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On the Number of Elementary Submodels of an Unsuperstable Homogeneous Structure

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Abstract. We show that if M is a stable unsuperstable homogeneous structure, then for most $\kappa < |M|$, the number of elementary submodels of M of power κ is 2^κ.

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Throughout this paper we assume that M is a stable unsuperstable homogeneous model such that $|M|$ is strongly inaccessible (= regular and strong limit). We can drop this last assumption if instead of all elementary submodels of M we study only suitably small ones. Notice also that we do not assume that $\mathrm{Th}(M)$ is stable. We assume that the reader is familiar with [3] and use all the notions and results of it freely. In [1] a strong nonstructure theorem was proved for the elementary submodels of M assuming the existence of Skolem-functions. In this paper we drop the assumption on the Skolem-functions and prove the following nonstructure theorem.

1 Theorem. *Let λ be the least regular cardinal $\geq \lambda(M)$. Assume κ is a cardinal ($< |M|$) such that $\kappa = \mathrm{cf}(\kappa) > \lambda$. Then there are models (=elementary submodels of M) $\mathcal{A}_i$, $i < 2^\kappa$, such that for all $i < 2^\kappa$, $|\mathcal{A}_i| = \kappa$ and for all $i < j < 2^\kappa$, $\mathcal{A}_i \not\cong \mathcal{A}_j$.*

Remark. By using the model construction of this paper and [5, Chapter III.5], we can improve Theorem 1. The assumption $\kappa = \mathrm{cf}(\kappa) > \lambda$ can be replaced by the assumption $\kappa > |\mathrm{Th}(M)|$.

See [1] for nonstructure results in the case M is unstable.

We prove Theorem 1 in a serie of lemmas. Let λ and κ be as in Theorem 1. By *λ-saturated*, *λ-primary* etc., we mean F^M_λ-saturated, F^M_λ-primary etc. Notice that M is λ-stable.

The notion *λ-construction* (i.e. F^M_λ-construction) is defined as general F-construction in [4].

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2 Lemma. *Assume $(C, \{a_i : i < \alpha\}, \{A_i : i < \alpha\})$ is a λ-construction and σ is a permutation of α. Let $b_i = a_{\sigma(i)}$ and $B_i = B_{\sigma(i)}$. If $B_i \subseteq C \cup \{b_j : j < i\}$ for all $i < \alpha$, then $(C, \{b_i : i < \alpha\}, \{B_i : i < \alpha\})$ is a λ-construction.*

Proof. Exactly as [4, IV Theorem 3.3]. □

We write $\kappa^{\leq\omega}$ for $\{\eta : \alpha \longrightarrow \kappa : \alpha \leq \omega\}$; $\kappa^{<\omega}$ and $\kappa^{=\omega}$ are defined similarly. Let $J \subseteq \kappa^{\leq\omega}$ be such that it is closed under initial segments. If η, $\xi \in J$, then by $r'(\eta, \xi)$ we mean the longest element of J which is an initial segment of both η and ξ. If $u, v \in I = P_\omega(J)$ (=the set of all finite subsets of J), then by $r(u, v)$ we mean the largest set R which satisfies

(i) $R \subseteq \{r'(\eta, \xi) : \eta \in u,\ \xi \in v\}$;

(ii) if $u, v \in R$ and u is an initial seqment of v, then $u = v$.

We order I by $u \leq v$ if for every $\eta \in u$ there is $\xi \in v$ such that η is initial seqment of ξ, i.e. $r(u, v) = r(u, u)$ $(= \{\eta \in u : \neg(\exists \xi \in u)\,(\eta$ is a proper initial segment of $\xi)\})$.

3 Definition. Assume $J \subseteq \kappa^{\leq\omega}$ is closed under initial segments and $I = P_\omega(J)$. Let $\Sigma = \{A_u : u \in I\}$ be an indexed family of subsets of $\boldsymbol{M}$ of power $< |\boldsymbol{M}|$. We say that Σ is *strongly independent* if

(i) for all u, $v \in I$, $u \leq v$ implies $A_u \subseteq A_v$;

(ii) if u, $u_i \in I$ for $i < n$, and $B \subseteq \bigcup_{i<n} A_{u_i}$ has power $< \lambda$, then there exists an automorphism $f = f^{\Sigma,B}_{(u,u_0,\ldots,u_{n-1})}$ of $\boldsymbol{M}$ such that $f \restriction (B \cap A_u) = \mathrm{id}_{B \cap A_u}$ and $f(B \cap u_i) \subseteq A_{r(u,u_i)}$.

