JOURNAL OF ALGEBRA **182,** 748–755 (1996)
ARTICLE NO. 0199

A Complete Boolean Algebra That Has No Proper Atomless Complete Subalgebra

Thomas Jech*

Department of Mathematics, The Pennsylvania State University, University Park, Pennsylvania 16802

and

Saharon Shelah†

School of Mathematics, The Hebrew University, Jerusalem, Israel, and Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903

Communicated by Leonard Lipshitz

Received July 11, 1995

There exists a complete atomless Boolean algebra that has no proper atomless complete subalgebra. © 1996 Academic Press, Inc.

An atomless complete Boolean algebra B is called *simple* [5] if it has no atomless complete subalgebra A such that $A \neq B$. We prove below that such an algebra exists.

The question whether a simple algebra exists was first raised in [8] where it was proved that B has no proper atomless complete subalgebra if and only if B is *rigid* and *minimal*. For more on this problem, see [4; 5; 1, p. 664].

Properties of complete Boolean algebras correspond to properties of generic models obtained by forcing with these algebra. (See [6, pp. 266–270]; we also follow [6] for notation and terminology of forcing and generic models.) When McAloon [7] constructed a generic model with all sets ordinally defined he noted that the corresponding complete Boolean algebra is *rigid*, i.e., admitting no nontrivial automorphisms. In [9] Sacks

*The first author was supported in part by an NSF Grant DMS-9401275. E-mail address:jech@math.psu.edu.

†The second author was partially supported by he U.S.–Israel Binational Science Foundation, Publication No. 566. E-mail address:shelah@math.huji.ac.il.

0021-8693/96 $18.00

gave a forcing construction of a real number of minimal degree of constructibility. A complete Boolean algebra B that adjoins a minimal set (over the ground model) is *minimal* in the following sense:

$$\begin{aligned}&\text{If } A \text{ is a complete atomless subalgebra of } B \text{ then there exists}\\ &\quad\text{a partition } W \text{ of } 1 \text{ such that for every } w \in W,\ A_w = B_w,\\ &\qquad\text{where } A_w = \{a \cdot w : a \in A\}.\end{aligned} \tag{1}$$

In [3], Jensen constructed, by forcing over L, a definable real number of minimal degree. Jensen's construction thus proves that in L there exists a rigid minimal complete Boolean algebra. This has been noted in [8] and observed that B is rigid and minimal if and only if it has no proper atomless complete subalgebra. McAloon then asked whether such an algebra can be constructed without the assumption that $V = L$. In [5] simple complete algebras are studied systematically, giving examples (in L) for all possible cardinalities.

In [10] Shelah introduced the (f, g)-bounding property of forcing and in [2] developed a method that modified Sacks' perfect tree forcing so that while one adjoins a minimal real, there remains enough freedom to control the (f, g)-bounding property. It is this method we use below to prove the following Theorem:

THEOREM. *There is a forcing notion $\mathscr{P}$ that adjoins a real number g minimal over V and such that $B(\mathscr{P})$ is rigid.*

COROLLARY. *There exists a countably generated simple complete Boolean algebra.*

The forcing notion $\mathscr{P}$ consists of finitely branching perfect trees of height ω. In order to control the growth of trees $T \in \mathscr{P}$, we introduce a *master tree* $\mathscr{T}$ such that every $T \in \mathscr{P}$ will be a subtree of $\mathscr{T}$. To define $\mathscr{T}$, we use the following fast growing sequences of integers $(P_k)_{k=0}^{\infty}$ and $(N_k)_{k=0}^{\infty}$:

$$P_0 = N_0 = 1, \qquad P_k = N_0 \cdot \cdots \cdot N_{k-1}, \qquad N_k = 2^{P_k} \tag{2}$$

(hence $N_k = 1, 2, 4, 256, 2^{2^{11}}, \ldots$).

