Psi_{a,b}(x, y)$ for $a < b$ in $M$. Let $\mathcal{U} = \{\Psi_a \mid a \in M\}$ and let $\mathcal{V} = \{\Psi_{a,b} \mid a < b \in M\}$. By Corollary 2.2, $\mathcal{U}$ and $\mathcal{V}$ are in $\mathbb{V}$ (although here they are sets of $L_{\mu^+, \omega}$-formulas) and their size is at most $\mu$ by the assumption on $P$. Enumerate (in $\mathbb{V}$) $\mathcal{U} = \{\psi_i \mid i \in u\}$ and $\mathcal{V} = \{\xi_j \mid j \in v\}$ where $u, v$ are subsets of $\mu$.

Let $L = \{<\} \cup \{P_i \mid i \in u\} \cup \{Q_j \mid j \in v\}$. Let $M'$ be an expansion of $M$ to $L$ defined by $P_i^{M'} = \{x \in M \mid x \models \psi_i\}$ and $Q_j^{M'} = \{(x, y) \in <^M \mid (x, y) \models \xi_j\}$. Note that $M'$ is atomic and that formulas of the form $\bigwedge_{i < n} Q_{j_i}(x_i, x_{i+1})$, if consistent, isolate a complete type, and the same is true for formulas of the form $P_i(x)$ (this follows from the fact that any automorphism of $M$ is also an automorphism of $M'$).

Let $T = \text{Th}(M')$. By Corollary 2.2, $T \in \mathbb{V}$. We claim that $T \in \mathbf{K}_\mu^*$. By Definition 5.1 and Remark 5.2, $T \in \mathbf{K}_\mu$ and $T \in \mathbf{K}_\mu^*$ as witnessed by $\xi_j$. By (3), $T$ has an atomic model $N' \in \mathbb{V}$. Hence in $\mathbb{V}[G]$, $N' \cong M'$. It follows that $N' \restriction \{<\}$ is isomorphic to $M$ in $\mathbb{V}[G]$. $\qquad\square$

## 5.2　More on $K_\mu^*$ and $K_\mu^+$

**Definition 5.6** Suppose that $T \in \mathbf{K}_\mu$. Suppose that $E$ is some equivalence relation on $v_T$. By induction on $\alpha < \mu^+$, define equivalence relations $\sim_T^{E, \alpha}$ as follows:

For $\alpha = 0$, $\sim_T^{E, 0} = E$.

For $\alpha$ limit, $j_1 \sim_T^{E, \alpha} j_2$ iff for all $\beta < \alpha$, $j_1 \sim_T^{E, \beta} j_2$.

For $\alpha = \beta + 1$, $j_1 \sim_T^{E, \alpha} j_2$ iff $j_1 \sim_T^{E, \beta} j_2$ and there exists a countable atomic model $M \in \mathbb{V}^{\text{Coll}(\mu, \aleph_0)}$ of $T$ such that

1. for any $(a, b) \in Q^M_{j_1}$, $(a', b') \in Q^M_{j_2}$, and for any $c \in M\backslash\{a, b\}$ there is some $c' \in M\backslash\{a', b'\}$ such that $abc$ and $a'b'c'$ have the same order type, and $Q(a, c) \sim^{E,\beta}_T Q(a', c')$ and $Q(c, b) \sim^{E,\beta}_T Q(c', b')$ and

2. the same, replacing $j_1$ with $j_2$.

Note that reflexivity of $\sim^{E,\alpha}_T$ follows from the fact that $Q_j(x, y)$ isolates a complete type in $T$ (and from the fact that a countable atomic model is homogeneous).

Let us say that an equivalence relation $E$ on $v_T$ is *additive* if for all $j_1, j_2$ and $j'_1, j'_2$ in $v_T$ such that $Q_{j_1}(x, y) \wedge Q_{j_2}(y, z)$ and $Q_{j'_1}(x, y) \wedge Q_{j'_2}(y, z)$ are both consistent, $j_1 \; E \; j'_1 \wedge j_2 \; E \; j'_2 \Rightarrow f_T(j_1, j_2) \; E \; f_T(j'_1, j'_2)$. Say that $E$ is *edge-preserving* if whenever $x < y$, $x' < y'$ and $Q(x, y) \; E \; Q(x', y')$ then $\text{tp}(x/\varnothing) = \text{tp}(x'/\varnothing)$ and $\text{tp}(y/\varnothing) = \text{tp}(y'/\varnothing)$.

