

# The stationary set splitting game

Paul B. Larson\*<sup>1</sup> and Saharon Shelah\*\*<sup>2</sup>

<sup>1</sup> Department of Mathematics and Statistics, Miami University, Oxford, Ohio 45056, USA

<sup>2</sup> The Hebrew University of Jerusalem, Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, Jerusalem 91904, Israel

and Department of Mathematics, Hill Center – Busch Campus, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, USA

Received 6 December 2006, revised 17 August 2007, accepted 29 August 2007

Published online 7 March 2008

**Key words** Definable determinacy, games of uncountable length.

**MSC (2000)** 03E35, 03E60

*The stationary set splitting game* is a game of perfect information of length  $\omega_1$  between two players, unsplit and split, in which unsplit chooses stationarily many countable ordinals and split tries to continuously divide them into two stationary pieces. We show that it is possible in ZFC to force a winning strategy for either player, or for neither. This gives a new counterexample to  $\Sigma_2^2$  maximality with a predicate for the nonstationary ideal on  $\omega_1$ , and an example of a consistently undetermined game of length  $\omega_1$  with payoff definable in the second-order monadic logic of order. We also show that the determinacy of the game is consistent with Martin's Axiom but not Martin's Maximum.

© 2008 WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim

## 1 Introduction

*The stationary set splitting game* ( $\mathcal{SG}$ ) is a game of perfect information of length  $\omega_1$  between two players, unsplit and split. In every round  $\alpha$ , unsplit either accepts or rejects  $\alpha$ . If unsplit accepts  $\alpha$ , then split puts  $\alpha$  into one of two sets  $A$  and  $B$ . If unsplit rejects  $\alpha$ , then split does nothing. After all  $\omega_1$  many rounds have been played, split wins if unsplit has not accepted stationarily often, or if both of  $A$  and  $B$  are stationary.

In this note we prove that it is possible to force a winning strategy for either player in  $\mathcal{SG}$ , or for neither, and we also show that the determinacy of  $\mathcal{SG}$  is consistent with Martin's Axiom but not Martin's Maximum [4]. We also present two guessing principles,  $\mathcal{C}_s$  (*club for split*) and  $\mathcal{D}_u$  (*diamond for unsplit*), which imply the existence of winning strategies for split and unsplit, respectively (and are therefore incompatible; see Theorems 2.5 and 2.8). These principles may be of independent interest.

## 2 Winning strategies

### 2.1 Strategies for split

A collection  $\mathcal{X}$  of countable sets is *stationary* if for every function  $F: [\bigcup \mathcal{X}]^{<\omega} \rightarrow \bigcup \mathcal{X}$  there is an element of  $\mathcal{X}$  closed under  $F$ . A set  $\mathcal{X}$  of countable sets is *projective stationary* [2] if for every stationary  $S \subset \omega_1$  the set of  $X \in \mathcal{X}$  with  $X \cap \omega_1 \in S$  is stationary. We note that a partial order  $P$  is said to be *proper* if forcing with  $P$  preserves the stationarity (in the sense above) of stationary sets from the ground model (see [11]).

The following statement holds in fine structural models such as  $L$ . It is a strengthening of the principle (+) used in [8]. Justin Moore has pointed out to us that his Mapping Reflection Principle [9] implies the failure of (+). We note also that in the statement of (+), “projective stationary” can be replaced with “club” without strengthening the statement. We do not know if that is the case for  $\mathcal{C}+$ .

\* Corresponding author: e-mail: larsonpb@muohio.edu

\*\* e-mail: shelah@math.huji.ac.il

**Definition 2.1** Let  $\mathcal{C}+$  be the statement that there is a projective stationary set  $\mathcal{X}$  consisting of countable elementary substructures of  $H(\aleph_2)$  such that for all  $X, Y$  in  $\mathcal{X}$  with  $X \cap \omega_1 = Y \cap \omega_1$ , either for each club  $C \subset \omega_1$  in  $X$  there is a club  $D \subset \omega_1$  in  $Y$  with  $D \cap Y \subset C \cap X$ , or for each club  $D \subset \omega_1$  in  $Y$  there is a club  $C \subset \omega_1$  in  $X$  with  $C \cap X \subset D \cap Y$ .

Given a partial run of  $\mathcal{SG}$  of length  $\alpha$ , we let  $E_\alpha$  be the set of  $\beta < \alpha$  accepted by unsplit, and we let  $A_\alpha, B_\alpha$  be the partition of  $E_\alpha$  chosen by split.

