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APPLICATIONS OF PCF THEORY

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Abstract. We deal with several pcf problems: we characterize another version of exponentiation: maximal number of κ-branches in a tree with λ nodes, deal with existence of independent sets in stable theories, possible cardinalities of ultraproducts and the depth of ultraproducts of Boolean Algebras. Also we give cardinal invariants for each λ with a pcf restriction and investigate further $T_D(f)$. The sections can be read independently, although there are some minor dependencies.

Annotated content. §1. T_D via true cofinality.

[Assume D is a filter on κ, $\mu = \mathrm{cf}(\mu) > 2^\kappa$, $f \in {}^\kappa\mathrm{Ord}$, and: D is $\aleph_1$-complete or $(\forall \sigma < \mu)(\sigma^{\aleph_0} < \mu)$. We prove that if $T_D(f) \geq \mu$ (i.e., there are $f_\alpha <_D f$ for $\alpha < \mu$ such that $f_\alpha \neq_D f_\beta$ for $\alpha < \beta < \mu$) then for some $A \in D^+$ and regular $\lambda_i \in (2^\kappa, f(i)]$ we have: μ is the true cofinality of $\prod_{i<\kappa} \lambda_i/(D+A)$. We end summing up conditions equivalent to $T_{D+A}(f) \geq \mu$ for some $A \in A^+$.]

§2. The tree revised power.

[We characterize more natural cardinal functions using pcf. The main one is $\lambda^{\kappa,\mathrm{tr}}$, the supremum on the number of κ-branches of trees with λ nodes, where κ is regular uncountable. If $\lambda > \kappa^{\kappa,\mathrm{tr}}$ it is the supremum on $\max \mathrm{pcf}\{\theta_\zeta : \zeta < \kappa\}$ for an increasing sequence $\langle \theta_\zeta : \zeta < \kappa\rangle$ of regular cardinals with $\zeta < \kappa \Rightarrow \lambda \geq \max \mathrm{pcf}\{\theta_\varepsilon : \varepsilon < \zeta\}$.]

§3. On the depth behaviour of ultraproducts.

[We deal with a problem of Monk on the depth of ultraproducts of Boolean algebras; this continues [Sh:506, §3]. We try to characterize for a filter D on κ and $\lambda_i = \mathrm{cf}(\lambda_i) > 2^\kappa$, and $\mu = \mathrm{cf}(\mu)$, when does $(\forall i < \kappa)[\lambda_i \leq \mathrm{Depth}^+(B_i)] \Rightarrow \mu < \mathrm{Depth}^+(\prod_{i<\kappa} B_i/D)$ (where $\mathrm{Depth}^+(B) = \bigcup\{\mu^+ :$ in B there is an increasing sequence of length $\mu\}$). When D is $\aleph_1$-complete or $(\forall \sigma < \mu)[\sigma^{\aleph_0} < \mu]$ the characterization is reasonable: for some $A \in D^+$ and $\lambda_i' = \mathrm{cf}(\lambda_i') < \lambda_i$ we have $\mu = \mathrm{tcf} \prod_{i<\kappa} \lambda_i'/(D+A)$. We then proceed to look at $\mathrm{Depth}_h^{(+)}$ (closing under homomorphic images), and with more work succeed. We use results from §1.]

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§4. On the existence of independent sets for stable theories.

[Bay has continued work in [Sh:c] on existence of independent sets (in the sense of non-forking) for stable theories. We connect those problems to pcf and shed some light. Note that the combinatorial Claim 4.1 continues [Sh:430, §3].]

§5. Cardinal invariants for general cardinals: restriction on the depth.

[We show that some (natural) cardinal invariants defined for any regular $\lambda(>\aleph_0)$, as functions of λ satisfies inequalities coming from pcf (more accurately norms for $\aleph_1$-complete filters). They are variants of depth, supremum of length of sequences from ${}^{\lambda}\lambda$ (increasing in a suitable sense) and also the supremum of sizes of λ-MAD families. Contrast this with Cummings Shelah [CuSh:541]. Also we connect pcf and the ideal $I[\lambda]$; see 5.19.]

§6. The class of cardinal ultraproducts mod D.

[Let D be an ultrafilter on κ and let

$$\operatorname{reg}(D) = \operatorname{Min}\{\theta : \text{ the filter } D \text{ is not } \theta\text{-regular}\},$$

so $\operatorname{reg}(D)$ is regular itself. We prove that if $\mu = \mu^{\operatorname{reg}(\theta)} + 2^\kappa$ then μ can be represented as $|\prod_{i<\kappa}\lambda_i/D|$, and for suitable μ's get μ-like such ultraproducts.]

We thank Todd Eisworth for doing much in corrections and improving presentation, and Andres Villaveces similarly for §4.

§1. T_D via true cofinality. We improve here results of [Sh:506, §3] but do not depend on them. See more related things in §6. Our main result is 1.6, which we will use in §3 in our analysis of ultraproducts of Boolean Algebras.

CLAIM 1.1. *Assume*

(a) *J is an $\aleph_1$-complete ideal on κ*
(b) *$f \in {}^{\kappa}\mathrm{Ord}$, each $f(i)$ an infinite ordinal*
(c) *$T^2_J(f) \geq \lambda = \mathrm{cf}(\lambda) > \mu \geq \kappa$ (see 1.2(1) below)*
(d) *$\mu = 2^\kappa$, or at least*
(d)$^-$(i) *if $\mathfrak{a} \subseteq \mathrm{Reg}$, and*

$$(\forall\theta \in \mathfrak{a})(\mu \leq \theta < \lambda \;\&\; \mu \leq \theta < \sup_{i<\kappa} f(i))$$

and $|\mathfrak{a}| \leq \kappa$, <u>then</u> $|\mathrm{pcf}(\mathfrak{a})| \leq \mu$
(ii) $|\mu^\kappa/J| < \lambda$
(iii) $2^\kappa < \lambda$.

<u>Then</u> for some $A \in J^+$ and $\bar{\lambda} = \langle \lambda_i : i \in A\rangle$ such that $\mu \leq \lambda_i = \mathrm{cf}(\lambda_i) \leq f(i)$ we have $\prod_{i\in A}\lambda_i/(J \restriction A)$ has true cofinality λ.

REMARK 1.2. (1) *Remember $T^2_J(f) = \mathrm{Min}\{|F| : F \subseteq \prod_{i<\kappa} f(i)$ and for every $g \in \prod_{i<\kappa} f(i)$ for some $g' \in F$ we have $\neg(g \neq_J g')\}$. See [Sh:506, §3] on the relationship of relatives of this definition; they agree when $> 2^\kappa$. The inverse of the claim is immediate, i.e., the conclusion implies that $\lambda \leq T^2_J(f)$.*

(2) *If $A_1 = \{i < \kappa : f(i) \geq \lambda\} \in J^+$ then the conclusion is immediate, with $\lambda_i = \lambda$.*

(3) *Note if $A_2 = \{i < \kappa : f(i) < (2^\kappa)^+\} \in J^+$ then $T^2_J(f) \leq 2^\kappa$. If in addition $\kappa \backslash A_2 \in J$ then any λ satisfying the conclusion satisfies $\lambda \leq 2^\kappa$.*

(4) *We can omit the assumption clause $(d)^-(iii)$ and weaken (here and in 2.7) the assumption "$|\mu^\kappa/J| < \lambda$" (in clause $(d)^-$) and just ask:*

$\bigoplus_{J,\mu,\lambda}$ *there is* $F \subseteq {}^{\kappa}\mu$ *of cardinality* $< \lambda$ *such that for every* $g \in {}^{\kappa}\mu$ *we can find* $F' \subseteq F$ *of cardinality* $\leq \mu$ *such that for every* $A \in J^+$ *for some* $f \in F'$ *we have* $\{i \in A : g(i) = f(i)\} \in J^+$, *or even*

$\bigoplus^-_{J,\mu,\lambda}$ *we require the above only for all* $g \in G$, *where* $G \subseteq {}^{\kappa}\mu$ *has cardinality* $< \lambda$ *and*: *if* $\langle \theta_i : i < \kappa \rangle$ *is a sequence of regulars in* $[\aleph_0, \mu]$ *and* $g' \in \prod_{i<\kappa} \theta_i$ *then for some* $g'' \in G$ *we have* $g' <_J g'' <_J \langle \theta_i : i < \kappa \rangle$.

Considering $(d)^-(iii)$ *in the proof we weaken* $g_n \restriction A \in N$ *to "for some* $g' \in G$, $A' \subseteq \kappa$ *we have* $g_n \restriction A =_J g' \restriction A'$."

(5) *Also in* 1.6 *and* 1.7 *we can replace the assumption* $\lambda > 2^{\kappa}$ *by the existence of a* μ *satisfying* $\lambda > \mu \geq \kappa$ *such that* $(d)^-$ *as weakened above holds.*

(6) *Note that we do not ask* $(\forall \alpha < \lambda)[|\alpha|^{<\mathrm{reg}(J)} < \lambda]$.

(7) *Of course, we can apply the claim to* $J \restriction A$ *for every* $A \in J^+$ *hence* $\{A/J : A \in J^+$, *and for some* $\bar{\lambda} = \langle \lambda_i : i \in A \rangle$ *such that* $\mu \leq \lambda_i = \mathrm{cf}(\lambda_i) \leq f(i)$ *we have* $\prod_{i \in A} \lambda_i/(J \restriction A)$ *has true cofinality* $\lambda\}$ *is dense in the Boolean Algebra* $\mathscr{P}(\kappa)/J$.

REMARK 1.3. *The changes in the proof of* 1.1 *below required for weakening in* 1.1 *the clause* $|\mu^{\kappa}/J| < \lambda$ *to* $\bigoplus^-_{J,\mu,\lambda}$ *from* 1.2(4) *are as follows.*

As $J, \mu, \lambda \in N$ *there are* $F \subseteq {}^{\kappa}\mu$, $G \subseteq {}^{\kappa}\mu$ *as required in* $\bigoplus^-_{J,\mu,\lambda}$ *belonging to* N (*hence* $\subseteq N$). *After choosing* $g^{n.1}$ *and* B_n *apply the assumption on* G *to* $g^{n.3} \in {}^{\kappa}\mu$ *when* $g^{n.3} \restriction B_n = (g^{n.2} \restriction B_n)$ *and* $g^{n.3} \restriction (\kappa \backslash B_n)$ *is constantly zero and* $\bar{\theta} = \langle \theta_i : i < \kappa \rangle$ *where* $\theta_i = \mathrm{cf}(g_n(i))$ *if* $i \in B_n$ *and* $\theta_i = \aleph_0$ *if* $i \in \kappa \backslash B_n$.

So we get some $g^{n.4} \in G$ *such that* $g^{n.3} <_J g^{n.4} <_J \langle \theta_i : i < \kappa \rangle$. *As* $G \in N$, $|G| < \lambda$ *clearly* $G \subseteq N$ *hence* $g^{n.4} \in G$. *Let* F'_n *be a subset of* F *of cardinality* $\leq \mu$ *such that*: *for every* $A \in J^+$ *for some* $f \in F'_n$ *we have* $\{i \in A : g^{n.4}(i) = f(i)\} \in J^+$.

Now continue as there but defining g_{n+1} *use* $g^{n.4}$ *instead* $g^{n.3}$ *and choose* $\mathscr{P}^1_{n+1}$ *as*

$$\{\{i < \kappa : g^{n.4}(i) = f(i)\} : f \in F'_n\}.$$

The rest is straight.

Remember

FACT 1.4. *Assume*

(a) $N \prec (\mathscr{H}(\chi), \in, <^*_{\chi})$ *and* $\mu < \lambda < \chi$ *and* $\{\mu, \lambda\} \in N$,
(b) $N \cap \lambda$ *is an ordinal*,
(c) $i^* \leq \mu$, *and for* $i < i^*$ *we have* $\mathfrak{a}_i \subseteq \mathrm{Reg} \backslash \mu^+$, $|\mathfrak{a}_i| \leq \mu$, $\theta_i \in pcf(\mathfrak{a}_i) \cap \lambda$ *and* $(\mathfrak{a}_i, \theta_i) \in N$, *and let* $\mathfrak{a} = \bigcup_{i<i^*} \mathfrak{a}_i$.

Then

(*) *for every* $g \in \Pi\mathfrak{a}$ *there is* f *such that*:
 (α) $g < f \in \Pi\mathfrak{a}$
 (β) $f \restriction \mathfrak{b}_{\theta_i}[\mathfrak{a}_i] \in N$, *and if* $\theta_i = \max \mathrm{pcf}(\mathfrak{a}_i)$ *we have* $f \restriction \mathfrak{a}_i \in N$.

PROOF. By [Sh:g, Chapter II,3.4] or [Sh:g, VIII,§1]. ⊣

PROOF OF 1.1. Note that assuming $2^{\kappa} < \lambda$ somewhat simplifies the proof, in this case we can demand $g_{A.n} = g_n \restriction A$. Assume toward contradiction that the conclusion fails. Let χ be large enough, and let N be an elementary submodel of

$(\mathscr{H}(\chi), \in, <^*_\chi)$ of cardinality $< \lambda$ such that $\{f, \lambda, \mu\}$ belongs to N and $N \cap \lambda$ is an ordinal and if we assume only clause (d)$^-$ then[1]

$\boxtimes$ for every $g \in {}^\kappa\mu$ there is $g' \in N \cap {}^\kappa\mu$ such that $g = g' \bmod J$ (if $J \in N$ this is immediate).

So we shall prove $F =: \left(\prod_{i<\kappa} f(i)\right) \cap N$ exemplifies that $T^2_J(f) \leq |F|(< \lambda)$, thus giving a contradiction

So it suffices to prove

(*) for every $g \in \prod_{i<\kappa} f(i)$ for some $g' \in F$ we have $\neg(g \neq_J g')$ i.e.,

$$\{i < \kappa : g'(i) = g(i)\} \in J^+.$$

Assume $g \in \prod_{i<\kappa} f(i)$ exemplifies the failure of $(*)$.

We now define by induction on $n < \omega$ the function g_n and the family $\mathscr{P}_n$ such that:

(i) $g_0 = f, g_n \in {}^\kappa\,\mathrm{Ord}$, and $g \leq g_n$

(ii) $g_{n+1} < g_n \bmod J$

(iii) $\mathscr{P}_n$ is a family of $\leq \mu$ members of J^+

(iv) if $A \in \mathscr{P}_n$ then $g_n \restriction A \in N$ hence $A \in N$ but if $2^\kappa \geq \lambda$ we just assume that for some $g_{A.n} \in \prod_{i \in A} f(i)$ we have $g_{A.n} = g_n \restriction A \bmod J$ and $g_{A.n} \in N$ hence $A \in N$

(v) $\mathscr{P}_0 = \{\kappa\}$

(vi) if $A \in \mathscr{P}_n$ and $B \subseteq A$ and $B \in J^+$ then for some $A' \in \mathscr{P}_{n+1}$ we have $A' \subseteq A \;\&\; A' \cap B \in J^+$

(vii) $g < g_n \bmod J$

(viii) $g(i) \leq g_n(i)$ and $g(i) < g_n(i) \Rightarrow g_{n+1}(i) < g_n(i)$ and $g(i) = g_n(i) \Rightarrow g(i) = g_{n+1}(i)$ (not necessary for 1.1).

If we succeed, as "J is $\aleph_1$-complete (see assumption (a))" then by clause (ii) we get a contradiction as $<_J$ is well founded. Also the case $n = 0$ is easy by (i)+(v).

(Note: Clause (vii) holds as $g \in \prod_{i<\kappa} f(i)$). So assume we have $g_n, \mathscr{P}_n$ and we shall define $g_{n+1}, \mathscr{P}_{n+1}$. In N there is a two-place function $\boldsymbol{e}$, written $\boldsymbol{e}_\delta(i)$ such that $\boldsymbol{e}_\delta(i)$ is defined if and only if $\delta \in \{\alpha : \alpha \text{ a non-zero ordinal } \leq \sup_{i<\kappa} f(i)\}$, and $i < \mathrm{cf}(\delta)$, and if δ is a limit ordinal, then $\langle \boldsymbol{e}_\delta(i) : i < \mathrm{cf}(\delta)\rangle$ is strictly increasing with limit δ and $\boldsymbol{e}_{\alpha+1}(0) = \alpha$; of course, $\mathrm{Dom}(\boldsymbol{e}_{\alpha+1}) = \{0\}$.

We also know by assumption (d) or (d)$^-$(i) that

$\otimes$ for every $A \in \mathscr{P}_n$ we have, letting $\mathfrak{a}^n_A =: \{\mathrm{cf}(g_{A.n}(i)) : i \in A\} \setminus \mu^+$, the set $\mathrm{pcf}(\mathfrak{a}^n_A)$ has at most μ members.

So $\mathscr{Y} =: \{(A, \mathfrak{a}^n_A, \theta) : A \in \mathscr{P}_n \text{ and } \theta \in \lambda \cap \mathrm{pcf}(\mathfrak{a}^n_A)\}$ has at most $|\mathscr{P}_n| \times \mu \leq \mu \times \mu = \mu$ members (as $|\mathscr{P}_n| \leq \mu$ and $|\mathrm{pcf}\; \mathfrak{a}^n_A| \leq \mu$ by $\otimes$ above) so let $\{(A^n_\varepsilon, \mathfrak{a}^n_\varepsilon, \theta^n_\varepsilon) : \varepsilon < \varepsilon^*_n\}$ list them with $\varepsilon^*_n \leq \mu$. Clearly $\mathfrak{a}^n_\varepsilon \in N$ (as $g_{A.n} \restriction A^n_\varepsilon \in N$), and since $\mu + 1 \subseteq N$ and $|\mathrm{pcf}(\mathfrak{a}^n_\varepsilon)| \leq \mu$, we have $\mathscr{Y} \subseteq N$. For each $\varepsilon < \varepsilon^*_n$ we define $h^n_\varepsilon \in \Pi\mathfrak{a}^n_\varepsilon$ by:

$$h^n_\varepsilon(\theta) = \mathrm{Min}\{\zeta < \theta : \text{ if } i \in A^n_\varepsilon, g(i) < g_n(i), \text{ and } \theta = \mathrm{cf}(g_n(i)) \text{ then } g(i) < \boldsymbol{e}_{g_n(i)}(\zeta)\}.$$

[1]Note we did not forget to ask $J \in N$, we just want to help reading this as a proof of 1.5 too, for the case $2^{|J|} \geq \lambda$, so there J' does not necessarily belong to N.

[Why is h^n_ε well defined? The number of possible i's is $\leq |A^n_\varepsilon| \leq \kappa \leq \mu$, for each relevant i, every $\zeta < \theta$ large enough is OK as $\langle e_{g_n(i)}(\zeta) : \zeta < \theta \rangle$ is increasing continuous with limit $g_n(i)$. Lastly, $\theta = \mathrm{cf}(\theta) > \mu$ (by the choice of $\mathfrak{a}^n_\varepsilon$) so all the demands together hold for every large enough $\zeta < \theta$.]

Let $\mathfrak{a}_n = \bigcup_{\varepsilon < \varepsilon^*_n} \mathfrak{a}^n_\varepsilon$ and let $h_n \in \Pi\mathfrak{a}_n$ be defined by

$$h_n(\theta) = \sup\{h^n_\varepsilon(\theta) : \varepsilon < \varepsilon^*_n \text{ and } \theta \in \mathfrak{a}^n_\varepsilon\},$$

it is well defined by the argument above. So by 1.4 there is a function $g^{n.1} \in \Pi\mathfrak{a}_n$ such that:

(α) $h_n < g^{n.1}$

(β) $g^{n.1} \restriction \mathfrak{b}_{\theta^n_\varepsilon}[\mathfrak{a}^n_\varepsilon] \in N$ (and $\theta^n_\varepsilon = \max \mathrm{pcf}(\mathfrak{a}^n_\varepsilon) \Rightarrow \mathfrak{b}_{\theta^n_\varepsilon}[\mathfrak{a}^n_\varepsilon] = \mathfrak{a}^n_\varepsilon$).

Also we can define $g^{n.2} \in {}^\kappa\,\mathrm{Ord}$ by:

$$g^{n.2}(i) = \mathrm{Min}\{\zeta < \mathrm{cf}(g_n(i)) : e_{g_n(i)}(\zeta) \geq g(i)\}.$$

So letting $B_n = \{i : 1 \leq \mathrm{cf}(g_n(i)) \leq \mu\}$ clearly $g^{n.2} \restriction B_n \in {}^{B_n}\mu$. Now if assumption (d) holds, then $\mu^\kappa/J < \lambda$, hence $\mu^\kappa \subseteq N$ so we can find $g^{n.3} \in N$ such that $g^{n.2} = g^{n.3} \bmod (J + (\kappa \backslash B_n))$; if assumption (d) fails we still can get such $g^{n.3}$ by $\boxtimes$ above. Lastly, we define $g_{n+1} \in {}^\kappa\,\mathrm{Ord}$:

$$g_{n+1}(i) = \begin{cases} e_{g_n(i)}\left(g^{n.1}(\mathrm{cf}(g_n(i)))\right) & \underline{\text{if}} \quad \mathrm{cf}(g_n(i)) > \mu \text{ and } g_n(i) > g(i) \\ e_{g_n(i)}\left(g^{n.3}(\mathrm{cf}(g_n(i)))\right) & \underline{\text{if}} \quad \mathrm{cf}(g_n(i)) \in [1, \mu] \text{ and } g_n(i) > g(i) \\ g_n(i) & \underline{\text{if}} \quad g(i) = g_n(i) \end{cases}$$

and $\mathscr{P}_{n+1} = (\mathscr{P}^0_{n+1} \cup \mathscr{P}^1_{n+1}) \backslash J$ where

$$\mathscr{P}^0_{n+1} = \{\{i \in A^n_\varepsilon : \mathrm{cf}(g_{A^n_\varepsilon.n}(i)) \in \mathfrak{b}_{\theta^n_\varepsilon}[\mathfrak{a}^n_\varepsilon]\} : \varepsilon < \varepsilon^*_n\}$$

and

$$\mathscr{P}^1_{n+1} = \{\{i \in A^* : \mathrm{cf}(g_{A^*.n}(i)) \leq \mu\} : A^* \in \mathscr{P}_n\}.$$

(Note: possibly $(\mathscr{P}^0_{n+1} \cup \mathscr{P}^1_{n+1}) \cap J \neq \emptyset$ but this does not cause problems.) $\dashv$

So let us check clauses (i)–(viii).

<u>Clause (i).</u> Trivial.

<u>Clause (ii).</u> By the definition of $g_{n+1}(i)$ above it is $< g_n(i)$ except when $g_n(i) = g(i)$, but by clause (vii) we know that $g < g_n \bmod J$ hence necessarily

$$\{i < \kappa : g_n(i) = g(i)\} \in J, \text{ so really } g_{n+1} < g_n \bmod J.$$

<u>Clause (iii).</u> $|\mathscr{P}_{n+1}| \leq |\mathscr{P}_n| + |\varepsilon^*_n| + \aleph_0$ and $|\mathscr{P}_n| \leq \mu$ by clause (iii) for n (i.e., the induction hypothesis) and during the construction we show that $|\varepsilon^*_n| = |\mathscr{Y}| \leq \mu$.

<u>Clause (iv).</u> Let $A \in \mathscr{P}_{n+1}$ so we have two cases.

<u>Case 1.</u> $A \in \mathscr{P}^0_{n+1}$.

So for some $\varepsilon < \varepsilon^*_n$ we have $(\theta^n_\varepsilon \in \lambda \cap \mathrm{pcf}(\mathfrak{a}^n_\varepsilon)$ and)

$$A =: \{i \in A^n_\varepsilon : \mathrm{cf}(g_{A^n_\varepsilon.n}(i)) \in \mathfrak{b}_{\theta^n_\varepsilon}[\mathfrak{a}^n_\varepsilon]\}.$$

Let $g_{A.n+1} \in \prod_{i \in A} f(i)$ be defined by $g_{A.n+1}(i) = e_{g_{A^n_\varepsilon.n(i)}(\varepsilon)}(g^{n.1}(\mathrm{cf}(g_{A^n_\varepsilon.n}(i))))$.

By the choice of $g^{n.1} \in \Pi\mathfrak{a}_n$ we have:

$$g^{n.1} \restriction \mathfrak{b}_{\theta^n_\varepsilon}[\mathfrak{a}^n_\varepsilon] \in N.$$

Now the set A is definable from $A^n_\varepsilon, g_{A^n_\varepsilon.n}$ and $\mathfrak{b}_{\theta^n_\varepsilon}[\mathfrak{a}^n_\varepsilon]$, all of which belong to N hence $A \in N$. Also $A^n_\varepsilon \in N$ and clearly $g_{A.n+1}$ is definable from the functions $g^{n.1} \restriction \mathfrak{b}_{\theta_\varepsilon}[\mathfrak{a}^n_\varepsilon], g^{n.1}, g_{A^n_\varepsilon.n}, A^n_\varepsilon$ and the function $\boldsymbol{e}$ (see the definition of g_{n+1} by cases), but all four are from N so $g_{A.n+1} \in N$. Lastly, $g_{n+1} \restriction A \equiv_J g_{A.n+1}$ as $i \in A$ & $g_{A^n_\varepsilon.n}(i) = g_n(i)$ & $g_n(i) > g(i) \Rightarrow g_{n+1}(i) = g_{A.n+1}(i)$ and each of the three assumptions fail only for a set of $i \in A$ that belongs to J.

<u>Case 2.</u> $A \in \mathscr{P}^1_{n+1}$.
So for some $A^* \in \mathscr{P}_n$ we have

$$A = \{i < \kappa : i \in A^* \text{ and } \mathrm{cf}(g_{A^*.n}(i)) \le \mu\}.$$

Let $g_{A.n+1}(i) \equiv \boldsymbol{e}_{g_{A.n}}(g^{n.3}(\mathrm{cf}(g_{A^*.n}(i)))$. Again, $g_{A.n+1} \in N, g_{A.n+1} \equiv_J g_{n+1} \restriction A$. Looking at the definition of $g_{A.n+1}$, clearly $g_{A.n}$ is definable from $g^{n.2} \in N, g_{A^*.n}$ and the function $\boldsymbol{e}$, all of which belong to N.

<u>Clause (v).</u> Holds trivially.

<u>Clause (vi).</u> Assume $A \in \mathscr{P}_n$ and $B \subseteq A$ satisfies $B \in J^+$ (so also $A \in J^+$), we have to find $A' \in \mathscr{P}_{n+1}$, such that $A' \subseteq A$ & $A' \cap B \in J^+$.

<u>Case 1.</u> $B_1 = \{i \in B : \mathrm{cf}(g_{A.n}(i)) \le \mu\} \in J^+$.
In this case $A' =: \{i \in A : \mathrm{cf}(g_{A.n}(i)) \le \mu\} \in \mathscr{P}^1_{n+1} \subseteq \mathscr{P}_{n+1}$ and $A' \cap B \in J^+$ by the assumption of the case.

<u>Case 2.</u> For some $\varepsilon < \varepsilon^*_n$ we have $A = A^n_\varepsilon$ and

$$B_{2.\varepsilon} = \{i \in B : \mathrm{cf}(g_{A.n}(i)) \in \mathfrak{b}_{\theta^n_\varepsilon}[\mathfrak{a}^n_\varepsilon]\} \in J^+.$$

In this case $A' =: \{i \in A : \mathrm{cf}(g_{A.n}(i)) \in \mathfrak{b}_{\theta^n_\varepsilon}[\mathfrak{a}^n_\varepsilon]\} \in J^+$ belongs to $\mathscr{P}^1_{n+1} \subseteq \mathscr{P}_{n+1}$, is $\subseteq A$ and $B_{2,\varepsilon} \cap A' \in J^+$ by the assumption of the case (remember $g <_J g_n$).

<u>Case 3.</u> Neither Case 1 nor Case 2.
So $B_3 = B \setminus B_1 \in J^+$ and let $\lambda_i = \mathrm{cf}(g_{A.n}(i))$.

We shall show that $\prod_{i \in B_3} \mathrm{cf}(g_{A.n}(i))/J$ is λ-directed. This suffices as letting $\lambda_i =: \mathrm{cf}(g_{A.n}(i)) \in (\mu, f(i)]$, by [Sh:g, II,1.4(1),pages 46,50] for some $\lambda'_i = \mathrm{cf}(\lambda'_i) \le \lambda_i$, we have $\lim\inf_{J \restriction B_3}\langle \lambda'_i : i \in B_3\rangle = \lim\inf_{J \restriction B_3}\langle \lambda_i : i \in B_3\rangle$ and $\lambda = \mathrm{tcf} \prod_{i \subseteq B_3} \lambda'_i/(J \restriction B_3)$ and this shows that the conclusion of 1.1 holds, contradicting our initial assumption, so the λ-directedness really suffices.

Now $i \in B \setminus B_1 \Rightarrow \lambda_i = \mathrm{cf}(g_n(i)) > \mu$; and if $\prod_{i \in B_3} \lambda_i/J$ is not λ-directed, by [Sh:g],I,§1 for some $B_4 \subseteq B_3$ and $\theta = \mathrm{cf}(\theta) < \lambda$ we have: $B_4 \in J^+$ and $\prod_{i \in B_4} \lambda_i/J$ has true cofinality θ. Hence $\theta \in \mathrm{pcf}\{\mathrm{cf}(g_{A.n}(i)) : i \in A \text{ and } \mathrm{cf}(g_n(i)) > \mu\}$, and as $\theta > \mu$, for some $\varepsilon < \varepsilon^*_n$ we have $A = A^n_\varepsilon$ and $\theta = \theta^n_\varepsilon$ so $A' = \{i \in A : \mathrm{cf}(g_{A.n}(i)) \in \mathfrak{b}_{\theta_\varepsilon}[\mathfrak{a}^n_\varepsilon]\}$ is as required in Case 2 on $B_{2.\varepsilon}$ (note: we could have restricted ourselves to θ's like that).

<u>Clause (vii).</u> By the choice of $g^{n.1}, g^{n.2}$ and g^n clearly $i < \kappa$ & $g(i) < g_n(i) \Rightarrow g(i) \le g_{n+1}(i)$. As $g < g_n \bmod J$ it suffices to prove $B =: \{i : g(i) = g_{n+1}(i)\} \in J$. If not, we choose by induction on $\ell \le n+1$ a member B_ℓ of $\mathscr{P}_\ell$ such that $B_\ell \cap B \in J^+$. For $\ell = 0$ let $B_\ell = \kappa \in \mathscr{P}_0$, for $\ell + 1$ apply clause (vi) for ℓ (even when $\ell = n$ we have just proved it). So $B_{n+1} \cap B \in J^+$ and $g_{n+1} \restriction (B_{n+1} \cap B) = g \restriction (B_{n+1} \cap B)$ hence $\neg(g_{n+1} \restriction B_{n+1} \neq_J g_n \restriction B_{n+1})$ but $g_{n+1} \restriction B_{n+1} \in N$ so we have contradicted the choice of g as contradicting $(*)$.

<u>Clause (viii).</u> Easy. $\dashv_{1.1}$

Claim 1.5. *Assume*

(a) J *is an ideal*[2] *on* κ

(b) $f \in {}^{\kappa}\mathrm{Ord}$, *each* $f(i)$ *an infinite ordinal*

(c) $T^2_J(f) \geq \lambda = \mathrm{cf}(\lambda) > \mu > \kappa$

(d) $\mu = (2^\kappa)^+$ *or at least*

(d)$^-$ (i) *if* $\mathfrak{a} \subseteq \mathrm{Reg}$, *and* $(\forall \theta \in \mathfrak{a})(\mu \leq \theta < \lambda \,\&\, \mu \leq \theta < f(i))$ *and* $|\mathfrak{a}| \leq \kappa$ *then* $|\mathrm{pcf}(\mathfrak{a})| \leq \mu$

(ii) $|\mu^\kappa/J| < \lambda \vee (\forall g \in {}^{\kappa}\mu)[|\Pi g/J| < \lambda]$ *and* μ *is regular*

(e) $\alpha < \lambda \Rightarrow |\alpha|^{\aleph_0} < \lambda$.

