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Modules Over Arbitrary Domains

Rüdiger Göbel[1] and Saharon Shelah[2]

[1] FB6 Mathematik, Universität Essen, Postfach, D-4300 Essen, Federal Republic of Germany

[2] Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem, Israel

§ 1. Introduction

For a discussion of our result, we will restrict to any non zero domain R with 1, i.e. R is a commutative ring without zero-divisors $\neq 0$. One interesting class of such rings are Dedekind domains or more generally noetherian domains, another one, which is quite attractive, are the valuation domains. There is an increasing number of papers on modules over such rings, some of them export results on abelian groups to these classes of modules. A special flavour of such investigations on valuation domains is due to the fact that R does no longer satisfy the maximum condition in general. This has the nice consequence, that many new ideas have to be developed for the study of such modules; and it is not surprising that such ideas are interesting mixtures of algebra and analysis; – see L. Fuchs [9] and next year also L. Fuchs and L. Salce [10] for further references.

One of the central problems in module categories is concerned with the existence of indecomposable modules. We are most grateful to Laszlo Fuchs, for drawing our attention to this question for valuation domains, when he was a visiting professor at Essen University in 1982. Answering this problem, it turns out that the methods are applicable to all domains which are not fields. Since we want to derive counter examples to possible Krull-Schmidt-type-theorems in these categories – or equivalently, examples for Kaplansky's famous test-problems – we shall derive a realization of algebras as endomorphism algebras of suitable modules in this category.

Suppose S is a multiplicatively closed subset of R representing a linear set of principal ideals. We will see that $\kappa = |S|$ may be chosen to be regular. If M is an R-module, which is *reduced* (with respect to S, i.e. $\bigcap_S Ms = 0$), then M has the natural Hausdorff S-topology, taking $\{Ms\colon s \in S\}$ as a system of neighbourhoods of $0 \in M$. Hence M can be *completed* to $\hat{M}$. If the module M is also *torsion-free* (with respect to S, i.e. $ms = 0$, $m \in M$, $s \in S$ implies $m = 0$), then $\hat{M}$ is torsion-free and reduced. The module G realizing some algebra A will be

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constructed between some basic submodule B depending on A and $\hat{B}$. We derive the following result:

If A is an R-algebra with torsion-free and reduced R-module structure and λ is a cardinal with $\lambda^\kappa = \lambda \geqq |A|$, then we find R-modules $G_i (i \in 2^\lambda)$ such that $\operatorname{End} G_i = A \oplus \operatorname{Ines} G_i$ *and* $\operatorname{Hom}(G_i, G_j) = \operatorname{Ines}(G_i, G_j)$ *for* $i \neq j \in 2^\lambda$.

The ideal Ines G of End G is the set of all endomorphisms which map the completion $\hat{G}$ of G into G. The definition of $\operatorname{Ines}(G_i, G_j)$ is similar. This result leads already to counter examples to "Krull-Schmidt-theorems" even over a complete ring R; such theorems may be derived by a method given in [5, 4, 7] §7.

If we restrict the algebras (and hence the rings) a little further, we obtain a strong realization theorem. The appropriate notion is cotorsion-freeness, cf. [11, 12]. – A module M is *cotorsion-free* if and only if M is torsion-free and reduced, and $\operatorname{Hom}(\hat{R}, M) = 0$.

If A is an R-algebra with cotorsion-free R-module structure and λ as above, then we find R-modules G_i, $i \in 2^\lambda$, such that $\operatorname{End} G_i = A$ *and* $\operatorname{Hom}(G_i, G_j) = 0$ *for all* $i \neq j \in 2^\lambda$.

If we now choose the algebra A properly, we get the desired results mentioned at the beginning.

Suppose R is a valuation domain and $S = R - 0$. If R is complete (or a field), then the indecomposable R-modules have rank 1. Hence we assume that R is neither a field nor a complete domain. From (5.8) we conclude that R is a cotorsion-free R-module. For cotorsion-free domains, and $\lambda = \lambda^{\kappa(R)} \geqq |R|$ we may apply the strong realization theorem for $R = A$ to obtain the following result.

There exists a rigid system of 2^λ indecomposable, cotorsion-free R-modules of cardinality λ; and λ is in a proper class of cardinals.

This answers Fuchs' question above; for a history of this problem (in particular for $R = \mathbb{Z}$) see [5, 8]. Next we assume R not a field (otherwise the following result fails by results on Linear Algebra). In [3, 4] we find an R-algebra with free R-module structure which can be applied to the split-realization theorem. Using arguments as in [7] if R is complete (or applying the strong realization theorem if R is not complete), the given algebra gives rise to the following.

For any abelian semi-group Γ there exist R-modules G_γ $(\gamma \in \Gamma)$ which are torsion-free and reduced such that for

$$\alpha, \beta, \gamma \in \Gamma, \quad G_\alpha \oplus G_\beta = G_\gamma \quad \textit{if and only if} \quad \alpha + \beta = \gamma.$$

For $R = \mathbb{Z}$ and the special $\Gamma = \mathbb{Z}/n\mathbb{Z}$ this may be found in [8], Vol. II, p. 145, Theorem 91.6.

Another remarkable class of modules can be constructed as follows. Assume that R is an incomplete domain and not a field; and apply the strong realization theorem to Corner's algebra [2] extended naturally to arbitrary domains; see also [8], Vol. II, p. 145. We conclude.

There are R-modules G of cardinality λ (λ as above) which do not have any indecomposable direct summands $\neq 0$.

The techniques of the proof of our main theorem are similar to methodes of [4, 6] and [17], and we shall follow the exposition given in [4]. However we shall need some substantial changes in order to deal with non-metrizable topologies.

§ 2. Preliminaries

We shall prove our results in an even more general category of rings than that of domains. Throughout this paper R will be a non-zero commutative ring with 1 and S a subset of non-zero divisors subject to the following conditions

(1) $\{Rs: s\in S\}$ is a descending chain with $\bigcap_S Rs=0$,

(2) $1\in S$ and $|S|$ is a regular cardinal κ.