**Claim 5.7** Suppose that $E$ is some equivalence relation on $v_T$, where $T \in \mathbf{K}_\mu$.

(1) For all $\alpha < \mu^+$, $\sim^{E,\alpha+1}_T \subseteq \sim^{E,\alpha}_T$. Hence for some $\alpha = \alpha(T, E)$, $\sim^{E,\alpha+1}_T = \sim^{E,\alpha}_T$.

(2) Definition 5.6 does not depend on the choice of a generic filter $G$ for $\text{Coll}(\mu, \aleph_0)$ because $\text{Coll}(\mu, \aleph_0)$ is weakly homogeneous (see above). Recall that $\text{Coll}(\mu, \aleph_0)$ is the collection of partial functions with finite domain from $\aleph_0$ to $\mu$. Also note that any permutation $\sigma$ of $\aleph_0$ induces an automorphism (which we shall also denote by $\sigma$) of $\text{Coll}(\mu, \aleph_0)$, and that we can expand $\sigma$ to act on the class of $\text{Coll}(\mu, \aleph_0)$-names. Now, for some such $\sigma$, $\sigma(p_1)$ and $p_2$ are compatible, and by the discussion above, $\sigma(p_1)$ still forces the same thing (because $j_1, j_2, E^\beta_T \in \mathbb{V}$), so we immediately get a contradiction.

(3) If $E$ is additive, then so is $\sim^{\alpha,E}_T$ for any $\alpha < \mu^+$.

(4) In Definition 5.6, we can do any of the following changes and get an equivalent definition:

(a) Replace "there exists a countable atomic model" (which exists since the isolated types are dense), by "for any countable atomic model" (as any two are isomorphic).

(b) Replace the items by "for some $(a, b) \in Q^M_{j_1}$, $(a', b') \in Q^M_{j_2}$ the following holds. For any $c \in M\backslash\{a, b\}$, there is some $c' \in M\backslash\{a', b'\}$ such that $abc$ and $a'b'c'$ have the same order type, $Q(a, c) \sim^{E,\beta}_T Q(a', c')$ and $Q(c, b) \sim^{E,\beta}_T Q(c', b')$, and vice versa."

(c) If $E$ is edge-preserving and additive, then we can replace $M\backslash\{a, b\}$ and $M\backslash\{a', b'\}$ by the intervals $(a, b)$ and $(a', b')$ in the items.

P r o o f . (1) is clear, (2) is explained in the claim. For (3) use induction on $\alpha$ and the fact that $T$ is additive. (4) (a) is explained above, (4) (b) follows by the fact that $Q_j$ isolates a complete type and by homogeneity, (4) (c) is proved by induction on $\alpha$ using (3). □

As usual, assume that $T \in \mathbf{K}_\mu$. Given an equivalence relation $E$ on $v_T$, say that $E$ is *definable* in some sub-language $\{<\} \subseteq L' \subseteq L_T$ if for every $E$-class $C \subseteq v_T$ there is some $L_{\mu^+,\aleph_0}$-formula $\psi_C$ in $L'$ such that if $M \in \mathbb{V}^{\text{Coll}(\mu,\aleph_0)}$ is a countable atomic model of $T$, then $\psi^M_C = C^M = \bigcup\{Q^M_j \mid j \in C\}$.

**Proposition 5.8** *Suppose that $T \in \mathbf{K}_\mu$ and $E$ is some equivalence relation on $v_T$. Assume that $E$ is definable in some sub-language $\{<\} \subseteq L' \subseteq L$. Then for every $\alpha < \mu^+$, $\sim^{E,\alpha}_T$ is also definable in $L'$.*

P r o o f . The proof is by induction on $\alpha < \mu^+$. For $\alpha = 0$ this is given.

Suppose $\alpha > 0$ is a limit. Then for each $\sim^{E,\alpha}_T$-class $C$ there is a sequence $\langle D_\beta \mid \beta < \alpha \rangle$ of $\sim^{E,\beta}_T$-classes such that $C = \bigcap_{\beta<\alpha} D_\beta$, and for any countable atomic model $M \in \mathbb{V}^{\text{Coll}(\mu,\aleph_0)}$, $(x, y) \in C^M$ iff $(x, y) \in D^M_\beta$ for all $\beta < \alpha$.