DEFINITION. The *master tree* $\mathscr{T}$ and the *index function* ind:

$$\begin{aligned}
&\text{(i)} && \mathscr{T} \subset [\omega]^{<\omega},\\
&\text{(ii)} && \text{ind is a one-to-one function of } \mathscr{T} \text{ onto } \omega,\\
&\text{(iii)} && \text{ind}(\langle\ \rangle) = 0,\\
&\text{(iv)} && \text{if } s, t \in \mathscr{T} \text{ and length}(s) < \text{length}(t) \text{ then ind}(s) < \text{ind}(t),\\
&\text{(v)} && \text{if } s, t \in \mathscr{T}, \text{length}(s) = \text{length}(t) \text{ and } s <_{lex} t \text{ then ind}(s)\\
& && < \text{ind}(t),
\end{aligned} \tag{3}$$

(vi) if $s \in \mathscr{T}$ and $\operatorname{ind}(s) = k$ then s has exactly N_k successors in $\mathscr{T}$, namely all $s \frown i$, $i = 0, \ldots, N_k - 1$.

The forcing notion $\mathscr{P}$ is defined as follows:

DEFINITION. $\mathscr{P}$ is the set of all subtrees T of $\mathscr{T}$ that satisfy the following:

$$\text{for every } s \in T \text{ and every } m \text{ there exists some } t \in T,\ t \supset s, \text{ such that } t \text{ has at least } {P^m}_{\operatorname{ind}(t)} \text{ successors in } T. \tag{4}$$

(We remark that $\mathscr{T} \in \mathscr{P}$ because for every m there is a K such that for all $k \geq K$, $P_i^m \leq 2^{P_k} = N_k$.)

When we need to verify that some T is in $\mathscr{P}$ we find it convenient to replace (4) by an equivalent property:

LEMMA. *A tree $T \subseteq \mathscr{T}$ satisfies* (4) *if and only if*

(i) *every $s \in T$ has at least one successor in T,*

(ii) *for every n, if $\operatorname{ind}(s) = n$ and $s \in T$ then there exists a k such that if $\operatorname{ind}(t) = k$ then $t \in T$, $t \supset s$, and t has at least ${P^n}_k$ successors in T.* (5)

Proof. To see that (5) is sufficient, let $s \in T$ and let m be arbitrary. Find some $\bar{s} \in T$ such that $\bar{s} \supset s$ and $\operatorname{ind}(\bar{s}) \geq m$, and apply (5)(ii). ∎

The forcing notion $\mathscr{P}$ is partially ordered by inclusion. A standard forcing argument shows that if G is a generic subset of $\mathscr{P}$ then $V[G] = V[g]$ where g is the *generic branch*, i.e., the unique function $g\colon \omega \to \omega$ whose initial segments belong to all $T \in G$. We shall prove that the generic branch is minimal over V, and that the complete Boolean algebra $B(\mathscr{P})$ admits no nontrivial automorphisms.

First we introduce some notation needed in the proof:

$$\text{For every } k,\ s_k \text{ is the unique } s \in \mathscr{T} \text{ such that } \operatorname{ind}(s) = k. \tag{6}$$

$$\text{If } T \text{ is a tree then } s \in \operatorname{trunk}(T) \text{ if for all } t \in T, \text{ either } s \subseteq t \text{ or } t \subseteq s. \tag{7}$$

$$\text{If } T \text{ is a tree and } a \in T \text{ then } (T)_a = \{s \in T\colon s \subseteq a \text{ or } a \subseteq s\}. \tag{8}$$

Note that if $T \in \mathscr{P}$ and $a \in T$ then $(T)_a \in \mathscr{P}$. We shall use repeatedly the following technique:

LEMMA. *Let $T \in \mathscr{P}$ and, let l be an integer and let $U = T \cap \omega^l$ (the lth level of T). Let $\dot{x}$ be a name for some set in V. For each $A \in U$ let $T_a \subseteq (T)_a$ and x_a be such that $T_a \in \mathscr{P}$ and $T_a \vdash \dot{x} = x_a$.*

Then $T' = \bigcup\{T_a \colon a \in U\}$ *is in* $\mathscr{P}$, $T' \subseteq T$, $T' \cap \omega^l = T \cap \omega^l = U$, *and* $T' \vdash \dot{x} \in \{x_a \colon a \in U\}$.