Condition (2) is no restriction for domains, but it will be convenient; in fact (2) follows from

(2*) S satisfies the intersection property, i.e.

If $X\subseteq S$, $|X|<cf|S|$, then $S\cap\bigcap_X Rx\neq 0$. Suppose $\kappa=cf|S|$, then we find a filtration $S=\bigcup_{\alpha\in\kappa} S_\alpha$ such that $|S_\alpha|<|S|$. By (2*) we have $s_\alpha\in S\cap\bigcap_{S_\alpha} Rs$ and therefore $\{1, s_\alpha\in S;\ \alpha\in\kappa\}$ will satisfy (1) and (2). Recall that (2*) replaces the standard assumption that S is multiplicatively closed. If R is a valuation domain, then (1) is automatic; more generally (1) and (2) can be arranged for *any* domain which is not a field, without loss of generality we shall assume that S is multiplicatively closed.

A module M over R is called *S-reduced*, if $\bigcap_S Ms=0$. The set S induces the natural S-topology on M, taking $\{Ms: s\in S\}$ as a basis of neighbourhoods of $0\in M$. Since $\bigcap_S Ms=0$, the S-topology is Hausdorff and we may consider the *completion* $\hat{M}$ of M in its S-topology. Furthermore, M is called *S-torsion-free* if $ms=0$ for $m\in M$, $s\in S$ implies $m=0$. A module M is *S-cotorsion-free*, if and only if M is S-reduced, S-torsion-free and $\mathrm{Hom}(\hat{R}, M)=0$; cf. [11, 12] and (2.4)(c).

Let A be a fixed R-algebra such that the R-module A is S-reduced and S-torsion-free for such an S. Then we will fix some cardinal λ with $\lambda^\kappa=\lambda\geq|A|$ and choose a sequence $\{a_i\in S,\ i\in\kappa\}$ representing a strictly descending chain of ideals Aa_i of A with $\bigcap_{i\in\kappa} Aa_i=0$. It follows from König's Theorem (cf. [13], pp. 45, 46) that

(2.0) $cf(\lambda)>\kappa$.

Hence, every subset of λ of cardinality $\leq\kappa$ has an upper bound in λ. Since S is fixed, we will now omit the prefixed S.

We are interested in torsion-free, reduced R-modules M; therefore we first recall some of their basic properties from [14, 15]:

(2.1) We can identify $\hat{M}$ with a submodule of $\prod_S M/Ms$, in fact

$$\hat{x} = \sum_{s \in S} x_s + Ms \in \hat{M} \quad \text{if and only if } x_s - x_{st} \in Ms \text{ for } s, t \in S.$$

We shall identify M with a submodule of $\hat{M}$, in the sense $M \subset \hat{M}$ that $x \to \sum_{s \in S} x + Ms$.

(2.2) (a) $\hat{M}$ is torsion-free and reduced
(b) $\hat{M}/M$ is torsion-free and divisible (S-divisible in the obvious sense).

Remark. Use (2.1) to prove (2.2).

(2.3) **Definition.** $U \subseteq M$ is *S-pure* if and only if $Us = U \cap Ms$ for all $s \in S$.

If $U \subseteq M$, then $U_* = \bigcap \{X \subseteq M : U \subset X,\ X S\text{-pure in } M\} = \{y \in M : s \in U$ for some $s \in S\}$ = minimal S-pure submodule of M containing U.

(2.4) Suppose R is a domain and the S-topology is the same as the R-topology (induced by $R - 0$). Then
(a) M is divisible if and only if M is injective.
(b) If $A \subseteq C$ such that C/A is torsion-free and divisible, then any h: $A \to D$ has a unique extension $\hat{h}$: $C \to D$ provided D is complete.
(c) M is complete if M is cotorsion, i.e. $\mathrm{Ext}^1(Q, M) = 0$ for the quotient field Q of R.

(2.5) If R is a valuation domain, then "all" notions of purity are equivalent. However, we stick to Warfield's RD-purity as in (2.3).

Finally we want to supply some rings for the module categories under discussion. If Γ is an arbitrary totally ordered abelian group and F a field, then we can find a valuation domain R with quotient field Q and a valuation map v: $Q \to \Gamma \cup \{\infty\}$ such that $R = \{a \in Q : v(a) \geqq 0\}$, $P = \{a \in Q,\ v(a) > 0\}$ the maximal ideal of R such that $R/P = F$. All valuation domains arise this way; see [1, 16]. The ideals of R correspond to the sections H of Γ, which are subsets of Γ with ($h \in H$, $h < g \Rightarrow g \in H$). So the chain of the ideals $\neq 0$ of valuation domains may be very long; hence κ above can be large.

The following example illustrates the limitation of our construction; this arises from the fact that the R-topology might be far away from being "linearly ordered" at 0.

Let X denote a set of commuting variables of size $\kappa > \aleph_0$. Then we consider the ring $R = \mathbb{Z}[X]$ of polynomials with coefficients in $\mathbb{Z}$. Take any enumeration of $R \setminus 0 = \{b_i : i < \alpha\}$ and consider a set of commuting variables $Y = \{x_{V,w} : V \subseteq \alpha$, $|V| < \kappa$, w a finite subset of $V\}$. Let R act on Y by $b_i x_{V,w} = x_{V,w \setminus \{i\}}$ for $i \in w \subseteq V$. Finally, construct the R-algebra $A = R[Y]$ which is generated on Y.