Then for some $A \in J^+$ *and* $\bar{\lambda} = \langle \lambda_i : i \in A\rangle$ *such that* $\mu \leq \mathrm{cf}(\lambda_i) = \lambda_i \leq f(i)$ *we have* $\prod_{i\in A} \lambda_i/J$ *has true cofinality* λ.

Proof. We repeat the proof of 1.1 but we choose N such that ${}^{\omega}N \subseteq N$, (possible by assumption (e) as λ is regular), and let $F =: (\prod_{i<\kappa} f(i)) \cap N$. If $2^\kappa < \lambda$ then clearly

$$F = \Big\{ g \in \prod_{i<\kappa} f(i) : \text{for some partition } \langle A_n : n < \omega\rangle \text{ of } \kappa \text{ and } g_n \in N \cap \prod_{i<\kappa} f(i) \text{ we have } g = \bigcup_{n<\omega} (g_n \restriction A_n)\Big\}.$$

Then assume $(*)$ (from the proof of 1.1) fails and $g \in \prod_{i<\kappa} f(i)$ exemplifies it and we let J' be the ideal $J' = \{A \subseteq \kappa : g \restriction A = g' \restriction A$ for some $g' \in F\}$.

Clearly J' is $\aleph_1$-complete, $J' \subseteq J$ (as g is a counterexample to $(*)$ and the representation of F above) and we continue as there getting the conclusion for J' hence for J.

If $2^\kappa \geq \lambda$, let $F' = N \cap \prod_{i<\kappa} f(i)$, then

$\bigotimes$ for $g \in \prod_{i<\kappa} f(i)$ and $A \in J^+$ we have (i) $\Leftrightarrow$ (ii) where:

(i) there are $g'_n \in F'$ for $n < \omega$ such that

$$\{i < \kappa : \bigvee_{n<\omega} g(i) = g'_n(i)\} \supseteq A \bmod J$$

(ii) for some $g' \in F'$ we have $\{i < \kappa : g(i) = g'(i)\} \supseteq A \bmod J$.

[Why? $\Leftarrow$ is trivial; now $\Rightarrow$ holds as $g_n \in N$ also $\langle g_n : n < \omega\rangle \in N$ hence $\langle\{g_n(i) : n < \omega\} : i < \kappa\rangle \in N$ and use $\omega^\kappa/J \leq \mu^\kappa/J < \lambda$ (or just $\bigoplus_{J.\mu.\lambda}$ from 1.2(4).]

Let $g \in \prod_{i<\kappa} f(i)$ be such that $g' \in N \cap \prod_{i<\kappa} f(i) \Rightarrow g \neq_J g'$. Now we repeat the proof of 1.1 with our $\kappa, f, \lambda, N, F, g$ this time using the demands in clause (viii) (i.e., $g(i) \leq g_n(i)$). The proof does not change except that we do not get a contradiction from $n < \omega \Rightarrow g_{n+1} <_J g_n$. However, for each $i < \kappa$, $\langle g_n(i) : n < \omega\rangle$ is non-increasing (by clause (viii)) hence eventually constant and by that clause eventually equal to $g(i)$. So clause (i) of $\bigotimes$ above holds hence clause (ii) so we are done. $\dashv_{1.5}$

[2] Compared to 1.1 we are omitting "J is $\aleph_1$-complete."

CONCLUSION 1.6. Assume J is an ideal on κ, $f \in {}^{\kappa}\mathrm{Ord}$, $i < \kappa \Rightarrow f(i) > 2^{\kappa}$, $\lambda = \mathrm{cf}(\lambda) > 2^{\kappa}$, and

(*) J is $\aleph_1$-complete or $(\forall \alpha < \lambda)(|\alpha|^{\aleph_0} < \lambda)$.

Then (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (b)$^+$ $\Leftrightarrow$ (c) $\Leftrightarrow$ (c)$^+$ where

(a) for some $A \in J^+$ we have $T^2_{J \restriction A}(f \restriction A) \geq \lambda$

(b) for some $A \in J^+$ and $\lambda_i = \mathrm{cf}(\lambda_i) \in (2^{\kappa}, f(i)]$ (for $i \in A$) we have $\prod_{i \in A} \lambda_i / (J \restriction A)$ is λ-directed

(b)$^+$ like (b) but $\prod_{i \in A} \lambda_i / (J \restriction A)$ has true cofinality λ

(c) for some $A \in J^+$, and $\bar{n} = \langle n_i : i < \kappa \rangle \in {}^{\kappa}\omega$ and ideal J^* on $A^* = \bigcup_{i \in A} (\{i\} \times n_i)$ satisfying

$$(\forall B \subseteq A)\left[B \in J \Leftrightarrow \bigcup_{i \in B} (\{i\} \times n_i) \in J^*\right]$$

and regular cardinals $\lambda_{(i,n)} \in (2^{\kappa}, f(i)]$ we have $\prod_{(i,n) \in A^*} \lambda_{(i,n)} / J^*$ is λ-directed

(c)$^+$ as in (c) but $\prod_{(i,n) \in A^*} \lambda_{(i,n)} / J^*$ has true cofinality λ.

PROOF. Clearly (b)$^+$ $\Rightarrow$(b), (b)$\Rightarrow$(c),(b)$^+$ $\Rightarrow$(c)$^+$ and (c)$^+$ $\Rightarrow$(c). Also (b)$\Rightarrow$(b)$^+$ by [Sh:g, Chapter II, 1.4(1)], and similarly (c)$\Rightarrow$(c)$^+$. Now we prove (c)$\Rightarrow$(a); let $\lambda_i = \max\{\lambda_{(i,n)} : n < n_i\}$ and let g_i be a one-to-one function from $\prod_{n < n_i} \lambda_{(i,n)}$ into λ_i and let $\langle f_\alpha : \alpha < \lambda \rangle$ be a $<_{J^*}$-increasing sequence in $\prod_{(i,n) \in A^*} \lambda_{(i,n)}$. Define $f^*_\alpha \in \prod_{i \in A} \lambda_i$ by $f^*_\alpha(i) = g_i\left(f_\alpha \restriction (\{i\} \times n_i)\right)$. So if $\alpha < \beta$, then

$$\left\{i \in A : f^*_\alpha(i) = f^*_\beta(i)\right\} = \left\{i : \bigwedge_{n < n_i} f_\alpha((i,n)) = f_\beta(i,n)\right\}$$

so by the assumption on J^* and the choice of $\langle f_\alpha : \alpha < \lambda \rangle$, for $\alpha < \beta < \lambda$ we get $f^*_\alpha \neq_J f^*_\beta$ hence $\{f^*_\alpha : \alpha < \lambda\}$ is as required in clause (a).

Lastly $(a) \Rightarrow (b)$ by 1.1 (in the case J is $\aleph_1$-complete) or 1.5 (in the case $(\forall \alpha < \lambda)(|\alpha|^{\aleph_0} < \lambda)$). We have gotten enough implications to prove the conclusions. $\dashv_{1.6}$

CONCLUSION 1.7. Let D be an ultrafilter on κ. If $|\prod_{i<\kappa} f(i)/D| \geq \lambda = \mathrm{cf}(\lambda) > 2^{\kappa}$ and $(\forall \alpha < \lambda)[|\alpha|^{\aleph_0} < \lambda]$, then for some regular $\lambda_i \leq f(i)$ (for $i < \kappa$) we have $\lambda = \mathrm{tcf}(\prod_{i<\kappa} \lambda_i / D)$.

REMARK 1.8. *On* $|\prod_{i<\kappa} \lambda_i / D|$, *see* [*Sh*:506, 3.9*B*] *and* §6 *here.*

§2. The tree revised power.

DEFINITION 2.1. For κ regular and $\lambda \geq \kappa$ let

$$\lambda^{\kappa.\mathrm{tr}} = \sup\{|\lim_\kappa(T)| : T \text{ a tree with } \leq \lambda \text{ nodes and } \kappa \text{ levels}\}$$

where $\lim_\kappa(T)$ is the set of κ-branches of T; and let when $\lambda \geq \mu \geq \kappa$ and $\theta \geq \kappa$

$$\lambda^{\langle\kappa,\theta\rangle} = \mathrm{Min}\Big\{\mu : \text{ if } T \text{ is a tree with } \lambda \text{ nodes and } \kappa \text{ levels, } \underline{\text{then}} \text{ there is } \mathscr{P} \in \big[[T]^{\theta}\big]^{\mu} \text{ such that } \eta \in \lim_\kappa(T) \Rightarrow (\exists A \in \mathscr{P})(\eta \subseteq A)\Big\}.$$

$$\lambda^{\langle\kappa\rangle} = \lambda^{\langle\kappa,\kappa\rangle}.$$

Recall $[A]^\kappa =: \{B : B \subseteq A \text{ and } |B| = \kappa\}$.

REMARK 2.2. (1) *Clearly* $\lambda^{\langle\kappa,\theta\rangle} \leq \lambda^{\kappa,\mathrm{tr}} \leq \lambda^{\langle\kappa,\theta\rangle} + \theta^{\kappa,2} \leq \lambda^{\langle\kappa,\theta\rangle} + \theta^{\kappa}$.

(2) *If* $\kappa = \aleph_0$ *then obviously* $\lambda^{\kappa,\mathrm{tr}} = \lambda^\kappa$.

(3) *Of course,* $\lambda^{\langle\kappa,\theta\rangle} \leq \mathrm{cov}(\lambda, \theta^+, \kappa^+, \kappa)$ *and* $\kappa \leq \theta \leq \sigma \leq \lambda \Rightarrow \lambda^{\langle\kappa,\theta\rangle} \leq \lambda^{\langle\kappa,\sigma\rangle} + cov(\lambda, \theta^+, \kappa^+, \kappa)$. (*See* [*Sh:g, Chapter II,* §5] *if these concepts are unfamiliar.*)

THEOREM 2.3. *Let* κ *be regular uncountable* $\leq \lambda$. Then *the following cardinals are equal*:

(i) $\lambda^{\langle\kappa\rangle}$

(ii) $\lambda + \sup\{\max \mathrm{pcf}(\mathfrak{a}) : \mathfrak{a} \subseteq \mathrm{Reg} \cap \lambda \backslash \kappa, \mathfrak{a} = \{\theta_\zeta : \zeta < \kappa\}$ *strictly increasing, and if* $\xi < \kappa$ *then* $\max \mathrm{pcf}(\{\theta_\zeta : \zeta < \xi\}) \leq \theta_\xi \leq \lambda\}$.

REMARK 2.4. *We can add*

(ii)$^-$ *like* (ii) *but we demand only* $\max \mathrm{pcf}(\{\theta_\zeta : \zeta < \xi\}) \leq \lambda$.

PROOF. First inequality. Cardinal of (i) (i.e., $\lambda^{\langle\kappa\rangle}$) is $\leq$ cardinal of (ii).

Assume not and let μ be the cardinal from clause (ii) so $\mu \geq \lambda$. Let T, a tree with κ levels and λ nodes, exemplify $\lambda^{\langle\kappa\rangle} > \mu$. Without loss of generality $T \subseteq {}^{\kappa>}\lambda$ and $<_T = \lhd \restriction T$. Let $\chi = \beth_7(\mu)$ and $\{T, \kappa, \lambda, \mu\} \in \mathfrak{B}_n \prec (\mathscr{H}(\chi), \in <^*_\chi)$, $\mu + 1 \subseteq \mathfrak{B}_n$, $\|\mathfrak{B}_n\| = \mu$, for $n < \omega$, $\mathfrak{B}_n \in \mathfrak{B}_{n+1}, \mathfrak{B}_n \prec \mathfrak{B}_{n+1}$ and let $\mathfrak{B} =: \bigcup_{n<\omega} \mathfrak{B}_n$. So $\mathscr{P} =: \mathfrak{B} \cap [T]^{\leq\kappa}$ cannot exemplify (i). So there is $\eta \in \lim_\kappa(T)$ such that $(\forall A \in \mathscr{P})[\{\eta \restriction \zeta : \zeta < \kappa\} \nsubseteq A]$.

We choose by induction on n, N^0_n, N^1_n such that:

(a) $N^0_n \prec N^1_n \prec \mathfrak{B}_n$.

(b) $N^1_0 = \mathrm{Sk}_{\mathfrak{B}_0}(\{\zeta : \zeta < \kappa\} \cup \{\eta \restriction \zeta : \zeta < \kappa\} \cup \{\kappa, \mu, \lambda, T\})$ and

$$N^0_0 = \mathrm{Sk}_{\mathfrak{B}_0}(\{\zeta : \zeta < \kappa\} \cup \{\kappa, \mu, \lambda, T\}).$$

(c) $\|N^\ell_n\| = \kappa$.

(d) $N^0_n \in \mathfrak{B}_{n+1}$.

(e) $N^1_n = \mathrm{Sk}_{\mathfrak{B}_n}(N^0_n \cup \{\eta \restriction \zeta : \zeta < \kappa\})$.

(f) $\theta \in \lambda^+ \cap \mathrm{Reg} \cap N^0_n \backslash \kappa^+ \Rightarrow \sup(N^0_{n+1} \cap \theta) > \sup(N^1_n \cap \theta)$.

(Here "Sk" denotes the Skolem hull.)

Let us carry the induction.

For $n = 0$: No problem.

For $n + 1$: Let $\mathfrak{a}^n =: N^0_n \cap \mathrm{Reg} \cap \lambda^+ \backslash \kappa^+$, so $\mathfrak{a}^n \in \mathfrak{B}_{n+1}$ and $\mathfrak{a}^n$ is a set of cardinality $\leq \kappa$ of regular cardinals $\in (\kappa, \lambda^+)$.

Let $g^n \in \Pi\mathfrak{a}^n$ be defined by $g^n(\theta) =: \sup(N^1_n \cap \theta)$. Let

$(*)_1$ $I^n = \{\mathfrak{b} \subseteq \mathfrak{a}^n :$ for some $f \in (\Pi\mathfrak{a}^n) \cap \mathfrak{B}_{n+1}$ we have $g^n \restriction \mathfrak{b} < f\}$,

so we need to show $\mathfrak{a}^n \in I^n$.

An easy induction on $\mathrm{pcf}(\mathfrak{a}^n)$ tells us that

$(*)_2$ $J_{<\mu}[\mathfrak{a}^n] \subseteq I^n$ (in particular all singletons are in I^n).

FACT. *There is* $f^* \in \mathfrak{B}_{n+1} \cap \Pi\mathfrak{a}^n$ *such that*:

$$\mathfrak{b}^n =: \{\theta \in \mathfrak{a}^n : f^*(\theta) < g^n(\theta)\}$$

satisfies

$$[\mathfrak{b}^n]^{<\kappa} \subseteq J_{\leq\lambda}[\mathfrak{a}^n]$$

(*yes*! *not* $J_{\leq\mu}[\mathfrak{a}^n]$).

PROOF. In $\mathfrak{B}_{n+1}$ there is a list $\langle a_{n.\varepsilon} : \varepsilon < \kappa\rangle$ of N_n^0. For each $\nu \in T$ let ν be of level ζ and let $N_{n.\nu}^1 = \mathrm{Sk}_{\mathfrak{B}_n}(\{(a_{n.\varepsilon}, \nu \restriction \varepsilon) : \varepsilon < \zeta\})$. So the function $\nu \mapsto N_{n.\nu}^1$ (i.e., the set of pairs $\{(\nu, N_{n.\nu}^1) : \nu \in T\}$) belongs to $\mathfrak{B}_{n+1}$. Clearly $\langle N_{n.\eta\restriction\zeta}^1 : \zeta < \kappa\rangle$ is increasing continuous with union N_n^1. Let $g_{n.\nu}^1 \in \Pi(\mathfrak{a}^n \cap N_{n.\nu}^1)$ be defined by $g_{n.\nu}^1(\theta) = \sup(\theta \cap N_{n.\nu}^1)$, so $\{(\mathfrak{a}^n \cap N_{n.\nu}^1, g_{n.\nu}^1) : \nu \in T\} \in \mathfrak{B}_{n+1}$. Now $\Pi\mathfrak{a}^n / J_{\leq\lambda}[\mathfrak{a}^n]$ is λ^+-directed, hence as $|T| \leq \lambda$ there is $f^* \in \Pi\mathfrak{a}^n$ such that:

$(*)_3$ $\nu \in T \Rightarrow g_{n.\nu}^1 <_{J_{\leq\lambda}[\mathfrak{a}^n]} f^*$, that is

$$\{\theta \in \mathrm{Dom}(g_{n.\nu}^1) : \neg(g_{n.\nu}^1(\theta) < f^*(\theta))\} \in J_{\leq\lambda}[\mathfrak{a}^n].$$

and by the previous sentence without loss of generality $f^* \in \mathfrak{B}_{n+1}$. Note that for $\theta \in \mathfrak{a}^n$ the sequence $\langle g_{n.\eta\restriction\zeta}^1(\theta) : \zeta < \kappa\rangle$ is non-decreasing with limit $g^n(\theta)$.

Let $\mathfrak{c} = \{\theta \in \mathfrak{a}^n : f^*(\theta) < g^n(\theta)\}$, now note

$(*)_4$ if $\theta \in \mathfrak{c}$ then for every $\zeta < \kappa$ large enough, $f^*(\theta) < g_{n.\eta\restriction\zeta}^1(\theta)$.

Hence $\mathfrak{c}' \in [\mathfrak{c}]^{<\kappa} \Rightarrow \mathfrak{c}' \in J_{\leq\lambda}[\mathfrak{a}^n]$ as required in the fact.

(Why the implication? Because if $\mathfrak{c}' \subseteq \mathfrak{c}$, $|\mathfrak{c}| < \kappa$ then by $(*)_4$ for some $\zeta < \kappa$ we have $f^* \restriction \mathfrak{c}' < g'_{n.\eta\restriction\zeta} \restriction \mathfrak{c}'$ which by $(*)_3$ gives $\mathfrak{c}' \in J_{\leq\lambda}[\mathfrak{a}^n]$); so let $\mathfrak{b}^n = \mathfrak{c}$. $\dashv_{\text{Fact}}$

Now if $\mathfrak{b}^n$ is in $J_{\leq\mu}[\mathfrak{a}^n]$, by $(*)_1 + (*)_2$ above we can finish the induction step.

If not, some $\tau^* \in \mathrm{Reg}\setminus\mu^+$ satisfies $\tau^* \in \mathrm{pcf}(\mathfrak{b}^n)$; let $\langle \mathfrak{c}_\zeta : \zeta < \kappa\rangle$ be an increasing continuous sequence of subsets of $\mathfrak{a}^n$ each of cardinality $< \kappa$ such that $\mathfrak{b}^n = \bigcup_{\zeta<\kappa} \mathfrak{c}_\zeta$ and so (by the fact above) $\zeta < \kappa \Rightarrow \tau^* > \lambda \geq \max\mathrm{pcf}(\mathfrak{c}_\zeta)$. We know that this implies that for some club E of κ and $\theta_\zeta \in \mathrm{pcf}(\mathfrak{c}_\zeta)$, for $\zeta \in E$, $\tau^* \in \mathrm{pcf}_{\kappa\text{-complete}}(\{\theta_\zeta : \zeta \in E\})$ and $\langle\theta_\zeta : \zeta \in E\rangle$ is strictly increasing and $\max\mathrm{pcf}\{\theta_\zeta : \zeta \in E \cap \xi\} \leq \theta_\xi$ for $\xi \in E$, by [Sh:g, Chapter VIII, 1.5(2),(3), page 317].

Now $\max\mathrm{pcf}\{\theta_\varepsilon : \varepsilon \in \zeta \cap E\} \leq \max\mathrm{pcf}(\mathfrak{c}_\zeta) \leq \lambda$ so $\mu < \tau^* \leq$ the cardinal from clause (i) of 2.3, against an assumption. So we have carried out the inductive step in defining N_n^0, N_n^1.

So N_n^0, N_n^1 are well defined for every n, clearly $\bigcup_{n<\omega} N_n^0 \cap \lambda = \bigcup_{n<\omega} N_n^1 \cap \lambda$ (see [Sh:g, Chapter IX,3.3A, page 379]) hence $\bigcup_{n<\omega} N_n^0 \cap T = \bigcup_{n<\omega} N_n^1 \cap T$, hence for some n, $N_n^0 \cap \{\eta \restriction \zeta : \zeta < \kappa\}$ has cardinality κ. Now

$$A = \{\nu \in T : \text{ for some } \rho \text{ we have } \nu \lhd \rho \in N_n^0\}$$

belongs to $\mathfrak{B}_{n+1} \cap [T]^\kappa$ and $\{\eta \restriction \zeta : \zeta < \kappa\} \subseteq A$, contradicting the choice of η. $\dashv$

<u>Second inequality</u> Cardinal of (ii) $\leq$ cardinal of (i).

By the proof of [Sh:g, II,3.5]. $\dashv_{2.3}$

DEFINITION 2.5. (1) Assume $I \subseteq J \subseteq \mathscr{P}(\kappa)$, I an ideal on κ, J an ideal or the complement of a filter on κ, e.g., $J = \mathscr{P}^-(\kappa) = \mathscr{P}(\kappa)\setminus\{\kappa\}$ stipulating $f \neq_J g \Leftrightarrow$

$\{i < \kappa : f(i) = g(i)\} \in J$. We let

$$T^+_{I,J}(f, \lambda) = \sup\{|F|^+ : F \in \mathscr{F}_{I,J}(f, \lambda)\}$$

and

$$T_{I,J}(f, \lambda) = \sup\{|F| : F \in \mathscr{F}_{I,J}(f, \lambda)\},$$

where

$$\mathscr{F}_{I,J}(f, \lambda) = \Big\{F \subseteq \prod_{i<\kappa} f(i) : f \neq g \in F \Rightarrow f \neq_J g \text{ and } A \in I \Rightarrow \lambda \geq |\{f \restriction A : f \in F\}|\Big\}.$$

(2) For J an ideal on κ, $\theta \geq \kappa$ and $f \in {}^\kappa(\mathrm{Ord} \setminus \{0\})$, we let

$$\boldsymbol{U}_J(f, \theta) = \mathrm{Min}\Big\{|\mathscr{P}| : \mathscr{P} \subseteq [\sup \mathrm{Rang}(f)]^\theta \text{ and for every } g \in \prod_{i<\kappa} f(i) \text{ for some } a \in \mathscr{P} \text{ we have } \{i < \kappa : g(i) \in a\} \in J^+\Big\}.$$

If $\theta = \kappa$ ($= \mathrm{Dom}(J)$), then we may omit θ. If f is constantly λ we may write λ instead of f.

(3) For $I \subseteq J$, I ideal on κ, J an ideal or complement of a filter on κ, $\mu \geq \theta \geq \kappa$ and $f \in {}^\kappa(\mathrm{Ord} \setminus \{0\})$ let

$$\boldsymbol{U}_{I,J}(f, \theta, \mu) = \sup\{\boldsymbol{U}_J(F, \theta) : F \in \mathscr{F}^-_I(f, \mu)\}$$

where

$$\mathscr{F}^-_I(f, \mu) = \Big\{F : F \subseteq \prod_{i<\kappa} f(i) \text{ and } A \in I \Rightarrow \mu \geq |\{f \restriction A : f \in F\}|\Big\}$$

and

$$\boldsymbol{U}_J(F, \theta) = \mathrm{Min}\Big\{|\mathscr{P}| : \mathscr{P} \subseteq [\sup \mathrm{Rang}(f)]^\theta \text{ and for every } f \in F \text{ for some } a \in \mathscr{P} \text{ we have } \{i < \kappa : f(i) \in a\} \in J^+\Big\}.$$

FACT 2.6. *Let* $\lambda \geq \theta \geq \kappa = \mathrm{cf}(\kappa) > \aleph_0$.
(1) $\lambda^{\kappa.\mathrm{tr}} = T_{J^{bd}_\kappa, \mathscr{P}-(\kappa)}(\lambda, \lambda)$ *and* $\lambda^{\langle\kappa,\theta\rangle} \leq \boldsymbol{U}_{J^{bd}_\kappa}(\lambda, \theta)$.
(2) *If* $\lambda \geq \mu$, *<u>then</u>* $\lambda^{\kappa.\mathrm{tr}} \geq \mu^{\kappa.\mathrm{tr}}$ *and* $\lambda^{<\kappa>} \geq \mu^{<\kappa>}$.
(3) $\lambda^{\kappa.\mathrm{tr}} = \lambda^{\langle\kappa\rangle} + \kappa^{\kappa.\mathrm{tr}}$.
(4) *Assume* $I \subseteq J$ *are ideals on* κ. *Then* $T^+_I(f, \lambda) > \mu$ *if*:
  (i) *each* $f(i)$ *is a regular cardinal* $\lambda_i \in (\kappa, \lambda)$
  (ii) $\prod_{i<\kappa} f(i)/J$ *is* μ*-directed*
  (iii) *for some* $A_\zeta \subseteq \kappa$ *for* $\zeta < \zeta^* < \mathrm{Min}_{j<\kappa} f(j)$ *we have*:

$$\max \mathrm{pcf}\{f(i) : i \in A_\zeta\} \leq \lambda$$

(*hence* $\mathrm{cf}(\prod_{i \in A_\zeta} f(i)) \leq \lambda$) *and* $\{A_\zeta : \zeta < \zeta^*\}$ *generates an ideal on* κ *extending* I *but included in* J.

(5) $\boldsymbol{U}_J(\lambda) \leq \boldsymbol{U}_J(\lambda, \theta) \leq \boldsymbol{U}_J(\lambda) + \mathrm{cf}([\theta]^\kappa, \subseteq) \leq \boldsymbol{U}_J(\lambda) + \theta^\kappa$ *and* $T_I(f) \leq \boldsymbol{U}_I(f) + 2^\kappa$ *and* $\boldsymbol{U}_{I,J}(f, \lambda) \leq T_{I,J}(f, \lambda) \leq \boldsymbol{U}_{I,J}(f, \lambda) + 2^\kappa$ *where* $I \subseteq J$ *are ideals on* κ.
Also obvious monotonicity properties (*in* I, J, λ, θ, f) *hold.*

PROOF. (1) Easy. Let us prove the first equation. First assume

$$F \in \mathscr{F}_{J_\kappa^{\mathrm{bd}},\mathscr{P}-(\kappa)}(\lambda,\lambda),$$

and we define a tree as follows: for $i < \kappa$ the ith level is

$$T_i = \{f \restriction i : f \in F\}$$

and

$$T = \bigcup_{i<\kappa} T_i, \text{ with the natural order } \subseteq .$$

Clearly T is a tree with κ levels, the ith level being T_i.

By the definition of $\mathscr{F}_{J_\kappa^{\mathrm{bd}},\mathscr{P}-(\kappa)}(\lambda,\lambda)$ as $i < \kappa \Rightarrow \{j : j < i\} \in J_\kappa^{\mathrm{bd}}$, clearly $|T_i| \leq \lambda$. Now for each $f \in F$, clearly $t_f =: \langle (f \restriction i) : i < \kappa\rangle$ is a κ-branch of T, and $f_1 \neq f_2 \in F \Rightarrow t_{f_1} \neq t_{f_2}$ so T has at least $|F|$ κ-branches.

The other direction is easy, too. Note that the proof gives $=^+$; i.e., the supremum is obtained in one side if and only if it is obtained in the other side.

(2) If T is a tree with μ nodes and κ levels then we can add λ nodes adding λ branches. Also the other inequality is trivial.

(3) First $\lambda^{\kappa,\mathrm{tr}} \geq \lambda^{\langle\kappa\rangle}$ because if T is a tree with λ nodes and κ levels, then we know $|\lim_\kappa(T)| \leq \lambda^{\kappa,\mathrm{tr}}$, hence $\mathscr{P} = \{t : t \text{ is a } \kappa\text{-branch of } T\}$ has cardinality $\leq \lambda^{\kappa,\mathrm{tr}}$ and satisfies the requirement in the definition of $\lambda^{<\kappa>}$.

Second $\lambda^{\kappa,\mathrm{tr}} \geq \kappa^{\kappa,\mathrm{tr}}$ by part (2) of 2.6.

Lastly, $\lambda^{\kappa,\mathrm{tr}} \leq \lambda^{<\kappa>} + \kappa^{\kappa,\mathrm{tr}}$ because if T is a tree with λ nodes and κ levels, we know by Definition 2.1 that there is $\mathscr{P} \subseteq [T]^\kappa$ of cardinality $\leq \lambda^{<\kappa>}$ such that every κ-branch of T is included in some $A \in \mathscr{P}$, without loss of generality $x <_T y \in A \in \mathscr{P} \Rightarrow x \in A$; so

$$\begin{aligned} |\lim_\kappa(T)| &= |\{t : t \text{ a } \kappa\text{-branch of } T\}| \\ &= |\bigcup_{A\in\mathscr{P}} \{t \subseteq A : t \text{ a } \kappa\text{-branch of } T\}| \\ &\leq \sum_{A\in\mathscr{P}} |\lim_\kappa(T \restriction A)| \\ &\leq |\mathscr{P}| + \kappa^{\kappa,\mathrm{tr}} \leq \lambda^{<\kappa>} + \kappa^{\kappa,\mathrm{tr}}. \end{aligned}$$

(4) Like the proof of [Sh:g, Chapter II,3.5].

(5) Left to the reader. $\dashv_{2.6}$

LEMMA 2.7. *Assume*

(a) $I \subseteq J$ *are ideals on* κ
(b) I *is generated by* $\leq \mu^*$ *sets,* $\mu^* \geq \kappa$
(c) $T^+_{I,J}(f,\lambda) > \mu = \mathrm{cf}(\mu) > \mu^* \geq T_{I,J}(\mu^*,\kappa)$
(d) κ *is not the union of countably many members of* I.

Then We can find $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n \subseteq \dots$ from I^+ with union κ, such that for each n there is $\langle \lambda_i^n : i \in A_n \rangle, \mu^* < \lambda_i^n = \mathrm{cf}(\lambda_i^n) \leq f(i)$ such that:

$$\prod_{i \in A_n} \lambda_i^n / J \text{ is } \mu\text{-directed}$$

$$A \subseteq A_n, A \in I \Rightarrow \mathrm{cf}\left(\prod_{i \in A} \lambda_i^n\right) \leq \lambda.$$

REMARK 2.8. *The point in the proof is that if I is generated by $\{B_\gamma : \gamma < \gamma^* \leq \mu^*\}$, and $\{\eta_\alpha : \alpha < \mu^+\}$ are distinct branches and $f \in {}^A(\lambda + 1 \backslash \{0\}), A \subseteq \kappa$ and $i \in A \Rightarrow \mathrm{cf}(f(i)) > \mu^*$, then for some $g < f$ for every $\gamma < \gamma^*$ and $\alpha < \mu^+$, $\{i < \gamma : \text{if } \eta_\alpha(i) < f(i) \text{ then } \eta_\alpha(i) < g(i)\} = \gamma \bmod J_{<\lambda^+}(f \restriction \gamma)$.*

PROOF. Similar to the proof of 1.1 adding the main point of the proof of 2.3, the "fact" there. ⊣

We can further generalize

DEFINITION 2.9. For $I \subseteq J \subseteq \mathscr{P}(\kappa)$, function $f^* \in {}^\kappa\mathrm{Reg}$ and λ, we let

$$\mathscr{F}^1_{(I,J,\lambda)}(f^*) = \Big\{F \subseteq \prod_{i<\kappa} f^*(i) : \text{if } A \in J \text{ then } \lambda \geq |\{(f \restriction A)/I : f \in F\}|\Big\}$$

(so I is without a loss of generality an ideal on κ and if $I = \{\emptyset\}$ this is just $\mathscr{F}_J^-(f^*, \lambda)$)

$$\mathscr{F}^2_{(I,J,\lambda)}(f^*) = \Big\{F \subseteq \prod_{i<\kappa} f^*(i) : \text{if } A \in J, \text{ and } f, g \in F \text{ are distinct then } \{i \in A : f(i) = g(i)\} \in I\Big\}$$

$$\mathscr{F}^3_{(I,J,\lambda,\bar\theta)}(f^*) = \Big\{F \subseteq \prod_{i<\kappa} f^*(i) : \text{if } A \in J, \text{ then for some } G \subseteq \prod_{i \in A} [f^*(i)]^{\theta_i} \text{ of cardinality } \leq \lambda \text{ we have } (\forall f \in F)(\exists g \in G)[\{i \in A : f(i) \notin g(i)\} \in I]\Big\}.$$

If Ξ is a set of such tuples, then we let $\mathscr{F}^\ell_\Xi(f^*) = \bigcap_{\Upsilon \in \Xi} \mathscr{F}^\ell_\Upsilon(f^*)$. If in all the tuples λ is the third element, we write triples and f^*, λ instead of f^*.