We shall combine this with *fusion*, in the form stated below:

LEMMA. *Let* $(T_n)_{n=0}^\infty$ *and* $(l_n)_{n=0}^\infty$ *be such that each* T_n *is in* $\mathscr{P}$, $T_0 \supseteq T_1 \supseteq \cdots \supseteq T_n \supseteq \ldots$, $l_0 < l_1 < \cdots < l_n < \ldots$, $T_{n+1} \cap \omega^{l_n} = T_n \cap \omega^{l_n}$, *and such that*

$$\text{for every } n\text{, if } s_n \in T_n \text{ then there exists some } t \in T_{n+1},\ t \supset s_n\text{, with } \operatorname{length}(t) < l_{n+1}\text{, such that } t \text{ has at least } P^n{}_{\operatorname{ind}(t)} \text{ successors in } T_{n+1}. \tag{9}$$

Then $T = \bigcap_{n=0}^\infty T_n \in \mathscr{P}$.

Proof. To see that T satisfies (5), note that if $s_n \in T$ then $s_n \in T_n$, and the node t found by (9) belongs to T. ∎

We shall now prove that the generic branch is minimal over V:

LEMMA. *If* $X \in V[G]$ *is a set of ordinals, then either* $X \in V$ *or* $G \in V[X]$.

Proof. The proof is very much like the proof for Sacks' forcing. Let $\dot{X}$ be a name for X and let $T_0 \in \mathscr{P}$ force that $\dot{X}$ is not in the ground model. Hence for every $T \leq T_0$ there exist $T', T'' \leq T$ and an ordinal α such that $T' \Vdash \alpha \in \dot{X}$ and $T'' \Vdash \alpha \notin \dot{X}$. Consequently, for any $T_1 \leq T$ and $T_2 \leq T$ there exist $T_1' \leq T_1$ and $T_2' \leq T_2$ and an α such that both T_1' and T_2' decide "$\alpha \in \dot{X}$" and $T_1' \Vdash \alpha \in \dot{X}$ if and only if $T_2' \Vdash \alpha \notin \dot{X}$.

Inductively, we construct $(T_n)_{n=0}^\infty$, $(l_n)_{n=0}^\infty$, $U_n = T_n \cap \omega^{l_n}$, and ordinals $\alpha(a, b)$ for all $a, b \in U_n$, $a \neq b$, such that

(i) $T_n \in \mathscr{P}$ and $T_0 \supseteq T_1 \supseteq \cdots \supseteq T_n \supseteq \ldots$,

(ii) $l_0 < l_1 < \cdots < l_n < \cdots$,

(iii) $T_{n+1} \cap \omega^{l_n} = T_n \cap \omega^{l_n} = U_n$,

(iv) for every n, if $s_n \in T_n$ then there exists some $t \in T_{n+1}$, $t \supset s_n$, with $\operatorname{length}(t) < l_{n+1}$, such that t has at least $P^n{}_{\operatorname{ind}(t)}$ successors in T_{n+1}, (10)

(v) for every n, for all $a, b \in U_n$, if $a \neq b$ then both $(T_n)_a$ and $(T_n)_b$ decide "$\alpha(a, b) \in \dot{X}$" and $(T_n)_a \Vdash \alpha(a, b) \in \dot{X}$ if and only if $(T_n)_b \Vdash \alpha(a, b) \in \dot{X}$.

When such a sequence has been constructed, we let $T = \bigcap_{n=0}^\infty T_n$. As (9) is satisfied, we have $T \in \mathscr{P}$ and $T \leq T_0$. If G is a generic such that $T \in G$ and if X is the G-interpretation of $\dot{X}$ then the generic branch g is in $V[X]$: for every n, $g \restriction l_n$ is the unique $a \in U_n$ with the property that for every $b \in U_n$, $b \neq a$, $\alpha(a, b) \in X$ if and only if $(T)_a \Vdash \alpha(a, b) \in \dot{X}$.