Suppose $\alpha = \beta + 1$. Then for each $\sim^{E,\alpha}_T$-class $C$ there is an $\sim^{E,\beta}_T$-class $D$ and a set $A$ of tuples of the form $(r, D_1, D_2)$ where $r$ is an order type of three points, and $D_1, D_2$ are $\sim^{E,\beta}_T$-classes such that for any countable atomic $M \in \mathbb{V}^{\text{Coll}(\mu,\aleph_0)}$, $(x, y) \in C^M$ iff $(x, y) \in D^M$ and:

For every $z \in M\backslash\{x, y\}$, there is some triple $(r, D_1, D_2) \in A$ such that $\text{otp}(xyz) = r$, $(x, z) \in D^M_1$ (if $x < z$, otherwise we ask that $(z, x) \in D^M_1$) and $(z, y) \in D^M_2$ (if $z < y$, else same as before) and for every triple $(p, E_1, E_2) \in A$ there is some $w \in M\backslash\{x, y\}$ such that $p = \text{otp}(xyw)$, $(x, w) \in E^M_1$ (if $x < w$, else see above) and $(w, y) \in E^M_2$ (if $w < y$, else see above).

Note that these parameters $(D, A)$ belong to $\mathbb{V}$ and depend only on $C$ and not on the choice of atomic model or generic filter. By the induction hypothesis this can be written in $L_{\mu^+, \aleph_0}$ (as there are at most $\mu$-many classes) using $L'$ (since $< \in L'$).  $\square$

Let $T \in \mathbf{K}_\mu$. Let $E_0$ be the following equivalence relation on $v_T$: $j_1 \ E_0 \ j_2$ iff for some $i_1, i_2 \in u_T$, $Q_{j_1}(x, y) \vdash P_{i_1}(x) \wedge P_{i_2}(y)$ and $Q_{j_2}(x, y) \vdash P_{i_1}(x) \wedge P_{i_2}(y)$.

Let $E_1$ be the trivial equivalence relation on $v_T$, i.e., $E_1 = v_T \times v_T$.

**Claim 5.9** Suppose $T \in \mathbf{K}_\mu$. The relations $E_0$ and $E_1$ are definable in the languages $L_0 = \{<\} \cup \{P_i \mid i \in u_T\}$ and $L_1 = \{<\}$ respectively and are additive. The relation $E_1$ is edge-preserving.

P r o o f.  The first assertion is easy to check. The second follows from the fact that $P_i(x)$ isolates a complete type.  $\square$

As $E_0 \subseteq E_1$, it follows that $\sim_T^{E_0, \alpha} \subseteq \sim_T^{E_1, \alpha}$ for all $\alpha < \mu^+$.

**Remark 5.10** We can also define "intermediate" equivalence relations between $E_0$ and $E_1$, taking into account a specific subset $s$ of $u_T$, and then define $j_1 \ E_s \ j_2$ iff $P(a) \equiv_s P(a')$ and $P(b) \equiv_s P(b')$ whenever $(a, b) \in Q_{j_1}$ and $(a', b') \in Q_{j_2}$ where $\equiv_s$ is the equivalence relation on $u_T$ whose classes are $\{s\} \cup \{\{i\} \mid i \in u_T \backslash s\}$. In this notation $E_1 = E_{u_T}$ and $E_0 = E_\varnothing$. These are less important but they are worth mentioning.

**Proposition 5.11** Let $\mu$ be a cardinal. Then $\mathbf{K}_\mu^+$ is the class of theories $T \in \mathbf{K}_\mu$ such that $\sim_T^{E_0, \alpha(T, E_0)}$ is equality (cf. Claim 5.7 (1)). Similarly, $\mathbf{K}_\mu^*$ is the class of theories $T \in \mathbf{K}_\mu$ such that $\sim_T^{E_1, \alpha(T, E_1)}$ is equality.

P r o o f.  The proofs for $\mathbf{K}_\mu^+$ and $\mathbf{K}_\mu^*$ are similar, so we shall do the $\mathbf{K}_\mu^*$ case.