The model construction in Lemma 4 below is a generalized version of the construction used in [4, XII.4].

4 Lemma. *Assume that $\Sigma = \{A_u : u \in I\}$, $I = P_\omega(J)$, is strongly independent. Then there are sets $\mathcal{A}_u \subseteq \boldsymbol{M}$, $u \in I$, such that*

(i) *for all u, $v \in I$, $u \leq v$ implies $\mathcal{A}_u \subseteq \mathcal{A}_v$;*

(ii) *for all $u \in I$, $\mathcal{A}_u$ is λ-primary over A_u, (and so by (i), $\bigcup_{u \in I} \mathcal{A}_u$ is a model);*

(iii) *if $v \leq u$, then $\mathcal{A}_u$ is λ-atomic $(= F^{\boldsymbol{M}}_\lambda$-atomic$)$ over $\bigcup_{u \in I} A_u$ and λ-primary over $\mathcal{A}_v \cup A_u$;*

(iv) *if $J' \subseteq J$ is closed under initial segments and $u \in P_\omega(J')$, then $\bigcup_{v \in P_\omega(J')} \mathcal{A}_v$ is λ-constructible over $\mathcal{A}_u \cup \bigcup_{v \in P_\omega(J')} A_v$.*

Proof. Let $\{u_i : i < \alpha^*\}$ be an enumeration of I such that $u \leq v$ and $v \not\leq u$ implies $i < j$. It is easy to see that we can choose α, $\gamma_i < \alpha$ for $i < \alpha^*$, a_γ and B_γ for $\gamma < \alpha$, and $s : \alpha \longrightarrow I$ so that

(a) $\gamma_0 = 0$ and $(\gamma_i)_{i<\alpha^*}$ is increasing and continuous;

(b) if $\gamma_i \leq \gamma < \gamma_{i+1}$, then $s(\gamma) = u_i$;

(c) if $\gamma < \alpha$, $|B_\gamma| < \lambda$ and we write for $\gamma \leq \alpha$, $A^\gamma_u = A_u \cup \{a_\delta : \delta < \gamma,\ s(\delta) \leq u\}$, then $B_\gamma \subseteq A^\gamma_{s(\gamma)}$;

(d) for all $\gamma < \alpha$, if we write $A^\gamma = \bigcup_{u \in I} A^\gamma_u$, then $t(a_\gamma, B_\gamma)$ λ-isolates $t(a_\gamma, A^\gamma)$;

(e) for all $i < \alpha^*$, there are no a and $B \subseteq A^{\gamma_i+1}_{u_i}$ of power $< \lambda$ such that $t(a, B)$ λ-isolates $t(a, A^{\gamma_i+1})$;

(f) if $a_\delta \in B_\gamma$, then $B_\delta \subseteq B_\gamma$.

For all $u \in I$, we define $A_u = A^\alpha_u$. We show that these are as wanted:

(i) follows immediately from the definitions and for (ii) it is enough to prove the following claim (Claim (III) implies (ii) easily).

Claim. *For all $i < \alpha^*$,*

(I) *$\Sigma_i = \{A^{\gamma_i}_u : u \in I\}$ is strongly independent, and we write $f^{i,B}_{(u,u_0,\ldots,u_{n-1})}$ instead of $f^{\Sigma_i,B}_{(u,u_0,\ldots,u_{n-1})}$;*

(II) *the functions $f^{i,B}_{(u,u_0,\ldots,u_{n-1})}$ can be chosen so that if $j < i$, u, $u_k \in I$, $k < n$, $B \subseteq \bigcup_{i<n} A^{\gamma_i}_{u_k}$ has power $< \lambda$ and $a_\gamma \in B$ implies $B_\gamma \subseteq B$, and $B' = B \cap A^{\gamma_j}$, then $f^{i,B}_{(u,u_0,\ldots,u_{n-1})} \restriction B' = f^{j,B'}_{(u,u_0,\ldots,u_{n-1})} \restriction B'$;*