For any $\mathscr{F}^\ell_\Upsilon$ we let $T^\ell_\Upsilon(f^*) = \sup\{|F| : F \in \mathscr{F}^\ell_\Upsilon(f^*)\}$

Remark. We have proof like $|\cdot|$, but: instead of T we have $F \in \mathscr{F}_I(f)$ exemplifying $\boldsymbol{U}_{I,J}(f, \lambda) > \mu$; i.e., $\boldsymbol{U}_{I,J}(F, \lambda) > \mu$. Then $\eta \in F$ satisfies $(\forall A \in \mathscr{P})[\{i : \eta(i) \in A\} \in J]$. We choose N_n^0, N_n^1 satisfying (a)–(f) with $\gamma_n = 1$.

§3. **On the depth behaviour of ultraproducts.** The problem originates from Monk [M] and see on it Roslanowski Shelah [RoSh:534] and then [Sh:506, §3] but the presentation is self-contained.

We would like to have (letting B_i denote Boolean algebra), for D an ultrafilter on κ:

$$\operatorname{Depth}\left(\prod_{i<\kappa} B_i/D\right) \geq \left|\prod_{i<\kappa} \operatorname{Depth}(B_i)/D\right|.$$

(If D is just a filter, we should use T_D instead of product in the right side). Because of the problem of attainment (serious, see Magidor Shelah [MgSh:433]), we rephrase the question:

$\otimes$ for D an ultrafilter on κ, does $\lambda_i < \operatorname{Depth}^+(B_i)$ for $i < \kappa$ imply

$$\left|\prod_{i<\kappa} \lambda_i/D\right| < \operatorname{Depth}^+\left(\prod_{i<\kappa} B_i/D\right)$$

at least when $\lambda_i > 2^\kappa$;

$\otimes'$ for D a filter on κ does $\lambda_i < \operatorname{Depth}^+(B_i)$ for $i < \kappa$ imply (assuming $\lambda_i > 2^\kappa$ for simplicity):

$$\begin{aligned}&\mu = \operatorname{cf}(\mu) < T^+_{D+A}(\langle \lambda_i : i < \kappa \rangle) \text{ for some } A \in D^+\\ &\qquad \Rightarrow \mu < \operatorname{Depth}^+\left(\prod_{i<\kappa} B_i/(D+A)\right) \text{ for some } A \in D^+.\end{aligned}$$

As found in [Sh:506], this actually is connected to a pcf problem, whose answer under reasonable restrictions is 1.6. So now we can clarify the connections.

Also, by changing the invariant (closing under homomorphisms, see [M]) we get a nicer result; this shall be dealt with here.

The results here (mainly 3.5) supercede [Sh:506, 3.26].

DEFINITION 3.1. (1) For a partial order P (e.g., a Boolean algebra) let

$$\begin{aligned}\operatorname{Depth}^+(P) = \min\{\lambda : &\text{ we cannot find } a_\alpha \in P\\ &\text{ for } \alpha < \lambda \text{ such that } \alpha < \beta \Rightarrow a_\alpha <_P a_\beta\}.\end{aligned}$$

(2) For a Boolean algebra B let

$$\begin{aligned}D_h^+(B) &= \operatorname{Depth}_h^+(B)\\ &= \sup\{\operatorname{Depth}^+(B') : B' \text{ is a homomorphic image of } B\}.\end{aligned}$$

(3)
$$\begin{aligned}\operatorname{Depth}(P) = \sup\{\mu :\ &\text{there are } a_\alpha \in P \text{ for } \alpha < \mu\\ &\text{such that } \alpha < \beta < \mu \Rightarrow a_\alpha <_P a_\beta\}.\end{aligned}$$

(4) $\operatorname{Depth}_h(P) = D_h(P) = \sup\{\operatorname{Depth}(B') : B' \text{ is a homomorphic image of } B\}$.

(5) We write D_r or $D_{h,r}$ or Depth_r if we restrict ourselves to regular cardinals. Of course we could have looked at the ordinals.

DEFINITION 3.2. (1) For a linear order $\mathscr{I}$, let the interval Boolean algebra, $BA[\mathscr{I}]$ be the Boolean algebra of subsets of $\mathscr{I}$ generated by $\{[s,t)_{\mathscr{I}} : s < t \text{ are from } \{-\infty\} \cup \mathscr{I} \cup \{+\infty\}\}$.

(2) For a Boolean algebra B and regular θ, let $\mathrm{com}_{<\theta}(B)$ be the $(<\theta)$-completion of B, that is the closure of B under the operations $-x$ and $\bigvee_{i<\alpha} x_i$ for $\alpha < \theta$ inside the completion of B.

FACT 3.3. (1) *If B is the interval Boolean algebra of the ordinal $\gamma \geq \omega$ then*

(a) $D_h^+(B) = |\gamma|^+$

(b) $Depth^+(B) = |\gamma|^+$.

(2) *If B' is a subalgebra of a homomorphic image of B, then $D_h^+(B) \geq D_h^+(B')$.*

(3) *If $D' \supseteq D$ are filters on κ and for $i < \kappa$, B_i' is a subalgebra of a homomorphic image of B_i then*:

(α) $\prod_{i<\kappa} B_i'/D'$ *is a subalgebra of a homomorphic image of* $\prod_{i<\kappa} B_i/D$, *hence*

(β) $D_h^+(\prod_{i<\kappa} B_i/D) \geq D_h^+(\prod_{i<\kappa} B_i'/D')$.

(4) *In parts* (2), (3) *we can replace D_h by D if we omit "homomorphic image."*

PROOF. Straightforward. ⊣

CLAIM 3.4. (1) *If D is a filter on κ and for $i < \kappa$, B_i a Boolean algebra, $\lambda_i <$* $\mathrm{Depth}_h^+(B_i)$ *then*

(a) $\mathrm{Depth}_h^+(\prod_{i<\kappa} B_i/D) \geq \sup_{D_1 \supseteq D} \left(\mathrm{tcf}(\prod_{i<\kappa} \lambda_i/D_1)\right)^+$ (*i.e.*, sup *on the cases tcf is well defined*)

(b) $\mathrm{Depth}_h^+(\prod_{i<\kappa} B_i/D)$ *is* $\geq \mathrm{Depth}_h^+(\mathscr{P}(\kappa)/D)$ *and is at least*

$$\sup\left\{\left[\mathrm{tcf}\left(\prod_{i<\kappa} \lambda_i'/D_1\right)\right]^+ : \lambda_i' < \mathrm{Depth}^+(B_i), D_1 \supseteq D\right\}.$$

(2) $\mu < \mathrm{Depth}_h^+(B)$ *if and only if for some $a_i \in B$ for $i < \mu$ we have that: $\alpha < \beta < \mu, n < \omega$, and $\alpha_\ell < \beta_\ell < \mu$ for $\ell < n$ together imply that*

$$B \models \text{“}(a_\beta - a_\alpha) - \bigcup_{\ell<n}(a_{\alpha_\ell} - a_{\beta_\ell}) > 0.\text{”}$$

(3) *Let $A \in D^+$ (D a filter on κ). In $\prod_{i<\kappa} B_i/D$ there is a chain of order type Υ if in $\prod_{i<\kappa} B_i/(D+A)$ there is such a chain. If $\Upsilon = \lambda$,* $\mathrm{cf}(\lambda) > 2^\kappa$ *also the inverse is true.*

(4) *If* $\mu < \mathrm{Depth}^+(\prod_{i<\kappa} B_i/D)$ *and* $\mathrm{cf}(\mu) > 2^\kappa$, *then we can find $A \in D^+$ and $f_\alpha \in \prod_{i<\kappa} B_i$ for $\alpha < \mu$ such that letting $D^* = D + A$*:

$$\alpha < \beta < \mu \Rightarrow \left(\prod_{i<\kappa} B_i/D^*\right) \models f_\alpha/D^* < f_\beta/D^* \text{ moreover } f_\alpha <_{D^*} f_\beta.$$

(5) *Like* (1) *replacing* Depth_h^+ *by* Depth^+, $D_1 \supseteq D$ *by* $\{D + A : A \in D^+\}$.

PROOF. Check, e.g.:

(2) The "if" direction:

Let I be the ideal of B generated by $\{a_\alpha - a_\beta : \alpha < \beta < \mu\}, h : B \to B/I$ the canonical homomorphism, so $\langle a_\alpha/I : \alpha < \mu\rangle$ is strictly increasing in B/I.

The "only if" direction:

Let h be a homomorphism from B onto B_1 and $\langle b_\alpha : \alpha < \mu \rangle$ be a (strictly) increasing sequence of elements of B_1. Choose $a_\alpha \in B$ such that $h(a_\alpha) = b_\alpha$, so $\alpha < \beta \Rightarrow a_\alpha \backslash a_\beta \in \operatorname{Ker}(h)$ but $a_\alpha \notin \operatorname{Ker}(h)$, moreover $\beta < \alpha \Rightarrow a_\alpha - a_\beta \notin \operatorname{Ker}(h)$.

(3) The first implication is trivial, the second follows from part (4).

(4) First, assume μ is regular. Let $\langle f_\alpha / D : \alpha < \mu \rangle$ exemplify

$$\mu < \operatorname{Depth}^+\left(\prod_{i<\kappa} B_i / D\right).$$

Then $\alpha < \beta < \mu \Rightarrow f_\alpha \leq_D f_\beta \;\&\; \neg(f_\alpha =_D f_\beta)$, so for each α,

$$\langle \{i < \kappa : f_\alpha(i) = f_\beta(i)\}/D : \beta < \mu, \beta \geq \alpha \rangle$$

is decreasing and $|2^\kappa / D| < \mu = \operatorname{cf}(\mu)$ hence for some $\beta_\alpha \in (\alpha, \mu)$ we have

$$(\forall \beta)(\beta_\alpha \leq \beta < \mu \Rightarrow \{i < \kappa : f_\alpha(i) \neq f_{\beta_\alpha}(i)\} \\ = \{i < \kappa : f_\alpha(i) \neq f_\beta(i)\} \bmod D$$

(as f_γ / D is increasing). So $\langle \{i : f_\alpha(i) = f_{\beta_\alpha}(i)\}/D : \alpha < \mu \rangle$ is decreasing and $|2^\kappa / D| \leq 2^\kappa < \mu$, hence for some $A^* \subseteq \kappa$ the set

$$E = \{\alpha < \mu : \{i < \kappa : f_\alpha(i) < f_{\beta_\alpha}(i)\} = A^* \bmod D\}$$

is unbounded and even stationary in μ. Let $D^* = D + A^*$, so for $\alpha < \beta < \mu$ we have $f_\alpha \leq_D f_\beta$ hence $f_\alpha \leq_{D^*} f_\beta$, but $\alpha \in E \;\&\; \beta \geq \beta_\alpha \Rightarrow f_\alpha \neq_{D^*} f_\beta$. Hence some is $E' \subseteq \{\delta \in E : (\forall \alpha < \delta \cap E)(\beta_\alpha < \delta)\}$ is unbounded in μ and clearly $(\forall \alpha, \beta)(\alpha < \beta \;\&\; \alpha \in E' \;\&\; \beta \in E' \Rightarrow f_\alpha <_{D^*} f_\beta)$.

So $\{f_\alpha : \alpha \in E'\}$ exemplifies the conclusion.

Second, if μ is singular, let $\mu = \sum_{\zeta < \operatorname{cf}(\mu)} \mu_\zeta, \mu_\zeta > 2^\kappa; \mu_\zeta$ strictly increasing and each μ_ζ is regular. So given $\langle f_\alpha : \alpha < \mu \rangle$, for each $\zeta < \operatorname{cf}(\mu)$ we can find $E_\zeta \subseteq \mu_\zeta^+$ of cardinality μ_ζ^+ and $A_\zeta \in D^+$ such that $\alpha \in E_\zeta \;\&\; \beta \in E_\zeta \;\&\; \alpha < \beta \Rightarrow f_\alpha <_{D + A_\zeta} f_\beta$. For some A, $\operatorname{cf}(\mu) = \sup\{\zeta : A_\zeta = A\}$; so A and the f_α's for $\alpha \in \bigcup\{E_\zeta \backslash \{\operatorname{Min}(E_\zeta)\} : \zeta < \operatorname{cf}(\mu)$ is such that $A_\zeta = A\}$ are as required. $\dashv_{3.4}$

We now give lower bound of depth of reduced products of Boolean algebras B_i from the depths of the B_i's.

First main lemma 3.5. *Let D be a filter on κ and $\langle \lambda_i : i < \kappa \rangle$ a sequence of cardinals ($> 2^\kappa$) and $2^\kappa < \mu = \operatorname{cf}(\mu)$. Then:*

(1) $(\alpha) \Leftrightarrow (\alpha)^+ \Leftrightarrow (\beta) \Leftrightarrow (\beta)^- \Leftrightarrow (\gamma)$ and $(\gamma)^+ \Rightarrow (\gamma) \Rightarrow (\delta)$.

(2) If in addition $(\forall \sigma < \mu)(\sigma^{\aleph_0} < \mu) \vee$ (D is $\aleph_1$-complete) we also have $(\gamma) \Leftrightarrow (\gamma)^+ \Leftrightarrow (\delta)$ so all clauses are equivalent, where:

(α) *if B_i is a Boolean algebra, $\lambda_i \leq \operatorname{Depth}^+(B_i)$ then $\mu < \operatorname{Depth}^+(\prod_{i<\kappa} B_i / D)$*

(β) *there are cardinals $\gamma_i < \lambda_i$ for $i < \kappa$ such that, letting B_i be*

$$\mathrm{BA}[\gamma_i] = \textit{the interval Boolean algebra of (the linear order) } \gamma_i,$$

we have $\mu < \operatorname{Depth}^+(\prod_{i<\kappa} B_i / D)$

(γ) *there are $\langle \langle \lambda_{i,n} : n < n_i \rangle : i < \kappa \rangle$ where $\lambda_{i,n} = \operatorname{cf}(\lambda_{i,n}) < \lambda_i$ and a non-trivial filter D^* on $\bigcup_{i<\kappa}(\{i\} \times n_i)$ such that:*

(i) $\mu = \operatorname{tcf}(\prod_{(i,n)} \lambda_{i,n} / D^*)$.

(ii) *for some* $A^* \in D^+$ *we have*

$$D + A^* = \left\{ A \subseteq \kappa : \textit{the set } \bigcup_{i \in A} (\{i\} \times n_i) \textit{ belongs to } D^* \right\}$$

(δ) *for some filter* $D' = D + A, A \in D^+$ *and cardinals* $\lambda'_i < \lambda_i$ *we have* $\mu \leq T_{D'}(\langle \lambda'_i : i < \kappa \rangle)$

$(\beta)'$ *like* (β) *we allow* γ_i *to be an ordinal*

$(\beta)^-$ *letting* B_i *be the disjoint sum of* $\{BA[\gamma] : \gamma < \lambda_i\}$ *we have*:

$$\mu < \mathrm{Depth}^+\left(\prod_{i<\kappa} B_i/D\right)$$

$(\gamma)^+$ *for some filter* D^* *of the form* $D + A$ *and* $\lambda'_i = \mathrm{cf}(\lambda'_i) < \lambda_i$ *we have* $\mu = \mathrm{tcf}\left(\prod_{i<\kappa} \lambda'_i/D^*\right)$

$(\alpha)^+$ *if* B_i *is a Boolean algebra,* $\lambda_i \leq \mathrm{Depth}^+(B_i)$ *then for some* $A \in D^+$ *we have, setting* $D^* = D + A$, *that* $\mu < \mathrm{Depth}^+\left(\prod_{i<\kappa} B_i, <_{D^*}\right)$; *moreover for some* $f_\alpha \in \prod_{i<\kappa} B_i$ *for* $\alpha < \mu$ *we have*

$$\alpha < \beta \Rightarrow \{i : B_i \models f_\alpha(i) < f_\beta(i)\} = \kappa \bmod D^*.$$

Proof. (1) We shall prove $(\alpha) \Leftrightarrow (\beta) \Rightarrow (\beta)' \Rightarrow (\beta)^- \Rightarrow (\beta)' \Rightarrow (\gamma) \Rightarrow (\beta)$ and $(\alpha)^+ \Leftrightarrow (\alpha)$ and $(\gamma)^+ \Rightarrow (\gamma) \Rightarrow (\delta)$.

This suffices.

Now for $(\alpha)^+ \Rightarrow (\alpha)$ note that if $(\lambda_i, B_i$ for $i < \kappa$ are given and) $A \in D^+$, $\langle f_\alpha : \alpha < \lambda \rangle$ exemplify $(\alpha)^+$ then letting $f'_\alpha = (f_\alpha \restriction A) \cup 0_{(\kappa \setminus A)}$; i.e., $f'_\alpha(i)$ is $f_\alpha(i)$ when $i \in A$ and 0_{B_i} if $i \in \kappa \setminus A$, easily $\langle f'_\alpha : \alpha < \lambda \rangle$ exemplifies (α). Next $(\alpha) \Rightarrow (\alpha)^+$ by 3.4(4).

Now $(\beta) \Rightarrow (\beta)' \Rightarrow (\beta)^-$ holds trivially and for $(\beta)' \Rightarrow (\gamma)$ repeat the proof of [Sh:506, 3.24, page 35] or the relevant part of the proof of 3.6 below (with appropriate changes, the case there is more complicated). Also $(\beta)^- \Rightarrow (\beta)'$ is proved in the proof of 3.6 below. Easily $(\gamma)^+ \Rightarrow (\beta)$; also $(\beta) \Rightarrow (\alpha)$ because

(i) if γ_i a cardinal $< \mathrm{Depth}^+(B_i)$, the Boolean Algebra $BA[\gamma_i]$ can be embedded into B_i, and

(ii) if B'_i is embeddable into B_i for $i < \kappa$ then $B' = \prod_{i<\kappa} B'_i/D$ can be embedded into $\prod_{i<\kappa} B_i/D$

(iii) if B' is embeddable into B then $\mathrm{Depth}^+(B') \leq \mathrm{Depth}^+(B)$.

Now $(\alpha) \Rightarrow (\beta)$ trivially. Also $(\gamma)^+ \Rightarrow (\gamma)$ trivially and $(\gamma) \Rightarrow (\delta)$ as in the proof of the implication "$(c) \Rightarrow (a)$" in the proof of 1.6. Also we note $(\beta) \Rightarrow (\delta)$, as if $B_i = BA[\gamma_i]$ and $\gamma_i < \lambda_i$ and $\mu < \mathrm{Depth}^+(\Pi B_i/D)$, then by 3.4(4) there is a sequence $\langle f_\alpha : \alpha < \mu \rangle$ satisfying $f_\alpha \in \prod_{i<\kappa} B_i$ and $A^* \in D^+$ such that $\alpha < \beta < \mu \Rightarrow f_\alpha <_{D+A} f_\beta$. So $\{f_\alpha : \alpha < \mu\}$ exemplifies that $T_{D+A}(\langle |B_i| : i < \kappa \rangle) \geq \mu$, as required in clause (δ).

(2) Assume $(\forall \sigma < \mu)(\sigma^{\aleph_0} < \mu)$.

Now 1.6 gives $(\delta) \Rightarrow (\gamma)^+$ hence $(\gamma) \Leftrightarrow (\gamma)^+ \Leftrightarrow (\delta)$. $\dashv_{3.5}$

Now we turn to the other variant, D^+_h.

Second main lemma 3.6. *Let* D *be a filter on* κ *and* $\langle \lambda_i : i < \kappa \rangle$ *be a sequence of cardinals* $(> 2^\kappa)$ *and* $2^\kappa < \mu = \mathrm{cf}(\mu)$. Then *(see below on* $(\alpha), \ldots$*)*:

(1) $(\alpha) \Leftrightarrow (\alpha)^+ \Leftrightarrow (\beta) \Leftrightarrow (\beta)' \Leftrightarrow (\beta)^- \Leftrightarrow (\gamma)$ *and* $(\gamma)^+ \Rightarrow (\gamma) \Leftrightarrow (\beta) \Rightarrow (\delta)$.

(2) *If* $(\forall \sigma < \mu)(\sigma^{\aleph_0} < \mu)$ *we also have* $(\beta) \Leftrightarrow (\gamma) \Leftrightarrow (\gamma)^+ \Leftrightarrow (\delta)$ (*so all clauses are equivalent*); *where*:

- (α) *if* B_i *is a Boolean algebra*, $\lambda_i \leq \mathrm{Depth}_h^+(B_i)$ *then* $\mu < \mathrm{Depth}_h^+(\prod_{i<\kappa} B_i/D)$
- (β) *there are cardinals* $\gamma_i < \lambda_i$ *for* $i < \kappa$ *such that, letting* B_i *be* $BA[\gamma_i]$ = *the interval Boolean algebra of* (*the linear order*) γ_i, *we have* $\mu < \mathrm{Depth}_h^+(\prod_{i<\kappa} B_i/D)$
- (γ) *there are* $\langle\langle \lambda_{i.n} : n < n_i\rangle : i < \kappa\rangle$ *where* $\lambda_{i.n} = \mathrm{cf}(\lambda_{i.n}) < \lambda_i$ *and a non-trivial filter* D^* *on* $\bigcup_{i<\kappa}\{i\} \times n_i$ *such that*:

$$\mu = \mathrm{tcf}\left(\prod_{(i.n)} \lambda_{i.n}/D^*\right) \text{ and } D \subseteq \left\{A \subseteq \kappa : \text{the set } \bigcup_{i\in A}\{i\} \times n_i \text{ belongs to } D^*\right\}$$

- (δ) *for some filter* $D^* \supseteq D$ *and cardinals* $\lambda_i' < \lambda_i$ *we have* $\mu \leq T_{D^*}(\langle \lambda_i : i < \kappa\rangle)$
- $(\beta)'$ *like* (β) *but allowing* γ_i *to be any ordinal* $< \lambda_i$
- $(\beta)^-$ *letting* B_i *be the disjoint sum of* $\{BA[\gamma] : \gamma < \lambda_i\}$ (*so* $\mathrm{Depth}^+(B_i) = \lambda_i$) *we have*: $\mu < \mathrm{Depth}_h^+(\prod_{i<\kappa} B_i/D)$
- $(\gamma)^+$ *there are* $\lambda_i' = \mathrm{cf}(\lambda_i') \in (2^\kappa, \lambda_i)$ *for* $i < \kappa$ *and filter* $D_1^* \supseteq D$ *such that* $\prod_{i\in A} \lambda_i'/D^*$ *has true cofinality* μ
- $(\alpha)^+$ *if* B_i *is a Boolean algebra*, $\lambda_i \leq \mathrm{Depth}_h^+(B_i)$ *then for some filter* $D^* \supseteq D$ *we have* $\mu < \mathrm{Depth}_h^+\left(\prod_{i<\kappa} B_i/D^*\right)$.

Proof. Now $(\beta) \Rightarrow (\beta)'$ trivially and $(\beta)' \Rightarrow (\beta)^-$ by 3.3(3) as $BA[\gamma_i]$ can be embedded into B_i, and similarly $(\beta) \Rightarrow (\alpha)$ by 3.3(3), and $(\alpha) \Rightarrow (\beta)$ trivially. Also $(\alpha) \Rightarrow (\alpha)^+$ trivially and $(\alpha)^+ \Rightarrow (\alpha)$ easily (e.g., by 3.3(3)).

Also $(\gamma)^+ \Rightarrow (\beta)$ trivially and $(\beta) \Rightarrow (\delta)$ easily (as in the proof of 3.5).

We shall prove below $(\gamma) \Rightarrow (\beta)$, $(\beta)' \Rightarrow (\gamma)$ and $(\beta)^- \Rightarrow (\beta)'$. Together we have $(\alpha) \Rightarrow (\alpha)^+ \Rightarrow (\alpha) \Rightarrow (\beta) \Rightarrow (\beta)' \Rightarrow (\beta)^- \Rightarrow (\beta)' \Rightarrow (\gamma) \Rightarrow (\beta) \Rightarrow (\alpha)$ and $(\gamma)^+ \Rightarrow (\gamma) \Rightarrow (\delta)$; this is enough for part (1).

Lastly, to prove part (2) of 3.6, by part (1) it is enough to prove $(\delta) \Rightarrow (\gamma)^+$ as in the proof of 3.5, that is we use 1.6. ⊣

$(\gamma) \Rightarrow (\beta)$

So we have $\lambda_{i.n}$ (for $n < n_i, i < \kappa$), D^* as in clause (γ) and let $\langle g_\varepsilon : \varepsilon < \mu\rangle$ be $<_{D^*}$-increasing cofinal in $\prod_{(i.n)} \lambda_{i.n}$ but abusing notation we may write $g_\varepsilon(i,n)$ for $g_\varepsilon((i,n))$. Let $\gamma_i =: \max\{\lambda_{i.n} : n < n_i\}$ and $B_i =: BA[\gamma_i]$, clearly $\gamma_i < \lambda_i$, a (regular) cardinal as by assumption $\lambda_{i.n} < \lambda_i \leq \mathrm{Depth}^+(B_i)$ is regular for $n < n_i$. In B_i we have a strictly increasing sequence of length γ_i. Without loss of generality $\{\lambda_{i.n} : n < n_i\}$ is with no repetition (see [Sh:g, I, 1.3(8)]) and $\lambda_{i.0} > \lambda_{i.1} > \cdots > \lambda_{i.n_i-1}$.

So for each i we can find $a_{i.n} \in B_i$ (for $n < n_i$) pairwise disjoint and $\langle a_{i.n.\zeta} : \zeta < \lambda_{i.n}\rangle$ (again in B_i) strictly increasing and $< a_{i.n}$.

Let $b_{i.\varepsilon} \in B_i$ be $\bigcup_{n<n_i} a_{i.n.g_\varepsilon(i.n)}$ (it is a finite union of members of B_i hence a member of B_i). Let $b_\varepsilon \in \prod_{i<\kappa} B_i/D$ be $b_\varepsilon = \langle b_{i.\varepsilon} : i < \kappa\rangle/D$. Let J be the ideal of $B =: \prod_{i<\kappa} B_i/D$ generated by $\{b_\varepsilon - b_\zeta : \varepsilon < \zeta < \mu\}$. Clearly $\varepsilon < \zeta < \mu \Rightarrow b_\varepsilon \leq b_\zeta \bmod J$, so by 3.4(2) what we have to prove is: assuming $\varepsilon < \zeta < \mu, k < \omega$ and $\varepsilon_m < \zeta_m < \mu$ for $m < k$, then $B \models$ "$b_\zeta - b_\varepsilon - \bigcup_{m<k}(b_{\varepsilon_m} - b_{\zeta_m}) \neq 0$".

Now

$$Y =: \Big\{(i,n) : g_\varepsilon(i,n) < g_\zeta(i,n) \text{ and } g_{\varepsilon_m}(i,n) < g_{\zeta_m}(i,n) \text{ for } m = 0,1,\dots,k-1\Big\}$$

is known to belong to D^*, hence it is not empty so let $(i^*,n^*) \in Y$. Now

$$B_{i^*} \models b_{i^*,\xi} \cap a_{i^*,n^*} = a_{i^*,n^*,g_\xi(i^*,n^*)},$$

for every $\xi < \mu$, in particular for ξ among $\varepsilon, \zeta, \varepsilon_m, \zeta_m$ (for $m < k$). As $(i^*,n^*) \in Y$ we have

$$B_{i^*} \models (b_{i^*,\zeta} - b_{i^*,\varepsilon}) \cap a_{i^*,n^*} \geq b_{i^*,\zeta} \cap a_{i^*,n^*} - b_{i^*,\varepsilon} \cap a_{i^*,n^*} = a_{i^*,n^*,g_\zeta(i^*,n^*)} - a_{i^*,n^*,g_\varepsilon(i^*,n^*)} > 0$$

(as $g_\zeta(i^*,n^*) > g_\varepsilon(i^*,n^*)$ since $(i^*,n^*) \in Y$) and similarly

$$B_{i^*} \models (b_{i^*,\varepsilon_m} - b_{i^*,\zeta_m}) \cap a_{i^*,n^*} = 0.$$

Hence

$$B_{i^*} \models \text{“}b_{i^*,\zeta} - b_{i^*,\varepsilon} - \bigcup_{m<k}(b_{i^*,\varepsilon_m} - b_{i^*,\zeta_m}) \neq 0.\text{”}$$

As this holds for every $(i^*,n^*) \in Y$ and $Y \in D^*$, by the assumptions on D^* we have

$$\Big\{i^* < \kappa : B_{i^*} \models \text{“}b_{i^*,\zeta} - b_{i^*,\varepsilon} - \bigcup_{m<k}(b_{i^*,\varepsilon_m} - b_{i^*,\zeta_m}) \neq 0\text{”}\Big\} \in D^+$$

hence in $B, b_\zeta - b_\varepsilon \notin J$ as required.

$(\beta)' \Rightarrow (\gamma)$

Let B_i be the interval Boolean algebra for γ_i, an ordinal $< \lambda_i$.

To prove clause (γ) we assume that our regular μ is $< \mathrm{Depth}^+_h(\prod_{i<\kappa} B_i/D)$, and we have to find $n_i < \omega, \lambda_{i,n} < \lambda_i$ for $i < \kappa, n < n_i$ and D^* as in the conclusion of clause (γ). So there are $f_\alpha \in \prod_{i<\kappa} B_i$ for $\alpha < \mu$ and an ideal J of the Boolean algebra $B =: \prod_{i<\kappa} B_i/D$ such that $f_\alpha/D < f_\beta/D \bmod J$ for $\alpha < \beta$.

Remember $\mu > 2^\kappa$. Let $f_\alpha(i) = \bigcup_{\ell<n(\alpha,i)}[j_{\alpha,i,2\ell}, j_{\alpha,i,2\ell+1})$ where $j_{\alpha,i,\ell} < j_{\alpha,i,\ell+1} \leq \gamma_i$ for $\ell < 2n(\alpha,i)$. As $\mu = \mathrm{cf}(\mu) > 2^\kappa$, without loss of generality $n(\alpha,i) = n_i$ for all $\alpha < \mu$. By [Sh:430, 6.6D] (better yet, see [Sh:513, 6.1] or [Sh:620, 7.0]) we can find $A \subseteq A^* =: \{(i,\ell) : i < \kappa, \ell < 2n_i\}$ and $\langle \gamma^*_{i,\ell} : i < \kappa, \ell < 2n_i\rangle$ such that $(i,\ell) \in A \Rightarrow \gamma^*_{i,\ell}$ is a limit ordinal of cofinality $> 2^\kappa$ and

(*) for every $f \in \prod_{(i,\ell)\in A} \gamma^*_{i,\ell}$ and $\alpha < \mu$ there is $\beta \in (\alpha,\mu)$ such that:

$$(i,\ell) \in A^* \backslash A \Rightarrow j_{\beta,i,\ell} = \gamma^*_{i,\ell}$$
$$(i,\ell) \in A \Rightarrow f(i,\ell) < j_{\beta,i,\ell} < \gamma^*_{i,\ell}.$$

For $(i,\ell) \in A^*$ define $\beta^*_{i,\ell}$ by

$$\beta^*_{i,\ell} =: \sup\{\gamma^*_{i,m} : (i,m) \in A^* \text{ and } \gamma^*_{i,m} < \gamma^*_{i,\ell} \text{ and } m < 2n_i \text{ (actually } m < \ell \text{ suffices)}\}.$$

Now $\beta^*_{i.\ell} < \gamma^*_{i,\ell}$ as the supremum is on a finite set, except the case $0 = \beta^*_{i.\ell} = \gamma^*_{i.\ell}$ which does not occur if $(i, \ell) \in A$. Let

$$Y = \{\alpha < \mu : \text{ if } (i,\ell) \in A^* \backslash A \text{ then } j_{\alpha.i.\ell} = \gamma^*_{i.\ell} \\ \text{and if } (i,\ell) \in A \text{ then } \beta^*_{i.\ell} < j_{\alpha,i,\ell} < \gamma^*_{i,\ell}\}.$$

Clearly $\{f_\alpha : \alpha \in Y\}$ satisfies $(*)$, so without loss of generality $Y = \mu$.