**Theorem 2.2** *If  $\mathcal{C}+$  holds, then split has a winning strategy in  $\mathcal{SG}$ .*

*Proof.* Let  $\mathcal{X}$  be a set of countable elementary submodels of  $H(\aleph_2)$  witnessing  $\mathcal{C}+$ , and for each  $\alpha < \omega_1$  let  $\mathcal{X}_\alpha$  be the set of  $X \in \mathcal{X}$  with  $X \cap \omega_1 = \alpha$ . Let  $Z$  be the set of  $\alpha < \omega_1$  such that  $\mathcal{X}_\alpha$  is nonempty (since  $\mathcal{X}$  is projective stationary, this set contains a club).

Play for split as follows. In round  $\alpha \in Z$ , if unsplit accepts  $\alpha$ , let  $\mathcal{Y}_\alpha$  be the set of all  $X \in \mathcal{X}_\alpha$  such that  $X$  contains a stationary subset of  $\omega_1$ ,  $E_X$ , such that  $E_X \cap \alpha = E_\alpha$ . If  $\mathcal{Y}_\alpha = \emptyset$ , put  $\alpha \in A_{\alpha+1}$ . Otherwise, since every club subset of  $\omega_1$  in every member of  $\mathcal{Y}_\alpha$  intersects  $E_\alpha$ , there cannot be two club subsets of  $\omega_1$  in  $\bigcup \mathcal{Y}_\alpha$ , one disjoint from  $A_\alpha$  and one disjoint from  $B_\alpha$ , since some club subset of  $\omega_1$  in  $\bigcup \mathcal{Y}_\alpha$  would be contained in both of these clubs. If any member of  $\mathcal{Y}_\alpha$  contains a club subset of  $\omega_1$  disjoint from  $A_\alpha$ , put  $\alpha$  in  $A_{\alpha+1}$ , and if any member of  $\mathcal{Y}_\alpha$  contains a club subset of  $\omega_1$  disjoint from  $B_\alpha$ , put  $\alpha$  in  $B_{\alpha+1}$ . If neither case holds, put  $\alpha \in A_{\alpha+1}$ .

Let  $E$  be the play by unsplit in a run of  $\mathcal{SG}$  where split has played by this strategy, and let  $A$  and  $B$  be the corresponding play by split. Let  $C$  be a club subset of  $\omega_1$  and supposing that  $E$  is stationary, fix  $X \in \mathcal{X}$  containing  $E$ ,  $A$ ,  $B$ , and  $C$  with  $X \cap \omega_1 \in E \cap C$ . Then we have that if  $A \cap C \cap X \cap \omega_1 = \emptyset$ , then  $X \cap \omega_1 \in A \cap C$ , and if  $B \cap C \cap X \cap \omega_1 = \emptyset$ , then  $X \cap \omega_1 \in B \cap C$ , which shows that  $C$  does not witness that unsplit won this run of the game.  $\square$

The following fact, in conjunction with Theorem 2.2, shows that Martin's Axiom is consistent with the existence of a winning strategy for split.

**Theorem 2.3** *The statement  $\mathcal{C}+$  is preserved by forcing with c. c. partial orders.*

*Proof.* Let  $P$  be a c. c. c. forcing and let  $\mathcal{X}$  witness  $\mathcal{C}+$ . Let  $\gamma$  be a regular cardinal greater than  $\aleph_2$  and  $2^{|\mathcal{P}|}$ . Let  $G \subset P$  be a  $V$ -generic filter, and let

$$\mathcal{X}[G] = \{X[G] \cap H(\aleph_2)^{V[G]} \mid X \prec H(\gamma)^V, X \cap H(\aleph_2)^V \in \mathcal{X}\}.$$

Since every club subset of  $\omega_1$  in  $V[G]$  contains one in  $V$ , in order to show that  $\mathcal{X}[G]$  witnesses  $\mathcal{C}+$  in  $V[G]$ , it suffices to show that  $\mathcal{X}[G]$  is projective stationary there. Fix a  $P$ -name  $\varrho$  for a function from  $[H(\aleph_2)^{V[G]}]^{<\omega}$  to  $H(\aleph_2)^{V[G]}$ . For any countable  $X \prec H(\gamma)$  with  $X \cap H(\aleph_2) \in \mathcal{X}$  and  $\varrho \in X$ ,  $X[G] \cap H(\aleph_2)^{V[G]}$  is in  $\mathcal{X}[G]$  and closed under the realization of  $\varrho$ . Fix a  $P$ -name  $\tau$  for a stationary subset of  $\omega_1$  and a condition  $p \in P$ . Let  $S$  be the set of countable ordinals forced to be in  $\tau$  by some condition below  $p$ . Then there exist a countable  $X \prec H(\gamma)$  with  $X \cap H(\aleph_2) \in \mathcal{X}$ ,  $X \cap \omega_1 \in S$ , and  $\varrho \in X$ , and a condition  $q$  below  $p$  forcing that  $X[\dot{G}] \cap \omega_1$  (where  $\dot{G}$  is the name for the generic filter) is in the realization of  $\tau$ . By genericity, then,  $\mathcal{X}[G]$  is projective stationary.  $\square$

We do not know how to force  $\mathcal{C}+$ , however, and use a different principle to force the existence of a winning strategy for split.