(III) *if $j < i$, then $A^{\gamma_j+1}_{u_j}$ is λ-saturated.*

Proof. Notice that if $a_\gamma \in A^\delta_u \cap A^\delta_v$, then $a_\gamma \in A^\delta_{r(u,v)}$. Similarly we see that the first half of (I) in the Claim is always true (i.e. if $u \leq v$, then for all $\delta < \alpha$, $A^\delta_u \subseteq A^\delta_v$.) We prove the rest by induction on $i < \alpha^*$. We notice first that it is enough to prove the existence of $f^{i,B}_{(u,u_0,\ldots,u_{n-1})}$ only in the case when B satisfies the condition

(∗) if $a_\gamma \in B$, then $B_\gamma \subseteq B$.

For $i = 0$, there is nothing to prove. If i is limit, then the Claim follows easily from the induction assumption (use (II) in the Claim). So we assume that the Claim holds for i and prove it for $i+1$. We prove first (I) and (II). For this let u, $u_k \in I$ for $k < n$, and $B \subseteq \bigcup_{k<n} A^{\gamma_i+1}_{u_k}$ be of power $< \lambda$ such that (∗) is satisfied. If for all $k < n$, $s(\gamma_i) \not\leq u_k$, then (I) and (II) in the Claim follow immediately from the induction assumption. So we may assume that $s(\gamma_i) \leq u_0$. Let $B' = B \cap (\bigcup_{k<n} A^{\gamma_i}_{u_k})$. By the induction assumption there is an automorphism $f = f^{i,B'}_{(u,u_0,\ldots,u_{n-1})}$ of $\boldsymbol{M}$ such that $f \restriction (B' \cap A^{\gamma_i}_u) = \mathrm{id}_{B' \cap A^{\gamma_i}_u}$ and $f(B' \cap A^{\gamma_i}_{u_k}) \subseteq A^{\gamma_i}_{r(u,u_k)}$. If $s(\gamma_i) \leq u$, then, by (∗) and (d) in the construction, we can find an automorphism $g = f^{i+1,B}_{(u,u_0,\ldots,u_{n-1})}$ of $\boldsymbol{M}$ such that $g \restriction B' = f \restriction B'$ and $g \restriction (B - B') = \mathrm{id}_{B-B'}$. Clearly this is as wanted.

So we may assume that $s(\gamma_i) \not\leq u$. Since $s(\gamma_i) \leq u_0$, $u_0 \not\leq r(u, u_0)$. By the choice of the enumeration of I there is $j < i$ such that $u_j = r(u, u_0)$. Then by the induction assumption (part (III)), $A^{\gamma_i+1}_{u_j} = A^{\gamma_i}_{u_j} = A^{\gamma_j+1}_{u_j}$ is λ-saturated, and by the choice of f, $f(B' \cap A^{\gamma_i}_{u_0}) \subseteq A^{\gamma_i}_{u_j}$. So by (d) in the construction and (∗) there are no difficulties in finding the required automorphism $f^{i+1,B}_{(u,u_0,\ldots,u_{n-1})}$.

So we need to prove (III): For this it is enough to show that $A^{\gamma_i+1}_{u_i}$ is λ-saturated. Assume not. Then there are a and B such that $B \subseteq A^{\gamma_i+1}_{u_i}$, $|B| < \lambda$ and $t(a, B)$ is not realized in $A^{\gamma_i+1}_{u_i}$. Since $\lambda \geq \lambda(\boldsymbol{M})$, there are b and C such that $B \subseteq C \subseteq A^{\gamma_i+1}_{u_i}$, $|C| < \lambda$, $t(b, B) = t(a, B)$ and $t(b, C)$ λ-isolates $t(b, A^{\gamma_i+1}_{u_i})$. But since (I) in the Claim holds for $i+1$, $t(b, C)$ λ-isolates $t(b, A^{\gamma_i+1})$. This contradicts (e) in the construction.