Clearly

$(*)_1$ $\langle \gamma^*_{i,\ell} : \ell < 2n_i \rangle$ is non-decreasing (for each i).

Let $u_i = \{\ell < 2n_i : (\forall m < \ell)[\gamma^*_{i.m} < \gamma^*_{i.\ell}]\}$.

For $i < \kappa, \ell < 2n_i$ define $b_{i,\ell} =: f_\alpha(i) \cap [\beta^*_{i,\ell}, \gamma^*_{i,\ell}) \in B_i$. Let

$$w_i =: \{\ell \in u_i : \text{for every (equivalently some) } \alpha < \mu \text{ we have} \\ B_i \models \text{“}[\beta^*_{i,\ell}, \gamma^*_{i,\ell}) \cap f_\alpha(i) \text{ is } \neq \emptyset \text{ and } \neq [\beta^*_{i,\ell}, \gamma^*_{i.\ell})\text{”}\}.$$

So

$(*)_2$ $f_\alpha(i) \backslash \bigcup_{\ell \in w_i} b_{i,\ell}$ does not depend on α, call it $c_i (\in B_i)$.

Let for $\ell \in w_i$

$$u_{i,\ell} =: \left\{ n < n_i : [j_{\alpha,i,2n}, j_{\alpha,i,2n+1}) \text{ is not disjoint to } [\beta^*_{i,\ell}, \gamma^*_{i,\ell}) \right. \\ \left. \text{for some (equivalently every) } \alpha < \mu \right\}.$$

$$A_0 = \left\{ (i,\ell) : i < \kappa, \ell \in w_i \text{ and for some } n \in u_{i,\ell} \text{ we have, for some} \right. \\ \left. (\equiv \text{ every}) \ \alpha < \mu \text{ that } j_{\alpha,i,2n} \leq \beta^*_{i,\ell} < j_{\alpha.i,2n+1} < \gamma^*_{i,\ell} \right\}.$$

$$A_1 = \left\{ (i,\ell) : i < \kappa, \ell \in w_i \text{ and for some } n \in u_{i,\ell} \text{ we have, for some} \right. \\ \left. (\equiv \text{ every}) \ \alpha < \mu \text{ that } \beta^*_{i.\ell} < j_{\alpha,i,2n} < \gamma^*_{i.\ell} \leq j_{\alpha.i,2n+1} \right\}.$$

Let

$$b^0_i =: \bigcup\{[\beta^*_{i.\ell}, \gamma^*_{i.\ell}) : \ell \in w_i \text{ and } (i,\ell) \in A_0\} \in B_i$$

$$b^1_i =: \bigcup\{[\beta^*_{i.\ell}, \gamma^*_{i.\ell}) : \ell \in w_i \text{ and } (i,\ell) \in A_1\} \in B_i$$

$$c^1_i = b^0_i \cap b^1_i, \qquad c^2_i = b^0_i \cap (1 - b^1_i),$$

$$c^3_i = (1 - b^0_i) \cap b^1_i, \qquad c^4_i = (1 - b^0_i) \cap (1 - b^1_i)$$

$$b_0 =: \langle b^0_i : i < \kappa \rangle / D \in B \qquad b_1 =: \langle b^1_i : i < \kappa \rangle / D \in B$$

$$c_t = \langle c^t_i : i < \kappa \rangle / D \in B$$

$$c = \langle c_i : i < \kappa \rangle / D \in B, \text{ see } (*)_2.$$

Let $J_1 = \{b \in B : \langle (f_\alpha/D) \cap b : \alpha < \mu\rangle$ is eventually constant modulo J, i.e., $(\exists \alpha < \mu)(\forall \beta)[\alpha \leq \beta < \mu \rightarrow (f_\alpha/D) \cap b - (f_\beta/D) \cap b \in J]\}$. Also $B \models c \leq f_\alpha/D$ hence $c \in J_1$.

Clearly J_1 is an ideal of B extending J and $1_B \notin J_1$. Also if $x \in J_1^+$ then for some closed unbounded $E \subseteq \mu$ we have: $\langle (f_\alpha/D) \cap x : \alpha \in E\rangle$ is strictly increasing modulo J.

Hence by easy manipulations without loss of generality:

$(*)_3$(a) if $c_t \in J_1^+$ then $\langle (f_\alpha/D) \cap c_t : \alpha < \mu\rangle$ is strictly increasing modulo J

(b) for at least one t, $c_t \in J_1^+$.

By $(*)$ we can find $0 < \alpha_0 < \alpha_1 < \alpha_2 < \mu$ such that:

$(*)_4$ if $i < \kappa, \ell < 2n_i, \bigwedge_{\alpha<\mu} \gamma^*_{i,\ell} > j_{\alpha,i,\ell}$ and $k < 2$ then

$$\sup\{j_{\alpha_k,i,\ell_1} : j_{\alpha_k,i,\ell_1} < \gamma^*_{i,\ell} \text{ and } \ell_1 < 2n_i\} < j_{\alpha_{k+1},i,\ell}.$$

Now if in $(*)_3$, $c_4 \in J_1^+$ occurs then

$$\begin{aligned} B_i \models \text{“} & f_{\alpha_0}(i) \cap f_{\alpha_1}(i) \cap c_i^4 - c_i \\ &= \bigcup\{(f_{\alpha_0}(i) \cap f_{\alpha_1}(i)) \cap [\beta^*_{i,\ell}, \gamma^*_{i,\ell}) : \ell \in w_i \\ &\qquad\qquad \text{and } (i,\ell) \notin A_0, (i,\ell) \notin A_1\} \\ &= \bigcup_{\ell \in w_i} 0_{B_i} = 0_{B_i}\text{”} \end{aligned}$$

(as for each $\ell \in w_i$ such that $(i, \ell) \notin A_0 \cup A_1$, the intersection is the intersection of two unions of intervals which are pairwise disjoint) whereas we know $(f_{\alpha_0}/D) \cap (f_{\alpha_1}/D) \cap c_4 - c =_J (f_{\alpha_0}/D) \cap c_4 - c \notin J$; contradiction.

Next if in $(*)_3$, $c_3 \in J_1^+$ holds then

$$\begin{aligned} B_i \models \text{“}&(f_{\alpha_1}(i) \cap c_i^3 - c_i) - (f_{\alpha_0}(i) \cap c_i^3 - c_i) \\ &= \bigcup\{(f_{\alpha_1}(i) \cap [\beta^*_{i,\ell}, \gamma^*_{i,\ell}) - f_{\alpha_0}(i) \cap [\beta^*_{i,\ell}, \gamma^*_{i,j})) : \ell \in w_i \text{ and } (i,\ell) \in A_1 \backslash A_0\} \\ &= \bigcup_{\ell \in w_i} 0_{B_i} = 0_{B_i}\text{”} \end{aligned}$$

(as for each $\ell \in w_i$ such that $(\ell, i) \in A_1 \backslash A_0$ the term is the difference of two unions of intervals but the first is included in the right most interval of the second) and we have a contradiction.

Now if in $(*)_3$, $c_1 \in J^+$ holds then

$$\begin{aligned} B_i \models \text{“}&(f_{\alpha_2}(i) \cap c_i^1 - c_i) - (f_{\alpha_1}(i) \cap c_i^1 - c_i) \cup (f_{\alpha_0}(i) \cap c_i^1 - c_i) \\ &= \bigcup_i\{((f_{\alpha_2}(i) - f_{\alpha_1}(i) \cup f_{\alpha_0}(i)) \cap [\beta^*_{i,\ell}, \gamma^*_{i,\ell})) : \ell \in w_i \text{ and } (i,\ell) \in A_0 \cap A_1\} \\ &= \bigcup_{\ell \in w_i} 0_{B_i} = 0_{B_i}\text{”} \end{aligned}$$

and we get a similar contradiction.

So

$(*)_5$ in $(*)_3$, $c_2 \in J_1^+$.

Without loss of generality

$(*)_6$ for $\alpha < \mu, i < \kappa$ and $\ell < 2n_i$ such that $(i, \ell) \in A$ we have

$$\sup\{j_{2\alpha.i.\ell_1} : \ell_1 < 2n_i \text{ and } j_{2\alpha.i.\ell_1} < \gamma^*_{i.\ell}\} < j_{2\alpha+1.i.\ell}.$$

Let $v_i = \{\ell \in w_i : (i, \ell) \in A_0, (i, \ell) \notin A_1\}$, so $c_i^2 = \bigcup\{[\beta^*_{i.\ell}, \gamma^*_{i.\ell}) : \ell \in v_i\}$. As $\ell \in v_i \Rightarrow (i, \ell) \in A_0$ necessarily

$(*)_7$ if $\ell \in v_i$ then ℓ is odd and $j_{\alpha.i.\ell-1} = \beta^*_{i.\ell} < j_{\alpha.i.2\ell+1} < \gamma^*_{i.\ell}$.

Now for every $\alpha < \mu$ define $f'_\alpha \in \prod_{i<\kappa} B_i$ by

$$f'_\alpha(i) = \bigcup_{\ell \in v_i} [\beta^*_{i.\ell}, j_{2\alpha.i.2\ell+1} :).$$

Clearly

$$B_i \models \text{“}f_{2\alpha}(i) \cap c_i^2 - c_i \leq f'_\alpha(i) \leq f_{2\alpha+1}(i) \cap c_i^2 - c_i.\text{”}$$

Let $Y^* =: \bigcup_{i<\kappa}(\{i\} \times v_i)$ and we shall define now a family D_0 of subsets of Y^*.

For $Y \subseteq Y^*$, and for $\alpha < \mu$ define $f_{\alpha.Y} \in \prod_{i<\kappa} B_i$ by

$$f_{\alpha.Y}(i) = \bigcup\{[\beta^*_{i.2\ell+1}, j_{\alpha.i.2\ell+1}) : 2\ell + 1 \in v_i \text{ and } (i, \ell) \notin Y\}.$$

For $g \in G =: \prod_{(i,\ell)\in Y^*}[\beta^*_{i.\ell}, \gamma^*_{i.\ell})$ define $f_g \in \prod_{i<\kappa} B_i$ by

$$f_g(i) = \bigcup_{\ell \in v_i} [\beta^*_{i.\ell}, g((i, \ell))),$$

now

$(*)_8$ for every $\alpha < \mu$ for some $g = g^*_\alpha \in G$ we have $f'_\alpha = f_g$.

[Why? By the previous analysis; in particular $(*)_7$.]

Let

$$D_0 = \Big\{ Y \subseteq Y^* : \text{for some } g_1 \in G \text{ for every } g \in G \text{ satisfying } [(i, \ell) \in Y \Rightarrow g(i, 0) = \beta^*_{i.\ell}] \text{ we have } f_g/D - f_{g_1}/D \text{ belongs to } J_1 \Big\}$$

it is a filter on Y^*.

$(*)_9$ if $g_1, g_2 \in G$ then

(a) $g_1 \leq_{D_0} g_2 \Leftrightarrow B \models (f_{g_1}/D) \cap c_2 \leq (f_{g_2}/D) \cap c_2$

(b) $g_1 <_{D_0} g_2 \Leftrightarrow B \models (f_{g_1}/D) \cap c_2 < (f_{g_2}/D) \cap c_2$

$(*)_{10}$ for every $g' \in G$ for some $\alpha(g') < \mu$ we have $g' < g^*_{\alpha(g')}$ (see $(*)_8$).

[Why? By $(*)$.]

Clearly

$(*)_{11}$ if $A \in D$ then $\bigcup\{\{i\} \times v_i : i \in A\} \in D_0$.

Now

$\bigotimes$ $\mathrm{cf}(\prod_{(i.\ell)\in Y^*} \gamma^*_{i.\ell}/D_0) \geq \mu$.

[Why? If not, we can find $G^* \subseteq G = \prod_{(i.\ell)\in Y^*}[\beta^*_{i.\ell}, \gamma^*_{i.\ell})$ of cardinality $< \mu$, cofinal in $\prod_{(i.\ell)\in Y^*} \gamma^*_{i.\ell}/D_0$. For each $g \in G^*$ for some $\alpha(g) < \mu$ we have $g < g^*_{\alpha(g)}$, hence $\alpha \in [\alpha(g), \mu) \Rightarrow g <_{D_0} g^*_\alpha$, let $\alpha(*) = \sup\{\alpha(g) : g \in G\}$ so $\alpha(*) < \mu$ so $\bigwedge_{g\in G} g <_{D_0} g^*_{\alpha(*)}$; contradiction, so $\bigotimes$ holds.]

So for some ultrafilter D^* on Y^* extending $D_0, \mu \leq \operatorname{tcf}\left(\prod_{(i,\ell)\in Y^*} \gamma^*_{i,\ell}/D^*\right)$, hence $\mu \leq \operatorname{tcf} \prod_{(i,\ell)\in Y^*} \operatorname{cf}(\gamma^*_{i,\ell})/D^*$ and by [Sh:g, II,1.3] for some $\lambda'_{i,\ell} = \operatorname{cf}(\lambda'_{i,\ell}) \leq \operatorname{cf}(\gamma^*_{i,\ell}) \leq \gamma_i < \lambda_i$ we have $\mu = \operatorname{tcf}\left(\prod_{(i,\ell)\in Y^*} \lambda'_{i,\ell}/D^*\right)$ as required (we could, instead of relying on this quotation, analyze more).

So we have proved $(\beta)' \Rightarrow (\gamma)$.

$\underline{(\beta)^- \Rightarrow (\beta)'}$

Let $B_{i,\gamma}$ be the interval Boolean algebra on γ for $\gamma < \lambda_i, i < \kappa$, and we let $B^*_{i,\gamma}$ be generated by $\{a^{i,\gamma}_j : j < \gamma\}$ freely except $a^{i,\gamma}_{j_1} \leq a^{i,\gamma}_{j_2}$ for $j_1 < j_2 < \gamma$. So without loss of generality B_i is the disjoint sum of $\{B^*_{i,\gamma} : \gamma < \lambda_i\}$. Let $e_{i,\gamma} = 1_{B_{i,\gamma}}$; so $\langle e_{i,\gamma} : \gamma < \lambda_i\rangle$ is a maximal antichain of B_i, $B_i \restriction \{x \in B_i : x \leq e_{i,\gamma}\}$ is isomorphic to $B_{i,\gamma}$ and B_i is generated by $\{x : (\exists \gamma < \lambda_i)(x \leq e_{i,\gamma})\}$. Let $\langle f_\alpha : \alpha < \mu\rangle$ and an ideal J of B exemplify clause $(\beta)^-$, that is $f_\alpha \in \prod_{i<\alpha} \beta_i$ and $\alpha < \beta \Rightarrow f_\alpha/D < f_\beta/D \bmod J$; for the proof of 3.5 we just fix $J = \{0_B\}$.

Let I_i be the ideal of B_i generated by $\{e_{i,\gamma} : \gamma < \lambda_i\}$, so it is a <u>maximal ideal</u>; let I be such that $(B, I) = \prod_{i<\kappa}(B_i, I_i)/D$ so clearly $|B/I| = |2^\kappa/D| \leq 2^\kappa < \operatorname{cf}(\mu)$ (actually $|B/I| = 2$ if D is an ultrafilter on κ), so without loss of generality $\alpha < \beta \leq \mu \Rightarrow f_\alpha/D = f_\beta/D \bmod I$. We can use $\langle f_{1+\alpha}/D - f_0/D : \alpha < \mu\rangle$, so without loss of generality $f_\alpha/D \in I$, hence without loss of generality $f_\alpha(i) \in I_i$ for $\alpha < \mu, i < \kappa$.

Let $f_\alpha(i) = \tau_{\alpha,i}(\dots, e_{i,\gamma(\alpha,i,\varepsilon)}, a^{i,\gamma(\alpha,i,\varepsilon)}_{j(\alpha,i,\varepsilon)}, \dots)_{\varepsilon<n_{\alpha,i}}$ where $n_{\alpha,i} < \omega$ and $\tau_{\alpha,i}$ is a Boolean term. As μ is regular $> 2^\kappa$, without loss of generality $\tau_{\alpha,i} = \tau_i$ and $n_{\alpha,i} = n_i$. Let $\gamma^0_{\alpha,i,\varepsilon} = \gamma(\alpha, i, \varepsilon)$ and $\gamma^1_{\alpha,i,\varepsilon} = j(\alpha, i, \varepsilon)$.

By [Sh:430, 6.6D] (or better [Sh:620, 7.0]) we can find a subset A of

$$A^* = \{(i, n, \ell) : i < \kappa \text{ and } n < n_i \text{ and } \ell < 2\}$$

and $\langle \gamma^*_{i,n,\ell} : i < \kappa \text{ and } n < n_i \text{ and } \ell < 2\rangle$ such that:

(*)(A) $(i, n, \ell) \in A \Rightarrow \operatorname{cf}(\gamma^*_{i,n,\ell}) > 2^\kappa$

(B) for every $g \in \prod_{(i,n,\ell)\in A} \gamma^*_{i,n,\ell}$ for arbitrarily large $\alpha < \mu$ we have

$$(i, n, \ell) \in A^* \backslash A \Rightarrow \gamma^\ell_{\alpha,i,n} = \gamma^*_{i,n,\ell}$$

$$(i, n, \ell) \in A \Rightarrow g(i, n, \ell) < \gamma^\ell_{\alpha,i,n} < \gamma^*_{i,n,\ell}.$$

Let

$$\beta^*_{i,n,\ell} = \sup\{\gamma^*_{i,n',\ell'} : n' < n_i, \ell' < 2 \text{ and } \gamma^*_{i,n',\ell'} < \gamma^*_{i,n,\ell}\}.$$

Without loss of generality

$$(i, n, \ell) \in A \;\&\; \alpha < \mu \Rightarrow \gamma^\ell_{\alpha,i,n} \in (\beta^*_{i,n,\ell}, \gamma^*_{i,n,\ell})$$

$$(i, n, \ell) \in A^* \backslash A \;\&\; \alpha < \mu \Rightarrow \gamma^\ell_{\alpha,i,n} = \gamma^*_{i,n,\ell}.$$

Also without loss of generality

(*) for $\alpha < \mu$ and $(i, n, \ell) \in A$ we have

$$\gamma^\ell_{2\alpha+1,i,n} > \sup\{\gamma^{\ell'}_{2\alpha,i,n'} : i < \kappa, \ell' < 2, n' < n_i, \text{and } \gamma^{\ell'}_{2\alpha,i,n'} < \gamma^*_{i,n,\ell}\}.$$

Let $\triangle_i = \{\gamma^*_{i,n,0} : n < n_i \text{ and } (i, n, 0) \in A^* \backslash A\}$ and

$$B'_i = B_i \restriction \sum\{e_{i,\gamma} : \gamma \in \triangle_i\}.$$

We define $f'_\alpha \in \prod_{i<\kappa} B'_i$ by $f'_\alpha(i) = f_{2\alpha+1}(i) \cap (\bigcup_{\gamma\in\triangle_i} e_{i,\gamma}) \in B'_i \subseteq B_i$. Obviously $\langle f'_\alpha/D : \alpha < \mu\rangle$ is $\leq_D$-increasing.

Now easily $f'_\alpha/D \leq f_{2\alpha+1}/D$ and for $\alpha < \mu$, $i < \kappa$ we have $f_{2\alpha}(i) - 1_{B'_i}$, $f_{2\alpha+1}(i) - 1_{B'_i}$ are disjoint (in B_i) hence (also in B)

$$f_{2\alpha}/D - f'_\alpha/D \leq f_{2\alpha}/D - f_{2\alpha+1}/D \in J,$$

hence $\langle f'_\alpha/D : \alpha < \lambda\rangle$ is strictly increasing modulo J. So $\langle B'_i : i < \kappa\rangle$, $\langle f'_\alpha : \alpha < \mu\rangle$ form a witness, too. But B'_i is isomorphic to the interval Boolean algebra of the ordinal $\gamma_i = \sum_{\gamma\in\triangle_i} \gamma < \lambda_i$, so we are almost done. Well, γ_i is an ordinal, not necessarily a cardinal, but we are proving $(\beta)'$ not (β). $\dashv_{3.6}$

§4. **On the existence of independent sets for stable theories.** The following is motivated by questions of Bays [Bay] which continues some investigations of [Sh:a] (better see [Sh:c]) dealing with questions on $\mathrm{Pr}_T(\mu)$, Pr^*_T for stable T (see Definition 4.3 below). We connect this to pcf, using [Sh:430, 3.17] and also [Sh:513, 6.12]). We assume basic knowledge on non-forking (see [Sh:c, Chapter III,I]) and we say some things on the combinatorics but the rest of the paper does not depend on this section. For simplicity, we concentrate on the regular case.

CLAIM 4.1. *Assume $\lambda > \theta \geq \kappa$ are regular uncountable. Then the following are equivalent*:

(A) *If $\mu < \lambda$ and $a_\alpha \in [\mu]^{<\kappa}$ for $\alpha < \lambda$ then for some $A \in [\lambda]^\lambda$ we have $\bigcup_{\alpha\in A} a_\alpha$ has cardinality $< \theta$*

(B) *if $\delta = \mathrm{cf}(\delta) < \kappa$ and $\eta_\alpha \in {}^\delta\lambda$ for $\alpha < \lambda$ and $|\{\eta_\alpha \upharpoonright i : \alpha < \lambda, i < \delta\}| < \lambda$ then for some $A \in [\lambda]^\lambda$ the set $\{\eta_\alpha \upharpoonright i : \alpha \in A, i < \delta\}$ has cardinality $< \theta$.*

REMARK 4.2. *Of course, if a_α is just a set of cardinality $< \kappa$, by renaming $a_\alpha \in [\lambda]^{<\kappa}$ and for some stationary $S \subseteq \lambda$ and $\alpha^* < \mu$, $\langle a_\alpha \backslash \alpha^* : \alpha \in S\rangle$ are pairwise disjoint, renaming $\alpha^* = \mu < \lambda$, etc., see more in* [Sh:430, §2].

PROOF. (A)$\Rightarrow$(B). Immediate.

$\neg$(A)$\Rightarrow$ $\neg$(B)

CASE 1. For some $\mu \in (\theta, \lambda)$ we have $\mathrm{cf}(\mu) < \kappa$ and $\mathrm{pp}(\mu) \geq \lambda$. Without loss of generality μ is minimal. So

(*) $\mathfrak{a} \subseteq \mathrm{Reg} \cap \mu \backslash \theta, |\mathfrak{a}| < \kappa, \sup(\mathfrak{a}) < \mu \Rightarrow \max \mathrm{pcf}(\mathfrak{a}) < \mu$.

Subcase 1a. $\lambda < \mathrm{pp}^+(\mu)$.

So by [Sh:g, Chapter VIII,1.6(2), page 321], (if $\mathrm{cf}(\mu) > \aleph_0$) and [Sh:430, 6.5] (if $\mathrm{cf}(\mu) = \aleph_0$) we can find $\langle \lambda_\alpha : \alpha < \mathrm{cf}(\mu)\rangle$, a strictly increasing sequence of regulars from (θ, μ) with limit μ and an ideal J on $\mathrm{cf}(\mu)$ satisfying $J^{\mathrm{bd}}_{\mathrm{cf}(\mu)} \subseteq J$ such that $\lambda = \mathrm{tcf}\left(\prod_{\alpha<\mathrm{cf}(\mu)} \lambda_\alpha/J\right)$ and $\max \mathrm{pcf}\{\lambda_\beta : \beta < \alpha\} < \lambda_\alpha$. By [Sh:g, II,3.5], there is $\langle f_\zeta : \zeta < \lambda\rangle$ which is $<_J$-increasing cofinal in $\prod_{\alpha<\mathrm{cf}(\mu)} \lambda_\alpha/J$ with

$$|\{f_\zeta \upharpoonright \alpha : \zeta < \lambda\}| < \lambda_\alpha.$$

Easily $\langle f_\zeta : \zeta < \lambda\rangle$ exemplifies $\neg(B)$: if $A \in [\lambda]^\lambda$ and $B =: \bigcup_{\zeta\in A} \mathrm{Range}(f_\zeta)$ has cardinality $< \mu$ let $g \in \prod_\alpha \lambda_\alpha$ be: $g(\alpha) = \sup(\lambda_\alpha \cap B)$ if $< \lambda_\alpha$, zero otherwise and let $\alpha_0 = \mathrm{Min}\{\alpha < \mathrm{cf}(\mu) : \lambda_\alpha > |B|\}$. So $\alpha_0 < \mathrm{cf}(\mu)$ and $\zeta \in A \Rightarrow f_\zeta \upharpoonright [\alpha_0, \mathrm{cf}(\mu)) < g$, contradiction to " $<_J$-cofinal."

Subcase 1b. $\mathrm{cf}(\mu) > \aleph_0$ and $\mathrm{pp}^+(\mu) = \mathrm{pp}(\mu) = \lambda$. Note that by [Sh:g, Chapter II, 5.4, page 88-7] we have $\mathrm{cov}(\mu, \theta, \kappa, \aleph_1) \leq \lambda$ and let $\{b_\alpha : \alpha < \lambda\} \subseteq [\mu]^{<\kappa}$ exemplify this. Try to choose by induction on $\alpha < \lambda$ a set $a_\alpha \in [\mu]^{\mathrm{cf}(\mu)}$ such that $(\forall \beta < \alpha)(|a_\alpha \cap b_\beta| < \mathrm{cf}(\mu))$; arriving to α, by [Sh:g, Chapter VIII, Section 1] and [Sh:g, Chapter II, 1.4(1)+(3), page 50] there is an increasing sequence $\langle \lambda_i : i < \mathrm{cf}(\mu)\rangle$ of regular cardinals $> \theta$ with limit μ, such that $\mathrm{tcf}(\prod_{i<\kappa} \lambda_i / J^{\mathrm{bd}}_{\mathrm{cf}(\mu)} = (|\alpha| + |\theta|)^{++}$, exemplified by $\langle f_\varepsilon : \varepsilon < (|\alpha| + \theta)^{++}\rangle$ which is μ-free. Necessarily for some ε the set $\mathrm{Range}(f_\varepsilon)$ is as required.

Subcase 1c. $\mathrm{cf}(\mu) = \aleph_0$ and $\lambda = \mathrm{pp}^+(\mu) = \mathrm{pp}(\mu) = \lambda$.

Let $\mu' < \lambda$ and $a_\alpha \in [\mu']^{<\kappa}$ for $\alpha < \lambda$ exemplify $\neg$(A). We can find (as in clause (b)) a sequence $\langle \lambda_n : n < \omega\rangle$ of regular cardinals in (θ, μ) and ideal J on ω containing the finite subsets of ω such that $\prod_{n<\omega} \lambda_n / J$ is $(\mu')^{++}$-directed, so we can find $f_\varepsilon \in \prod_{n<\omega} \lambda_n$ for $e < \mu'$, $<_J$-increasing and $\{f_\varepsilon : \varepsilon < \mu'\}$ is μ-free (see [Sh:g, Chapter II, 1.4]). Define $b_\alpha = \bigcup\{\mathrm{Rang}(f_\varepsilon) : \varepsilon \in a_\alpha\} \in [\mu]^{<\kappa}$ for $\alpha < \lambda$. Easily also μ, $\langle b_\alpha : \alpha < \lambda\rangle$ form a counterexample to clause (A). Also in Case 2 below and the choice of μ we have $\theta \leq \chi < \mu \Rightarrow \mathrm{cov}(\chi, \theta, \kappa, \aleph_1) < \lambda$ and we can proceed as there.

Case 2. Not Case 1.

So by [Sh:g, Chapter II, 5.4, page 88-9], we have $\theta \leq \mu < \lambda \Rightarrow \mathrm{cov}(\mu, \theta, \kappa, \aleph_1) < \lambda$.

As we are assuming $\neg(A)$, we can find $\mu_0 < \lambda, a_\alpha \in [\mu_0]^{<\kappa}$ for $\alpha < \lambda$ such that $A \in [\lambda]^\lambda \Rightarrow |\bigcup_{\alpha \in A} a_\alpha| \geq \theta$, but by the previous sentence we can find $\mu_1 < \lambda$ and $\{b_\beta : \beta < \mu_1\} \subseteq [\mu_0]^{<\theta}$ such that: every $a \in [\mu_0]^{<\kappa}$ is included in the union of $\leq \aleph_0$ sets from $\{b_\beta : \beta < \mu_1\}$. So we can find $c_\alpha \in [\mu_1]^{\aleph_0}$ for $\alpha < \lambda$ such that $a_\alpha \subseteq \bigcup_{\beta \in c_\alpha} b_\beta$. Now for $A \in [\lambda]^\lambda$, if $|\bigcup_{\alpha \in A} c_\alpha| < \theta$ then

$$\begin{aligned}
\Big|\bigcup\{a_\alpha : \alpha \in A\}\Big| &\leq \Big|\bigcup\Big\{\bigcup_{\beta \in c_\alpha} b_\beta : \alpha \in A\Big\}\Big| \\
&= \Big|\bigcup\Big\{b_\beta : \beta \in \bigcup_{\alpha \in A} c_\alpha\Big\}\Big| \\
&< \min\{\sigma : \sigma = \mathrm{cf}(\sigma) > |b_\beta| \text{ for } \beta < \mu_1\} \\
&\qquad + |\bigcup_{\alpha \in A} c_\alpha|^+ \leq \theta + \theta = \theta
\end{aligned}$$

contradicting the choice of $\langle a_\alpha : \alpha < \lambda\rangle$.

So

(*) $c_\alpha \in [\mu_1]^{\leq \aleph_0}$, for $\alpha < \lambda, \mu_1 < \lambda$ and $A \in [\lambda]^\lambda \Rightarrow |\bigcup_{\alpha \in A} c_\alpha| \geq \theta$.

Let η_α be an ω-sequence enumerating c_α, so $\langle \eta_\alpha : \alpha < \lambda\rangle$ is a counterexample to clause (**B**). $\dashv_{4.1}$

We concentrate below on λ, θ, κ regular (others can be reduced to it).

Definition 4.3. Let $\boldsymbol{T}$ be a complete first order theory; which is stable ($\mathfrak{C}$ the monster model of $\boldsymbol{T}$ and $A, B, \ldots$ denote subsets of $\mathfrak{C}^{\mathrm{eq}}$ of cardinality $< \|\mathfrak{C}^{\mathrm{eq}}\|$).

(1) $\mathrm{Pr}_{\boldsymbol{T}}(\lambda, \chi, \theta)$ means:

(*) if $A \subseteq \mathfrak{C}^{eq}$, $|A| = \lambda$ then we can find $A' \subseteq A, |A'| = \chi$ and $B', |B'| < \theta$ such that A' is independent over B' (i.e., $a \in A' \Rightarrow \text{tp}(a, B' \cup (A' \setminus \{a\}))$ does not fork over B').