Let $S \subseteq R \setminus 0$ be any multiplicatively closed subset of R with $|S| < \kappa$. If $V = \{i < \alpha,\ \exists s \in S,\ s = b_i\}$, then $|V| < \kappa$ and $b_i x_{V,\{i\}} = x_{V,\emptyset}$. Therefore $x_{V,\emptyset} \in \bigcap_S As \neq 0$. Since R has obviously no descending chains of principal ideals of length $\aleph_1$, the algebra A is not S-reduced for a *suitable* S, satisfying condition (1). On the other hand, let S be the multiplicative closure of $\{x_i,\ i<\kappa\}$ in R. If $0 \neq a \in A$, we may write $a = \sum_{i=1}^{n} r_i x_{V(i),w(i)}$ with $0 \neq r_i \in R \setminus 0$, $w(i) \subseteq V(i) \subseteq \alpha$ such that $|V(i)| < \kappa$. Using the relations in A, we may assume that the presentation of a is reduced, i.e. if $i \leqq r$ and $j \in w(i)$, then $b_j \nmid r_i$. The element $r_i x_{V(i),w(i)}$ is only divisible by all b_i with $i \in V(i)$ and b_j with b_j/r_i. The cardinality of these elements is $<\kappa$. Therefore a is divisible by $<\kappa$ elements and $\bigcap_S As = 0$ by $|S| = \kappa$, i.e. A is S-reduced for *some* "bad" $S \subseteq R$. Nevertheless, for a fixed domain R, not a field, there are many algebras including R itself which are S-reduced for some S – even in this case.

Added in Proof (Nov. 1984): Using different techniques, we will derive a Realization Theorem which also covers any "bad" $S \subseteq R$; see a forthcomming part II of this paper (submitted to Fundamenta Mathematicae).

§ 3. Construction of the R-modules

(3.1) *The Tree* $T = {}^{\kappa >}\lambda$

We will consider the tree $T = {}^{\kappa >}\lambda$ consisting of all functions $\tau\colon \nu \to \lambda\ (\nu < \kappa)$ ordered by set theoretical containment, i.e.

$\sigma \leqq \tau$ if and only if $\sigma \subseteq \tau$ if and only if $(\operatorname{dom}\sigma \subseteq \operatorname{dom}\tau$ and $\sigma = \tau \restriction \operatorname{dom}\sigma)$.

Clearly $|T| = \lambda$, for $\tau \in T$ define the length $l(\tau) = \operatorname{dom}\tau \in \kappa$, which has cardinality $<\kappa$. Maximal linearly ordered subsets of T are called *branches*; $Br(T)$ shall denote the set of all branches of T. We also fix a continuous, strictly increasing function

$$\rho\colon cf(\lambda) + 1 \to \lambda + 1$$

such that $\rho(0) = 0$ and $\rho(cf(\lambda)) = \lambda$. This will be used to define norms of the basic module B.

(3.2) *The "Basic Module"* B

We start with the observation that direct sums of cyclic modules might be complete in the S-topology. Hence we shall substitute the "standard basic subgroups" by a more suitable module.

Let $\tau \in T$ be a generator of a cyclic A-module τA with $\operatorname{Ann}(\tau) = 0$. If $C = \prod_{\tau \in T} \tau A$ denotes the cartesian product, then its subset $\{f \in C, |[f]| < \kappa\}$ is the κ-direct sum $B = \bigoplus_{\tau \in T}^{\kappa} \tau A$, which is a pure submodule of C; $[f] = \{\tau \in T\colon f(\tau) \neq 0\}$ denotes the *support* of f. Since A is a reduced R-module, also B is reduced and we can construct its completion $\hat{B}$. According to § 2 we have that $B \subset \hat{B}$ are reduced and torsion-free and $\hat{B}/B$ is torsion-free and divisible. We call B *the basic submodule* and the modules G we want to construct will satisfy $B \subseteq G \subseteq \hat{B}$. First we want to know how the elements in $\hat{B}$ look: If $g \in \hat{B}$, then $g_\tau \in \prod_{\tau \in T} \tau \hat{A}$ and we may write $g = \sum_{\tau \in [g]} \tau g_\tau$ with $g_\tau \in \hat{A}$ and $[g]$ the support of g as above. Then we have from the definition of B

$$(3.2)(a)\quad g=\sum_{\tau\in[g]}\tau g_\tau\in\hat{B}\Leftrightarrow\begin{cases}\text{(i) } g_\tau\in\hat{A} \quad \text{for all } \tau\in[g]\\ \text{(ii) If } s\in S, \text{ then there exists a set } X\subseteq[g] \text{ with } |X|<\kappa\\ \quad\ \text{and } g_\tau\in s\hat{A} \text{ for all } \tau\in[g]\setminus X.\end{cases}$$

This can be seen as follows:

If $g_s=g\restriction\{\tau\in T, g_\tau\notin s\hat{A}\}$, then $|[g_s]|<\kappa$ and $g_s\in\bigoplus_T^\kappa\tau\hat{A}$. Then $(g-g_s)\underset{S}{\to}0$ and $g=\lim_S g_s\in\hat{B}$.

If $X\subseteq\hat{B}$, we let $[X]=\bigcup\{[g]:g\in X\}=\{\tau\in T:\exists g\in X \text{ with } g_\tau\neq 0\}$.

We define the norm $\|g\|=\min\{\nu\leqq cf(\lambda), [g]\subseteq^{\kappa>}\rho(\nu)\}$ and similarly $\|X\|=\sup\{\|g\|:g\in X\}$. Since $|[g]|\leqq\kappa$ from (3.2)(a), we have $\|X\|\in cf(\lambda)$ for all $|X|\leqq\kappa$ from (2.6).

The following convention will be used quite often.

(3.2)(b) If $X\subseteq T$, $\nu<cf(\lambda)$, let ${}_\nu X=\{\tau\in X:\|\tau\|>\nu\}$.

(3.3) *Canonical Submodules*

This concept is used to find modules of sufficiently small cardinality. For finding pretty large modules we could also adopt the construction in [5] and then we could omit canonical submodules. If $\tau\in T$, the submodules of τA purely generated by at most κ elements are called *canonical* submodules P_τ. If $Y\subseteq T$ and $|Y|\leqq\kappa$ and P_τ $(\tau\in Y)$ are canonical submodules, then

$$P=\bigoplus_{\tau\in Y}^{\kappa}P_\tau\subseteq B \quad \text{is a } \textit{canonical submodules} \text{ as well.}$$

We have the following immediate consequences:

(a) If P is a canonical submodule, $X\subseteq B$, $|X|\leqq\kappa$, then there exists a canonical submodule P' such that $P\cup X\subseteq P'$.