□ Claim

Conditions (iii) and (iv) in Lemma 4 follow immediately from the construction, Claim (III) and Lemma 2. □

Since M is unsuperstable, by [3, Lemma 5.1], there are a and $\lambda(M)$-saturated models $\mathcal{A}_i$, $i < \omega$, such that

(i) if $j < i < \omega$, then $\mathcal{A}_j \subseteq \mathcal{A}_i$;

(ii) for all $i < \omega$, $a \not\downarrow_{\mathcal{A}_i} \mathcal{A}_{i+1}$.

It is easy to see that we may choose the models $\mathcal{A}_i$ so that they are λ-saturated and of power λ. Let $\mathcal{A}_\omega$ be λ-primary over $a \cup \bigcup_{i<\omega} \mathcal{A}_i$. As in [1, Section 1], for all $\eta \in \kappa^{\leq\omega}$, we can find $\mathcal{A}_\eta$ such that

(a) for all $\eta \in \kappa^{\leq\omega}$, there is an automorphism f_η of M such that $f_\eta(\mathcal{A}_{\text{length}(\eta)}) = \mathcal{A}_\eta$;

(b) if η is an initial segment of ξ, then $f_\xi \restriction \mathcal{A}_{\text{length}(\eta)} = f_\eta \restriction \mathcal{A}_{\text{length}(\eta)}$;

(c) if $\eta \in \kappa^{<\omega}$, $\alpha \in \kappa$ and X is the set of those $\xi \in \kappa^{\leq\omega}$ such that $\eta^\frown(\alpha)$ is an initial segment of ξ, then $\bigcup_{\xi\in X} \mathcal{A}_\xi \downarrow_{\mathcal{A}_\eta} \bigcup_{\xi\in(\kappa^{\leq\omega}-X)} \mathcal{A}_\xi$.

For all $\eta \in \kappa^{=\omega}$, we let $a_\eta = f_\eta(a)$.

5 Lemma. *Assume $\eta \in \kappa^{<\omega}$, $\alpha \in \kappa$ and X is the set of those $\xi \in \kappa^{<\omega}$ such that $\eta^\frown(\alpha)$ is an initial segment of ξ. Let $B \subseteq \bigcup_{\xi\in(\kappa^{\leq\omega}-X)} \mathcal{A}_\xi$ and $C \subseteq \bigcup_{\xi\in X} \mathcal{A}_\xi$ be of power $< \lambda$. Then there is $C' \subseteq \mathcal{A}_\eta$ such that $t(C', B) = t(C, B)$.*

Proof. By [2, Lemma 8] (or [3, Lemma 3.15] plus little work) we can find $D \subseteq \mathcal{A}_\eta$ of power $< \lambda$ such that for all $b \in B$, $t(b, \mathcal{A}_\eta \cup C)$ does not split over D. So if we choose $C' \subseteq \mathcal{A}_\eta$ so that $t(C', D) = t(C, D)$, then C' is as wanted. □

6 Lemma. *Assume $J \subseteq \kappa^{\leq\omega}$ and $I = P_\omega(J)$. For all $u \in I$ let A_u be the set $\bigcup_{\eta\in u} \mathcal{A}_\eta$. Then $\{A_u : u \in I\}$ is strongly independent.*

Proof. This follows immediately from Lemma 5. □

For each $\alpha < \kappa$ of cofinality ω, let $\eta_\alpha \in \kappa^{=\omega}$ be a strictly increasing sequence such that $\bigcup_{i<\omega} \eta_\alpha(i) = \alpha$. Let $S \subseteq \{\alpha < \kappa : \text{cf}(\alpha) = \omega\}$. By J_S we mean the set $\kappa^{<\omega} \cup \{\eta_\alpha : \alpha \in S\}$. Let $I_S = P_\omega(J_S)$ and $\mathcal{A}_S$ be the model given by Lemmas 4 and 6 for $\{A_u : u \in I_S\}$.