Suppose that $T \in \mathbf{K}_\mu^*$ as witnessed by $\psi_j$ for $j \in v_T$. Suppose $j_1 \sim_T^{E_1, \alpha(T, E_1)} j_2$. Let $M$ be a countable atomic model of $T$ after the collapse, and let $(a, b) \in Q_{j_1}^M$, $(a', b') \in Q_{j_2}^M$. Let $F$ be the set of all finite order-preserving partial functions $g$ from $M$ to $M$ which map $a, b$ to $a', b'$ and such that for $x < y$ in its domain, $Q(x, y) \sim_T^{E_1, \alpha(T, E_1)} Q(g(x), g(y))$. Then $F$ is a back and forth system by the choice of $\alpha(T, E_1)$ and by the additivity of $\sim_T^{E_1, \alpha}$ (cf. Claim 5.7(3)) and hence there is an automorphism of $M {\restriction} \{<\}$ taking $a, b$ to $a', b'$, so both $(a, b)$ and $(a', b')$ satisfy $\psi_{j_1}$ and $\psi_{j_2}$. But the $Q_j$'s are disjoint, and $\psi_j$ defines $Q_j$ in $M$, so $j_1 = j_2$.

Conversely, by Proposition 5.8, for every $j \in v_T$, there is some formula $\psi_j$ in $L_{\mu^+, \omega}$ which defines (in the sense of said proposition) the class of $j$ in $E_{1, \alpha(T, E_1)}$ which is just $\{j\}$.  $\square$

### 5.3  An example of a theory in $K_\mu$ without an atomic model

**Theorem 5.12** Let $\mu = (2^{\aleph_0})^+$. There is some $T \in \mathbf{K}_\mu$ without an atomic model.

P r o o f.  Let $\mathbb{Q}^+$ be the set of positive rationals, and let $v_T = \{(i_1, i_2, (n, q)) \mid i_1, i_2 \in \mu, n < \omega, q \in \mathbb{Q}^+\}$. Let $L = L_T = \{<\} \cup \{P_i \mid i \in \mu\} \cup \{Q_j \mid j \in v_T\}$.

Let $P = \text{Coll}(\mu, \aleph_0)$, and let $G$ be a generic filter. Work in $\mathbb{V}[G]$, where $\mu$ is countable. Let $M$ be the following structure. Its universe $X$ is the set of all functions $\eta$ from $\omega$ to $\mathbb{Q}$ such that for some $n < \omega$, $\eta(m) = 0$ for all $m \geq n$ (so that $X$ is countable). The order is: $\eta_1 > \eta_2$ iff for $n = \min\{k \mid \eta_1(k) \neq \eta_2(k)\}$, $\eta_1(n) > \eta_2(n)$. As an order, this is just a dense linear order with no endpoints.

Choose $P_i^M$ in a dense way. More precisely, choose $P_i^M$ in such a way that each $P_i^M$ is dense and unbounded from above and below, and such that the $P_i$'s form a partition of $M$. It is easy to see that this is possible to construct "by hand", or one can see this as the countable atomic model of the model completion of the theory of linear order and infinitely many colors.

Now we must define $Q_{i_1, i_2, (n, q)}$ where $i_1, i_2 \in \mu$, $n \in \omega$ and $q \in \mathbb{Q}^+$. Suppose $\eta_1 < \eta_2$. Then $(\eta_1, \eta_2) \in Q_{i_1, i_2, (n, q)}^M$ iff $\eta_1 \in P_{i_1}^M$, $\eta_2 \in P_{i_2}^M$, $n = \min\{k \mid \eta_1(k) \neq \eta_2(k)\}$ and $q = \eta_2(n) - \eta_1(n)$ (which must be > 0). Note that the $Q_j$'s form a partition of $<^M$.

As above, write $P^M(\eta) = i$ to denote that $\eta \in P_i^M$ and similarly $Q^M(\eta_1, \eta_2) = j$.

**Claim 5.13** The theory of $M$ is additive: given $j_1, j_2 \in v_T$ there is a unique $j_3 = f_{\text{Th}(M)}(j_1, j_2)$ such that $Q_{j_1}(x, y) \wedge Q_{j_2}(y, z) \vdash Q_{j_3}(x, z)$.

P r o o f. There are several cases to check. Suppose that $j_1 = (i_1, i_2, (n, q))$, $j_2 = (k_1, k_2, (m, r))$. Then, if $n > m$, then $j_3 = (i_1, k_2, (m, r))$, and if $n < m$, $j_3 = (i_1, k_2, (n, q))$. If $n = m$, then $j_3 = (i_1, k_2, (n, q + r))$. □

**Claim 5.14** Suppose that $N$ is an $L$-structure (perhaps in some transitive model of set theory containing $\mathbb{V}^P$) with universe $X$ such that $<^N = <^M$, the sets $P_i^N$ form a partition of $X$ for $i \in \mu$ such that $P_i^N$ is dense and unbounded, and $Q_j^N$ defined in the same way as $Q_j^M$ for $j \in v_T$. Then $M \cong N$. Moreover, $M$ is atomic, and the formulas of the form $P_i(x)$, $\bigwedge_{i<n} Q_{j_i}(x_i, x_{i+1})$ isolate complete types which are dense in $\mathrm{Th}(M)$.