(2) $\text{Pr}^*_{\boldsymbol{T}}(\lambda, \mu, \chi, \theta)$ means:

(**) if $A \subseteq \mathfrak{C}^{eq}$ is independent over B where $|A| = \lambda$ and $|B| < \mu, B \subseteq \mathfrak{C}^{eq}$ then there are $A' \subseteq A, |A'| = \chi$ and $B' \subseteq B$ satisfying $|B'| < \theta$ such that $\text{tp}(A', B)$ does not fork over B' (hence A' is independent over B').

(3) $\text{Pr}^*_{\boldsymbol{T}}(\lambda, \chi, \theta)$ means $\text{Pr}^*_{\boldsymbol{T}}(\lambda, \lambda, \chi, \theta)$.

FACT 4.4. *Assume λ is regular $> \theta \geq \kappa_r(\boldsymbol{T})$ then*

(1) *if $\chi = \lambda$ then $Pr_{\boldsymbol{T}}(\lambda, \chi, \theta) \Leftrightarrow Pr^*_{\boldsymbol{T}}(\lambda, \lambda, \chi, \theta)$*

(2) *if $\lambda \geq \chi \geq \mu \geq \theta$ then $Pr_{\boldsymbol{T}}(\lambda, \chi, \theta) \Rightarrow Pr^*_{\boldsymbol{T}}(\lambda, \mu, \chi, \theta)$.*

PROOF. (1) The direction $\Leftarrow$ is by the proof in [Sh:a, III].

[In detail, let A, B be given (the B is not really necessary), such that $\lambda = |A| > |B| + \kappa_r(\boldsymbol{T})$ so let $A = \{a_i : i < \lambda\}$; define

$$A_i =: \{a_j : j < i\}, S = \{i < \lambda : \text{cf}(i) \geq \kappa_r(\boldsymbol{T})\},$$

so by the definition of $\kappa_r(\boldsymbol{T})$ for $\alpha \in S$ there is $j_\alpha < \alpha$ such that $\text{tp}(a_\alpha, A_\alpha \cup B)$ does not fork over $A_{j_\alpha} \cup B$ so for some j^* the set $S' = \{\delta \in S : j_\delta = j^*\}$ is stationary, now apply the right side with $\{a_\delta : \delta \in S'\}, A_j \;\; \cup B$, here standing for A, B there].

The other direction $\Rightarrow$ follows by part (2).

(2) This is easy, too, by the non-forking calculus [Sh:a, III,Theorem 0.1 + (0)–(4), pages 82–84] but we give details. So we are given a set $A \subseteq \mathfrak{C}^{eq}$ independent over B, where $|A| = \lambda$ and $|B| < \mu$. As we are assuming $\text{Pr}_{\boldsymbol{T}}(\lambda, \chi, \theta)$ there is $A' \subseteq A, |A'| = \chi$ and $B', |B'| < \theta$ such that A' is independent over B'. So for every finite $\bar{c} \subseteq B$ for some $A_{\bar{c}} \subseteq A'$ of cardinality $< \kappa(\boldsymbol{T})\ (\leq \kappa_r(\boldsymbol{T}))$ we have: $A' \setminus A_{\bar{c}}$ is independent over $B' \cup \bar{c}$. So $A^* = \bigcup\{A_{\bar{c}} : \bar{c} \subseteq B \text{ finite}\}$ has cardinality $< \kappa_r(\boldsymbol{T}) + |B|^+ \leq \chi$ so necessarily $A' \setminus A^*$ has cardinality χ and it is independent over $\bigcup\{\bar{c} : \bar{c} \subseteq B \text{ finite}\} \cup B' = B \cup B'$. We can find a set $B^* \subseteq B$ of cardinality $< |B|^+ + \kappa_T(\boldsymbol{T})$ such that $\bar{c} \in B' \Rightarrow \text{tp}(\bar{c}, B)$ does not fork over B^*. Now B^*, $A' \setminus A^*$ are as required.] $\dashv_{4.4}$

Discussion 4.5. So in order to understand the model theoretic property it suffices to prove the equivalence

$$\text{Pr}^*_{\boldsymbol{T}}(\lambda, \mu, \chi, \theta) \Leftrightarrow \text{Pr}(\lambda, \mu, \chi, \theta, \kappa) \text{ with } \kappa = \kappa_r(\boldsymbol{T}),$$

where

DEFINITION 4.6. Assume

(*) $\lambda \geq \max\{\mu, \chi\} \geq \min\{\mu, \chi\} \geq \theta \geq \kappa > \aleph_0$ and $\mu > \theta$ and for simplicity λ, θ, κ are regular if not said otherwise (as the general case can be reduced to this case).

(1) $\text{Pr}(\lambda, \mu, \chi, \theta, \kappa)$ is defined as follows: if $u_\alpha \in [\mu]^{<\kappa}$ for $\alpha < \lambda$ and $|\bigcup_{\alpha<\lambda} u_\alpha| < \mu$ then there is $Y \in [\lambda]^\chi$ such that $|\bigcup_{\alpha \in Y} u_\alpha| < \theta$;

(2) $\text{Pr}^{\text{tr}}(\lambda, \mu, \chi, \theta, \kappa)$ is defined similarly but for some tree T each u_α is a branch of T.

(3) We write $\text{Pr}(\lambda, \leq \mu, \chi, \theta, \kappa)$ for $\text{Pr}(\lambda, \mu^+, \chi, \theta, \kappa)$ and similarly for Pr^{tr} and $\text{Pr}^*_{\boldsymbol{T}}$.

FACT 4.7. *Assume* $\lambda, \mu, \chi, \theta, \kappa = \kappa_r(\boldsymbol{T})$ *satisfies* $(*)$ *of Definition* 4.6. *Then*
(1) $Pr(\lambda, \mu, \chi, \theta, \kappa_r(\boldsymbol{T})) \Rightarrow Pr^*_{\boldsymbol{T}}(\lambda, \mu, \chi, \theta) \Rightarrow Pr^{\mathrm{tr}}(\lambda, \mu, \chi, \theta, \kappa_r(\boldsymbol{T}))$.
(2) $\mathrm{Pr}(\lambda, \lambda, \chi, \theta, \kappa_r(\boldsymbol{T})) \Rightarrow \mathrm{Pr}_{\boldsymbol{T}}(\lambda, \chi, \theta) \Rightarrow \mathrm{Pr}^{\mathrm{tr}}(\lambda, \lambda, \chi, \theta, \kappa_r(\boldsymbol{T}))$.
(3) *We have obvious monotonicity properties.*

PROOF. Straight.

(1) First we prove the first implication so assume $\mathrm{Pr}(\lambda, \mu, \chi, \theta, \kappa_r(\boldsymbol{T}))$, let $\kappa = \kappa_r(\boldsymbol{T})$, hence $(*)$ of 4.6 holds and we shall prove $\mathrm{Pr}^*_{\boldsymbol{T}}(\lambda, \mu, \chi, \theta)$. So (see Definition 4.3(2)) we have $A \subseteq \mathfrak{C}^{\mathrm{eq}}$ is independent over $B \subseteq \mathfrak{C}^{\mathrm{eq}}, |A| = \lambda$ and $|B| < \mu$. Let $A = \{a_\alpha : \alpha < \lambda\}$ with no repetitions and $B = \{b_j : j < j(*)\}$ so $j(*) < \mu$. For each $\alpha < \lambda$, there is a subset u_α of $j(*)$ of cardinality $< \kappa_r(\boldsymbol{T}) = \kappa$ such that $\mathrm{tp}(a_\alpha, B)$ does not fork over $\{b_j : j \in u_\alpha\}$. So $u_\alpha \in [\mu]^{<\kappa}$ and $|\bigcup_{\alpha<\lambda} u_\alpha| \leq |j(*)| < \mu$ hence as we are assuming $\mathrm{Pr}(\lambda, \mu, \chi, \theta, \kappa)$, there is $Y \in [\lambda]^\chi$ such that $|\bigcup_{\alpha \in Y} u_\alpha| < \theta$. Let $B' = \{b_j : j \in \bigcup_{\alpha \in Y} u_\alpha\}, A' = \{a_\alpha : \alpha \in Y\}$ so $B' \subseteq B, |B'| < \theta$ and $A' \subseteq A, |A'| = \chi$ and by the nonforking calculus, $\mathrm{tp}(A', B)$ does not fork over B' (even $\{a_\alpha : \alpha \in Y\}$ is independent over (B, B')).

Second, we prove the second implication, so we assume $\mathrm{Pr}^*_{\boldsymbol{T}}(\lambda, \mu, \chi, \theta)$ and we shall prove $\mathrm{Pr}^{\mathrm{tr}}(\lambda, \mu, \chi, \theta, \kappa_r(\boldsymbol{T}))$. Let $\kappa = \kappa_r(\boldsymbol{T})$.

Let T be a tree and for $\alpha < \lambda, u_\alpha$ a branch, $|u_\alpha| < \kappa, |\bigcup_{\alpha<\lambda} u_\alpha| < \mu$. Without loss of generality $T = \bigcup_{\alpha<\lambda} u_\alpha, \lambda = \bigcup_{\zeta<\kappa} A_\zeta$, where $A_\zeta = \{\alpha : \mathrm{otp}(u_\alpha) = \zeta\}$. Without loss of generality $T \subseteq {}^{\kappa>}\mu, T = \bigcup_{\zeta<\kappa} T_\zeta$ where $T_\zeta = \bigcup\{u_\alpha : \alpha \in A_\zeta\}$ and

$$\eta \in T_\zeta \backslash \{\langle\rangle\} \Rightarrow \eta(0) = \zeta.$$

Now T_ζ can be replaced by $\{\eta \upharpoonright C_\zeta : \eta \in T_\zeta\}$ where $0 \in C_\zeta$, $\mathrm{otp}(C_\zeta) = 1 + \mathrm{cf}(\zeta), \sup(C) = \zeta$. So without a loss of generality

$$T = \bigcup\{T_\sigma : \sigma \in \mathrm{Reg} \cap \kappa\}$$
$$\langle\rangle \neq \eta \in T_\sigma \Rightarrow \eta(0) = \sigma.$$

Without loss of generality $\lambda = \bigcup\{A_\sigma : \sigma \in \mathrm{Reg} \cap \kappa\}$ and $\bigcup_{\alpha \in A_\sigma} u_\alpha = T_\sigma$. It is enough to take care of one σ (otherwise a little more work is required). So without a loss of generality:

$$\alpha < \lambda \Rightarrow \mathrm{otp}(u_\alpha) = \sigma.$$

As $\sigma = \mathrm{cf}(\sigma) < \kappa$ there are $A_i \subseteq \mathfrak{C}^{\mathrm{eq}}$ such that $\langle A_i : i \leq \sigma\rangle$ increases continuously and $p \in S(A_\sigma)$ and for each $i < \sigma$ the type $p \upharpoonright A_{i+1}$ forks over A_i say $\varphi(x, c_i) \in p \upharpoonright A_{i+1}$ forks over A_i and $A_i = \{c_j : j < i\}$, (recall we work in $\mathfrak{C}^{\mathrm{eq}}$).

By the nonforking calculus we can find $\langle f_\eta : \eta \in T\rangle, f_\eta$ elementary mapping

$$\mathrm{Dom}(f_\eta) = A_{\ell g(\eta)}$$

$\langle f_\eta : \eta \in T\rangle$ nonforking tree, that is

$$\nu \lhd \eta \Rightarrow \mathrm{tp}(\mathrm{Rang}(f_\eta), \bigcup\{\mathrm{Rang}(f_\rho) : \rho \in T, \rho \upharpoonright (\ell g(\nu) + 1) \ntrianglelefteq \eta\})$$

does not fork over A_ν. For $\alpha < \lambda$, let

$$g_\alpha = \bigcup\{f_\nu : \nu \in a_\alpha\}, \quad A_\alpha = \bigcup_{\nu \in a_\alpha} \mathrm{Rang}(f_\nu) = g_\alpha(A_\sigma) \quad \text{and} \quad p_\alpha = g_\alpha(p).$$

Let $b_\alpha \in \mathfrak{C}$ realize p_α for $\alpha < \lambda$ be such that:

$$\operatorname{tp}\left(b_\alpha, \bigcup_{\eta \in T} \operatorname{Rang}(f_\eta) \cup \{b_\beta : \beta \neq \alpha\}\right) \text{ does not fork over } A_\alpha.$$

Now we apply $\Pr^*_{\boldsymbol{T}}(\lambda, \mu, \chi, \theta)$ on

$$A = \{b_\alpha : \alpha < \lambda\}$$
$$B = \bigcup_{\eta \in T} \operatorname{Rang}(f_\eta).$$

So there are $A' \subseteq A, |A'| = \chi$ and $B' \subseteq B, |B'| < \theta$, $\operatorname{tp}(A', B)$ does not fork over B', hence (for some $Y \in [\lambda]^\chi$) we have $A' = \{a_\alpha : \alpha \in Y\}$ independent over B'. So there is $T' \subseteq T_\alpha$ subtree such that $|T'| = |B'| + \sigma < \theta$ and such that $B' \subseteq \bigcup_{\rho \in T'} A_\rho$. Throwing "few" ($< |B'|^+ + \kappa_r(\boldsymbol{T})$) members of A' that is of Y we get A' independent over B' as by the nonforking calculus, if $\alpha \in Y$ then $\operatorname{tp}(b_\alpha, \bigcup\{\operatorname{Rang}(f_\eta) : \eta \in T\})$ does not fork over $\bigcup_{\eta \in T'} \operatorname{Rang}(f_\eta)$ hence $u_\alpha \subseteq T'$. So clearly Y is as required.

(2) By part (1) and 4.4.

(3) Left to the reader. $\dashv_{4.7}$

<u>Discussion</u> So by 4.7(1) if Pr and $\Pr^{\mathrm{tr}}$ are equivalent, $\kappa = \kappa_r(\boldsymbol{T})$ then $\Pr^*_{\boldsymbol{T}}$ is equivalent to them (for the suitable cardinal parameter), so we would like to prove such equivalence. Now Claim 4.1 gives the equivalence when $\theta = \kappa_r(\boldsymbol{T}), \lambda = \chi = \operatorname{cf}(\lambda)$ and "for every $\mu < \lambda$." We give below more general cases; e.g., if λ is a successor of regular or $\{\delta < \lambda : \operatorname{cf}(\delta) = \theta^*\} \in I(\lambda)$ or ...

FACT 4.8. *Assume $\lambda, \mu, \chi, \theta, \kappa$ are as in* $(*)$ *of Definition* 4.6 *and $\mu^* \in [\theta, \mu)$ and* $\operatorname{cf}(\mu^*) < \kappa$.

(0) *$Pr(\lambda, \mu, \chi, \theta, \kappa) \Rightarrow \Pr^{\mathrm{tr}}(\lambda, \mu, \chi, \theta, \kappa)$ and if $\lambda > |\alpha|^\kappa$ for $\alpha < \mu$, both hold.*

[*Why? Straight.*]

(1) *If $\kappa < \lambda$ and $\mu < \lambda$ and $\operatorname{cf}(\mu) \geq \kappa$, <u>then</u>*

$$\Pr(\lambda, \mu, \chi, \theta, \kappa) \Leftrightarrow (\forall \mu_1 < \mu) \Pr(\lambda, \leq \mu_1, \chi, \theta, \kappa);$$

similarly for $\Pr^{\mathrm{tr}}$.

(2) *If $pp(\mu^*) > \lambda$ <u>then</u> $\neg \Pr^{\mathrm{tr}}(\lambda, \mu, \chi, \theta, \kappa)$ (by* [*Sh*:355, 1.5*A*], *see* [*Sh*:513, 6.10]).

(3) *If $pp(\mu^*) \geq \lambda$ and*

(a) $\{\delta < \lambda : \operatorname{cf}(\delta) = \theta\} \in I[\lambda]$ *or just*

(a)$^-$ *for some $S \in I[\lambda], (\forall \delta \in S)[\operatorname{cf}(\delta) = \theta]$ and*

(a)$_S$ *for every closed $e \subseteq \lambda$ of order type χ, $e \cap S \neq \emptyset$.*

<u>Then</u> $\neg \Pr^{\mathrm{tr}}(\lambda, \mu, \chi, \theta, \kappa)$.

[*Why? As in* [*Sh*:*g*, *Chapter VIII*,6.4] *based on* [*Sh*:*g*, *Chapter II*,5.4] *better still* [*Sh*:*g*, *Chapter II*,3.5].]

(4) *If λ is a successor of regular and $\theta^+ < \lambda$, <u>then</u> the assumption $(a)^-$ of part* (3) *holds (see* [*Sh*:*g*, *Chapter VIII*,6.1] *based on* [*Sh*:351, §4]).

(5) *If $\mu < \lambda$ and $cov(\mu, \theta, \kappa, \aleph_1) < \lambda$ (equivalently*

$$(\forall \tau)[\theta < \tau \leq \mu \ \& \ \operatorname{cf}(\tau) < \kappa \rightarrow \operatorname{pp}_{\aleph_1\text{-}complete}(\tau) < \lambda],$$

<u>then</u> $\neg Pr(\lambda, \mu^+, \chi, \theta, \kappa)$ implies that for some $\mu_1 \in (\mu, \lambda)$ we have $\neg \Pr(\lambda, \mu_1, \chi, \theta, \aleph_1)$ (as in Case 2 in the proof of 4.1).

(6) $\Pr(\lambda, \mu, \chi, \theta, \aleph_1) \Leftrightarrow \Pr^{\mathrm{tr}}(\lambda, \mu, \chi, \theta, \aleph_1)$.

(7) $\Pr(\lambda, \mu, \lambda, \theta, \kappa)$ *if and only if for every* $\tau \in [\theta, \mu)$ *we have* $\Pr(\lambda, \leq \tau, \lambda, \tau, \kappa)$; *similarly for* $\Pr^{\mathrm{tr}}$.

(8) $\Pr(\lambda, \leq \mu, \lambda, \theta, \kappa)$ *if and only if* $\Pr^{\mathrm{tr}}(\lambda, \leq \mu, \lambda, \theta, \kappa)$ (*by* 4.1).

CLAIM 4.9. *Under GCH we get equivalence*: $\Pr(\lambda, \mu, \chi, \theta, \kappa) \Leftrightarrow \Pr^{\mathrm{tr}}(\lambda, \mu, \chi, \theta, \kappa)$.

PROOF. $\Pr \Rightarrow \Pr^{\mathrm{tr}}$ is trivial; so let us prove $\neg \Pr \Rightarrow \neg \Pr^{\mathrm{tr}}$, so assume

$$\{a_\alpha : \alpha < \lambda\} \subseteq [\mu]^{<\kappa}$$

exemplifies $\neg\Pr(\lambda, \mu, \chi, \theta, \kappa)$. Without loss of generality

$$|a_\alpha| = \kappa^* < \kappa.$$

By 4.8(1) without loss of generality $\lambda = \mu = \mu_1^+$, $\mu_1^{\kappa^*} = \lambda$, so necessarily

(a) $\lambda = \mu_1^+, \mu_1 > \kappa \geq \mathrm{cf}(\mu_1)$ or
(b) $\lambda = \mu_1^+, \kappa = \lambda$.

Let T be the set of increasing sequences of bounded subsets of μ each of cardinality $\leq \kappa^*$ of length $< \mathrm{cf}(\mu) \leq \kappa^*$. For each $\alpha < \lambda$ let $\bar{b}^\alpha = \langle b_{\alpha,\varepsilon} : \varepsilon < \mathrm{cf}(\mu)\rangle$ be a sequence, every initial segment is in T and $a_\alpha = \bigcup_{\varepsilon<\mathrm{cf}(\mu)} b_{\alpha,\varepsilon}$, so

$$t_\alpha = \{\bar{b}^\alpha \restriction \zeta : \zeta < \mathrm{cf}(\mu)\}$$

is a $\mathrm{cf}(\mu)$-branch of T, and it should be clear. ⊣

REMARK 4.10. *We can get an independence result by instances of Chang's Conjecture (so the consistency strength seems somewhat more than huge cardinals, see Foreman* [For], *Levinski–Magidor–Shelah* [*LMSh*:198]).

§5. Cardinal invariants for general regular cardinals: restrictions on the depth. Cummings and Shelah [CuSh:541] prove that there are no non-trivial restrictions on some cardinal invariants like $\mathfrak{b}_\lambda$ and $\mathfrak{d}_\lambda$, even for all regular cardinals simultaneously; i.e., on functions like $\langle \mathfrak{b}_\lambda : \lambda \in \mathrm{Reg}\rangle$. But not everything is independent of ZFC. Consider the cardinal invariants $\mathfrak{dp}_\lambda^{\ell+}$, defined below, and also $\mathfrak{a}'$ (see 5.13, 5.14).

DEFINITION 5.1. (1) We are given an ideal J on a regular cardinal λ.
If $\lambda > \aleph_0$ let

$$\mathfrak{dp}_\lambda^{1+} = \mathrm{Min}\{\mu : \text{there is no sequence } \langle C_\alpha : \alpha < \mu\rangle \text{ such that:}$$
$$\text{(a)} \quad C_\alpha \text{ is a club of } \lambda,$$
$$\text{(b)} \quad \beta < \alpha \Rightarrow |C_\alpha \backslash C_\beta| < \lambda,$$
$$\text{(c)} \quad C_{\alpha+1} \subseteq \mathrm{acc}(C_\alpha)\},$$

where $\mathrm{acc}(C)$ is the set of accumulation points of C.
If $\lambda \geq \aleph_0$ let

$$\mathfrak{dp}_{\lambda,J}^{2+} = \mathrm{Min}\{\mu : \text{ there are no } f_\alpha \in {}^\lambda\lambda \text{ for } \alpha < \mu \text{ such that } \alpha < \beta < \mu \Rightarrow f_\alpha <_J f_\beta\}.$$

If $\lambda \geq \aleph_0$ let

$$\mathfrak{dp}^{3+}_{\lambda,J} = \mathrm{Min}\{\mu : \text{there is no sequence } \langle A_\alpha : \alpha < \mu\rangle \text{ such that: } A_\alpha \in J^+ \text{ and } \alpha < \beta < \mu \Rightarrow [A_\beta \backslash A_\alpha \in J^+ \;\&\; A_\alpha \backslash A_\beta \in J]\}.$$

If $J = J^{\mathrm{bd}}_\lambda$, we may omit it. We can replace J by its dual filter.

(2) For $\ell \in \{1, 2, 3\}$ let $\mathfrak{dp}^\ell_\lambda = \sup\{\mu : \mu < \mathfrak{dp}^{\ell+}_\lambda\}$.

(3) For a regular cardinal λ let

$$\mathfrak{d}_\lambda = \mathrm{Min}\{|F| : F \subseteq {}^\lambda\lambda \text{ and } (\forall g \in {}^\lambda\lambda)(\exists f \in F)(g <_{J^{\mathrm{bd}}_\lambda} f)\}$$

(equivalently $g < f$)

$$\mathfrak{b}_\lambda = \mathrm{Min}\{|F| : F \subseteq {}^\lambda\lambda \text{ and } \neg(\exists g \in {}^\lambda\lambda)(\forall f \in F)[f <_{J^{\mathrm{bd}}_\lambda} g]\}.$$

We shall prove here that in the "neighborhood" of singular cardinals there are some connections between the $\mathfrak{dp}^{\ell+}_\lambda$'s (hence by monotonicity, also with the $\mathfrak{b}_\lambda$'s).

We first note connections for "one λ."

FACT 5.2. (1) *If* $\lambda = \mathrm{cf}(\lambda) > \aleph_0$ *then*

$$\mathfrak{b}_\lambda < \mathfrak{dp}^{1+}_\lambda \leq \mathfrak{dp}^{2+}_\lambda \leq \mathfrak{dp}^{3+}_\lambda.$$

(2) $\mathfrak{b}_{\aleph_0} < \mathfrak{dp}^{2+}_{\aleph_0} = \mathfrak{dp}^{3+}_{\aleph_0}$.

(3) *In the definition of* $\mathfrak{dp}^{1+}_\lambda$, $C_{\alpha+1} \subseteq \mathrm{acc}(C_\alpha) \bmod J^{bd}_\lambda$ *suffices.*

PROOF. (1) First inequality: $\mathfrak{b}_\lambda < \mathfrak{dp}^{1+}_\lambda$.

We choose by induction on $\alpha < \mathfrak{b}_\lambda$, a club C_α of λ such that

$$\beta < \alpha \Rightarrow |C_\alpha \backslash C_\beta| < \lambda \text{ and } C_{\beta+1} \subseteq \mathrm{acc}(C_\beta).$$

For $\alpha = 0$ let $C_\alpha = \lambda$, for $\alpha = \beta + 1$ let $C_\alpha = \mathrm{acc}(C_\beta)$, and for α limit let, for each $\beta < \alpha$, $f_\beta \in {}^\lambda\lambda$ be defined by $f_\beta(i) = \mathrm{Min}(C_\alpha \backslash (i + 1))$. So $\{f_\beta : \beta < \alpha\}$ is a subset of ${}^\lambda\lambda$ of cardinality $\leq |\alpha| < \mathfrak{b}_\lambda$, so there is $g_\alpha \in {}^\lambda\lambda$ such that $\beta < \alpha \Rightarrow f_\beta <_{J^{\mathrm{bd}}_\lambda} g_\alpha$.

Lastly, let $C_\alpha = \{\delta < \lambda : \delta \text{ a limit ordinal such that } (\forall\zeta < \delta)[g_\alpha(\zeta) < \delta]\}$, now C_α is as required.

So $\langle C_\alpha : \alpha < \mathfrak{b}_\lambda\rangle$ exemplifies $\mathfrak{b}_\lambda < \mathfrak{dp}^{1+}_\lambda$.

Second inequality: $\mathfrak{dp}^{1+}_\lambda \leq \mathfrak{dp}^{2+}_\lambda$

Assume $\mu < \mathfrak{dp}^{1+}_\lambda$. Let $\langle C_\alpha : \alpha < \mu\rangle$ exemplify it, and let us define for $\alpha < \mu$ the function $f_\alpha \in {}^\lambda\lambda$ by: $f_\alpha(\zeta)$ is the $(\zeta + 1)$-th member of C_α; clearly $f_\alpha \in {}^\lambda\lambda$ and f_α is strictly increasing. Also, if $\beta < \alpha$ then $C_\alpha \backslash C_\beta$ is a bounded subset of λ, say by δ_1, and there is $\delta_2 \in (\delta_1, \lambda)$ such that $\mathrm{otp}(\delta_2 \cap C_\beta) = \delta_2$. So for every $\zeta \in [\delta_2, \lambda)$ clearly $f_\beta(\zeta) =$ the $(\zeta + 1)$-th member of $C_\beta =$ the $(\zeta + 1)$-th member of $C_\beta \backslash \delta_1 \leq$ the $(\zeta + 1)$-th member of C_α. So $\beta < \alpha \Rightarrow f_\beta \leq_{J^{\mathrm{bd}}_\lambda} f_\alpha$. Lastly, for $\alpha < \mu$, $C_{\alpha+1} \subseteq \mathrm{acc}(C_\alpha)$ hence $f_\alpha(\zeta) =$ the $(\zeta + 1)$-th member of $C_\alpha <$ the $(\zeta + \omega)$-th member of $C_\alpha \leq$ the $(\zeta + 1)$-th member of $\mathrm{acc}(C_\alpha) \leq$ the $(\zeta + 1)$-th member of $C_{\alpha+1}$. So $\beta < \alpha \Rightarrow f_\beta <_{J^{\mathrm{bd}}_\lambda} f_{\beta+1} \leq_{J^{\mathrm{bd}}_\lambda} f_\alpha$, so $\langle f_\alpha : \alpha < \lambda\rangle$ exemplifies $\mu < \mathfrak{dp}^{2+}_\lambda$.

Third inequality: $\partial\mathfrak{p}^{2+}_\lambda \le \partial\mathfrak{p}^{3+}_\lambda$

Assume $\mu < \partial\mathfrak{p}^{2+}_\lambda$ and let $\langle f_\alpha : \alpha < \mu\rangle$ exemplify this.

Let $c : \lambda \times \lambda \to \lambda$ be one to one and let

$$A_\alpha = \{c(\zeta, \xi) : \zeta < \lambda \text{ and } \xi < f_\alpha(\zeta)\}.$$

Now $\langle A_\alpha : \alpha < \mu\rangle$ exemplifies $\mu < \partial\mathfrak{p}^{3+}_\lambda$.

(2), (3) Easy. $\dashv_{5.2}$

OBSERVATION 5.3. Suppose $\lambda = \mathrm{cf}(\lambda) > \aleph_0$.

(1) If $\langle f_\alpha : \alpha \le \gamma^*\rangle$ is $<_{J^{\mathrm{bd}}_\lambda}$-increasing then we can find a sequence $\langle C_\alpha : \alpha < \gamma^*\rangle$ of clubs of λ, such that $\alpha < \beta \Rightarrow |C_\alpha \setminus C_\beta| < \lambda$ and $C_{\alpha+1} \subseteq \mathrm{acc}(C_\alpha) \bmod J^{\mathrm{bd}}_\lambda$.

(2) $\partial\mathfrak{p}^{1+}_\lambda = \partial\mathfrak{p}^{+2}_\lambda$ or for some $\mu, \partial\mathfrak{p}^{1+}_\lambda = \mu^+, \partial\mathfrak{p}^{2+}_\lambda = \mu^{++}$ (moreover though there is in $({}^\lambda\lambda, <_{J^{\mathrm{bd}}_\lambda})$ an increasing sequence of length μ, there is none of length $\mu + 1$).

PROOF. (1) Let

$$C^* = \big\{\delta < \lambda : \delta \text{ a limit ordinal and } (\forall \beta < \delta) f_{\gamma^*}(\beta) < \delta \text{ and } \omega^\delta = \delta \text{ (ordinal exponentiation)}\big\};$$

this is a club of λ.

For each $\alpha < \gamma^*$ let

$$C_\alpha = \{\delta + \omega^{f_\alpha(\delta)} \cdot \beta : \delta \in C^* \text{ and } \beta < f_{\gamma^*}(\delta) \text{ and } f_{\gamma^*}(\delta) < f_\alpha(\delta)\}.$$

(2) Follows. $\dashv_{5.3}$

Now we come to our main concern.

THEOREM 5.4. *Assume*

(a) κ *is regular uncountable,* $\ell \in \{1, 2, 3\}$

(b) $\langle \mu_i : i < \kappa\rangle$ *is (strictly) increasing continuous with limit* μ,

$$\lambda_i = \mu_i^+,\ \lambda = \mu^+$$

(c) $2^\kappa < \mu$ *and* $\mu_i^\kappa < \mu$

(d) D *a normal filter on* κ

(e) $\theta_i < \partial\mathfrak{p}^{\ell+}_{\lambda_i}$ *and* $\theta = \mathrm{tcf}(\prod_{i<\kappa} \theta_i/D)$ *or just*

$$\theta < \mathrm{Depth}^+\left(\prod_{i<\kappa} \theta_i/D\right).$$

Then $\theta < \partial\mathfrak{p}^{\ell+}_\lambda$.

PROOF. By 5.15, 5.16, 5.6 below for $\ell = 1, 2, 3$ respectively (the conditions there are easily checked). $\dashv_{5.4}$

REMARK 5.5. (1) *Concerning assumption (e), e.g., if* $2^{\mu_i} = \mu_i^{+5}$ *and* $2^\mu = \mu^{+5}$, *then necessarily* $\mu^{+\ell} = \mathrm{tcf}(\prod_{i<\kappa} \mu_i^{+\ell}/D)$ *for* $\ell = 1, \dots, 5$ *and so* $\bigwedge_{i<\kappa} \partial\mathfrak{p}^{\ell+}_{\lambda_i} = 2^{\mu_i} \Rightarrow \partial\mathfrak{p}_\lambda = 2^\mu$ *and we can use* $\mu_i = (2^\kappa)^{+i}, \lambda_i = \mu_i^+, \theta_i = \mu_i^{+5}, \theta = \mu^{+5}$.

