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A ZFC DOWKER SPACE IN $\aleph_{\omega+1}$: AN APPLICATION OF PCF THEORY TO TOPOLOGY

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Abstract. The existence of a Dowker space of cardinality $\aleph_{\omega+1}$ and weight $\aleph_{\omega+1}$ is proved in ZFC using pcf theory.

1. Introduction

A Dowker space is a normal Hausdorff topological space whose product with the unit interval is not normal. The problem of existence of such spaces was raised by C. H. Dowker in 1951. C. H. Dowker characterized Dowker spaces as normal Hausdorff and not countably paracompact [4].

Exactly two Dowker spaces were constructed in ZFC prior to the construction of the space below. The existence of a Dowker space in ZFC was first proved by M. E. Rudin in 1971 [6], and her space was the only known Dowker space in ZFC for over two decades. Rudin's space is a subspace of $\prod_{n\geq 1}(\aleph_n + 1)$ and has cardinality $\aleph_\omega^{\aleph_0}$. The problem of finding a Dowker space of smaller cardinality in ZFC was referred to as the "small Dowker space problem".

Z. T. Balogh constructed [1] a Dowker space in ZFC whose cardinality is $2^{\aleph_0}$. Another, screenable, Dowker space of size $2^{\aleph_0}$ was constructed by Balogh recently [2].

While both Rudin's and Balogh's spaces are constructed in ZFC, their respective cardinalities are not decided in ZFC, as is well known by the independence results of P. Cohen: both $2^{\aleph_0}$ and $\aleph_\omega^{\aleph_0}$ have no bound in ZFC (and may be equal to each other).

The problem of which is the first $\aleph_\alpha$ in which ZFC proves the existence of a Dowker space remains thus unanswered by Rudin's and Balogh's results.

In this paper we prove that there is a Dowker space of cardinality $\aleph_{\omega+1}$. A non-exponential bound is thus provided for the cardinality of the smallest ZFC Dowker space. We do this by exhibiting a Dowker subspace of Rudin's space of that cardinality. Our construction avoids the exponent which appears in the cardinality of Rudin's space by working with only a fraction of $\aleph_\omega^{\aleph_0}$. It remains open whether $\aleph_{\omega+1}$ is the *first* cardinal at which there is a ZFC Dowker space.

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We shall describe shortly the cardinal arithmetic developments which enable this result. The next three paragraphs are not necessary for understanding the proofs in this paper.

In the last decade there has been a considerable advance in the understanding of the infinite exponents of singular cardinals, in particular the exponent $\aleph_\omega^{\aleph_0}$. This exponent is the product of two factors: $2^{\aleph_0} \times \mathrm{cf}\langle[\aleph_\omega]^{\aleph_0}, \subseteq\rangle$. The second factor, the cofinality of the partial ordering of inclusion over all countable subsets of $\aleph_\omega$, is the least number of countable subsets of $\aleph_\omega$ needed to cover every countable subset of $\aleph_\omega$; the first factor is the number of subsets of a single countable set. Since $\aleph_\omega^{\aleph_0}$ is the number of countable subsets of $\aleph_\omega$, the equality $\aleph_\omega = 2^{\aleph_0} \times \mathrm{cf}\langle[\aleph_\omega]^{\aleph_0}\rangle$ is clear.

While for $2^{\aleph_0}$ it is consistent with ZFC to equal any cardinal of uncountable cofinality, the second author's work on Cardinal Arithmetic provides a ZFC bound of $\aleph_{\omega_4}$ on the factor $\mathrm{cf}\langle[\aleph_\omega]^{\aleph_0}, \subseteq\rangle$.