7 Lemma.

(i) *Assume $\eta \in \kappa^{<\omega}$, $u \in I_S$, $\alpha < \kappa$, $\{\eta\} \leq u$ and $\{\eta^\frown(\alpha)\} \not\leq u$. Let X be the set of those $\xi \in J_S$ such that $\eta^\frown(\alpha)$ is an initial segment of ξ. Then*

$$\bigcup_{\xi\in X} \mathcal{A}_\xi \downarrow_{\mathcal{A}_u} \bigcup_{\xi\in J_S-X} \mathcal{A}_\xi.$$

(ii) *Assume $\alpha \in \kappa$, $u \in I_S$ and $v \in P_\omega(J_S \cap \alpha^{\leq\omega})$ is maximal such that $v \leq u$. Then*

$$\mathcal{A}_u \downarrow_{\mathcal{A}_v} \bigcup_{w\in P_\omega(J_S\cap\alpha^{\leq\omega})} \mathcal{A}_w.$$

Proof.

(i) Let $C = \bigcup_{\xi\in X} \mathcal{A}_\xi$. By (c) in the definition of $\mathcal{A}_\xi$, $\xi \in \kappa^{\leq\omega}$, there is C' such that $t(C', \bigcup_{\xi\in J_S-X} \mathcal{A}_\xi) = t(C, \bigcup_{\xi\in J_S-X} \mathcal{A}_\xi)$ and $C' \downarrow_{\mathcal{A}_\eta} \mathcal{A}_u \cup \bigcup_{\xi\in J_S-X} \mathcal{A}_\xi$. So the claim follows from the first half of Lemma 4(iii).

(ii) By (i), $A_u \downarrow_{\mathcal{A}_v} \bigcup_{w\in P_\omega(J_S\cap\alpha^{\leq\omega})} A_w$, from which the claim follows by Lemma 4(iii) and (iv). □

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8 Lemma. *Assume S, $R \subseteq \{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega\}$ are such that $(S-R)\cup(R-S)$ is stationary. Then $\mathcal{A}_S$ is not isomorphic to $\mathcal{A}_R$.*

Proof. Assume $\mathcal{A}_S \cong \mathcal{A}_R$. Let $f : \mathcal{A}_S \longrightarrow \mathcal{A}_R$ be an isomorphism. We write I_S^α for the set of those $u \in I_S$, which satisfy that for all $\xi \in u$, $\bigcup_{i<\mathrm{length}(\xi)} \xi(i) < \alpha$. The set I_R^α is defined similarly. Then we can find α and α_i, $i < \omega$, such that $(\alpha_i)_{i<\omega}$ is strictly increasing, for all $i < \omega$, $f(\bigcup_{u\in I_S^{\alpha_i}} \mathcal{A}_u) = \bigcup_{u\in I_R^{\alpha_i}} \mathcal{A}_u$, and $\alpha = \bigcup_{i<\omega} \alpha_i$ belongs to $(S-R)\cup(R-S)$. Without loss of generality we may assume that $\alpha \in S-R$, and so $\eta_\alpha \in J_S$. Let $\mathcal{A}_S^{\alpha_i} = \bigcup_{u\in I_S^{\alpha_i}} \mathcal{A}_u$ and $\mathcal{A}_R^{\alpha_i} = \bigcup_{u\in I_R^{\alpha_i}} \mathcal{A}_u$. Then it easy to see that for all $i < \omega$ there is $j < \omega$ such that $a_{\eta_\alpha} \not\downarrow_{\mathcal{A}_S^{\alpha_i}} \mathcal{A}_S^{\alpha_j}$ (use [3, Lemma 3.8(iii)]). So there is $u \in I_R$ such that for all $i < \omega$ there is $j < \omega$ such that $\mathcal{A}_u \not\downarrow_{\mathcal{A}_R^{\alpha_i}} \mathcal{A}_R^{\alpha_j}$. Since $\alpha \notin R$, this contradicts Lemma 7(ii). $\square$

We can now prove Theorem 1. By [4, Appendix 1, Theorem 1.3(2) and (3)], there are stationary $S_i \subseteq \{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega\}$, $i < \kappa$, such that for all $i < j < \kappa$, $S_i \cap S_j = \emptyset$. For all $X \subseteq \kappa$, let $\mathcal{A}_X = \mathcal{A}_{\cup_{i\in X} S_i}$. Then by Lemma 8, if $X \neq X'$, then $\mathcal{A}_X$ is not isomorphic to $\mathcal{A}_{X'}$. Since clearly $|J_{\cup_{i\in X} S_i}| = \kappa$, $|\mathcal{A}_X| = \kappa$. $\square$

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