(b) If P is a canonical submodule, then $|[P]|\leqq\kappa$.

(c) The set of canonical submodules is closed under unions of chains.

(3.4) *Divisibility Chains*

In order to determine pure submodules explicitly, we use the

Definition. $g^s\in\hat{B}$ $(s\in S)$ is a *divisibility chain* of $g=g^1$ if and only if

(a) $g^s-t\,g^{st}\in B$ for all $s,t\in S$

(b) If $s\in S$, there exists $\nu<\|g^s\|$ such that ${}_\nu[g^s]\subseteq[g]$.

We have the following Lemma to ensure the existence of divisibility chains.

Lemma. (a) *If P is a canonical submodule of B and $g\in\hat{P}$, then there exists a divisibility chain $g^s\in\hat{P}$ $(s\in S)$ of $g=g^1$.*

(b) *If $v\in Br(T)$ and $v^s=\sum_{\sigma\in v,\,a_{l(\sigma)}\in Rs}a_{l(\sigma)}s_\sigma^{-1}$ for $s\in S$, then v^s $(s\in S)$ is a divisibility chain of v^1.*

Proof. (a) The canonical submodule P is designed to ensure $g^s\in\hat{P}$ if $g\in\hat{P}$. It is enough to construct $g^s\in\hat{B}$. Let $g=\sum\tau g_\tau$, $s\in S$ and $X=\{\tau\in T:\ g_\tau\notin s\hat{A}\}$. Then $|X|<\kappa$ from (3.2)(a), and since A is dense in $\hat{A}$ [(2.2)(b)] we find $g_\tau^s\in\hat{A}$, $g_\tau'\in A$ such that $g_\tau=s\,g_\tau^s+g_\tau'$ for all $\tau\in X$. If $g'=\sum_X\tau\,g_\tau'$, then $g'\in B$ from $|X|<\kappa$ and the

Definition (3.2) of B. If $\tau\in Y=T\setminus X$, then $g_\tau=s g_\tau^s$ for some $g_\tau^s\in A$. Then $g^s=\sum \tau g_\tau^s$ is an element of B and

$$g-sg^s=\sum \tau g_\tau \sum \tau s g_\tau^s=\sum_Y \tau s g_\tau^s+\sum_X \tau s g_\tau^s+\sum_X \tau g_\tau'-\sum \tau s g_\tau^s=g'\in B.$$

If $s,t\in S$ and also $g-st g^{st}\in B$ by definition of g^{st}, then $s(g^s-t g^{st})\in B$ and also $g^s-t g^{st}\in B$ by purity; the relation $[g^s]\subseteq[g]$ is immediate.

(b) Is straight forward.

Remark. If $g\notin B$, then $\langle g^s+B\colon s\in S\rangle\cong S^{-1}R$.

Lemma (a) can also be derived from (2.2); however the proof given allows natural generalizations.

(3.5) *The Combinatorial Black Box*

This is also needed to decrease the size of our final modules.

Definition. (f,P,φ) is called a *trap* if $f\colon {}^{\kappa>}\kappa\to{}^{\kappa>}\lambda$ is a tree embedding, P a canonical submodule with $\varphi\in\mathrm{End}_R P$ and the following properties,

(a) $\mathrm{Im} f\subseteq P$,

(b) $[P]\subseteq P$ and $\bigoplus^\kappa[P]\subseteq P$,

(c) $[P]$ is subtree of T,

(d) $cf\|P\|=\kappa$,

(e) $\|v\|=\|P\|$ for all $v\in\mathrm{Br}(\mathrm{Im} f)$.

The following Lemma can be derived from [4], Appendix, with a few minor changes. It is based on [17].

Black Box. There are an ordinal λ^* and traps $(f_\alpha,P_\alpha,\varphi_\alpha)$ for all $\alpha<\lambda^*$ with the following properties:

(i) If $\beta\leqq\alpha<\lambda^*$, then $\|P_\beta\|\leqq P_\alpha\|$.

(ii) If β, $\alpha<\lambda^*$ are different, then $\mathrm{Br}(\mathrm{Im} f_\alpha\cap\mathrm{Im} f_\beta)=\emptyset$.

(iii) If $\beta+2^\kappa\leqq\alpha<\lambda^*$, then $\mathrm{Br}(\mathrm{Im} f_\alpha\cap[P_\beta])=\emptyset$.

(iv) If $X\subseteq\hat B$, $|X|\leqq\kappa$ and $\varphi\in\mathrm{End}\,\hat B$, there exists an $\alpha<\lambda^*$ which catches X, φ i.e. $X\subseteq\hat P_\alpha$, $\|X\|<\|P_\alpha\|$ and $\varphi\restriction P_\alpha=\varphi_\alpha$.

§ 4. Construction of the Module G

(4.1) *Construction.* We want G to be the union of a continuous ascending chain of submodules G_α $(\alpha<\lambda^*)$ with λ^* from the Black Box (3.5). In addition, we define elements $b_\alpha\in\hat B\cup\{\infty\}$ such that

(i) $b_\beta\notin G_\alpha$ for all $\beta<\alpha<\lambda^*$

subject to various conditions, depending on the transfinite construction of the submodules G_α.