This is done by approximating $\mathrm{cf}\langle[\aleph_\omega]^{\aleph_0}, \subseteq\rangle$ by an interval of regular cardinals, whose first element is $\aleph_{\omega+1}$ and whose last element is $\mathrm{cf}\langle[\aleph_\omega]^{\aleph_0}, \subseteq\rangle$, and so that every regular cardinal λ in this interval is the *true cofinality* of a reduced product $\prod B_\lambda / J_{<\lambda}$ of a set $B_\lambda \subseteq \{\aleph_n : n < \omega\}$ modulo an ideal $J_{<\lambda}$ over ω. The theory of reduced products of small sets of regular cardinals, known now as *pcf theory*[1], is used to put a bound of ω_4 on the length of this interval.

Back to topology now, it turns out that the pcf approximations to $\aleph_\omega^{\aleph_0}$ are concrete enough to "commute" with Rudin's construction of a Dowker space. Rudin defines a topology on a subspace of the functions space $\prod_{n>1}(\aleph_n + 1)$. What is gotten by restricting Rudin's definition to the first approximation of $\aleph_\omega^{\aleph_0}$ is a *closed* and *cofinal* Dowker subspace X of the Rudin space X^R of cardinality $\aleph_{\omega+1}$. The fact that X is Dowker follows, actually, from its closure and cofinality in X^R.

Hardly any background is needed to state the pcf theorem we are using here. However, an interested reader can find presentations of pcf theory in either [3], the second author's [8] or the first author's [5]. The pcf theorem used here is covered in detail in each of those three sources.

2. Notation and pcf

In this section we present a few simple definitions needed to state the pcf theorem used in proving the existence of an $\aleph_{\omega+1}$-Dowker space.

Suppose $B \subseteq \omega$ is a subset of the natural numbers.

Definition 1. 1. $\prod_{n\in B} \aleph_n = \{f : \mathrm{dom} f = B \wedge f(n) < \aleph_n \text{ for } n \in B\}$.

2. $\prod_{n\in B}(\aleph_n + 1) = \{f : \mathrm{dom} f = B \wedge f(n) \le \aleph_n \text{ for } n \in B\}$.
3. For $f, g \in \prod_{n\in B}(\aleph_n + 1)$ let:
   (a) $f < g$ iff $\forall n \in B\, [f(n) < g(n)]$,
   (b) $f \le g$ iff $\forall n \in B\, [f(n) \le g(n)]$,
   (c) $f \le^* g$ iff $\{n : f(n) > g(n)\}$ is finite,
   (d) $f <^* g$ iff $\{n : f(n) \ge g(n)\}$ is finite,
   (e) $f =^* g$ iff $\{n : f(n) \ne g(n)\}$ is finite.
4. A sequence $\langle f_\alpha : \alpha < \lambda\rangle$ of functions in $\prod_{n\in B} \aleph_n$ is *increasing in* $<$ $(\le, <^*, \le^*)$ iff $\alpha < \beta < \lambda \Rightarrow f_\alpha < f_\beta$ ($f_\alpha \le f_\beta$, $f_\alpha <^* f_\beta$, $f_\alpha \le^* f_\beta$).
5. $g \in \prod_{n\in B}(\aleph_n + 1)$ is an *upper bound* of $\{f_\alpha : \alpha < \delta\} \subseteq \prod_{n\in B} \aleph_n$ if and only if $f_\alpha \le^* g$ for all $\alpha < \delta$.

[1] pcf means *possible cofinalities*.

6. $g \in \prod_{n\in B}(\aleph_n + 1)$ is a *least upper bound* of $\{f_\alpha : \alpha < \delta\} \subseteq \prod_{n\in B} \aleph_n$ if and only if g is an upper bound of $\{f_\alpha : \alpha < \delta\} \subseteq \prod_{n\in B} \aleph_n$, and if g' is an upper bound of $\{f_\alpha : \alpha < \delta\}$, then $g \leq^* g$.

Theorem 1 (Shelah)**.** *There is a set $B = B_{\aleph_{\omega+1}} \subseteq \omega$ and a sequence $\overline{f} = \langle f_\alpha : \alpha < \aleph_{\omega+1}\rangle$ of functions in $\prod_{n\in B} \aleph_n$ such that:*

- *$\overline{f}$ is increasing in $<^*$,*
- *$\overline{f}$ is cofinal: for every $f \in \prod_{n\in B} \aleph_n$ there is an $\alpha < \aleph_{\omega+1}$ so that $f <^* f_\alpha$.*

A sequence as in the theorem above will be referred to as an "$\aleph_{\omega+1}$-scale".