So this theorem really says that the function $\lambda \mapsto \partial\mathfrak{p}_\lambda$ *has more than the cardinality exponentiation restrictions.*

(2) *Note that Theorem 5.4 is trivial if* $\prod_{i<\kappa} \lambda_i = 2^\mu = \lambda$, *so (see [Sh:g, V]) it is natural to assume* $E =: \{D' : D' \text{ a normal filter on } \kappa\}$ *is nice, but this will not be used.*

(3) *Note that the proof of* 5.16 (*i.e., the case* $\ell = 2$) *does not depend on the longer proof of* 5.6, *whereas the proof of* 5.15 *does.*

(4) *Recall that for an* $\aleph_1$*-complete filter* D, *say on* κ, *and* $f \in {}^{\kappa}\mathrm{Ord}$ *we define* $\|f\|_D$ *by* $\|f\|_D = \bigcup\{\|g\|_D + 1 : g \in {}^{\kappa}\mathrm{Ord}$ *and* $g <_D f\}$.

(5) *Below we shall use the assumption*

(*) $\|\lambda\|_{D+A} = \lambda$ *for every* $A \in D^+$.
This is not a strong assumption as
(a) *if SCH holds, then the only case of interest is if* $\langle \chi_i : i < \kappa\rangle$ *is increasing continuous with limit* χ *and* $\|\langle \chi_i^+ : i < \kappa\rangle\|_D = \chi^+$ *for any normal filter* D *on* κ; *so our statements degenerate and say nothing,*
(b) *if SCH fails, there are nice filters for which this phenomenon is "popular" see* [*Sh*:*g*, *V*, 1.13, 3.10] (*see more in* 5.17).

THEOREM 5.6. *Assume*

(a) D *is an* $\aleph_1$*-complete filter on* κ
(b) $\langle \lambda_i : i < \kappa\rangle$ *is a sequence of regular cardinals* $> (2^\kappa)^+$
(c) $\|\langle \lambda_i : i < \kappa\rangle\|_{D+A} = \lambda$ *for* $A \in D^+$, λ *regular*
(d) $\mu_i < \mathfrak{dp}^{3+}_{\lambda_i}$
(e) $\mu = \mathrm{tcf}(\Pi \mu_i / D)$ *or at least*
(e$^-$) $\mu < \mathrm{Depth}^+(\Pi\mu_i, <_D)$ *and* $\mu > 2^\kappa$.

<u>Then</u> $\mu < \mathfrak{dp}^{3+}_{\lambda}$.

Remark. Why not assume just $\|f\|_D = \lambda$ for $f =: \langle \lambda_i : i < \kappa\rangle$? Note that $\mathrm{cla}^{\alpha}_{I}(f, A)$, see below, does not make much sense in this case.

We delay the proof of 5.6 until we complete some preliminary work.

FACT 5.7. *Assuming* 5.6(*a*), *for any* $f \in {}^{\kappa}(\mathrm{Ord} \setminus (2^\kappa)^+)$ *we have*: $T_D(f)$ *is smaller or equal to the cardinality of* $\|f\|_D$ *remembering* (5.5(4) *above and*)

$$T_D(f) = \sup\left\{|F| : F \subseteq \prod_{i<\kappa} f(i) \text{ and } f \neq g \in F \Rightarrow f \neq_D g\right\}.$$

PROOF. Why? Let F be as in the definition of $T_D(f)$, note: $f_i \neq_D f_j$ & $f_i \leq_D f_j \Rightarrow f_i <_D f_j$. Note that as $i < \kappa \Rightarrow f(i) \geq (2^\kappa)^+$, necessarily $|F| > 2^\kappa$. Now for each ordinal α let $F^{[\alpha]} =: \{f \in F : \|f\|_D = \alpha\}$. Clearly $F^{[\alpha]}$ has at most 2^κ members, as otherwise some $f_i \in F^{[\alpha]}$ for $i < (2^\kappa)^+$ are pairwise distinct so for some $i < j$, $f_i <_D f_j$ (by [Sh:111, §2] or simply use Erdös–Rado on $c(i, j) = \min\{\zeta < \kappa : f_i(\zeta) > f_j(\zeta)\}$).

So

$$\begin{aligned}\|f\|_D &\geq \sup\{\|g\|_D : g \in F\} \geq \mathrm{otp}\{\alpha : F^{[\alpha]} \neq \emptyset\}\\ &\geq |\{\alpha : F^{[\alpha]} \neq \emptyset\}| \geq |F|/2^\kappa = |F|.\end{aligned}$$

So $\|f\|_D \geq T_D(f)$. $\dashv_{5.7}$

DEFINITION 5.8. For $f \in {}^{\kappa}\mathrm{Ord}$ (natural to be mainly interested in the case $0 \notin \mathrm{Rang}(f)$) and D an $\aleph_1$-complete filter on κ let

$$\prod_{i<\kappa}^{*} f(i) = \{g : \mathrm{Dom}(g) = \kappa, f(i) > 0 \Rightarrow g(i) < f(i)$$
$$\text{and } f(i) = 0 \Rightarrow g(i) = 0\}$$

and

(1) $$\mathrm{cla}(f, D) = \left\{(g, A) : g \in \prod_{i<\kappa}^{*} f(i) \text{ and } A \in D^+\right\}$$
$$\mathrm{cla}^{\alpha}(f, D) = \{(g, A) \in \mathrm{cla}(f, D) : \|g\|_{D+A} = \alpha\}.$$

Here "cla" abbreviates "class."

(2) For $(g, A) \in \mathrm{cla}(f, D)$ let

$$J_D(g, A) = \{B \subseteq \kappa : \text{ if } B \in (D + A)^+ \text{ then } \|g\|_{(D+A)+B} > \|g\|_{D+A}\}.$$

(3) We say $(g', A') \approx (g'', A'')$ if (both are in $\mathrm{cla}(f, D)$ and) $A' = A'' \bmod D$ and $J_D(g', A') = J_D(g'', A'')$ and $g' = g'' \bmod J_D(g', A')$.

(4) For I an ideal on κ disjoint to D we let

$$I * D = \{A \subseteq \kappa : \text{for some } X \in D \text{ we have } A \cap X \in I\},$$

(usually we have $\{\kappa \backslash A : A \in D\} \subseteq I$ so $I * D = I$) and let

$$\mathrm{cla}_I(f, D) = \left\{(g, A) : g \in \prod_{i<\kappa}^{*} f(i) \text{ and } A \in (I * D)^+\right\}.$$

(5) On $\mathrm{cla}_I(f, D)$ we define a relation $\approx_I$

$$(g_1, A_1) \approx_I (g_2, A_2) \text{ if:}$$

(a) $A_1 = A_2 \bmod D$ and
(b) there is $B_0 \in I$ such that: if $B_0 \subseteq B \in I$ then

$$\|g_1\|_{(D+A_1)+(\kappa \backslash B)} = \|g_2\|_{(D+A_2)+(\kappa \backslash B)} \text{ and}$$
$$J_{(D+A_1)+(\kappa \backslash B)}(g_1, A_1) = J_{(D+A_1)+(\kappa \backslash B)}(g_2, A_2).$$

6) $$J_{D.I}(g_1, A_1) = \{A \subseteq \kappa : \text{for some } B_0 \in I \text{ if } B_0 \subseteq B \in I$$
$$\text{we have } A \in J_{(D+A_1)+(\kappa \backslash B)}(g_1, A_1)\}.$$

$$\mathrm{cla}_i^{\alpha}(g, D) = \left\{(h, A)/I : h \in \prod_{i<\kappa}^{*} g(i), A \in I * D)^+,\right.$$
$$\left.\text{and for some } B_0 \in I \text{ if } B_0 \subseteq B \in I \text{ then } \|h\|_{D+(A \backslash B)} = \alpha\right\}.$$

(7) Let $\mathrm{com}(D)$ be <u>the maximal</u> θ such that D is θ-complete.

FACT 5.9. *For $f \in {}^\kappa \mathrm{Ord}$ and D an $\aleph_1$-complete filter on κ and $A \in D^+$:*

(0) *If $f_1 \le f_2$ then $\mathrm{cla}(f_1, D) \subseteq \mathrm{cla}(f_2, D)$ and for $g', g'' \in \prod^*_{i<\kappa} f_1(i)$, $A \in D^+$ we have $(g', A) \approx (g'', A)$ in $\mathrm{cla}(f_1, D)$ if and only if $(g', A) \approx (g'', A)$ in $\mathrm{cla}(f_2, D)$ (so we shall be careless about this).*

(1) *$J_D(g, A)$ is an ideal on κ, com(D)-complete, and normal if D is normal.*

(2) *A does not belong to $J_D(g, A)$, which includes $\{B \subseteq \kappa : B = \emptyset \bmod (D + A)\}$. If $B \in J^+_D(g, A)$ then $A \cap B \in D^+$ and $\|g\|_{D+(A\cap B)} = \|g\|_{D+A}$.*

(3) *$\approx$ is an equivalence relation on $\mathrm{cla}(f, D)$, similarly $\approx_I$ on $\mathrm{cla}_I(f, D)$.*

(4) *Assume*

(i) *$(g, A) \in \mathrm{cla}^\alpha(f, D)$, $g' \in \prod^*_{i<\kappa} f(i)$ and*

(ii) (a) *$g' = g \bmod (D + A)$ or*

(b) *for some $B \in J_D(g, A)$ we have: (i) $\alpha \in B \Rightarrow g'(\alpha) > \|g\|_D$ (or just $\|g\|_{D+A}$ · $\|g'\|_{D+B}$) and (ii) $g' \restriction (\kappa \backslash B) = g \restriction (\kappa \backslash B) \bmod D$.*

Then $(g', A) \approx (g, A)$.

(5) *For each α, in $\mathrm{cla}^\alpha(f, D)/\approx$ there are at most 2^κ classes.*

(6) *For $f \in {}^\kappa(\mathrm{Ord})$, in $\mathrm{cla}(f, D)/\approx$ there are at most $2^\kappa + \sup_{A\in D^+} \|f\|_{D+A}$ classes.*

PROOF. (0) Easy.

(1) Straight (e.g., it is an ideal as for $B \subseteq \kappa$ we have

$$\|g\|_D = \mathrm{Min}\{\|g\|_{D+A}, \|g\|_{D+(\kappa-A)}\},$$

where we stipulate $\|g\|_{\mathscr{P}(\kappa)} = \infty$, see [Sh:71]).

(2) Check.

(3) Check.

(4) Check.

(5) We can work also in $\mathrm{cla}^\alpha(f + 2, D)$ (this change gives more elements and by (0) it preserves $\approx$). Assume α is a counterexample (note that "$\le 2^{2^\kappa}$" is totally immediate). Let χ be large enough; choose $N \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ of cardinality 2^κ such that $\{f, D, \kappa, \alpha\} \in N$ and ${}^\kappa N \subseteq N$. So necessarily there is $(g, A) \in \mathrm{cla}^\alpha(f, D)$ such that the equivalence class $(g, A)/\approx$ does not belong to N, by the definition of cla^α, clearly $\|g\|_{D+A} = \alpha$. Let $B =: \{i < \kappa : g(i) \notin N\}$. ⊣

CASE 1. $B \in J_D(g, A)$.

Let $g' \in \prod_{i<\kappa}(f(i) + 2)$ be defined by: $g'(i) = g(i)$ if $i \in \kappa\backslash B$ and $g'(i) = f(i) + 1$ if $i \in B$. By part (4) we have $(g', A) \approx (g, A)$ and by the choice of N we have $(g', A) \in N$ as $A \in \mathscr{P}(\kappa) \subseteq N, g' \in N$ (as $\mathrm{Rang}(g') \subseteq N$ & ${}^\kappa N \subseteq N$) and, of course, $D \in A$. Thus, there is $(g', A) \in N$ such that $(g', A) \approx (g, A)$ as required.

CASE 2. $B \notin J_D(g, A)$.

Let $g' \in {}^\kappa \mathrm{Ord}$ be: $g'(i) = \mathrm{Min}(N \cap (f(i)+1)\backslash g(i)) \le f(i)$ if $i \in B$, $g'(i) = g(i)$ if $i \notin B$ (note: $f(i) \in N, g(i) \le f(i)$ so g' is well defined).

Clearly $g' \in N$, (as Rang $(g') \subseteq N$ and ${}^\kappa N \subseteq N$), and

$$(\mathscr{H}(\chi), \in, <^*_\chi) \models (\exists x)\Bigg(x \in \prod^*_{i<\kappa} f(i) \wedge (\forall i \in \kappa\backslash B)(x(i) = g'(i)) \wedge (\forall i \in B)(x(i) < g'(i)) \wedge \|x\|_{D+(A\cap B)} = \alpha\Bigg).$$

(Why? Because $x = g$ is like that, last equality as $B \notin J_D(g, A)$.) So there is such x in N, call it g''. So $g'' \in \prod_{i<\kappa}(f(i)+1)$ and $\|g''\|_{D+(A\cap B)} = \alpha$ and for $i \in B$ we have $g''(i) \in g'(i) \cap N$ hence $g''(i) < g(i)$ by the definition of $g'(i)$.

So $g'' < g \bmod D+(A\cap B)$, but this contradicts $\|g''\|_{D+(A\cap B)} = \alpha = \|g\|_{D+(A\cap B)}$, the last equality as $B \notin J_D(g, A)$.

(6) Immediate from (5). $\dashv_{5.9}$

FACT 5.10. *Assume $f \in {}^{\kappa}\mathrm{Ord}$ and D an $\aleph_1$-complete filter on κ and I an $\mathrm{com}(D)$-complete ideal on κ.*

(1) *If $(g, A) \in \mathrm{cla}_I(f, D)$ then $J_{D,I}(g, A)$ is an ideal on κ, which is $\mathrm{com}(D)$-complete and normal if D, I are normal.*

For some $B_0 \in I$, if $B \in (J_{D,I}(g, A))^+$ then $\|g\|_{D+(A\cap B\setminus B_0)} = \|g\|_{D+(A\setminus B_0)}$, and $(D + (A \cap B)) \cap I = \emptyset$.

(2) *$\approx_I$ is an equivalence relation on $\mathrm{cla}(f, D)$.*

(3) *If $(g, A) \in \mathrm{cla}(f, D)$ and $g' \in \prod^*_{i<\kappa} f(i)$ and $g' = g \bmod J_{D,I}(g, A)$ <u>then</u> for some A' we have $(g', A') \approx_I (g, A')$ so $(g', A') \in \mathrm{cla}(f, D)$ and $\|g'\|_{D+A'} = \|g\|_{D+A'}$ (in fact $A' = \{i \in A : g'(i) = g(i)\}$ is O.K.).*

PROOF. Easy. $\dashv$

FACT 5.11. *Let κ, f, D be as in 5.10.*

(1) *If $f_\zeta \in {}^{\kappa}\mathrm{Ord}$, for $\zeta \le \delta$, $\mathrm{cf}(\delta) > \kappa$ and for each i the sequence $\langle f_\zeta(i) : \zeta \le \delta\rangle$ is increasing $(\le)$ continuous <u>then</u> $\|f_\delta\|_D = \sup_{\zeta<\delta} \|f_\zeta\|_D$.*

(2) *If $\delta = \|f\|_D$, $\mathrm{cf}(\delta) > 2^\kappa$ <u>then</u> $\{i : \mathrm{cf}(f(i)) \le 2^\kappa\} \in J_D(f, \kappa)$.*

(3) *If $\|f\|_D = \delta$, $A \in J^+_D(f, \kappa)$ <u>then</u> $\prod^*_{i<\kappa} f(i)/(D + A)$ is not $(\mathrm{cf}(\delta))^+$-directed.*

(4) *If $\|f\|_D = \delta$ and $A \in J^+_D(f, \kappa)$ <u>then</u> $\mathrm{cf}(\delta) \le \mathrm{cf}(\prod^*_{i<\kappa} \mathrm{cf}(f(i))/(D + A))$.*

(5) *If $\|f\|_D = \delta$ and $A \subseteq \kappa$, $(\forall i \in A)\, \mathrm{cf}(f(i)) > \kappa$ and*

$$\max \mathrm{pcf}\{f(i) : i \in A\} < \mathrm{cf}(\delta)$$

(or just $\mathrm{cf}(\delta) > \max\{\mathrm{cf} \prod^*_{i<\kappa} f(i)/D' : D'$ an ultrafilter extending $D + A\}$) <u>then</u> $A \in J_D(f, \kappa)$.

(6) *If $\|f\|_D = \delta$, $\mathrm{cf}(\delta) > 2^\kappa$, <u>then</u> $\prod^*_{i<\kappa} f(i)/J_D(f, \kappa)$ is $\mathrm{cf}(\delta)$-directed.*

(7) *If $\|f\|_D = \delta$, $\mathrm{cf}(\delta) > 2^\kappa$, <u>then</u> for some $A \in J^+_D(f, \kappa)$ we have*

$$\prod^*_{i<\kappa} f(i)/(J_D(f, \kappa) + (\kappa\setminus A)) \text{ has true cofinality } \mathrm{cf}(\delta).$$

(8) *Assume $\|f\|_D = \lambda = \mathrm{cf}(\lambda) > 2^\kappa$.*
Then $(\forall A \in D^+)(\|f\|_{D+A} = \lambda)$ <u>implies</u> $\mathrm{tcf}(\prod^*_{i<\kappa} f(i)/D) = \lambda$.

(9) *If $\|f\|_D = \delta$, $\mathrm{cf}(\delta) > 2^\kappa$ <u>then</u> tcf $\prod^*_{i<\kappa} f(i)/J_D(f, \kappa) = \mathrm{cf}(\delta)$.*

PROOF. (1) Let $g <_D f_\delta$, so $A = \{i < \kappa : g(i) < f_\delta(i)\} \in D$, now for each $i \in A$ we have $g(i) < f_\delta(i) \Rightarrow (\exists \alpha < \delta)(g(i) < f_\alpha(i)) \Rightarrow$ there is $\alpha_i < \delta$ such that $(\forall\alpha)[\alpha_i \le \alpha \le \delta \Rightarrow g(i) < f_{\alpha_i}(i)]$. Hence $\alpha(*) =: \sup\{\alpha_i : i \in A\} < \delta$ as $\mathrm{cf}(\delta) > \kappa$, so $g <_D f_{\alpha(*)}$ hence $\|g\|_D < \|f_{\alpha(*)}\|_D$; this suffices for one inequality, the other is trivial.

(2) Let $A = \{i : \mathrm{cf}(i) \le 2^\kappa\}$, and assume toward contradiction that $A \in J^+_D(f, \kappa)$. For each $i \in A$ let $C_i \subseteq f(i)$ be unbounded of order type $\mathrm{cf}(f(i)) \le 2^\kappa$.

Let $F = \{g \in \prod^*_{i<\kappa}(f(i) + 1)$: if $i \in A$ then $g(i) \in C_i$, if $i \in \kappa\setminus A$ then $g(i) = f(i)\}$. So $|F| \le 2^\kappa$ and:

(*) if $g <_{D+A} f$ then for some $g' \in F, g <_{D+A} g'$,

hence $\delta = \|f\|_{D+A} = \sup\{\|g\|_{D+A+1} : g \in F\}$ but the supremum is on $\leq |F| < \mathrm{cf}(\delta)$ ordinals each $< \delta$ because $g' \in F \Rightarrow g' <_{D+A} f$ as $\|f\|_D = \delta \Rightarrow f \neq_D 0_\kappa$, and δ is a limit ordinal contradiction to $\mathrm{cf}(\delta) > 2^\kappa$.

(3) Assume this fails, so $\|f\|_D = \delta, A \in J_D^+(f,\kappa)$ and $\prod^*_{i<\kappa} f(i)/(D+A)$ is $(\mathrm{cf}(\delta))^+$-directed. Let $C \subseteq \delta$ be unbounded of order type $\mathrm{cf}(\delta)$; as $\|f\|_{D+A} = \delta$ (because $A \in J_D^+(f,A)$) for each $\alpha \in C$ there is $f_\alpha <_{D+A} f$ such that $\|f_\alpha\|_{D+A} \geq \alpha$ (even $= \alpha$ by the definition of $\|-\|_{D+A}$). As $\prod^*_{i<\kappa} f(i)/(D+A)$ is $(\mathrm{cf}(\delta))^+$-directed there is $f' <_{D+A} f$ such that $\alpha \in C \Rightarrow f_\alpha <_{D+A} f'$. By the first inequality $\|f'_{D+A}\| < \|f\|_{D+A} = \delta$, and by the second inequality $\alpha \in C \Rightarrow \alpha \leq \|f_\alpha\|_{D+A} \leq \|f'\|_{D+A}$ hence $\delta = \sup(C) \leq \|f'\|_{D+A}$, a contradiction.

(4) Same proof as part (2).

(5) By part (4) and [Sh:g, Chapter II,3.1].

(6) Follows.

(7) Toward contradiction assume that not; by part (2) without loss of generality $\forall i[\mathrm{cf}(f(i)) > 2^\kappa]$; let $C \subseteq \delta$ be unbounded, $\mathrm{otp}(C) = \mathrm{cf}(\delta)$. For each $\alpha \in C$ and $A \in J_D^+(f,\kappa)$ choose $f_{\alpha,A} <_D f$ such that $\|f_{\alpha,A}\|_{D+A} \geq \alpha$. Let f_α be $f_\alpha(i) = \sup\{f_{\alpha,A}(i) : A \in J_D^+(f,\kappa)\}$. As $\left(\prod^*_{i<\kappa} f_\alpha(i), <_{J_D(f,\kappa)}\right)$ is $\mathrm{cf}(\delta)$-directed (see part (6)), by the assumption toward contradiction and the pcf theorem we have $\prod^*_{i<\kappa} f(i)/J_D(f,\kappa)$ is $(\mathrm{cf}(\delta))^+$-directed. Hence we can find $f^* < f$ such that $\alpha \in C \Rightarrow f_\alpha <_{J_D(f,\kappa)} f^*$. Let $\beta = \sup\{\|f^*\|_{D+B} : B \in J_D^+(f,A)\}$, it is $< \delta$ as $\mathrm{cf}(\delta) > 2^\kappa$; hence there is $\alpha, \beta < \alpha \in C$, so by the choice of f^* we have $f_\alpha <_{J_D(f,\kappa)} f^*$, and let $A =: \{i < \kappa : f_\alpha(i) < f^*(i)\}$ so $A \in J_D^+(f,\kappa)$, so $f_{\alpha,A} \leq f_\alpha <_{D+A} f^*$ hence $\alpha \leq \|f_{\alpha,A}\|_{D+A} \leq \|f_\alpha\|_{D+A} \leq \|f^*\|_{D+A} \leq \beta$ contradicting the choice of α.

(8) For every $\alpha < \lambda$ we can choose $f_\alpha <_D f$ such that $\|f_\alpha\|_D \geq \alpha$. Let $a_\alpha = \{\|f_\alpha\|_{D+A} : A \in D^+\}$, as $A \in D^+ \Rightarrow \alpha \leq \|f_\alpha\|_D \leq \|f_\alpha\|_{D+A} < \|f\|_{D+A} = \lambda$, clearly a_α is a subset of $\lambda \backslash \alpha$, and its cardinality is $\leq 2^\kappa < \lambda$. So we can find an unbounded $E \subseteq \lambda$ such that $\alpha < \beta \in E \Rightarrow \sup(a_\alpha) < \beta$. So if $\alpha < \beta, \alpha \in E, \beta \in E$, let $A = \{i < \kappa : f_\alpha(i) \geq f_\beta(i)\}$, and if $A \in D^+$, then $\|f_\beta\|_{D+A} \leq \|f_\alpha\|_{D+A} \leq \sup(a_\alpha) < \beta$, contradiction. Hence $A = \emptyset \bmod D$, that is $f_\alpha <_D f_\beta$. Also if $g <_D f$, then $a =: \{\|g\|_{D+A} : A \in D^+\}$ is again a subset of λ of cardinality $\leq 2^\kappa$ hence for some $\beta < \lambda$, $\sup(a) < \beta$, so as above $g <_D f_\beta$. Together $\langle f_\alpha : \alpha \in E\rangle$ exemplify $\lambda = \mathrm{tcf}(\Pi f(i), <_D)$.

(9) Similar proof (to part (8)), using parts (6), (7). $\dashv_{5.11}$

REMARK 5.12. *We think Claims* 5.9, 5.10, 5.11 (*and Definition* 5.8) *can be applied to the problems from* [*Sh*:497] *probably saving some uses of niceness so weakening some assumptions*; *but we have not checked.*

PROOF OF 5.6. Fix $f \in {}^\kappa\mathrm{Ord}$ as $f(i) = \lambda_i$ and let $\approx, \approx_I$ be as in Definition 5.8. For each $i < \kappa$ let $\bar{X}^i = \langle X^i_\alpha : \alpha < \mu_i\rangle$ be a sequence of members of $[\lambda_i]^{\lambda_i}$ such that

$$\alpha < \beta < \mu_i \Rightarrow X^i_\alpha \backslash X^i_\beta \in J^{\mathrm{bd}}_{\lambda_i} \;\&\; X^i_\beta \backslash X^i_\alpha \notin J^{\mathrm{bd}}_{\lambda_i}.$$

(it exists by assumption (d)).

Let $\bar{g}^* = \langle g^*_\zeta : \zeta < \mu\rangle$ be a $<_D$-increasing sequence of members of $\prod_{i<\kappa} \mu_i$, it exists by assumption (e) or (e)$^-$.

Let $I =: \{B \subseteq \kappa : \text{if } B \in D^+ \text{ then } \|f\|_{D+B} > \lambda\}$, it is a com$(D)$-complete ideal on κ disjoint to D, i.e., $I = J_D(\bar{\lambda}, \kappa) \supseteq \{\kappa \backslash A : A \in D\}$, and $\approx_I, \approx$ are equal because I is the ideal on κ dual to D which holds by assumption (c). For any sequence $\bar{X} = \langle X_i : i < \kappa \rangle \in \prod_{i<\kappa} [\lambda_i]^{\lambda_i}$, let

$$Y[\bar{X}] =: \left\{ \|h\|_{D+A} : h \in \prod_{i<\kappa} X_i \text{ and } A \in I^+ \right\}$$

and

$$\mathscr{Y}[\bar{X}] =: \left\{ (h, A)/\approx : h \in \prod_{i<\kappa} X_i \text{ and } (h, A) \in \operatorname{cla}^\alpha(\bar{\lambda}, D) \text{ for some } \alpha < \lambda \right\}.$$

Note: $Y[\bar{X}] \subseteq \lambda$ and $\mathscr{Y}[\bar{X}] \subseteq \mathscr{Y}^* =: \bigcup_{\alpha<\lambda} \operatorname{cla}^\alpha(\bar{\lambda}, D)/\approx$.

Note that by 5.9(6)

$\boxtimes$ $Y = \bigcup_{\alpha<\lambda} \operatorname{cla}^\alpha(f, D)/\approx$ has cardinality $\leq \lambda$.

$(*)_0$ for $\bar{X} \in \prod_{i<\kappa}[\lambda_i]^{\lambda_i}$, the mapping $(g, A)/\approx_I \mapsto \|g\|_{D+A}$ is from $\mathscr{Y}[\bar{X}]$ onto $Y[\bar{X}]$ with every $\alpha \in Y[\bar{X}]$ having at most 2^κ preimages

[Why? By 5.9(5)]

$(*)_1$ if $\bar{X} \in \prod_{i<\kappa}[\lambda_i]^{\lambda_i}$ then $\mathscr{Y}[\bar{X}]$ has cardinality λ (hence also Y^* has).

[Why? By the definition of $\| - \|_D$ for every $\alpha < \lambda$ for some $g \in \prod_{i<\kappa} \lambda_i/D$ we have $\|g\|_D = \alpha$; as $\sup(X_i) = \lambda_i > g(i)$ we can find $g' \in \prod_{i<\kappa}(X_i \backslash g(i))$ such that $g \leq g' < \langle \lambda_i : i < \kappa \rangle$, so $\alpha = \|g\|_D \leq \|g'\|_D < \|\langle \lambda_i : i < \kappa \rangle\|_D = \lambda$. Clearly for some α' and A, $(g', A) \in \operatorname{cla}^{\alpha'}(f, A)$, so $A \in I^+ \subseteq D^+$, and $\alpha \leq \alpha' = \|g'\|_{D+A} < \|f\|_{D+A} = \lambda$ (as $A \in I^+$). So $\alpha' \in Y[\bar{X}]$ hence $Y[\bar{X}] \nsubseteq \alpha$; as $\alpha < \lambda$ was arbitrary and λ is regular, clearly $Y[\bar{X}]$ has cardinality $\geq \lambda$, by $\boxtimes$ equality holds hence (by $(*)_0$) also $\mathscr{Y}[\bar{X}]$ has cardinality λ.]

$(*)_2$ if $\bar{X}', \bar{X}'' \in \prod_{i<\kappa}[\lambda_i]^{\lambda_i}$, and $\{i < \kappa : X'_i \subseteq X''_i \bmod J^{\mathrm{bd}}_{\lambda_i}\} \in D$ then

(a) $Y[\bar{X}'] \subseteq Y[\bar{X}''] \bmod J^{\mathrm{bd}}_\lambda$

(b) $\mathscr{Y}[\bar{X}'] \backslash \mathscr{Y}[\bar{X}'']$ has cardinality $< \lambda$.

[Why? Define $g \in \prod_{i<\kappa} \lambda_i$ by $g(i) = \sup(X'_i \backslash X''_i)$ if

$$i \in A^* =: \{i < \kappa : X'_i \subseteq X''_i \bmod J^{\mathrm{bd}}_{\lambda_i}\}$$

and $g(i) = 0$ otherwise. Let $\alpha(*) = \sup\{\|g\|_{D+A} + 1 : A \in I^+\}$, as λ is regular $> 2^\kappa$ clearly $\alpha(*) < \lambda$ (see assumption (c) or definition of I). Assume $\beta \in Y[\bar{X}'] \backslash \alpha(*)$ and we shall prove that $\beta \in Y[\bar{X}'']$, moreover, $\mathscr{Y}[\bar{X}'] \cap (\operatorname{cla}^\beta(\bar{X}, D)/\approx_I) \subseteq \mathscr{Y}[\bar{X}'']$, this clearly suffices for both clauses. We can find $f^* \in \prod_{i<\kappa}((X'_i \cap X''_i) \cup \{0\})$ such that $\|f^*\|_D > \beta$.

So let a member of $\mathscr{Y}[\bar{X}'] \cap (\operatorname{cla}^\beta(\bar{\lambda}, D)/\approx)$ have the form $(h, A)/\approx_I$, where $A \in I^+, h \in \prod_{i<\kappa} X'_i$ and $\beta = \|h\|_{D+A}$ and let $A_1 =: \{i < \kappa : h(i) \leq g(i)\}$. We know $\beta = \|h\|_{D+A} = \operatorname{Min}\{\|h\|_{D+(A\cap A_1)}, \|h\|_{D+(A\backslash A_1)}\}$ (if $A \cap A_1 = \emptyset \bmod D$, then $\|h\|_{D+A\cap A_1}$ can be considered ∞).

If $\beta = \|h\|_{D+(A\cap A_1)}$ then note $h \leq_{D+(A\cap A_1)} g$ hence $\beta = \|h\|_{D+(A\cap A_1)} \leq \|g\|_{D+(A\cap A_1)} < \alpha(*)$, contradicting an assumption on β. So $\beta = \|h\|_{D+(A\backslash A_1)}$ and $A \cap A_1 \in J_{D,I}(h, A)$. Now define $h' \in \prod_{i<\kappa} f(i)$ by: $h'(i)$ is $h(i)$ if $i \in A \backslash A_1$ and $h'(i)$ is $f^*(i)$ if $i \in \kappa \backslash (A \backslash A_1)$. So $h' \in \prod_{i<\kappa} f(i)$ and $h' =_{D+(A\backslash A_1)} h$

hence $\|h'\|_{D+(A\setminus A_1)} = \|h\|_{D+(A\setminus A_1)} = \beta$, and clearly $\beta = \|h'\|_{D+(A\setminus A_1)} \in Y[\bar{X}'']$, as required for clause (a), moreover $(h, A) \approx (h', A)$ so $((h', A)/\approx) \in \mathscr{Y}[\bar{X}'']$ as required for clause (b).]