To construct $(T_n)_{n=0}^\infty$, $(l_n)_{n=0}^\infty$, and $\alpha(a,b)$, we let $l_0 = 0$ (hence $U_0 = \{s_0\}$) and proceed by induction. Having constructed T_n and l_n, we first find $l_{n+1} > l_n$ as follows: If $s_n \in T_n$, we find $t \in T_n$, $t \supset s_n$, such that t has at least ${P^n}_{\mathrm{ind}(t)}$ successors in T_n. Let $l_{n+1} = \mathrm{length}(t) + 1$. (If $s_n \notin T_n$, let $l_{n+1} = l_n + 1$.) Let $U_{n+1} = T_n \cap \omega^{l_{n+1}}$.

Next we consider, in succession, all pairs $\{a, b\}$ of district elements of U_{n+1}, eventually constructing conditions T_a, $a \in U_{n+1}$, and ordinals $\alpha(a,b)$, $a, b \in U_{n+1}$, such that for all a, $T_a \leq (T_n)_a$ and if $a \neq b$ then either $T_a \Vdash \alpha(a,b) \in \dot{X}$ and $T_b \Vdash \alpha(a,b) \notin \dot{X}$, or $T_a \Vdash \alpha(a,b) \notin \dot{X}$ and $T_b \Vdash \alpha(a,b) \in \dot{X}$. Finally, we let $T_{n+1} = \bigcup\{T_a\colon a \in U_{n+1}\}$. It follows that $(T_n)_{n=0}^\infty$, $(l_n)_{n=0}^\infty$, and $\alpha(a,b)$ satisfy (10). ∎

Let B be the complete Boolean algebra $B(\mathscr{P})$. We shall prove that B is rigid. Toward a contradiction, assume that there exists an automorphism π of B that is not the identity. First, there is some $u \in B$ such that $\pi(u) \cdot u = 0$. Let $p \in \mathscr{P}$ be such that $p \leq u$ and let $q \in \mathscr{P}$ be such that $q \leq \pi(p)$. Since $q \nleq p$, there is some $s \in q$ such that $s \notin p$. Let $T_0 = (q)_s$.

Note that for all $t \in T_0$, if $t \supseteq s$ then $t \notin p$. Let

$$A = \{\mathrm{ind}(t)\colon t \in p\},$$

and consider the following property $\varphi(x)$ (with parameters in V):

$$\begin{aligned}\varphi(x) \leftrightarrow\ & \text{if } x \text{ is a function from } A \text{ into } \omega \text{ such that } x(k) < N_k \\ & \text{for all } k, \text{ then there exists a function } u \text{ on } A \text{ in the ground} \\ & \text{model } V \text{ such that the values of } u \text{ are finite sets of integers} \\ & \text{and for every } k \in A,\ u(k) \subseteq \{0, \ldots, N_k - 1\} \text{ and } |u(k)| \\ & \leq P_k, \text{ and } x(k) \in u(k).\end{aligned} \tag{11}$$

We will show that

$$p \Vdash \exists x \neg \varphi(x), \tag{12}$$

and

$$\text{there exists a } T \leq T_0 \text{ such that } T \Vdash \forall x \varphi(x). \tag{13}$$

This will yield a contradiction: the Boolean value of the sentence $\exists x \neg \varphi(x)$ is preserved by π, and so

$$T_0 \leq q \leq \pi(p) \leq \pi(\|\exists x \neg \varphi(x)\|) = \|\exists x \neg \varphi(x)\|,$$

contradicting (13).

In order to prove (12), consider the following (name for a) function $\dot{x}\colon A \to \omega$. For every $k \in A$, let

$$\dot{x}(k) = \dot{g}\big(\mathrm{length}(s_k) + 1\big) \text{ if } s_k \subset \dot{g}, \qquad \text{and} \qquad \dot{x}(k) = 0 \text{ otherwise}.$$

Now if $p_1 < p$ and $u \in V$ is a function on A such that $u(k) \subseteq \{0, \dots, N_k - 1\}$ and $|u(k)| \le P_k$ then there exists a $p_2 < p_1$ and some $k \in A$ such that $s_k \in p_2$ has at least P_k^2 successors, and there exist in turn a $p_3 < p_2$ and some $i \notin u(k)$ such that $s_k \frown i \in \mathrm{trunk}(p_3)$. Clearly, $p_3 \vdash \dot{x}(k) \notin u(k)$.