By Theorem 1 we can find $B \subseteq \omega$ and an $\aleph_{\omega+1}$-scale $\overline{g} = \langle g_\alpha : \alpha < \aleph_{\omega+1}\rangle$ in $\prod_{n\in B} \aleph_n$. The set B is clearly infinite. Restricting every $g_\alpha \in \overline{g}$ to a fixed co-finite set of coordinates does not matter, so we assume without loss of generality that $0, 1 \notin B$. For notational simplicity we pretend that $B = \omega - \{0, 1\}$; if this is not the case, we need to replace $\aleph_n$ in what follows by the n-th element of B. We sum up our assumptions in the following:

Claim 2. We can assume without loss of generality that there is an $\aleph_{\omega+1}$-scale $\overline{g} = \langle g_\alpha : \alpha < \aleph_{\omega+1}\rangle$ in $\prod_{n>1} \aleph_n$.

Claim 3. There is an $\aleph_{\omega+1}$-scale $\overline{f} = \langle f_\alpha : \alpha < \aleph_{\omega+1}\rangle$ in $\prod_{n>1} \aleph_n$ so that for every $\delta < \aleph_{\omega+1}$, if $\mathrm{cf}\delta > \aleph_0$ and a least upper bound of $\overline{f}\restriction\delta$ exists, then f_δ is a least upper bound of $f\restriction\delta$.

Proof. Fix an $\aleph_{\omega+1}$-scale $\overline{g} = \langle g_\alpha : \alpha < \aleph_{\omega+1}\rangle$ in $\prod_{n>1} \aleph_n$ as guaranteed by Claim 2. Define f_α by induction on $\alpha < \aleph_{\omega+1}$ as follows: If α is a successor or limit of countable cofinality, let f_α be g_β for the first $\beta \in (\alpha, \aleph_{\omega+1})$ for which $g_\beta >^* f_\gamma$ for all $\gamma < \alpha$. If $\mathrm{cf}\alpha > \aleph_0$, then let g_α be a least upper bound to $\overline{f}\restriction\delta := \langle f_\beta : \beta < \alpha\rangle$, if such least upper bound exists; else, define f_α as in the previous cases.

The sequence $\overline{f} = \langle f_\alpha : \alpha < \aleph_{\omega+1}\rangle$ is increasing cofinal in $\prod_{n>1} \aleph_n$ and by its definition satisfies the required condition. $\square$

Claim 4. Suppose $0 < m \leq k < \omega$. Let $\langle \alpha(\zeta) : \zeta < \aleph_m\rangle$ be strictly increasing with $\sup\{\alpha(\zeta) : \zeta < \aleph_m\} = \delta < \aleph_{\omega+1}$. If $\langle g_\zeta : \zeta < \aleph_m\rangle$ is a sequence of functions in $\prod_{n>k} \aleph_n$ which is increasing in $<$, and $g_\zeta =^* f_{\alpha(\zeta)}$ for every $\zeta < \aleph_m$, then:

- $g := \sup\{g_\zeta : \zeta < \aleph_m\} \in \prod_{n>k} \aleph_n$ is a least upper bound of $\overline{f}\restriction\delta$,
- $\mathrm{cf} g(n) = \aleph_m$ for all $n > k$,
- $g =^* f_\delta$.

Proof. Let $g := \sup\{g_\zeta : \zeta < \aleph_m\}$. Since $\langle g_\zeta : \zeta < \aleph_m\rangle$ is increasing in $<$, necessarily $\mathrm{cf} g(n) = \aleph_m$ for all $n \geq k$, and since $g(n) \leq \aleph_n$, it follows that $g(n) < \aleph_n$ for $n > k$ and therefore $g \in \prod_{n>k} \aleph_n$.