$(*)_3$ If $\bar{X}', \bar{X}'' \in \prod_{i<\lambda}[\lambda_i]^{\lambda_i}$ and $\{i < \kappa : X_i'' \not\subseteq X_i' \bmod J^{\mathrm{bd}}_{\lambda}\} \in D$ then[3]

$$\mathscr{Y}[\bar{X}'']\backslash\mathscr{Y}[\bar{X}'] \text{ has cardinality } \lambda.$$

[Why? Let $\alpha < \lambda$, it is enough to find $\beta \in [\alpha, \lambda)$ such that

$$(\mathscr{Y}[\bar{X}'']\backslash\mathscr{Y}[\bar{X}']) \cap (\mathrm{cla}^{\beta}(f, D)/\approx) \neq \emptyset.$$

We can find $g \in \prod_{i<\kappa}\lambda_i$ such that $\|g\|_D = \alpha$. Define $g' \in \prod_{i<\kappa} X_i''$ by: $g'(i)$ is $\mathrm{Min}(X_i''\backslash X_i'\backslash g(i))$ when well defined, $\mathrm{Min}(X_i'')$ otherwise. By assumption $g \le_D g'$ and, of course, $g' \in \prod_{i<\kappa} X_i'' \subseteq \prod_{i<\kappa}\lambda_i$, so $\|g'\|_D \ge \alpha$. So

$$((g', \kappa)/\approx) \in \mathscr{Y}[\bar{X}'']$$

but trivially $((g', \kappa)/\approx) \notin \mathscr{Y}[\bar{X}']$, so we are done.]

Together $(*)_0 - (*)_3$ give that $\langle \mathscr{Y}[\langle X^i_{g^*_\zeta(i)} : i < \kappa\rangle] : \zeta < \mu\rangle$ is a sequence of subsets of $\mathscr{Y}^*$ of length μ (see $(*)_1$), $|\mathscr{Y}^*| = \lambda$, which is increasing modulo $[\mathscr{Y}^*]^{<\lambda}$ (by $(*)_2$), and in fact, strictly increasing (by $(*)_3$, see choice of $\langle g^*_\zeta : \zeta < \mu\rangle$ in the beginning of the proof). So modulo changing names we have finished. (In fact, also $\langle Y[\langle X^i_{g^*_\zeta(i)} : i < \kappa\rangle] : \zeta < \mu\rangle$ is as required.) $\dashv_{5.6}$

A related theorem

DEFINITION 5.13.

$$\mathfrak{a}'_\lambda = \mathrm{Min}\Big\{\mu : \text{there is no } \mathscr{P} \subseteq [\lambda]^\lambda \text{ of cardinality } \mu \ A \neq B \in \mathscr{P} \Rightarrow |A \cap B| < \lambda\Big\}.$$

THEOREM 5.14. *Assume*

(a) *D is an $\aleph_1$-complete filter on κ*
(b) *$\langle \lambda_i : i < \kappa\rangle$ is a sequence of regular cardinals $> (2^\kappa)^+$*
(c) *$\|\langle \lambda_i : i < \kappa\rangle\|_{D+A} = \lambda$ for $A \in D^+$*
(d) *$\mu_i < \mathfrak{a}'_{\lambda_i}$*
(e) *$\mu = \mathrm{tcf}(\Pi\mu_i/D)$ or at least*
(e^-) *$\mu < \mathrm{Depth}^+(\Pi\mu_i, <_D)$ and $\mu > 2^\kappa$.*

Then $\mu < \mathfrak{a}'_\lambda$.

PROOF OF 5.14. Similar to the proof of 5.6. $\dashv$

THEOREM 5.15. *Assume*

(a) *D an $\aleph_1$-complete filter on κ*
(b) *$\bar{\lambda} = \langle \lambda_i : i < \kappa\rangle$ is a sequence of regular cardinals $> 2^\kappa$*
(c) *$\lambda = \|\bar{\lambda}\|_{D+A}$ for $A \in D^+$ and λ is regular*
(d) *$\mu_i < \mathfrak{dp}^{1+}_{\lambda_i}$*
(e) *$\mu < \mathrm{Depth}^+(\prod_{i<\kappa}\mu_i, <_D)$.*

Then $\mu < \mathfrak{dp}^{1+}_\lambda$.

[3] In fact, just $\in I^*$ suffices here.

PROOF. Let $\mathrm{Club}(\lambda) = \{C : C \text{ a club of } \lambda\}$ so $\mathrm{Club}(\lambda) \subseteq [\lambda]^\lambda$ for $\lambda = \mathrm{cf}(\lambda) > \aleph_0$.

For any sequence $\bar{C} \in \prod_{i<\kappa} \mathrm{Club}(\lambda_i)$ let $\mathscr{C}(\bar{C})$ be the set $\mathrm{acc}(c\ell(Y(\bar{C}))$ where $Y[\bar{C}] =: \{\|g\|_D : g \in \prod_{i<\kappa} C_i\}(\subseteq \lambda)$; i.e., $\mathscr{C}(\bar{C}) = \{\delta < \lambda : \delta = \sup(\delta \cap Y[\bar{C}])\}$. Clearly

$(^*)_1$ for $\bar{C} \in \prod_{i<\kappa} \mathrm{Club}(\lambda_i)$ we have $\mathscr{C}(\bar{C}) \in \mathrm{Club}(\lambda)$
[the question is why it is unbounded, and this holds as $\|\bar{\lambda}\|_D = \lambda$ by its definition]

$(^*)_2$ if $\bar{C}', \bar{C}'' \in \prod_{i<\lambda} \mathrm{Club}(\lambda_i), g^* \in \Pi\lambda_i$, and $C_i'' = C_i' \backslash g^*(i)$ then
$\mathscr{C}(\bar{C}') = \mathscr{C}(\bar{C}'') \bmod J_\lambda^{\mathrm{bd}}$.

[Why? Let $\alpha(*) = \sup\{\|g^*\|_{D+A} : A \in D^+ \text{ and } \|g^*\|_{D+A} < \lambda\} + 1$, so as $2^\kappa < \lambda = \mathrm{cf}(\lambda)$ clearly $\alpha(*) < \lambda$. We shall show $\mathscr{C}(\bar{C}')\backslash\alpha(*) = \mathscr{C}(\bar{C}'')\backslash\alpha(*)$; for this it suffices to prove $Y(\bar{C}')\backslash\alpha(*) = Y(\bar{C}'')\backslash\alpha(*)$. If $\alpha \in Y(\bar{C}')\backslash\alpha(*)$ let $\alpha = \|h\|_D$ where $h \in \prod_i C_i'$, and let $A = \{i < \kappa : h(i) < g^*(i)\}$, so if $A \in (J_D(\bar{\lambda}, \kappa))^+$ then $\alpha \leq \|h\|_{D+A} < \lambda$ and $\|h\|_{D+A} \leq \|g^*\|_{D+A} < \alpha(*)$ but $\alpha \geq \alpha(*)$, a contradiction. So $A \in J_D(\bar{\lambda}, \kappa)$ hence $A \notin D^+$ by clause (c) of the assumption, so $g^* \leq_D h$. Now clearly there is $h' =_D h$ with $h' \in \prod_{i<\kappa} C_i''$, so $\alpha = \|h\|_D = \|h'\|_D \in \mathscr{C}(\bar{C}'')$. The other inclusion is easier.]

$(^*)_3$ if $\bar{C}', \bar{C}'' \in \prod_{i<\kappa} \mathrm{Club}(\lambda_i)$ and $\{i < \kappa : C_i'' \subseteq \mathrm{acc}(C_i')\} \in D$ then

$$\mathscr{C}(\bar{C}'') \subseteq \mathrm{acc}(\mathscr{C}(\bar{C}')).$$

[Why? Let $\beta \in \mathscr{C}[\bar{C}'']$ but $\beta \notin \mathrm{acc}(\mathscr{C}(\bar{C}'))$ and we shall get a contradiction. Clearly $\beta > \sup(\mathscr{C}(\bar{C}') \cap \beta)$ (as $\beta \notin \mathrm{acc}(\mathscr{C}(\bar{C}'))$. As $\mathscr{C}[\bar{C}'']$ is $\mathrm{acc}(c\ell Y[\bar{C}''])$, clearly there is $\alpha \in Y[\bar{C}'']$ such that $\beta > \alpha > \sup(\mathscr{C}(\bar{C}') \cap \beta)$, but

$$Y[\bar{C}''] = \{\|g\|_D : g \in \prod_{i<\kappa} C_i''\},$$

so there is $g \in \prod_{i<\kappa} C_i''$ such that $\|g\|_D = \alpha$. As $\{i : C_i'' \subseteq \mathrm{acc}(C_i')\} \in D$, clearly

$$B =: \{i < \kappa : g(i) \in \mathrm{acc}(C_i')\} \in D.$$

So if $h \in \prod_{i<\lambda} \lambda_i, h <_D g$ then we can find $h' \in \prod_{i<\kappa} C_i'$ such that $h <_D h' <_D g$ (just $h'(i) = \mathrm{Min}(C_i' \backslash (h(i) + 1)$ noting $B \in D$) hence

$$\alpha = \|g\|_D = \sup\{\|h\|_D : h(i) \in g(i) \cap C_i' \text{ when } i \in B, \\ h(i) = \mathrm{Min}(C_i') \text{ otherwise}\}$$

and in this set there is no last element and it is included in $Y[\bar{C}']$, so necessarily $\alpha \in \mathscr{C}(\bar{C}')$, contradicting the choice of $\alpha : \beta > \alpha > \sup(\mathscr{C}(\bar{C}') \cap \beta)$.]

$(^*)_4$ if $\bar{C}', \bar{C}'' \in \prod_{i<\kappa} \mathrm{Club}(\lambda_i)$ and $\{i : C_i'' \subseteq \mathrm{acc}(C_i') \bmod J_{\lambda_i}^{\mathrm{bd}}\} \in D$ then

$$\mathscr{C}(\bar{C}'') \subseteq \mathrm{acc}(\mathscr{C}(\bar{C}')) \bmod J_\lambda^{\mathrm{bd}}.$$

[Why? By $(*)_2 + (*)_3$, i.e., define C_i''' to be $C_i'' \backslash g(i)$ where

$$g(i) =: \sup(C_i'' \backslash \mathrm{acc}(C_i')) + 1)$$

when $C_i'' \subseteq \mathrm{acc}(C_i')$ and the empty set otherwise. Now by $(*)_2$ we know $\mathscr{C}(\bar{C}'') = \mathscr{C}(\bar{C}''') \bmod J_\lambda^{\mathrm{bd}}$ and by $(*)_3$ we know $\mathscr{C}(\bar{C}''') \subseteq \mathrm{acc}(\mathscr{C}(\bar{C}'))$.]

Now we can prove the conclusion of 5.15. Let $\langle C^i_\alpha : \alpha < \mu_i \rangle$ witness $\mu_i < \partial\mathfrak{p}^{1+}_{\lambda_i}$ and $\langle g_\alpha : \alpha < \mu \rangle$ witness $\mu < \mathrm{Depth}^+(\prod_{i<\kappa} \lambda_i, <_D)$. Let $C_\alpha =: \mathscr{C}(\langle C^i_{g_\alpha(i)} : i < \kappa \rangle)$ for $\alpha < \mu$. So $\langle C_\alpha : \alpha < \mu \rangle$ witnesses $\mu < \partial\mathfrak{p}^{1+}_\lambda$. $\dashv_{5.15}$

THEOREM 5.16. *Assume*

(a) κ *is regular uncountable*
(b) $\bar\lambda = \langle \lambda_i : i < \kappa \rangle$ *is a sequence of regular cardinals* $> \kappa$
(c) D *is a normal filter on* κ (*or just* $\aleph_1$*-complete*)
(d) $\lambda = \|\bar\lambda\|_D = \mathrm{tcf}(\prod_{i<\kappa} \lambda_i / D)$, λ *regular*
(e) $\mu_i < \partial\mathfrak{p}^{2+}_{\lambda_i}$
(f) $\mu < \mathrm{Depth}^+(\prod_{i<\kappa} \mu_i, <_D)$.

Then $\mu < \partial\mathfrak{p}^{2+}_\lambda$.

PROOF. Let $\langle f^i_\alpha : \alpha < \mu_i \rangle$ exemplify $\mu_i < \partial\mathfrak{p}^{+2}_{\lambda_i}$, let $\langle g_\alpha : \alpha < \mu \rangle$ exemplify $\mu < \mathrm{Depth}^+(\prod_{i<\kappa} \mu_i, <_D)$, and let $\langle h_\zeta : \zeta < \lambda \rangle$ exemplify $\lambda = \mathrm{tcf}(\prod_{i<\kappa} \lambda_i, <_D)$.

Now for each $\alpha < \mu$ we define $f_\alpha \in {}^\lambda\lambda$ as follows:

$$f_\alpha(\zeta) = \|\langle f^i_{g_\alpha(i)}(h_\zeta(i)) : i < \kappa \rangle\|_D.$$

Clearly $f_\alpha(\zeta)$ is an ordinal and as $f^i_{g_\alpha(i)} \in {}^{(\lambda_i)}\lambda_i$ clearly $\langle f^i_{g_\alpha(i)}(h_\zeta(i)) : i < \kappa \rangle <_D \langle \lambda_i : i < \kappa \rangle$ hence $f_\alpha(\zeta) < \|\bar\lambda\|_D = \lambda$, so

$(*)_1$ $f_\alpha \in {}^\lambda\lambda$.

The main point is to prove $\beta < \alpha < \mu \Rightarrow f_\beta <_{J^{\mathrm{bd}}_\lambda} f_\alpha$.

Suppose $\beta < \alpha < \mu$, then $g_\beta <_D g_\alpha$ hence $A =: \{i < \kappa : g_\beta(i) < g_\alpha(i)\} \in D$ so $i \in A \Rightarrow f^i_{g_\beta(i)} <_{J^{bd}_{\lambda_i}} f^i_{g_\alpha(i)}$. We can define $h \in \prod_{i<\kappa} \lambda_i$ by:

$h(i)$ is $\sup\{\zeta + 1 : f^i_{g_\beta(i)}(\zeta) \geq f^i_{g_\alpha(i)}(\zeta)\}$ if $i \in A$, and $h(i)$ is zero otherwise.

But $\langle h_\zeta : \zeta < \lambda \rangle$ is $<_D$-increasing and cofinal in $(\prod_{i<\kappa} \lambda_i, <_D)$ hence there is $\zeta(*) < \lambda$ such that $h <_D h_{\zeta(*)}$.

So it suffices to prove:

$$\zeta(*) \leq \zeta < \lambda \Rightarrow f_\beta(\zeta) < f_\alpha(\zeta).$$

So let $\zeta \in [\zeta(*), \lambda)$, so

$$B =: \{i < \kappa : h(i) < h_{\zeta(*)}(i) \leq h_\zeta(i) \text{ and } i \in A\}$$

belongs to D and by the definition of A and B and h we have

$$i \in B \Rightarrow f^i_{g_\beta(i)}(h_\zeta(i)) < f^i_{g_\alpha(i)}(h_\zeta(i)).$$

So

$$\langle f^i_{g_\beta(i)}(h_\zeta(i)) : i < \kappa \rangle <_D \langle f^i_{g_\alpha(i)}(h_\zeta(i)) : i < \kappa \rangle$$

hence (by the definition of $\| - \|_D$)

$$\|\langle f^i_{g_\beta(i)}(h_\zeta(i)) : i < \kappa \rangle\|_D < \|\langle f^i_{g_\alpha(i)}(h_\zeta(i)) : i < \kappa \rangle\|_D$$

which means

$$f_\beta(\zeta) < f_\alpha(\zeta).$$

As this holds for every $\zeta \in [\zeta(*), \lambda)$ clearly

$$f_\beta <_{J^{\mathrm{bd}}_\lambda} f_\alpha.$$

So $\langle f_\alpha : \alpha < \mu\rangle$ is $<_{J^{\mathrm{bd}}_\lambda}$-increasing, so we have finished. $\dashv_{5.16}$

5.1. Discussion. Now assumption (c) in 5.15 (and in 5.6) is not so serious once we quote [Sh:g, V] (to satisfy the assumption in the usual case we are given $\lambda = \mathrm{cf}(\lambda), \mu < \lambda \leq \mu^\kappa, \mathrm{cf}(\mu) = \kappa, (\forall\alpha < \mu)(|\alpha|^\kappa < \mu)$ and we like to find $\langle\lambda_i : i < \kappa\rangle$, and normal D such that $\|\langle\lambda_i : i < \kappa\rangle\|_{D+A} = \lambda$). E.g., ([Sh:g, Chapter V]) if SCH fails above 2^{2^θ}, θ regular uncountable, D a normal filter on θ, $\|f\|_D \geq \lambda = \mathrm{cf}(\lambda) > 2^{2^\theta}$, (so if $\mathscr{E}$ = family of normal filters on θ, so $\mathscr{E}$ is nice and $\mathrm{rk}^3_E(f) \geq \|f\|_D \geq \lambda$), so g_κ from [Sh:g, Chapter V,3.10, page 244] is as required.

Still we may note

Fact 5.17. *Assume*

(a) *D is an $\aleph_1$-complete filter on κ*
(b) *$f^* \in {}^\kappa\mathrm{Ord}$ and $\mathrm{cf}(f^*(i)) > 2^\kappa$ for $i < \kappa$.*

Then for any $\bar C = \langle C_i : i < \kappa\rangle, C_i$ a club of $f^*(i)$ and $\alpha < \|f^*\|_D$ we can find $f \in \prod_{i<\kappa} C_i$ such that:

(α) $A \in (J_D(f^*, \kappa))^+ \Rightarrow \alpha < \|f\|_{D+A} = \|f\|_D < \|f^*\|_D$
(β) $A \in J_D(f^*, \kappa) \cap D^+ \Rightarrow \|f\|_{D+A} \geq \|f^*\|_D$.

Proof. We choose by induction on $\zeta \leq \kappa^+$ a function f_ζ and

$$\langle f_{\zeta.A} : A \in (J_D(f^*, \kappa))^+\rangle$$

such that:

(a) $f_\zeta \in \prod_{i<\kappa} C_i$
(b) $\varepsilon < \zeta \Rightarrow \bigwedge_i f_\varepsilon(i) < f_\zeta(i)$
(c) for ζ limit $f_\zeta(i) = \sup_{\varepsilon<\zeta} f_\varepsilon(i)$
(d) for $A \in (J_D(f^*, A))^+$, letting $\alpha_{\zeta.A} =: \|f_\zeta\|_{D+A}$ we have
$f_{\zeta.A} \in \prod_{i<\kappa} f^*(i), \|f_{\zeta.A}\|_D > \alpha_{\zeta.A}$ and $f_{\zeta.A}(i) \geq f_\zeta(i)$ for $i < \kappa$
(e) $f_{\zeta.A}(i) < f_{\zeta+1}(i)$ for $i < \kappa, A \in (J_D(f^*, A))^+$
(f) $\|f_0\|_D \geq \alpha$ and $A \in J_D(f^*, \kappa) \Rightarrow \|f_0\|_{D+A} \geq \|f^*\|_D$.

There is no problem to carry out the definition: for defining f_0 for each $A \in J_D(f^*, \kappa)$ choose $g_A <_{D+A} f^*$ such that $\|g_A\|_{D+A} \geq \|f^*\|_D$ (possible as $\|f^*\|_{D+A} > \|f^*\|_D$ by the assumption on A). Let $g^* < f^*$ be such that $\|g^*\|_D \geq \alpha$, (possible as $\alpha < \|f^*\|_D$) and let $f_0 \in \prod_{i<\kappa} f^*(i)$ be defined by

$$f_0(i) = \mathrm{Min}(C_i \setminus \sup\{g^*(i), g_A(i) : A \in J_D(f^*, \kappa)\}).$$

For ζ limit there is no problem to define f_ζ; and also for ζ successor. If f_ζ is defined, we should choose $f_{\zeta,A}$. For clause (d) note that $\|f^*\|_{D+A} = \|f^*\|_D$ as $A \in (J_D(f^*, A))^+$ and use the definition of $\|f\|_D$. We use, of course, $\bigwedge_i \mathrm{cf}(f^*(i)) > 2^\kappa$.

Now f_{κ^+} is as required. Note: $f <_D f_{\kappa^+} \Rightarrow \bigvee_{\zeta<\kappa^+} f <_D f_\zeta$, and for $A \in (J_D(f^*, \kappa))^+$, we have

$$\begin{aligned}\|f_{\kappa^+}\|_{D+A} &= \sup_{\zeta<\kappa^+} \|f_\zeta\|_{D+A}\\ &= \sup_{\zeta<\kappa^+} \alpha_{\zeta.A} \leq \sup_{\zeta<\kappa^+} \|f_{\zeta+1}\|_D = \|f_{\kappa^+}\|_D.\end{aligned}$$

$\dashv_{5.17}$

Conclusion 5.18. (1) In 5.15 we can weaken assumption (c) to

(c)$^-$ $\|\langle\lambda_i : i < \kappa\rangle\|_D = \lambda$, λ regular.

(2) In 5.6 we can weaken assumption (c) to (c)$^-$.

Proof. (1) In the proof of 5.15, choose $g^{**} \in \prod_{i<\kappa} \lambda_i$ satisfying (exists by 5.17):

$(^*)_0$ $A \in J_D(\bar{\lambda}, \kappa) \cap D^+ \Rightarrow \|g^{**}\|_{D+A} \geq \lambda$ (which is $\|\bar{\lambda}\|_D$).

We redefine $Y[\bar{C}]$ as $\{\|g\|_D : g \in \prod_{i<\kappa} C_i$ but $g(i) > g^{**}(i)$ for $i < \kappa\}$. The only change is during the proof of $(*)_2$ there, we let

$$\alpha(*) = \sup\{\|g\|_{D+A} : A = (J_D(\lambda, \kappa))^+\}.$$

Now if $\alpha \in Y[\bar{C}']\backslash\alpha(*)$ then there is $h \in \prod_{i<\kappa} \lambda_i$ such that $[i < \kappa \Rightarrow h(i) \geq g^{**}(i)]$ and $\|h\|_D = \alpha$ and let $A = \{i < \kappa : h(i) < g^*(i)\}$. Now if $A \in (J_D(\bar{\lambda}, \kappa))^+$ we get a contradiction as there and if $A = \emptyset \bmod D$ we finish as there. So we are left with the case $A \in J_D(\bar{\lambda}, \kappa) \cap D^+$, $\|\bar{\lambda}\|_{D+A} > \|\bar{\lambda}\|_D \geq \lambda$ hence $\|g^{**}\|_{D+A} \geq \lambda$ hence $\|h\|_{D+A} \leq \lambda > \alpha$ hence necessarily $\|h\|_{D+(\kappa\backslash A)} = \alpha$ (as $\|h\|_D = \mathrm{Min}\{\|h\|_{D+A}, \|h\|_{D+(\kappa\backslash A)}\}$). Now choose $h' \in \prod_{i<\kappa} \lambda_i$ by $h' \restriction (\kappa\backslash A) = h \restriction (\kappa\backslash A)$ and $[i \in A \Rightarrow h'(i) = \mathrm{Min}(C_i''\backslash h(i))]$ so $h' \in \prod_{i<\kappa} C_i''$, $h \leq h' < \bar{\lambda}$, $\lambda \leq \|h\|_{D+A} \leq \|h'\|_{D+A} \leq \|h'\|_{D+A}$ and so

$$\|h'\|_D = \mathrm{Min}\{\|h'\|_{D+A}, \|h'\|_{D+(\kappa\backslash A)}\} = \alpha.$$

Also in the proof of $(^*)_3$ choose g such that $g > g^{**}$. So we are done.

(2) Let g^{**} be as in the proof of part (1). In the proof of 5.6 we let

$$Y[\bar{X}] =: \left\{\|h\|_{D+A} : h \in \prod_{i<\kappa}(X_i\backslash g^{**}(i)) \text{ and } A \in I^+\right\}$$

remembering $I = J_D(\bar{\lambda}, \kappa)$.

$$\mathscr{Y}[\bar{X}] =: \left\{(h, A)/\approx_I : h \in \prod_{i<\kappa}(X_i\backslash g^{**}(i)) \right.$$
$$\left. \text{and } (h, A) \in \mathrm{cla}_I^\alpha(\lambda, D) \text{ for some } \alpha < \lambda\right\}$$

and we can restrict ourselves to sequences $\bar{X}$ such that $X_i \cap g^{**}(i) = \emptyset$. In the proof of $(*)$ make $g > g^{**}$ and in $(^*)_3$, $g'(i) > g^{**}(i)$. $\dashv_{5.18}$

Claim 5.19. *Assume*

(a) *J is a filter on κ*
(b) *λ a regular cardinal, $\lambda_i > 2^\kappa, \theta > 2^\kappa$*
(c) *$\prod_{i<\kappa} \lambda_i/J$ is λ-like, i.e.,*
  (i) $\lambda = \mathrm{tcf}\,\Pi\lambda_i/J$
  (ii) $T_J(\langle\lambda_i : i < \kappa\rangle) = \lambda$ (*follows from* (i) + (iii) *actually*) *and*
  (iii) *if $\mu_i < \lambda_i$ then $T_J(\langle\mu_i : i < \kappa\rangle) < \lambda$*
(d) *$\kappa < \theta = \mathrm{cf}(\theta) < \lambda_i$ for $i < \kappa$*
(e) *$i < \kappa \Rightarrow S_\theta^{\lambda_i} = \{\delta < \lambda_i : \mathrm{cf}(\delta) = \theta\} \in I[\lambda_i]$ (see below)*
(f) $(\forall\alpha < \theta)[|\alpha|^\kappa < \theta]$.

Then $S_\theta^\lambda = \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\} \in I[\lambda]$.

REMARK 5.20. *Remember that for λ regular uncountable*

$$I[\lambda] = \Big\{ A \subseteq \lambda : \text{for some club } E \text{ of } \lambda \text{ and } \bar{\mathscr{P}} = \langle \mathscr{P}_\alpha : \alpha < \lambda \rangle \text{ with } \mathscr{P}_\alpha \subseteq \mathscr{P}(\alpha), |\mathscr{P}| < \lambda, \text{ for every } \delta \in A \cap E, \mathrm{cf}(\delta) < \delta \text{ and for some closed unbounded subset } a \text{ of } \delta \text{ of order type } < \delta, (\forall \alpha < \delta)(\exists \beta < \delta)(a \cap \alpha \in \mathscr{P}_\beta) \Big\}.$$

On finding $\bar{\lambda}$ as in clause (c) see [Sh:g, Chapter V].

PROOF. Clearly each λ_i is a regular cardinal and $\lambda = \mathrm{tcf}(\prod_{i<\kappa} \lambda_i / J)$, so let $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$ be a $<_J$-increasing sequence of members of $\prod_{i<\kappa} \lambda_i$, which is cofinal in $\prod_{i<\kappa} \lambda_i / J$. So without loss of generality if $\bar{f} \restriction \delta$ has a $<_J$-eub f' then $f_\delta =_J f'$.

For each $i < \kappa$ (see the references above) we can find $\bar{e}^i = \langle e^i_\alpha : \alpha < \lambda_i \rangle$ and E_i such that:

(i) E_i is a club of λ_i
(ii) $e^i_\alpha \subseteq \alpha$ and $\mathrm{otp}(e^i_\alpha) \leq \theta$
(iii) if $\beta \in e^i_\alpha$ then $e^i_\beta = e^i_\alpha \cap \beta$
(iv) if $\delta \in E_i$ and $\mathrm{cf}(\delta) = \theta$, then $\delta = \sup(e^i_\delta)$.

Choose $\bar{N} = \langle N_i : i < \lambda \rangle$ such that $N_i \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ where, e.g., $\chi = \beth_8(\lambda)^+$, $\|N_i\| < \lambda$, N_i is increasing continuous, $\bar{N} \restriction (i+1) \in N_{i+1}$, $N_i \cap \lambda$ is an ordinal, and $\{\bar{f}, J, \lambda, \langle \lambda_i : i < \kappa \rangle, \langle \bar{e}^i : i < \kappa \rangle\} \in N_0$. Let $E = \{\delta < \lambda : N_\delta \cap \lambda = \delta\}$, so it suffices to prove

(*) if $\delta \in E \cap S^\lambda_\theta$ then there is a such that:
  (i) $a \subseteq \delta$
  (ii) $\delta = \sup(a)$
  (iii) $|a| < \lambda$
  (iv) $\alpha < \delta \Rightarrow a \cap N_\alpha \in N_\delta$.

By clause (b) in the assumption necessarily $\bar{f} \restriction \delta$ has a $<_J$-eub ([Sh:g, Chapter II,§1]) so necessarily f_δ is a $<_J$-eub of $\bar{f} \restriction \delta$. Moreover, $A^* = \{i < \kappa : \mathrm{cf}(f_\delta(i)) = \theta \text{ and } f_\delta(i) \in E_i\} = \kappa \bmod J$ by clause (f) of the assumption. So for each $i \in A^*$, $e^i_{f_\delta(i)}$ is well-defined, and let $e^i_{f_\delta(i)} = \{\alpha^i_\zeta : \zeta < \theta\}$ with α^i_ζ increasing with ζ. For each $\zeta < \theta$ we have $\langle \alpha^i_\zeta : i < \kappa \rangle <_J f_\delta$ hence for some $\gamma(\zeta) < \delta$ we have $\langle \alpha^i_\zeta : i < \kappa \rangle <_J f_{\gamma(\zeta)}$, but $T_D(f_{\gamma(\zeta)}) < \lambda$ and $\gamma(\zeta) \in N_{\gamma(\zeta)+1}$ hence $f_{\gamma(\zeta)} \in N_{\gamma(\zeta)+1}$ hence for some $g_\zeta <_J f_{\gamma(\zeta)}$ we have: $g_\zeta \in N_{\gamma(\zeta)+1}$ and $A_\zeta = \{i < \kappa : g_\zeta(i) = \alpha^i_\zeta\} \neq \emptyset \bmod J$. As $\theta = \mathrm{cf}(\theta) > 2^\kappa$ for some $A \subseteq \kappa$ we have $B =: \{\zeta < \theta : A_\zeta = A\}$ is unbounded in θ.

Now for $\zeta < \theta$ let

$$a_\zeta = \Big\{ \mathrm{Min}\{\gamma < \lambda : \neg(f_\gamma \leq_{J+(\kappa \setminus A)} g)\} : g \in \prod_{i<\kappa} \{\alpha^i_\varepsilon : \varepsilon < \zeta\} = \prod_{i<\kappa} e^i_{(\alpha^i_\zeta)} \Big\}.$$

Clearly $\zeta < \xi < \theta \Rightarrow a_\zeta \subseteq a_\xi$. Also for $\zeta < \theta, a_\zeta$ is definable from $\bar{f}$ and $g_\zeta \restriction A$, hence belongs to $N_{\gamma(\zeta)+1}$, but its cardinality is $\leq \theta + 2^\kappa < \lambda$ hence it is a subset of $N_{\gamma(\zeta)+1}$. Moreover, also $\langle a_\xi : \xi < \zeta \rangle$ is definable from $\bar{f}$ and $\langle\langle\{\alpha^i_\varepsilon : \varepsilon < \xi\} : i < A\rangle : \xi \leq \zeta\rangle$ hence from $\bar{f}$ and $g_\zeta \restriction A$ and $\langle \bar{e}^i : i < \kappa\rangle$, all of which belong to $N_0 \prec N_{\gamma(\zeta)+1}$, hence $\zeta \in B \Rightarrow \langle a_\xi : \xi \leq \zeta\rangle \in N_{\gamma(\zeta)+1}$ & a_ζ is a bounded subset of δ. Now

(*) $\bigcup_{\xi<\theta} a_\xi$ is unbounded in δ.