We let $G_0=B$ and assume that G_α for $\alpha<\mu$ and b_α for $\alpha+1<\mu$ are constructed for some $\mu<\lambda^*$. If μ is a limit cardinal, then $G_\mu=\bigcup_{\alpha<\mu} G_\alpha$ and (i) holds. If $\mu=\alpha+1$, then we consider three cases:

(A) Suppose we find $v_\alpha \in \mathrm{Br}(\mathrm{Im}\, f_\alpha)$ and $g_\alpha \in \hat{P}_\alpha$ such that

$$\|g_\alpha^s - v_\alpha^s\| < \|P_\alpha\| (= \|v_\alpha\|) \quad (s \in S)$$

and $b_\alpha = g_\alpha \varphi_\alpha$ with $b_\beta \notin G_{\alpha+1} + \sum_S g_\alpha^s A$ for all $\beta < \alpha + 1$, then we choose these elements and call α a *strong ordinal.*

(B) Suppose (A) is not possible, but we find $v_\alpha \in \mathrm{Br}(\mathrm{Im}\, f_\alpha)$ and $g_\alpha \in \hat{P}_\alpha$ such that $\|g_\alpha^s - v_\alpha^s\| < \|v_\alpha\|$ $(s \in S)$, then we choose these elements and $G_{\alpha+1} = G_\alpha + \sum_S g_\alpha^s A$, $b_\alpha = \infty$ and call α *weak.*

(C) If (A) and (B) are not possible, we let $g_\alpha = 0$, $b_\alpha = \infty$ and $G_{\alpha+1} = G_\alpha$ and call α *useless.* □

Remark. We shall show in (4.4) that there are no useless ordinals!

If $g \in G$, then

$$g = b + x + \sum_{i \leq n} g_{\alpha(i)}^s a_i \qquad (b \in B,\ \|x\| < \|P_\alpha\|,\ \|g_{\alpha(i)}^s\| = \|P_\alpha\|)$$

if α is the least ordinal with $g \in G_{\alpha+1} \setminus G_\alpha$. If $b \in B$, then $|[b]| < \kappa$; and since $cf\, \|P_\alpha\| = \kappa$ by (3.5), there is no element in B with norm $\|P_\alpha\|$. Hence, the representation of g is plausible. We summarize its properties in the

(4.2) **Recognition Lemma.** *Let $g \in G \setminus B$, then*

(a) *there is a unique $\alpha < \lambda^*$ with $g \in G_{\alpha+1} \setminus G_\alpha$;*

(b) *there exist $\alpha(n) < \ldots < \alpha(0) = \alpha$ with $\|P_{\alpha(i)}\| = \|P_\alpha\|$, $v < \|P_\alpha\|$ such that ${}_v[g] = F \cup \bigcup_{i \leq n} {}_v[V_{\alpha(i)}]$ a disjoint union with $F \subseteq T$, $|F| < \kappa$ and each element of F has norm $> \|P_\alpha\|$;*

(c) *if $\beta < \lambda^*$ and $\|P_\beta\| = \|P_\alpha\|$, there exist $a \in A$, $s \in S$ such that for κ many $\sigma \in v_\alpha$; $g \restriction \sigma = \sigma\, a_{l(\sigma)}\, s^{-1} a$.*

(4.3) **Lemma.** *Let $\alpha < \lambda^*$, $v < \|P_\alpha\|$ such that for each $v \in \mathrm{Br}(\mathrm{Im}\, f_\alpha)$ there is a divisibility chain $g_v^s \in \hat{B}$ with ${}_v[g_v^s - v^s] = \emptyset$ for all $s \in S$. Then there exists $v \in \mathrm{Br}(\mathrm{Im}\, f_\alpha)$ such that $b_\beta \notin G_{\alpha+1}(v)$ $(\beta < \alpha)$ where $G_{\alpha+1}(v) = G_\alpha + \sum_S g_v^s A$.*

The Lemma has the immediate

(4.4) **Corollary.** *There are no useless ordinals.*

The proof of (4.3) is very similar to [4] (Lemma 3.9). Since it is also crucial in the proof of our main theorem, we sketch it: If (4.3) does not hold, then for each $v \in \mathrm{Br}(\mathrm{Im}\, f_\alpha)$ there is $\beta = \beta(v) < \alpha$ such that $b_\beta \in G_{\alpha+1}(v)$. Consequently $b_\beta = g_\beta \varphi_\beta \in P_\beta$; and we find $a = a(v) \in A$, $s = s(v) \in S$ with $b_\beta - g_v^s a \in G_\alpha$. Since $b_\beta \notin G_\alpha$, also $a \neq 0$ and ${}_v[g_v^s a] \subseteq v$ are sets of cardinality κ. By the Black Box (3.5)(ii) and the Recognition Lemma (4.2) we find a subset of size κ of v to be contained in $[b_\beta] \subseteq [P_\beta]$. By the Trap Definition (3.5)(c) we conclude $v \subseteq [P_\beta]$; and (iii) of the Black Box implies $\beta < \alpha < \beta + 2^\kappa$. If β_0 is the least of the $\beta = \beta(v)$, then $\beta_0 \leq \beta < \beta_0 + 2^\kappa$ and $|\mathrm{Br}(\mathrm{Im}\, f_\alpha)| = 2^\kappa$ implies that there are two distinct branches v, $w \in \mathrm{Br}(\mathrm{Im}\, f_\alpha)$ with $\beta = \beta(v) = \beta(w)$. Subtracting $b_{\beta(v)} - g_v^{s(v)} a(v) \in G_\alpha$ and $b_{\beta(w)} - g_w^s a(w) \in G_\alpha$, we have $g_v^s a(v) - g_w^{s(w)} a(w) \in G_\alpha$, which is impossible by the supports of the summands. □

§ 5. Proof of the Theorem

We will distinguish some subsets of the tree $T = {}^{\kappa >}\lambda$.

(5.1) **Definition.** For $\eta \in \lambda$, let $w(\eta)$ be the *constant branch* of η which is the set of all functions ${}^{\kappa >}\{\eta\}$. A set $\{\sigma_\nu \in T, \nu < \kappa\}$ is an *antibranch* if and only if $l(\sigma_\nu) = \nu$, $\|\sigma_\nu\| < \|\sigma_\mu\|$ but $\{\sigma_\nu, \sigma_\mu\}$ are not comparable in T for all $\nu < \mu < \kappa$.

The following two algebraic definitions are relevant in this section.

(5.2) *Convention.* $\Delta = \{(a,s) \in A \times S : Rs = R \text{ or } a \notin sA\}$. Compare [6].