Suppose that $\gamma < \delta$ is arbitrary. There exists $\zeta < \aleph_m$ such that $\gamma < \alpha(\zeta)$, hence $f_\gamma <^* f_{\alpha(\zeta)} =^* g_\zeta \leq g$. Thus g is an upper bound of $\overline{f}\restriction\delta$.

To show that g is a least upper bound suppose that g' is an upper bound of $\overline{f}\restriction\delta$. Let $X := \{n > k : g'(n) < g(n)\}$. For every $n \in X$ find $\zeta(n) < \aleph_m$ such that $g_{\zeta(n)}(n) > g'(n)$. Such $\zeta(n)$ can be found because $g = \sup\{g_\zeta : \zeta < \aleph_m\}$. Let $\zeta^* := \sup\{\zeta(n) : n > 1\}$. Since $\aleph_m > \aleph_0$, $\zeta^* < \aleph_m$. Since $\langle g_\zeta : \zeta < \aleph_m\rangle$ is increasing in $<$, it holds that $f_{\zeta(*)} \geq f_{\zeta(n)}(n) > g'(n)$ for every $n \in X$. But g' is an upper bound of $\overline{f}\restriction\delta$, so $f_{\zeta(*)} \leq^* g'$ and X is therefore finite.

By the definition of $\overline{f}$ we conclude that f_δ is a least upper bound of $\overline{f}\restriction\delta$. Since both g and f_δ are least upper bounds of $\overline{f}\restriction\delta$, it follows that $g =^* f_\delta$. $\square$

3. The space

Definition 5. Let $X^R = \{h \in \prod_{n>1}(\aleph_n + 1) : \exists m \,\forall n\, [\aleph_0 < \mathrm{cf}h(n) < \aleph_m]\}$.

The space X^R is *the Rudin space* from [6] with the Hausdorff topology defined by letting, for every $f < g$ in $\prod_{n>1}(\aleph_n + 1)$,

$$(f, g] := \{h \in X^R : f < h \leq g\} \tag{1}$$

be a basic open set (see [6]).

Recall that a normal Hausdorff space is *countably paracompact* iff for every decreasing sequence $\langle D_n : n < \omega\rangle$ of closed sets that satisfies $\bigcap D_n = \emptyset$ there are open sets $U_n \supseteq D_n$ with $\bigcap U_n = \emptyset$.

Definition 6. $D_n := \{h \in X^R : \exists m \geq n \left[h(m) = \aleph_m\right]\}$.

M. E. Rudin defined in [6] the closed subsets $D_n \subseteq X^R$ above and proved:

Theorem 2 (Rudin)**.** 1. *X^R is collectionwise normal.*
2. *If $U_n \subseteq X^R$ is open and $D_n \subseteq U_n$ for all $n > 1$, then $\bigcap_{n>1} U_n$ is not empty.*

These two facts establish by [4] that X^R is Dowker.

Let $\overline{f} = \langle f_\alpha : \alpha < \aleph_{\omega+1}\rangle$ be as provided by Claim 3. We use this scale to extract a closed Dowker subspace of cardinality $\aleph_{\omega+1}$ from Rudin's space.

Definition 7. $X = \{h \in X^R : \exists\alpha < \aleph_{\omega+1}\, [h =^* f_\alpha]\}$.

Since $|\{h \in X^R : h =^* f_\alpha\}| = \aleph_\omega$ for every $\alpha < \aleph_{\omega+1}$, it is obvious that $|X| = \aleph_{\omega+1}$.

Since $\overline{f}$ is totally ordered by $<^*$, for every $h \in X$ there exists a *unique* $\alpha < \aleph_{\omega+1}$ such that $h =^* f_\alpha$. Consequently, the space X is *totally quasi-ordered* by $<^*$, namely the following trichotomy holds:

$$\forall h, k \in X \left[h <^* k \,\vee\, k <^* h \,\vee\, h =^* k\right]. \tag{2}$$

Claim 4 translates to a property of X:

Claim 8. Suppose that $0 < m \leq k < \omega$ and that $\langle h_\zeta : \zeta < \aleph_m\rangle$ is a sequence of elements of X such that $\langle h_\zeta \restriction (k, \omega) : \zeta < \aleph_m\rangle$ is increasing in $<$. Denote $g = \sup\{h_\zeta : \zeta < \aleph_m\}$. Then there is some $h \in X$ such that $h =^* g$.