Property (13) will follow from this lemma:

LEMMA. *Let $T_1 \le T_0$ and $\dot{x}$ be such that T_1 forces that $\dot{x}$ is a function from A into ω such that $x(k) < N_k$ for all $k \in A$. There exists sequences $(T_n)_{n=1}^\infty$, $(l_n)_{n=1}^\infty$, $(j_n)_{n=1}^\infty$, $(U_n)_{n=1}^\infty$ and sets z_a, $a \in U_n$, such that*

(i) $T_n \in \mathscr{P}$ *and* $T_1 \supseteq T_2 \supseteq \cdots \supseteq T_n \supseteq \cdots$,

(ii) $l_1 < l_2 < \cdots < l_n < \cdots$,

(iii) $T_{n+1} \cap \omega^{l_n} = T_n \cap \omega^{l_n} = U_n$,

(iv) *for every n, if $s_n \in T_n$ then there exists some $t \in T_{n+1}$, $t \supset s_n$, with* $\mathrm{length}(t) < l_{n+1}$, *such that t has at least ${P^n}_{\mathrm{ind}(t)}$ successors in T_{n+1},* (14)

(v) $j_1 < j_2 < \cdots < j_n < \dots$,

(vi) *for every $a \in U_n$, $(T_n)_a \Vdash \langle \dot{x}(k)\colon k \in A \cap j_n \rangle = z_a$,*

(vii) *for every $k \in A$, if $k \ge j_n$ then $|U_n| < P_k$,*

(viii) *for every $k \in A$, if $k < j_n$ then $\big|\{z_a(k)\colon a \in U_n\}\big| \le P_k$.*

Granted this lemma, (13) will follow: If we let $T = \bigcap_{n=1}^\infty T_n$, then $T \in \mathscr{P}$ and $T \le T_1$ and for every $k \in A$, $T \Vdash \dot{x}(k) \in u(k)$ where $u(k) = \{z_a(k)\colon a \in U_n\}$ (for any and all $n > k$).

Proof of Lemma. We let $l_1 = j_1 = \mathrm{length}(s)$, $U_1 = \{s\}$, and strengthen T_1 if necessary so that T_1 decides $\langle \dot{x}(k)\colon k \in A \cap j_1 \rangle$, and let z_s be the decided value. We also assume that $\mathrm{length}(s) \ge 2$ so that $|U_1| = 1 < P_k$ for every $k \in A$, $k \ge j_1$. Then we proceed by induction.

Having constructed T_n, l_n, j_n, etc., we first find $l_{n+1} > l_n$ and $j_{n+1} > j_n$ as follows: If $s_n \notin T_n$ (Case I), we let $l_{n+1} = l_n + 1$ and $j_{n+1} = j_n + 1$. Thus assume that $s_n \in T_n$ (Case II).

Since $\mathrm{length}(s_n) \le n \le l_n$, we choose some $v_n \in U_n$ such that $s_n \subseteq v_n$. By (4) there exists some $t \in T_n$, $t \supset v_n$, so that if $\mathrm{ind}(t) = m$ then t has at least P_m^{n+1} successors in T_n. Moreover we choose t so that $m = \mathrm{ind}(t)$ is big enough so that there is at least one $k \in A$ such that $j_n \le k < m$. We let $l_{n+1} = \mathrm{length}(t) + 1$ and $j_{n+1} = m = \mathrm{ind}(t)$.

Next we construct U_{n+1}, $\{z_a: a \in U_{n+1}\}$, and T_{n+1}. In Case I, we choose for each $u \in U_n$ some successor $a(u)$ of u and let $U_{n+1} = \{a(u): u \in U_n\}$. For every $a \in U_{n+1}$ we find some $T_a \subseteq (T_n)_a$ and z_a so that $T_a \Vdash \langle \dot{x}(k): k \in A \cap j_{n+1} \rangle = z_a$, and let $T_{n+1} = \bigcup\{T_a: a \in U_{n+1}\}$. In this case $|U_{n+1}| = |U_n|$ and so (vii) holds for $n+1$ as well, while (viii) for $n+1$ follows either from (viii) or from (vii) for n (the latter if $j_n \in A$).