(5.3) **Definition.** Let M, M' be reduced R-modules; then $\operatorname{Ines}(M,M') = \{\sigma \in \operatorname{Hom}(M,M'), \hat{\sigma}(\hat{M}) \subseteq M'\}$, where $\hat{M}$ is the completion of M and $\hat{\sigma}$ is the unique extension of σ. We shall identify σ and $\hat{\sigma}$. In particular $\operatorname{Ines} M = \operatorname{Ines}(M,M)$ is a two-sided ideal of $\operatorname{End} M$; compare [5].

We have the following crucial

(5.4) **Lemma.** *Let* $\varphi \in \operatorname{End} \hat{B} \setminus A \oplus \operatorname{Ines} G$.

(a) *There exists a canonical submodule* $P \subseteq B$ *such that*

$$\hat{P}(s\varphi - a) \nsubseteq G \quad \text{for all } (a, s) \in \Delta.$$

(b) *There are a canonical submodule* $P^* \subseteq B$ *and a divisibility chain* $g^s \in P^*$ $(s \in S)$ *such that*

$$g\varphi \notin G + \sum_s g^s A.$$

Proof. (a) We consider first the case $(a, s) \in \Delta$ with $Rs = R$.

Let P' be a canonical submodule of B which contains an element $w^1 \in \hat{P}'$ for some constant branch $w = w(\eta)$. Suppose $P = P'$ does not satisfy (a) under the restriction $Rs = R$. Then we find $a' \in A$ such that $\hat{P}'(s\varphi - a') \subseteq G$. If $sr = 1$, $a = ra' \in A$, this implies $\hat{P}'(\varphi - a) \subseteq G$. Since $\varphi - a \notin \operatorname{Ines} G$, there exists $x \in \hat{B}$ such that $x(\varphi - a) \notin G$. Take any canonical submodule $P \supseteq P'$ with $x \in \hat{P}'$ – cf. (3.3)(a).

Suppose also $\hat{P}(s\varphi - b') \subseteq G$ with $Rs = R$. Then as above $\hat{P}(\varphi - b) \subseteq G$ for some $b \in A$. Subtracting leads to $\hat{P}'(a - b) \subseteq G$. Since $w^1 \in \hat{P}'$, also $w^1(a-b) \in G$ and $a = b$ by the Recognition Lemma (4.2). We derive $x(\varphi - b) = x(\varphi - a) \in G$, a contradiction. Next we consider *any* $s \in S$ and let $a_s \in A$ be so that

$$(a_s, s) \in \Delta \quad \text{and} \quad \hat{P}'(s\varphi - a_s) \subseteq G$$

where P' is chosen as before. We shall fix s and construct a converging sequence in $\hat{P}$ such that its image under $\psi = s\varphi - a_s$ contains an antibranch. The sequence will converge rapidly compared with s and the S-topology. First we define a sequence $c_\nu \in S$ $(\nu \in \kappa)$ by transfinite induction. Let $c_0 = 1$ and assume $c_\alpha \in S$ to be defined for all $\alpha < \nu$.

If ν is a limit $< \kappa$, we take any $c_\nu \in S$ with $c_\nu \in \bigcap_{\alpha < \nu} Rc_\alpha \cap Ra_\nu$ and if

$$\alpha + 1 = \nu, \qquad \text{let } c_\nu = c_\alpha \cdot a_\nu \cdot s.$$

Next we define inductively an antibranch consisting of elements $\sigma_\nu \in T$ and sums

(i) $h^\nu = \sum_{\alpha<\nu} c_\alpha \sigma_\alpha \in B$

subject to the following conditions

(ii) $\|h^\nu \psi\| < \|\sigma_\nu\|$

(iii) No two of the σ_α $(\alpha<\nu)$ are comparable in T.

In order to satisfy (iii), we let $\sigma_\alpha(0)=\alpha$. Choose $\sigma_0=(0)$ and assume that h^α, σ_α $(\alpha<\nu)$ are constructed. If ν is a limit, h^ν is defined by (i). Since $cf(\lambda)>\kappa$ (cf. (2.0)), $\|h^\nu \psi\| = \mu \in cf(\lambda)$ and we let $\sigma_\nu(1)=\rho^{-1}(\mu+1)$; cf. (3.1). The other values of σ_ν can be taken arbitrarily. Therefore $\|h^\nu \psi\| < \mu+1 \leqq \|\sigma_\nu\|$ and (ii) holds. If $\nu = \alpha+1$, we can choose σ_ν similarly such that h^ν, σ_ν satisfy (ii) and (iii). Finally let $h = \lim_{\nu\to\kappa} h^\nu = \sum_{\alpha\in\kappa} c_\alpha \sigma_\alpha$. Since c_α $(\alpha\in\kappa)$ is a nullsequence and $h^\nu \in B$, by (3.2)(a) the element h is in $\hat{B}$. If $r^\nu = h - h^{\nu+1}$, then $r^\nu \in s c_\nu \hat{B}$ and $h = h^\nu + c_\nu \sigma_\nu + r^\nu$. Therefore $h \equiv h^\nu + c_\nu \sigma_\nu \bmod s c_\nu \hat{B}$, and $\bmod s c_\nu \hat{B}$:

$$h\psi \equiv h^\nu \psi + c_\nu \sigma_\nu \psi \equiv h^\nu \psi + c_\nu \sigma_\nu s\varphi - c_\nu \sigma_\nu a_s \equiv h^\nu \psi - c_\nu \sigma_\nu a_s.$$

The contribution of $h\psi$ to σ_ν becomes by (ii)

$$h\psi \restriction \sigma_\nu \equiv -c_\nu a_s \sigma_\nu \bmod s c_\nu \hat{B}.$$

Suppose $h\psi \restriction \sigma_\nu \equiv 0 \bmod s c_\nu \hat{B}$; then $c_\nu a_s \in s c_\nu \hat{A}$ and we find $c \in \hat{A}$ such that $c_\nu a_s = s c_\nu c$ and equivalently $c_\nu(a_s - s c) = 0$. Since $c_\nu \in S$ and $\hat{A}$ is torsion-free by (2.2), also $a_s - s c = 0$, which implies $a_s \in s\hat{A} \cap A = sA$ by purity. Therefore $(a_s, s) \notin \Delta$, a contradiction. Hence $\sigma_\nu \in [h\psi]$ and $[h\psi]$ contains an antibranch of norm $\|h\psi\|$. The Recognition Lemma (4.2) implies $h\psi \notin G$.