Proof. For every $\zeta < \aleph_m$ there is a unique $\alpha(\zeta) < \aleph_{\omega+1}$ for which $h_\zeta =^* f_{\alpha(\zeta)}$. Since $\langle h_\zeta : \zeta < \aleph_m\rangle$ is increasing in $<^*$, the sequence $\langle\alpha(\zeta) : \zeta < \aleph_m\rangle$ is strictly increasing. Let $\delta = \sup\{\alpha(\zeta) : \zeta < \aleph_m\}$. By Claim 4, $\mathrm{cf}g(n) = \aleph_m$ for all $n \in (k, \omega)$ and $g =^* f_\delta$.

Let $h \in \prod_{n>1}(\aleph_n + 1)$ be defined by $h(n) = \aleph_n$ for $n \leq k$ and $h(n) = g(n)$ for $n > k$. Then $h \in X^R$ and $h =^* f_\delta$. Thus $h \in X$ and $h =^* g$ as required. □

Claim 9. X is a closed subspace of X^R.

Proof. Suppose $t \in \mathrm{cl}X$ and $t \in X^R$. For every $h \in X$ let $E(h, t) := \{n > 1 : h(n) = t(n)\}$.

Claim 10. If $h \leq t$ and $h \in X$, then $E(h, t)$ is either finite or co-finite.

Proof. Suppose to the contrary that $h \le t$, $h \in X$ and $|E(h,t)| = |\omega - E(h,t)| = \aleph_0$. Let, for $n > 1$,

$$f(n) = \begin{cases} 0 & \text{if } n \in E(h,t), \\ \\ h(n) & \text{if } n \in (\omega - E(h,t)). \end{cases}$$

Clearly $f < t$. We argue that $X \cap (f,t]$ is empty, contrary to $t \in \mathrm{cl}X$. Indeed, if $k \in X$ and $k(n) > h(n)$ for all $n \in (w - E(h,t))$, then $k \not<^* h \wedge k \neq^* h$ because $w - E(h,t)$ is infinite and so $h <^* k$ by the trichotomy (2). Since $E(h,t)$ is infinite and $\{n > 1 : k(n) \le h(n)\}$ is finite, there is an $n \in E(h,t)$ such that $k(n) > h(n) = t(n)$ and therefore $k \notin (f,t]$. □

We need a definition:

Definition 11. $\mathbf{W} := \{w \subseteq \omega : \forall f < t\ \exists h \in (f,t]\ \Big[E(h,t) = w\Big]\}$.

By Claim 10 if $w \in \mathbf{W}$, then w is finite or w is co-finite.

Claim 12. $\mathbf{W} \neq \emptyset$.

Proof. Assume that $\mathbf{W}$ is empty. This is equivalent, by Claim 10, to assuming that every finite and every co-finite $w \subseteq \omega$ is not in $\mathbf{W}$. For every finite or co-finite $w \subseteq \omega$ fix a function $f_w < t$ such that $h \in (f_w, t] \cap X \Rightarrow E(h,t) \neq w$. Let f be the supremum of f_w taken over all finite and co-finite $w \subseteq \omega$. Since there are countably many f_w and $\mathrm{cf}t(n) > \aleph_0$ for all $n > 1$ it follows that $f < t$. If $h \le t$ is in X and $w = E(h,t)$, then $h \notin (f_w, t]$ and hence $h \notin (f,t]$. Thus $(f,t] \cap X = \emptyset$, contrary to $t \in \mathrm{cl}X$. □

Let us denote $M_m = \{n > 1 : \mathrm{cf}t(n) = \aleph_m\}$. Likewise, $M_{<m} = \bigcup_{1<i<m} M_i$.