P r o o f. Note that $\mathrm{Th}(N)$ is also additive, and that $f_{\mathrm{Th}(N)} = f_{\mathrm{Th}(M)}$. We do a back and forth argument. Suppose that $g : M \to N$ is a partial finite isomorphism from some finite subset $s \subseteq X$ to $g(s)$, and we are given $\eta \in X$ which we want to add to its domain. Enumerate $s = \{\eta_i \mid i < n\}$ where $\eta_0 < \ldots < \eta_{n-1}$, and suppose $\eta_i < \eta < \eta_{i+1}$ (where $-1 \leq i < n$ and $\eta_{-1} = \infty$, $\eta_n = \infty$). Let $g(s) = \{v_i \mid i < n\}$ where $v_0 < \ldots < v_{n-1}$. By additivity, it is enough to find some $v \in X$ such that $v_i < v < v_{i+1}$, $P^N(v) = P^M(\eta)$, $Q^M(\eta_i, \eta) = Q^N(v_i, v)$ and $Q^M(\eta, \eta_{i+1}) = Q^N(v, v_{i+1})$. This follows easily by observing that for all $i < \mu$ and any $\eta \in X$ and $n < \omega$, there is some $\eta' \in X$ such that $\eta \restriction n = \eta' \restriction n$ and $P(\eta') = i$: consider $\eta \restriction n \frown \bar{0}$ (where $\bar{0}$ is just an infinite sequence of zeros), and $\eta \frown \langle 1 \rangle \frown \bar{0}$. Find $\eta'$ between these two.

The moreover part follows from the previous paragraph, the fact that $Q_j(x, y) \vdash P_{i_1}(x) \wedge P_{i_2}(y)$ for some $i_1, i_2 < \mu$ and the additivity. □

Let $\dot{\tau}$ be a $P$-name for $M$. By Claim 5.14, $(P, L, \dot{\tau})$ has the isomorphism property, and hence $T = \mathrm{Th}(M) \in \mathbb{V}$. We also get that $T \in \mathbf{K}_\mu$.

**Claim 5.15** The theory $T$ has no atomic model in $\mathbb{V}$.

P r o o f. Suppose $N$ is an atomic model of $T$. As $N$ is atomic, by Claim 5.14, $\{Q_j^N \mid j \in v_T\}$ partition $<^N$. Define a coloring of increasing pairs, $c : <^N \to \omega \times \mathbb{Q}$ by $c(x, y) = (n, q)$ iff $Q^N(x, y) = (i_1, i_2, (n, q))$ for some $i_1, i_2 < \mu$. By Erdős-Rado, for some infinite set $A \subseteq N$, and some $n, q, c(x, y) = (n, q)$ for all $x < y$ in $A$. Since $A$ is infinite, we can find $x < y < z$ in $A$. But $T$ forbids a triple $x < y < z$ with $(n, q) = c(x, y) = c(y, z) = c(x, z)$: a contradiction. □

This concludes the proof of the theorem. □

**Problem 5.16** What more can be said about theories from $\mathbf{K}_\mu$, $\mathbf{K}_\mu^+$ and $\mathbf{K}_\mu^*$? When do they have atomic models? We saw that for $\mu = (2^{\aleph_0})^+$, there is a theory in $\mathbf{K}_\mu$ without an atomic model. Is the same true for some $\mu$ and $\mathbf{K}_\mu^+$ (equivalently, by Theorem 5.5, $\mathbf{K}_\mu^*$)?

After this paper was finished, it came to our attention that a recent paper by Knight, Montalban and Schweber [6] deals with similar notions and proves some similar results, though for different purposes and with different methods. Their notion of a generically presentable structure is very close to our isomorphism property (cf. Definition 1.2). There is some overlapping between the two papers (e.g., our Corollary 4.3 and their Theorem 3.12). Their proofs also use Scott sentences, but our approach is different (they use Fraïssé limits and we use atomic models), and the focuses of the two papers are completely different. Two other recent papers, one by Baldwin, Friedman, Koerwien and Laskowski [1], and another by Larson [10] have some overlapping with [6] (they all give a new proof of a result of Harrington regarding Vaught's conjecture using different methods).

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