Thus consider Case II. For each $u \in U_n$ other than v_n we choose some $a(u) \in T_n$ of length l_{n+1} such that $a(u) \supset u$, and find some $T_{a(u)} \subseteq (T_n)_{a(u)}$ and $z_{a(u)}$ so that $T_{a(u)} \Vdash \langle \dot{x}(k): k \in A \cap m \rangle = z_{a(u)}$.

Let S be the set of all successors of t (which has been chosen so that $|S| \geq P_m^{n+1}$ where $m = \operatorname{ind}(t)$); every $a \in S$ has length l_{n+1}. For each $a \in S$ we choose $T_a \subseteq (T_n)_a$ and z_a, so that $T_a \Vdash \langle \dot{x}(k): k \in A \cap m \rangle = z_a$. If we denote $K = \max(A \cap m)$ then we have

$$\left|\{z_a: a \in S\}\right| \leq \prod_{i \in A \cap m} N_i \leq \prod_{i=0}^{K} N_i = P_{K+1} \leq P_m,$$

while $|S| \geq P_m^{n+1}$. Therefore there exists a set $U \subset S$ of size $P_m{}^n$ such that for every $a \in U$ the set z_a is the same. Therefore if we let

$$U_{n+1} = U \cup \{a(u): u \in U_n - \{v_n\}\},$$

and $T_{n+1} = \bigcup\{T_a: a \in U_{n+1}\}$, T_{n+1} satisfies property (iv). It remains to verify that (vii) and (viii) hold.

To verify (vii), let $k \in A$ be such that $k \geq j_{n+1} = m$. Since $m = \operatorname{ind}(t)$, we have $m \notin A$ and so $k > m$. Let $K \in A$ be such that $j_n \leq K < m$. Since $|U_n| < P_K$, we have

$$|U_{n+1} < |U_n| + |U| < P_K + N_m < P_m \cdot N_m = P_{m+1} \leq P_k.$$

To verify (viii), it suffices to consider only those $k \in A$ such that $j_n \leq k < m$. But then $|U_n| < P_k$ and we have

$$\left|\{z_a(k): a \in U_{n+1}\}\right| \leq \left|\{z_a: a \in U_{n+1}\}\right| \leq |U_n| + 1 \leq P_k. \qquad \blacksquare$$

REFERENCES

1. M. Bekkali and R. Bonnett, Rigid Boolean algebras, *in* "Handbook of Boolean Algebras" (J. D. Monk, Ed.), Vol. 2, pp. 637–678.
2. M. Goldstern and S. Shelah, Many simple cardinal invariants, *Arch. Math. Logic* **32** (1993), 203–221.
3. R. B. Jensen, Definable sets of minimal degrees *in* "Mathematical Logic and Foundations of Set Theory" (Y. Bar-Hillel, Ed.), pp. 122–128, North-Holland, Amsterdam, 1970.

4. T. Jech, A propos d'algèbres de Boole rigides et minimales, *C. R. Acad. Sci. Paris Sér. A* **274** (1972), 371–372.
5. T. Jech, Simple complete Boolean algebras, *Israel J. Math.* **18** (1974), 1–10.
6. T. Jech, "Set Theory," Academic Press, New York, 1978.
7. K. McAloon, Consistency results about ordinal definability, *Ann. Math. Logic* **2** (1971), 449–467.
8. K. McAloon, Les algèbres de Boole rigides et minimales, *C. R. Acad. Sci. Paris Sér. A* **272** (1971), 89–91.
9. G. Sacks, Forcing with perfect closed sets, *in* "Axiomatic Set Theory" (D. Scott, Ed.), pp. 331–355, Proc. Symp. Pure Math., Vol. 13, I, Amer. Math. Soc., Providence, 1971.
10. S. Shelah, Vive la différence. I. Nonisomorphism of ultrapowers of countable models, *in* "Set Theory of the Continuum" (H. Judah *et al.*, Eds.), pp. 357–405, MSRI, Springer-Verlag, New York/Berlin, 1992.