Such an $h = h_s$ can be contructed for any $s \in S$; i.e.

(iv) $h_s \in \hat{B}$, $h_s(s\varphi - a_s) \notin G$.

Since $|S| = \kappa$, we find a canonical submodule P of B with $P' \subseteq P$ and $h_s \in \hat{P}$ for all $s \in S$; see (3.3)(a). Suppose $\hat{P}(s\varphi - b) \subseteq G$ for some $b \in A$. Since $\hat{P}'(s\varphi - a_s) \subseteq G$, also $\hat{P}'(b - a_s) \subseteq G$ and in particular $w(b - a_s) \in G$. The Recognition Lemma (4.2) implies $b = a_s$ and therefore $\hat{P}(s\varphi - a_s) = \hat{P}(s\varphi - b) \subseteq G$. Since $h_s \in \hat{P}$, also $h_s(s\varphi - a_s) \in G$, which contradicts (iv); we conclude (a). □

(b) Let P be as in (a) and pick any $\eta < \lambda$ such that $\|P\| < \eta$, $\|P\varphi\| < \eta$; which is possible by (2.0). Let $w = w(\eta)$ be the constant branch of η. Any divisibility chain is contained in the completion of some canonical submodule. So we have to concentrate on the second condition of (b) only. If $g = w^1$ does not satisfy (b), we find some $a \in A$, $s \in S$ with

$$w^1 \varphi - w^s a \in G. \tag{$*$}$$

If $(a, s) \notin \Delta$, then $a \in sA$ and we may write $a = s a'$. Then $B \subseteq G$ and $(*)$ imply $w^1 \varphi - w^1 a' \in G$ and certainly $(a', 1) \in \Delta$. Hence we may assume $(a, s) \in \Delta$. From (5.4)(a) we find some $h \in \hat{P}$ with

$$h(s\varphi - a) \notin G. \tag{$**$}$$

We claim that $g=w^1+h$ satisfies (b). If this is not the case, then

$$w^1\varphi+h\varphi-w^t a'-h^t a'=g\varphi-g^t a'\in G \quad \text{for some } a'\in A,\ t\in S.$$

We obtain from (*) that $(h\varphi-h^t a')+(w^s a-w^t a')\in G$.

Assume without restriction $t=s\times r$ for some $r\in R$. Then $(h\varphi-h^t a')+w^t(a'-r a)\in G$. Comparing supports, we derive $a'=r a$ and $h\varphi-h^t ra=h\varphi-h^t a'\in G$. Multiplication by s leads to

$$hs\varphi-h^t sra=hs\varphi-(h^t t)a\in G \quad \text{and} \quad h(s\varphi-a)\in G, \quad \text{contradicting } (**).$$

(5.5) **Theorem.** *If A is an S-reduced and S-torsion-free R-algebra and the R-module as in* (4.1), *then* $\operatorname{End} G=A\oplus \operatorname{Ines} G$.

Proof. Since $A\oplus\operatorname{Ines} G\subseteq \operatorname{End} G$ holds trivially, we consider any $\varphi\in\operatorname{End} G\backslash A\oplus\operatorname{Ines} G$ and want to derive a contradiction.

From (5.4)(b) we find some $g\in\hat{B}$ with

$$(*) \qquad g\varphi\notin G+\sum_S g^s A$$

and from the Black Box (3.5) we have some $\alpha<\lambda^*$ such that g, $g\varphi\in\hat{P}_\alpha$,

$$\sup_S \{\|g^s\|, \|g^s\varphi\| : s\in S\}<\|P_\alpha\| \quad \text{and} \quad \varphi\restriction P_\alpha=\varphi_\alpha.$$

If α is a strong ordinal, then $g_\alpha\varphi=g_\alpha\varphi_\alpha=b_\alpha\notin G$ by construction (4.1) of G.

Since $g_\alpha\in G$, we conclude $\varphi\notin\operatorname{End} G$.

Therefore it is enough to show that α is strong:

Let $v\in\operatorname{Br}(\operatorname{Im} f_\alpha)$ and $\varepsilon\in\{0, 1\}$. Suppose there exist

$$a_\varepsilon\in A,\ s\in S \quad \text{with} \quad (v^1+\varepsilon g^s)\varphi-(v^s+\varepsilon g^s)a_\varepsilon\in G_\alpha,$$

then $(g\varphi-g^s a_1)+v^s(a_0-a_1)\in G_\alpha$. This is only possible if $a_0=a_1$ and therefore $g\varphi-g^s a_0\in G_\alpha$ contradicting (*).

Therefore we found $\varepsilon=\varepsilon_v$ and $g_v=v^1+\varepsilon_v g$ such that $g_v\varphi_\alpha\notin G+\sum_S g_v^s A$ which allow us to apply (4.3). So one of the branches v implies $b_\beta\notin G_\alpha+\sum_S g_v^s A$ and α is strong. □

Recall the definition of "cotorsion-free" from §2; a module M has this property if and only if M is torsion-free, reduced and $\operatorname{Hom}(\hat{R}, M)=0$. Then we have the

(5.6) **Proposition.** *A is cotorsion-free if and only if G is cotorsion-free.*

Proof (this is the same as in [4] (Lemma 6.2). We shall only sketch it here). The relevant part is to consider any non-zero homorphism φ: $\hat{R}\to G$. We let $g=1\varphi$. If $g\in B$, then also $\hat{R}\varphi\subseteq B$. Since B is obviously cotorsion-free, we have $\varphi=0$. Therefore $g\notin B$ and we find $\alpha<\lambda^*$ such that $g\in G_{\alpha+1}\backslash G_\alpha$. We may assume $g-g_\alpha a\in G_\alpha$ for some $a\in A$. Now it is easy to see that $rg-g_\alpha a'\in G_\alpha$ for $r\in\hat{R}$, which induces a non-zero homomorphism $\hat{R}\to A\,(r\to a')$. □

From (5.5) and (5.6) we have the immediate Corollary 5.7.