[Why? Let $\beta < \delta$, so for some $\zeta < \theta$ we have:

$$f_\beta(i) < f_\delta(i) \Rightarrow f_\beta(i) < \alpha^i_\zeta < f_\delta(i)$$

so

$$\mathrm{Min}\{\gamma : \neg(f_\gamma \leq_{J+(\kappa\setminus A)} \langle\alpha^i_\zeta : i < \kappa\rangle)\} \in (\beta, \delta) \cap a_{\zeta+1}.$$

Let $w = \{\zeta < \theta : a_\zeta \text{ is bounded in } a_{\zeta+1}\}$

$$a'_\zeta = \big\{\mathrm{Min}\{\gamma \in a_{\xi+1} : \gamma \text{ is an upper bound of } a_\xi\} : \xi < \zeta\big\}.$$

So $\bigcup\{a'_\zeta : \zeta < \theta\}$ is as required. $\dashv_{5.19}$

REMARK 5.21. (1) *If we want to weaken clause* (c) *in claim* 5.19 *retaining only* (i) *there* (*and omitting* (ii) + (iii)), *it is enough if we add*:

(g) *for each* $i < \kappa$ *and* $\delta \in S^{\lambda_i}_\theta$, $\{\gamma < \delta : \mathrm{cf}(\gamma) > \kappa$ *and* $\gamma \in e^i_\delta\}$ *is a stationary subset of* δ.

(2) *In part* (1) *of this remark, we can replace* $\mathrm{cf}(\gamma) > \kappa$ *by* $\mathrm{cf}(\gamma) = \sigma$, *if* D *is* σ^+*-complete or at least not* σ*-incomplete.*

(3) *This is particularly interesting if* $\lambda = \mu^+ = \mathrm{pp}(\mu)$.

§6. The class of cardinal ultraproducts modulo D.

We presently concentrate on ultrafilters (for filters: two versions). This continues [Sh:506, §3], see history there and in [CK], [Sh:g].

Recall

DEFINITION 6.1. (1) A filter D is θ-regular if there are $A_\varepsilon \in D$ for $\varepsilon < \theta$ such that the intersection of any infinitely many A_ε's is empty.

(2) For a filter D, let $\mathrm{reg}(D) = \min\{\theta : D \text{ is not } \theta\text{-regular}\}$. Note that $\mathrm{reg}(D)$ is a regular cardinal.

FACT 6.2. *Assume*

(a) *D is an ultrafilter on κ and* $\theta = \mathrm{reg}(D)$

(b) $\mu = \mathrm{cf}(\mu)$ *and* $\alpha < \mu \Rightarrow |\alpha|^{<\mathrm{reg}(D)} < \mu$

(c) $\bar{n} = \langle n_i : i < \kappa\rangle, 0 < n_i < \omega, A^* = \bigcup_{i<\kappa}(\{i\} \times n_i)$

(d) *for each* $i < \kappa, n < n_i$ *we have* $\lambda_{(i,n)}$ *is regular* $> \kappa$, $< \mu$ *strictly increasing with* n, *stipulating* $\lambda_{(i,n_i)} = \mu$.

(e) *if* $B \in D$ *then* $\mu \leq \max\mathrm{pcf}\{\lambda_{\langle i,n\rangle} : i \in B$ *and* $n < n_i\}$

Then for some $\langle m_i : i < \kappa\rangle \in \prod_{i<\kappa}(n_i + 1)$ *and* $B \in D$ *we have*:

(α) $\mu \leq \mathrm{tcf}(\prod_{i<\kappa} \lambda_{(i,m_i)}/D)$

(β) $\mu > \max\mathrm{pcf}\{\lambda_{(i,n)} : i \in B$ *and* $n < m_i\}$.

PROOF. We try to choose by induction on $\zeta < \mathrm{reg}(D)$, B_ζ and $\langle n^\zeta_i : i < \kappa\rangle$ such that:

(i) $B_\zeta \in D$
(ii) $n_i^\zeta \leq n_i$ non-decreasing in ζ
(iii) $B_\zeta = \{i : n_i^\zeta < n_i^{\zeta+1}\}$ and
(iv) $\max \operatorname{pcf}\{\lambda_{(i,n)} : i < \kappa \text{ and } n < n_i^\zeta\} < \mu$.

If we succeed, then $\{B_\zeta : \zeta < \operatorname{reg}(D)\}$ exemplifies D is $\operatorname{reg}(D)$-regular, contradiction. During the induction we choose B_ζ in step $\zeta + 1$. For $\zeta = 0$ try $n_i^\zeta = 0$, this cannot fail as clause (iv) holds trivially. For ζ limit let $n_i^\zeta = n_i^\xi$ for every $\xi < \zeta$ large enough, this is O.K. as

$$\max \operatorname{pcf}\{\lambda_{(i,n)} : i < \kappa \text{ and } n < n_i^\zeta\} \leq \prod_{\xi<\zeta} \max \operatorname{pcf}\{\lambda_{(i,n)} : i < \kappa \text{ and } n \leq n_i^\xi\} < \mu$$

by assumption (b). Lastly, for $\zeta = \xi + 1, \{i < \kappa : n_i^\xi < n_i\} \in D$ (otherwise contradiction as $\lambda_{(i,n_i)} = \mu$ and clause (iv) contradict assumption (e)), and if $\mu \leq \operatorname{tcf}(\prod_{i<\kappa} \lambda_{n_i^\xi}/D)$ we are done with $m_i = n_i^\xi$, if not there is $B \in D$ such that $\max \operatorname{pcf}\{\lambda_{n_i^\xi} : i \in B\} < \mu$ and let $B_\xi = \{\in B : n_i^\xi < n_i\}$

$$n_i^\zeta = \begin{cases} n_i^\xi + 1 & \underline{\text{if}} \quad i \in B_\xi, n_i^\xi < n_i \\ n_i^\xi & \underline{\text{if}} \quad \text{otherwise.} \end{cases} \qquad \dashv_{6.2}$$

LEMMA 6.3. *Assume*

(i) *D is an ultrafilter on κ*
(ii) *$\mu = \operatorname{cf}(\mu)$ and $\alpha < \mu \Rightarrow |\alpha|^{<\operatorname{reg}(D)} < \mu$*
(iii) *at least one of the following occurs*:
 (α) $\alpha < \mu \Rightarrow |\alpha|^{\operatorname{reg}(D)} < \mu$
 (β) *D is closed under decreasing sequences of length* $\operatorname{reg}(D)$.

__Then__ there is a minimal g/D such that:
$\mu = \operatorname{tcf}\left(\prod_{i<\kappa} g(i)/D\right)$ *and* $\bigwedge_{i<\kappa} \operatorname{cf}(g(i)) > \kappa$.

We shall prove it somewhat later.

REMARK 6.4. (1) *Note that necessarily* (*in* 6.3)

$$\{i < \kappa : g(i) \text{ a regular cardinal}\} \in D.$$

(2) *g is also $<_D$-minimal under*: $\mu \leq \operatorname{tcf}\left(\prod_{i<\kappa} g(i)/D\right)$ & $\{i : \operatorname{cf}(g(i)) > \kappa\} \in D$.

[*Why? assume $g' <_D g_\beta, \mu \leq \operatorname{tcf}\left(\prod_{i<\kappa} g'(i)/D\right)$, and*

$$X = \{i : \operatorname{cf}(g(i)) \leq \kappa\} = \emptyset \operatorname{mod} D;$$

clearly $\mu \leq \operatorname{tcf}\left(\prod_{i<\kappa} \operatorname{cf}(g'(i))/D\right)$. If $Lim_D \operatorname{cf}(g'(i))$ is singular, by [Sh:g, *II*,1.4(1), *page* 50] *for some $\langle \lambda_i : i < \kappa\rangle$, we have $\mu = \operatorname{tcf}(\Pi\lambda_i/D)$ and*

$$\operatorname{Lim}_D \lambda_i = \operatorname{Lim}_D \operatorname{cf}(g(i)), \quad \lambda_i \leq \operatorname{cf}(g(i))$$

and $(\forall i)[\operatorname{cf}(g(i)) > \kappa \rightarrow \lambda_i \geq \kappa]$, so again without loss of generality $\bigwedge_{i<\kappa} \lambda_i > \kappa$. Now $\langle \lambda_i : i < \kappa\rangle$ contradicts the choice of g. If $\operatorname{Lim}_D \operatorname{cf}(g'(i))$ is regular, it is $\leq \kappa$ and this contradicts an assumption on g'.]

(3) *If* $|\kappa^\kappa/D| < \mu$ *then we can omit* (*in the conclusion of* 6.3 *and of* 6.4(2)) *the clause* "$\{i : \mathrm{cf}(g(i)) > \kappa\} \in D$."

CONCLUSION 6.5. If assumptions (i)–(iii) of 6.3 hold and

(iv) $\mu > 2^\kappa$

then without loss of generality (each $g(i)$ is a regular cardinal) and

$$\left(\prod_{i<\kappa} g(i)/D, <_D\right)$$

is μ-like (i.e., of cardinality μ but every proper initial segment has smaller cardinality).

REMARK 6.6. *We use* $\mu > 2^\kappa$ *in* 6.5 *rather than* $\mu > |\kappa^\kappa/D|$ *as in* 6.4(3) (*which concerns* 6.3, 6.4(3)) *as the proof of* 6.5 *uses* 1.5.

PROOF OF 6.5. If D is $\aleph_1$-complete this is trivial, so assume not hence $\mathrm{reg}(D) > \aleph_0$.

Let $g \in {}^\kappa(\mu+1)$ be as in 6.3, so without loss of generality as in 6.4(2), and remember 6.4(1) so without loss of generality each $g(i)$ is a regular cardinal. Clearly $\prod_{i<\kappa} g(i)$ has cardinality $\geq \mu$. Assume first $\mu = \chi^+$.

Let $g' \in \prod_{i<\kappa} g(i)$, then by 6.4(3) and the choice of g

$$\sup\{\mathrm{tcf}\,\Pi\lambda_i/D : \lambda_i \leq g'(i) \text{ for } i < \kappa\} \leq \chi.$$

But as $\mathrm{reg}(D) > \aleph_0$ by clause (ii) of the assumption we have $\alpha < \mu \Rightarrow |\alpha|^{\aleph_0} < \mu$ so 1.5 applies (say for $J = \{\kappa \backslash A : A \in D\}$, as D is an ultrafilter clearly $T^2_J(f) = (\prod_{i<\kappa} f(i)/D)$ and by assumption (ii), clause (e) of 1.5 holds. So we get $|\prod_{i<\kappa} g'(i)/D| \leq \chi$, so really $\prod_{i<\kappa} g(i)/D$ is μ-like.

If μ is not a successor, then it is weakly inaccessible and $\mu = \sup(Z)$, where

$$Z = \{\chi^+ : |\kappa^\kappa/D| < \chi^{\aleph_0} = \chi < \mu\},$$

so for each $\chi \in Z$ by 6.3 we can find $g_\chi \in {}^\kappa(\mu+1)$ such that $\prod_{i<\kappa} g_\chi(i)/D$ is χ-like so necessarily for $\chi_1 < \chi_2$ in Z we have $g_{\chi_1} <_D g_{\chi_2}$. It is enough to find a $<_D$-lub for $\langle f_\chi : \chi \in Z\rangle$, and as $\mu > 2^\kappa$ this is immediate. $\dashv_{6.5}$

PROOF OF 6.3. First try to choose, by induction on α, f_α such that:

(A) $f_\alpha \in {}^\kappa(\mu+1)$

(B) $\mu = \mathrm{tcf}\left(\prod_{i<\kappa} f_\alpha(i)/D\right)$

(C) $\beta < \alpha \Rightarrow f_\alpha <_D f_\beta$

(D) each $f_\alpha(i)$ is a regular cardinal $> \kappa$.

Necessarily for some α^* we have: f_α is well-defined if and only if $\alpha < \alpha^*$. Now α^* cannot be zero as the constant function with value μ can serve as f_0. Also if α^* is a successor ordinal, say $\alpha^* = \beta + 1$, then f_β is as required in the desired conclusion (by 6.4(2)'s proof).

So α^* is a limit ordinal, and by passing to a subsequence, without loss of generality $\alpha^* = \mathrm{cf}(\alpha^*)$ and call it θ.

Without loss of generality

(E) $\mu = \max\mathrm{pcf}\{f_\alpha(i) : i < \kappa\}$.

We now try to choose by induction on $\zeta < \mathrm{reg}(D)$ the objects $\alpha_\zeta, A_\zeta, \mathfrak{b}_\zeta$ such that:

(a) $\alpha_\zeta < \theta$ is strictly increasing with ζ

(b) $A_\zeta \in D$
(c) $\mathfrak{b}_\zeta \subseteq \{f_{\alpha_\xi}(i) : \xi \leq \zeta, \text{ and } i \in A_\xi\}$
(d) $\mathfrak{b}_\zeta$ is increasing with ζ
(e) $\max \mathrm{pcf}(\mathfrak{b}_\zeta) < \mu$
(f) for each i the sequence
$\langle f_{\alpha_\xi}(i) : \xi \leq \zeta \text{ and } i \in A_\xi \text{ and } f_{\alpha_\xi}(i) \notin \mathfrak{b}_\zeta\rangle$ is strictly decreasing
(g) $\alpha_0 = 0, A_0 = \kappa, \mathfrak{b}_\zeta = \emptyset$
(h) $\alpha_{\zeta+1} = \alpha_\zeta + 1$ and $A_{\zeta+1} = \{i \in A_\zeta : f_{\alpha_{\zeta+1}}(i) < f_{\alpha_\zeta}(i) \text{ and } f_{\alpha_\zeta}(i) \notin \mathfrak{b}_\zeta\}$
(i) for ζ limit, α_ζ is the first $\alpha < \theta$ which is $\geq \bigcup_{\varepsilon<\zeta} \alpha_\varepsilon$ such that for some $B \in D$ we have:
$\mu > \max \mathrm{pcf}\{f_{\alpha_\xi}(i) : \xi < \zeta, i \in A_\xi \text{ and } i \in B \text{ and } f_{\alpha_\xi}(i) \leq f_\alpha(i)\}$
(j) $\mathfrak{b}_{\zeta+1} = \mathfrak{b}_\zeta$
(k) for ζ limit A_ζ satisfies the requirements on B in clause (i) and

$$\mathfrak{b}_\zeta = \bigcup_{\varepsilon<\zeta} \mathfrak{b}_\varepsilon \cup \bigcup \{f_\xi(i) : \xi < \zeta \text{ and } i \in A_\xi \cap A_\zeta \text{ and } f_{\alpha_\xi}(i) \leq f_{\alpha_\zeta}(i)\}$$

(ℓ) for $\xi \leq \zeta$ we have $\{i \in A_\xi : f_{\alpha_\xi}(i) \notin \mathfrak{b}_\zeta\} \in D$.

So for some $\zeta^* \leq \mathrm{reg}(D)$ we have $(\alpha_\zeta, A_\zeta, \mathfrak{b}_\zeta)$ is well defined if and only if $\zeta < \zeta^*$.

We check the different cases and get a contradiction in each (so α^* must have been a successor ordinal giving the desired conclusion). ⊣

<u>Case 1.</u> $\zeta^* = 0$.
We choose $\alpha_0 = 0$,

$$A_0 = \kappa, \mathfrak{b}_0 = \emptyset;$$

so clause (g) holds, first part of clause (a) (i.e., $\alpha_\zeta < \theta$) holds, clause (b) and clause (c) are totally trivial, clause (e) holds as $\max \mathrm{pcf}(\emptyset) = 0$ (formally we should have written $\sup \mathrm{pcf}(\mathfrak{b}_\zeta)$), clause (f) speaks on the empty sequence, and the other clauses are empty in this case.

<u>Case 2.</u> $\zeta^* = \zeta + 1$.
We choose $\alpha_{\zeta^*} = \alpha_{\zeta+1} = \alpha_\zeta + 1, A_{\zeta^*} = \{i \in A_\zeta : f_{\alpha_\zeta+1}(i) < f_{\alpha_\zeta}(i) \text{ and } f_{\alpha_\zeta}(i) \notin \mathfrak{b}_\zeta\}$ and $\mathfrak{b}_{\zeta+1} \supseteq \mathfrak{b}_\zeta$ is defined by clause (j). Clearly $\alpha_\zeta < \alpha_{\zeta+1} < \theta$ and $A_{\zeta+1} \in D$ as $A_\zeta \in D$ and $f_{\alpha_\zeta+1} <_D f_{\alpha_\zeta}$ and $\{i : f_{\alpha_\zeta}(i) \notin \mathfrak{b}_\zeta\} \in D$ by clause (ℓ); so clause (b) holds. Now clause (a) holds trivially and clauses (g) and (i) are irrelevant. Clause (h) holds by our choice.

For clause (f), the new cases are when $f_{\alpha_{\zeta+1}}(i)$ appears in the sequence, i.e., $i \in A_{\zeta+1}$ such that $f_{\alpha_{\zeta+1}}(i) \notin \bigcup_{\xi\leq\zeta+1} \mathfrak{b}_\xi = \mathfrak{b}_{\zeta+1} = \mathfrak{b}_\zeta$ but $i \in A_{\zeta+1} \Rightarrow i \in A_\zeta$ & $f_{\alpha_\zeta}(i) \notin \mathfrak{b}_\zeta$ so also $f_{\alpha_\zeta}(i)$ appears in the sequence and as $i \in A_{\zeta+1} \Rightarrow f_{\alpha_\zeta}(i) > f_{\alpha_\zeta+1}(i) = f_{\alpha_{\zeta+1}}(i)$ plus the induction hypothesis; we are done.

As for clause (ℓ) for $\xi \leq \zeta + 1$, if $\xi \leq \zeta$ this holds by the induction hypothesis (as $\mathfrak{b}_{\zeta+1} = \mathfrak{b}_\zeta$) so assume $\xi = \zeta + 1$. Clearly

$$\{i \in A_\xi : f_{\alpha_\xi}(i) \notin \mathfrak{b}_{\zeta+1}\} = A_\xi \cap \{i < \kappa : f_{\alpha_\xi}(i) \notin \mathfrak{b}_{\zeta+1}\}.$$

Now the first belongs to D by clause (b) proved above and the second belongs to D as $\max \mathrm{pcf}(\mathfrak{b}_{\zeta+1}) < \mu$ by clause (e) proved below as $\mathrm{tcf}\left(\prod_{i<\kappa} f_{\alpha_\xi}(i)/D\right) = \mu$ by clause (B).

We have chosen $\mathfrak{b}_{\zeta+1} = \mathfrak{b}_\zeta$, so (using the induction hypothesis) clauses (c), (d), (e) trivially hold and also clause (j) holds by the choice of $\mathfrak{b}_{\zeta^*}$, and clause (k) is irrelevant so we are done.

CASE 3. $\zeta^* = \zeta$ is a limit ordinal $< \operatorname{reg}(D)$.
Let $\mathfrak{b}^*_\zeta = \bigcup_{\xi<\zeta} \mathfrak{b}_\xi$, so by basic pcf:

$$\boxtimes \qquad \max \operatorname{pcf}(\mathfrak{b}^*_\zeta) \leq \prod_{\xi<\zeta} \max \operatorname{pcf}(\mathfrak{b}_\xi) < \mu$$

as

$$\mu = \operatorname{cf}(\mu) \;\&\; (\forall \alpha < \mu)[|\alpha|^{<\operatorname{reg}(D)} < \mu)] \;\&\; \zeta < \operatorname{reg}(D).$$

Now we try to define α_ζ by clause (i).

Subcase 3A. α_ζ is not well defined.
Let $w_i = \{\xi < \zeta : i \in A_\xi \text{ and } f_{\alpha_\xi}(i) \notin \mathfrak{b}^*_\zeta\}$. Note that by the induction hypothesis (clause (f)) for each $\varepsilon < \zeta$ and $i < \kappa$ we have the sequence $\langle f_{\alpha_\xi}(i) : \xi < \varepsilon \text{ and } i \in A_\xi \text{ and } f_{\alpha_\xi}(i) \notin \mathfrak{b}_\varepsilon\rangle$ is strictly decreasing, so as $\mathfrak{b}_\varepsilon \subseteq \mathfrak{b}^*_\zeta$ clearly $\langle f_{\alpha_\xi}(i) : \xi < \varepsilon \text{ and } \xi \in w_i\rangle$ is strictly decreasing. As this holds for each $\varepsilon < \zeta$ and ζ is a limit ordinal, clearly $\langle f_{\alpha_\xi}(i) : \xi \in w_i\rangle$ is strictly decreasing hence w_i is finite.

Now for each $B \in D$ we have (first inequality by clause (E) and clause (b) on the induction hypothesis on ζ, second by the definition of the w_i's)

$$\begin{aligned}\mu &\leq \max \operatorname{pcf}\left\{ f_\xi(i) : \xi < \zeta, i \in A_\xi \text{ and } i \in B \right\}\\ &\leq \max\left\{ \max \operatorname{pcf}(\mathfrak{b}_\zeta), \max \operatorname{pcf}\{f_\xi(i) : \xi \in w_i \text{ and } i \in B\}\right\},\end{aligned}$$

and $\max \operatorname{pcf}(\mathfrak{b}^*_\zeta) < \mu$ as said above, hence necessarily

(*) $B \in D \Rightarrow \mu \leq \max \operatorname{pcf}\{f_{\alpha_\xi}(i) : \xi \in w_i \text{ and } i \in B\}$.

As w_i is finite and each $f_\alpha(i)$ is a regular cardinal $> \kappa$ we have $\{i : w_i \neq \emptyset\} \in D$.

By Claim 6.2 (the case there of $\{i : m_i = n_i\} \in D$ is impossible by $(*)$ above) we can find $g \in \prod_{i<\kappa} w_i/D$, more exactly $g \in {}^\kappa\operatorname{Ord}$, $w_i \neq \emptyset \Rightarrow g(i) \in w_i$ and $B \in D$ such that:

(α) $\mu \leq \operatorname{tcf}\left(\prod_{i<\kappa} g(i)/D\right)$
(β) $\mu > \max \operatorname{pcf}\{f_{\alpha_\xi}(i) : \xi \in w_i \text{ and } i \in B \text{ and } f_{\alpha_\xi}(i) < g(i)\}$.

Now by the choice of $\langle f_\alpha : \alpha < \theta\rangle$ and clause (α) necessarily (and [Sh:g, Chapter II, 1.4(1), page 50]) for some $\alpha < \theta$ we have $f_\alpha <_D g$. Now for $\xi < \zeta$, let $B^\xi_\alpha = \{i < \kappa : f_\alpha(i) \geq f_{\alpha_\xi}(i)\}$, if $B^\xi_\alpha \in D$ then $B^* =: \{i < \kappa : \xi \in w_i \text{ and } i \in B \text{ and } g(i) > f_{\alpha_\xi}(i)\} \supseteq B \cap \{i < \kappa : f_\alpha(i) < g\alpha(i)\} \cap \{i < \kappa : i \in A_\xi\} \cap \{i < \kappa : f_\alpha(i) \notin \mathfrak{b}^*_\zeta\} \cap \{i < \kappa : f_\alpha(i) \geq f_{\alpha_\xi}(i)\}$ which is the intersection of five members of D hence belongs to D, but $\{f_{\alpha_\xi}(i) : i \in B^*\}$ is included in the set in the right side of clause (β) hence $\mu > \max \operatorname{pcf}\{f_{\alpha_\xi}(i) : i \in B^*\}$ contradicting $B^* \in D$, $\operatorname{tcf}(\prod_{i<\kappa} f_{\alpha_\xi}(i)/D) = \mu$. So necessarily $B^\xi_\alpha \notin D$, hence $f_\alpha <_D f_{\alpha_\xi}$ hence $\alpha > \alpha_\xi$. So $\bigcup_{\xi<\zeta} \alpha_\xi \leq \alpha < \theta$. Let $B' = B \cap \{i < \kappa : f_\alpha(i) < g(i)\}$ so $B' \in D$ and [first inclusion by the choice of B', second inclusion by the choice of $\mathfrak{b}^*_\zeta$]

$$\begin{aligned}\{f_{\alpha_\xi}(i) : \xi < \zeta, i \in A_\xi \text{ and } i \in B' \text{ and } f_{\alpha_\xi}(i) \le f_\alpha(i)\} \\ \subseteq \{f_{\alpha_\xi}(i) : \xi < \zeta, i \in A_\xi \text{ and } i \in B \text{ and } f_{\alpha_\xi}(i) < g(i)\} \\ \subseteq \mathfrak{b}^*_\zeta \cup \{f_{\alpha_\xi}(i) : \xi \in w_i \text{ and } i \in B \text{ and } f_{\alpha_\xi}(i) < g(i)\}\end{aligned}$$

hence

$$\begin{aligned}\max \mathrm{pcf}\Big\{ f_{\alpha_\xi}(i) :\xi < \zeta, i \in A_\xi \text{ and } i \in B' \text{ and } f_{\alpha_\xi}(i) \le f_\alpha(i)\Big\} \\ \le \max\Big\{\max \mathrm{pcf}(\mathfrak{b}_\zeta), \max \mathrm{pcf}\{f_{\alpha_\xi}(i) : \xi \in w_i \\ \text{and } i \in B \text{ and} f_{\alpha_\xi}(i) < g(i)\}\Big\} < \mu\end{aligned}$$

(the first term is $< \mu$ as the statement $\boxtimes$ was proved in the beginning of Case 3, the second term is $< \mu$ by clause (β)). So α is as required in clause (i) so α_ζ is well defined; contradiction to our case assumption.

CASE 3B. α_ζ is well defined.

Let $B \in D$ exemplify it. We choose A_ζ as B and we define $\mathfrak{b}_\zeta$ by clause (k).

Now clause (a) follows from clause (i) (which holds by the assumption of the subcase), clause (b) holds by the choice of B (and of A_ζ), clause (c) by the choice of $\mathfrak{b}_\zeta$, clause (d) by the choice of $\mathfrak{b}_\zeta$, clause (e) by the choice of $\mathfrak{b}_\zeta$, that is, by $\boxtimes$ above and the choice of A_ζ (see clause (i)). Now for clause (f) by the induction hypothesis and clause (d) we should consider only $f_{\alpha_\xi}(i) > f_{\alpha_\zeta}(i)$ when $\xi < \zeta, i \in A_\xi \cap A_\zeta$ and $f_{\alpha_\xi}(i), f_{\alpha_\zeta}(i) \notin \mathfrak{b}_\zeta$, but clauses (i)+(k) (i.e., the choice of $\mathfrak{b}_\zeta$) take care of this, clauses (g), (h), (j) are irrelevant, clause (i)+(k) holds by the choice of $\alpha_\zeta, A_\eta, B_\zeta$ and clause (ℓ) follows from clause (e).

So we are done.

CASE 4. $\zeta^* = \mathrm{reg}(D)$.

The proof is split according to the two cases in the assumption (iii).

Subcase 4A. $\alpha < \mu \Rightarrow |\alpha|^{\mathrm{reg}(D)} < \mu$.

Let $\mathfrak{b} = \bigcup\{\mathfrak{b}_\xi : \xi < \zeta^*\}$ so $\max \mathrm{pcf}(\mathfrak{b}) < \mu$, hence for each $\xi < \zeta^*$ we have $A'_\xi =: \{i \in A_\xi : f_{\alpha_\xi}(i) \notin \mathfrak{b}\} \in D$. Let $w_i = \{\xi < \zeta^* : i \in A'_\xi$, so $f_{\alpha_\xi}(i) \notin \mathfrak{b}\}$. Now for any $\zeta < \zeta^*$ and $i < \kappa$ the sequence $\langle f_{\alpha_\xi}(i) : \xi < \zeta$ and $\xi \in w_i\rangle$ is strictly decreasing (by clause (f)) hence $\langle f_{\alpha_\xi}(i) : \xi < \zeta^*$ and $\xi \in w_i\rangle$ is strictly decreasing hence w_i is finite. Also for each $\xi < \zeta^*$ the set A'_ξ belongs to D, so $\{A'_\xi : \xi < \zeta^*\}$ exemplifies D is $|\zeta^*|$-regular, but $\zeta^* = \mathrm{reg}(D)$, contradiction.

Subcase 4B. D is closed under decreasing sequences of length $\mathrm{reg}(D)$.

Let $\mathfrak{b} = \bigcup_{\zeta<\zeta^*} \mathfrak{b}_\zeta$.

In this case, for each $\xi < \zeta^*$, the sequence $\langle\{i \in A_\xi : f_{\alpha_\xi}(i) \notin \mathfrak{b}_\zeta\} : \zeta \in [\xi, \zeta^*]\}\rangle$ is a decreasing sequence of length $\zeta^* = \mathrm{reg}(D)$ of members of D so the intersection, $A'_\xi = \{i \in A_\xi : f_{\alpha_\zeta}(i) \notin \mathfrak{b}\} \in D$, and we continue as in the first subcase. $\dashv_{6.3}$

DEFINITION 6.7. (1) For an ultrafilter D on κ let $\mathrm{reg}'(D)$ be: $\mathrm{reg}(D)$ if D is closed under intersection of decreasing sequences of length $\mathrm{reg}(D)$ and $(\mathrm{reg}(D))^+$ otherwise.

(2) $\text{reg}''(D)$ is: $\text{reg}(D)$ <u>if</u> (a)$^-$ below holds and $(\text{reg}(D))^+$ otherwise

(a) $\text{reg}'(D) = \text{reg}(D)$ or just

(a)$^-$ letting $\theta = \text{reg}(D)$, in θ^κ/D there is a $<_D$-first function above the constant functions.

THEOREM 6.8. *If D is an ultrafilter on κ and $\theta = \text{reg}'(D)$ <u>then</u>* $\mu = \mu^{<\theta} \geq |\theta^\kappa/D| \Rightarrow \mu \in \{\prod_i \lambda_i/D : \lambda_i \in \text{Card}\}$.

PROOF. Apply Lemma 6.5 with D, κ, μ^+ here standing for D, κ, μ there; note that assumption (iii) there holds as the definition of $\text{reg}'(D)(= \theta)$ was chosen appropriately.

Let $g^*/D = \langle \lambda_i^* : i < \kappa \rangle$ be as there, so as $\left(\prod_{i<\kappa} \lambda_i^*/D\right)$ is μ^+-like, for some $f \in \prod_{i<\kappa} \lambda_i$, we have $|\prod_{i<\kappa} f(i)/D| = \mu$ as required. $\dashv_{6.8}$

REMARK 6.9. *Can $\text{reg}'(D) \neq \text{reg}(D)$? This is equivalent to: D is not closed under intersections of decreasing sequences of length $\theta = \text{reg}(D)$. So if $\text{reg}'(D) \neq \text{reg}(D) = \theta$ then θ is regular and for some function $\boldsymbol{i} : \kappa \to \theta$ the ultrafilter $D' = \{A \subseteq \theta : \boldsymbol{i}^{-1}(A) \in D\}$ is an ultrafilter on θ, with $\text{reg}(D') = \theta$ so D' is not regular.*

This leads to the well known problem (Kanamori [Kn]): if D is a uniform ultrafilter on κ with $\text{reg}(D) = \kappa$ does κ^κ/D have a first function above the constant ones?

Note that

FACT 6.10. *If $\theta = \text{reg}(D) < \text{reg}'(D)$, $\mu = \sum_{i<\theta} \mu_i$, $\mu_i^\kappa = \mu_i < \mu_{i+1}$ and*

$$\left|\prod_{i<\kappa} f(i)/D\right| \geq \mu \ \underline{\text{then}} \ \left|\prod_{i<\kappa} f(i)/D\right| \geq \mu^\theta = \mu^\kappa.$$

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