Claim 13. If there exists $h \in X$ for which $E(h,t)$ is co-finite, then $t \in X$.

Proof. Clear. □

Claim 14. There exists some $h \in X$ for which $E(h,t)$ is co-finite.

Proof. By Claim 10 it suffices to find $h \in X$ with infinite $E(h,t)$. Let m be the least integer for which M_m is infinite. Such m must exist, since $t \in X^R$.

Fix $w \in \mathbf{W}$. If w is infinite, then we are done; so assume w is finite. Let $k = \max\{M_{<m}, \max w\}$.

For every $n \in M_m$ fix an increasing sequence $\langle \gamma^n_\zeta : \alpha < \aleph_m \rangle$ with supremum $t(n)$. By induction of $\zeta < \aleph_m$ find a sequence $\langle h_\zeta : \zeta < \aleph_m \rangle$ so that:

1. $h_\zeta \le t$ is in X and $E(h_\zeta, t) = w$,
2. $\xi < \zeta < \aleph_m \Rightarrow h_\xi \restriction (k,\omega) < gh_\zeta \restriction (k,\omega) < t \restriction (k,\omega)$,
3. $h_\zeta(n) \ge \gamma^n_\zeta$ for all $n \in (k,\omega) \cap M_m$.

At stage ζ let $f = \sup\{h_\xi \restriction (k,\omega) : \xi < \zeta\}$. Since for every $\xi < \zeta$ it follows by $E(h_\xi, t) = w$ that $h_\xi \restriction (k,\omega) < t \restriction (k,\omega)$, and since $\mathrm{cf}t(n) \ge \aleph_m$ for all $n \in (k,\omega)$, we have $f < t$. By definition of $w \in \mathbf{W}$ we can find $h_\zeta \le t$ in X with $E(h_\zeta, t) = w$ such that $h_\zeta \restriction (k,\omega) > f \restriction (k,\omega)$. Without loss of generality we can choose h_ζ so that $h_\zeta(n) > \gamma^n_\zeta$ for all $n \ge k$ in M_m.

By Claim 8 there is some $h \in X$ with $h(n) =^* \sup\{h_\zeta(n) : \zeta < \aleph_m\}$. In particular, $h(n) = t(n)$ for all but finitely many $n \ge k$ in M_m. Since M_m is infinite, $E(h,t)$ is infinite, and we are done. □

Claim 15. X is collectionwise normal.

Proof. Clear from Claim 9 and Theorem 2. $\square$

We show next that X is not countably paracompact. Let $D_n^X = \{f \in X : \exists m \geq n\,[f(m) = \aleph_m]\}$ for $n > 1$. It is straightforward that D_n^X is closed and that $\bigcap_n D_n^X = \emptyset$.

Claim 16. If $U_n \subseteq X$ is open, and $D_n^X \subseteq U_n$ for all $n > 1$, then $\bigcap U_n$ is not empty.

The truth of the matter is that this follows trivially from the analogous property in X^R and the closedness of X: Suppose that $D_n^X \subseteq U_n \subseteq X^R$ and U_n is open for all n. The definition of D_n above is absolute between X and X^R, so $V_n := U_n \cup (X^R - X)$ is open and contains D_n. Rudin's proof in [6] shows that if $D_n \subseteq V_n$ and V_n is open for all n, then there is some $f \in \prod_{n>1} \aleph_n$ such that $h \in \bigcap_n V_n$ for all $h > f$ in X^R. Since for every such f there is an $h \in X$ with $h > f$, we see that $\bigcap U_n \cap X$ is not empty.

For the sake of completeness, though (but not less, for the reader's amusement) we shall prove this property directly for X using elementary submodels.

Proof of Claim 16. Suppose that $U_n \supseteq D_n^X$ is open for $n > 1$. We need to prove that $\bigcap_n U_n$ is not empty.

We shall prove that there is some $f \in \prod_{n>1} \aleph_n$ such that every $h > f$ in X belongs to this intersection.