(5.7) **Corollary.** *Let A be a cotorsion-free R-algebra and λ cardinal $\geqq |A|$ with $\lambda^{\kappa(R)}=\lambda$. Then we can find R-modules $G_i (i \in 2^\lambda)$ of cardinality λ such that* $\mathrm{Hom}(G_i, G_j)=0$ *and* $\mathrm{End}\, G_i = A$ *for all* $i \neq j \in 2^\lambda$.

Proof. If $\sigma \in \mathrm{Ines}\, G_i$, then $\hat{\sigma}$: $\hat{B} \to G_i$, and this extends to an homorphism γ of a free R – resolution of $\hat{B}$ into G_i. Since G_i is cotorsion-free by (5.6), we have $\gamma = 0$. Therefore $\mathrm{End}\, G_i = A$ follows from (5.5). The existence of a rigid system follows similarly, using arguments from [4].

We shall close with a few applications of our results. In order to apply (5.7) the following observation will be quite useful. We consider R as an R-module.

(5.8) **Proposition.** *Let R be a domain and $S = R \setminus \{0\}$. Then the following conditions on R are equivalent:*

(1) *R is neither a field nor a complete ring,*
(2) *R is cotorsion-free.*

Proof. If R is a field, then R is certainly not reduced, hence not cotorsion-free. If R is complete, then $0 \neq \mathrm{id} \in \mathrm{Hom}(\hat{R}, R)$ and R is not cotorsion-free. Therefore suppose that R is not cotorsion-free. If R is not reduced, we find a (maximal) divisible submodule $0 \neq D \subseteq R$, which has a complement $C \subseteq R$ by (2.4)(a). Choose $0 \neq d \in D$ and any $c \in C$; then $dc \in C \cap D = 0$ and c is a zero-divisor, which implies $C = 0$ and $D = R$ is a field. We now assume that R is reduced and we can build the completion $\hat{R}$. By assumption $\mathrm{Hom}(\hat{R}, R) \neq 0$; and let $0 \neq B = \hat{R}\sigma \subseteq R$ some image of $\hat{R}$ in R. From (2.4)(a) we have B cotorsion. If $0 \neq b \in B$, then $A = R/B$ is bounded by b, hence strongly cotorsion (in the sense of E. Matlis [1, p. 4]) and therefore cotorsion. Extensions of cotorsion modules by cotorsion modules are cotorsion; cf. [15], p. 5, Theorem 1.5(1). Since $0 \to B \to R \to A \to 0$, also R is cotorsion. Therefore R is complete by (2.4)(c). □

If R is a complete domain or a field, then indecomposable modules (vector spaces) have rank 1. Therefore, we naturally assume that R is neither a field nor a complete domain. For valuation domains with $S = R \setminus 0$, this is equivalent to saying that R is cotorsion-free; cf. (5.8). Realizing $A = R$ we obtain a rigid system of R-modules G_i with $\mathrm{End}\, G_i = R$ from (5.7).

Since R is a domain, R has only trivial idempotents. Consequently G_i is an indecomposable R-module. If we want a rigid system over an arbitrary incomplete ring $R \neq 0$, we simply localize R at some prime P so that R_P is still incomplete, not a field and apply the last result.

For the second application mentioned in §1 we only recall that the R-algebras A constructed in [3], [4] have a free R-module structure. Hence A is cortorsion-free iff R is cotorsion-free. If R is complete, we argue similarly to [7], since A can be chosen $\aleph_0$-cotorsion-free in the sense of [7], see also [4].

References

1. Bourbaki, N.: Commutative Algebra. Reading, MA: Addison-Wesley 1972
2. Corner, A.L.S.: Every countable reduced torsion-free ring is an endomorphism ring. Proc. London Math. Soc. **13**, 688–710 (1963)

3. Corner, A.L.S.: A generalized Leavitt-theorem, unpubl. manuscript see also: Corner, Additive categories and a theorem of W.G. Leavitt. Bull. Amer. Math. Soc. **75**, 78–82 (1969)
4. Corner, A.L.S., Göbel, R.: Prescribing Endomorphism Algebras – A unified Treatment. To appear in Proc. Lond. Math. Soc.
5. Dugas, M., Göbel, R.: Every cotorsion-free algebra is an endomorphism algebra. Math. Z. **181**, 451–470 (1982)
6. Dugas, M., Göbel, R.: Endomorphism algebras of torsion modules II, pp. 400–411, Abelian Group Theory, Proceedings, Honolulu 1982/3. Springer Lecture Notes in Mathematics **1006** (1983)
7. Dugas, M., Göbel, R., Goldsmith, B.: Representation of Algebras over a Complete Discrete Valuation Ring. Quart. Journ. Math. (Oxford) **35**, 131-146 (1984)
8. Fuchs, L.: Infinite Abelian Groups, Vol. I and Vol. II. New York: Academic Press 1970 and 1971
9. Fuchs, L.: Modules over valuation domains. Vorlesungen aus dem Fachbereich Mathematik der Universität Essen, Vol. 9 (1983) Essen
10. Fuchs, L., Salce, L.: Modules over valuation domains. Text book, in preparation
11. Göbel, R.: On stout and slender groups. J. Algebra **35**, 39–55 (1975)
12. Göbel, R., Wald, B.: Wachstumstypen und schlanke Gruppen. Symposia Mathematica **23**, 201–239 (1979)
13. Jech, T.: Set Theory. New York: Academic Press 1978
14. Matlis, E.: Cotorsion Modules. Memoirs AMS **49** (1964)
15. Matlis, E.: 1-Dimensional Cohen-Macauley Rings. Springer Lecture Notes in Mathematics **327** (1973)
16. Nagata, M.: Local Rings. New York: Interscience Publ. 1962
17. Shelah, S.: A combinatorial principle and endomorphism rings of p-groups. To appear in Israel Journ. Math.

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