It suffices to show that for each $n > 1$ there is some $f_n \in \prod_{n>1} \aleph_n$ such that $\forall h \in X \left[h > f_n \Rightarrow h \in U_n\right]$, because then $f = \sup\{f_n : 1 < n < \omega\}$ is as required.

Suppose to the contrary that $m > 1$ is fixed and for every function $f \in \prod_{n>1} \aleph_n$ there is some function $h_f > f$ in $X - U_m$. Since $h_f \notin D_m$, it follows that $h_f(n) < \aleph_n$ for all $n \geq m$.

For a given f, let $g_f = \sup\{h_{f'} : f' \in \prod_{n>1} \aleph_n \wedge (m, \omega) \subseteq E(f', f)\}$. Since this supremum is taken over $\aleph_m$ many functions $h_{f'}$, it follows from the above that $g_f(n) < \aleph_n$ for all $n > m$. Also, clearly $g_f(i) = \aleph_i$ for $1 < i \leq m$.

Let $\langle M_\zeta : \zeta \leq \omega_1 \rangle$ be an elementary chain of submodels of $H(\theta)$ for large enough regular θ so that:

- $\overline{f}$, X and the functions $f \mapsto h_f$ and $f \mapsto g_f$ belong to M_0,
- M_ζ has cardinality $\aleph_1$ and $\langle M_\xi : \xi < \zeta \rangle \in M_{\zeta+1}$ for all $\zeta < \omega_1$.

For every ζ let $\chi_\zeta(n) := \sup(M_\zeta \cap \aleph_n)$ for all $n > 1$. Since $|M_\zeta| = \aleph_1$, it follows that $\chi_\zeta(n) < \aleph_n$ for all n and hence $\chi_\zeta \in \prod_{n>1} \aleph_n$.

Since $\chi_\xi \in M_\zeta$ for $\xi < \zeta < \omega_1$, by elementarity also h_{χ_ξ} and g_{χ_ξ} belong to M_ζ and consequently $h_{\chi_\xi}, g_{\chi_\xi} < \chi_\zeta$.

Therefore, if $\xi < \zeta < \omega_1$, then $\chi_\xi < h_{\chi_\xi} < \chi_\zeta < h_{\chi_\zeta} < \chi_{\omega_1}$. Thus $\langle h_{\chi_\zeta} : \zeta < \omega_1 \rangle$ is a sequence in X, increasing in $<$ with supremum χ_{ω_1}. By Claim 8, $\chi_{\omega_1} \in X$.

Let χ' be so that $\chi'(n) = \chi_{\omega_1}(n)$ for all $n > m$ and $\chi'(i) = \aleph_i$ for $1 < i \leq m$. So $\chi' \in D_m^X \subseteq U_m$ and therefore $(f, \chi'] \subseteq U_n$ for some $f < \chi'$, as U_m is open.

Find some $\zeta < \omega_1$ such that $f{\restriction}(m, \omega) < \chi_\zeta{\restriction}(m, \omega)$. Let $f' = f{\restriction}(m+1) \cup \chi_\zeta{\restriction}(m, \omega)$. By the definition of g_{χ_ζ} we see that $f' < h_{f'} \leq g_{\chi_\zeta} \leq \chi'$ and, of course, $h_{f'} \notin U_m$. This contradicts $h_{f'} \in (f, \chi'] \subseteq U_m$. $\square$

The space X defined in 7 is normal and not countably paracompact by Claim 15 and Claim 16 respectively, and is therefore Dowker by [4]. Since $|X| = \aleph_{\omega+1}$ we have proved:

Theorem 3. *There is a ZFC Dowker space of cardinality* $\aleph_{\omega+1}$.

It is straightforward to verify that the space X constructed above has weight $\aleph_{\omega+1}$ and character $\aleph_\omega$.

Problem 17. Is $\aleph_{\omega+1}$ the *first* cardinal in which one can prove the existence of a Dowker space in ZFC?

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