**Definition 2.4** Let  $\mathcal{C}_s$  be the statement that there exist  $c_\alpha$  ( $\alpha < \omega_1$  limit) such that each  $c_\alpha$  is a sequence

$$\langle a_\beta^\alpha : \beta < \gamma_\alpha \rangle$$

(for some countable  $\gamma_\alpha$ ) of cofinal subsets of  $\alpha$  of ordertype  $\omega$  and

1. for all limit  $\alpha < \omega_1$  and all  $\beta < \beta' < \gamma_\alpha$ ,  $a_{\beta'}^\alpha \setminus a_\beta^\alpha$  is finite;
2. for every club  $C \subset \omega_1$  and every stationary  $E \subset \omega_1$  there exists  $a_\beta^\alpha$  with  $\alpha \in E$  such that  $a_\beta^\alpha \setminus C$  is finite and  $a_\beta^\alpha \cap E$  is infinite.

The principle  $\mathcal{C}_s$  also holds in fine structural models such as  $L$ . The winning strategy for split given by  $\mathcal{C}_s$  is very similar to the one given by  $\mathcal{C}+$ .

**Theorem 2.5** *If  $\mathcal{C}_s$  holds, then split has a winning strategy in  $\mathcal{SG}$ .*

*Proof.* Let  $a_\beta^\alpha$  ( $\alpha < \omega_1$  limit,  $\beta < \gamma_\alpha$ ) witness  $\mathcal{C}_s$ . Play for split as follows. In round  $\alpha$ ,  $\alpha$  a limit, if unsplit has accepted  $\alpha$  and some  $a_\beta^\alpha$  intersects  $A_\alpha$  infinitely and  $B_\alpha$  finitely, then put  $\alpha$  in  $B_{\alpha+1}$ . If some  $a_\beta^\alpha$  intersects  $B_\alpha$  infinitely and  $A_\alpha$  finitely, then put  $\alpha$  in  $A_{\alpha+1}$ . Since the  $a_\beta^\alpha$ 's ( $\beta < \gamma_\alpha$ ) are  $\subset$ -decreasing mod finite<sup>1)</sup>, both cases cannot occur. If neither case occurs, put  $\alpha$  in  $A_{\alpha+1}$ .

Let  $E$  be the play by unsplit in a run of  $\mathcal{SG}$  where split has played by this strategy, and let  $A$  and  $B$  be the corresponding play by split. Let  $C$  be a club subset of  $\omega_1$  and supposing that  $E$  is stationary, fix  $a_\beta^\alpha$  with  $\alpha \in E$  such that  $a_\beta^\alpha \setminus C$  is finite and  $a_\beta^\alpha \cap E$  is infinite. Then if  $A \cap a_\beta^\alpha$  is finite, then  $\alpha \in A \cap C$ , and if  $B \cap a_\beta^\alpha$  is finite, then  $\alpha \in B \cap C$ , which shows that  $C$  does not witness that unsplit won this run of the game.  $\square$

A partial order  $P$  is said to be *strategically  $\omega$ -closed* if there exists a function  $f: P^{<\omega} \rightarrow \mathcal{P}(P)$  such that whenever  $\langle p_i : i \leq n \rangle$  is a finite descending sequence in  $P$ ,  $f(\langle p_i : i \leq n \rangle)$  is a dense subset below  $p_n$ , and whenever  $\langle p_i : i < \omega \rangle$  is a descending sequence in  $P$  such that for each  $n$  there is  $j$  with  $p_j \in f(\langle p_i : i \leq n \rangle)$ , the sequence has a lower bound in  $P$ . It is easy to see that strategic  $\omega$ -closure is equal to the property that for every countable  $X \prec H((2^{|P|})^+)$  and every  $(X, P)$ -generic filter  $g$  contained in  $X$  there is a condition in  $P$  extending  $g$ .

Let us say that a set  $a$  captures a pair  $E, C$  if  $a \setminus C$  is finite and  $a \cap E$  is infinite. Given  $A \subset \omega_1$ , let  $\mathbb{C}(A)$  be the partial order which adds a club subset of  $A$  by initial segments. We force  $\mathcal{C}_s$  by first adding a potential  $\mathcal{C}_s$ -sequence by initial segments, and then iterating to kill off every counterexample.

We refer the reader to [11] for background on countable support iterations of proper forcing.

**Theorem 2.6** *Suppose that CH and  $2^{\aleph_1} = \aleph_2$  hold. Let  $\bar{P} = \langle P_\eta, Q_\eta : \eta < \omega_2 \rangle$  be a countable support iteration such that  $P_0$  is the partial order consisting of sequences  $\langle c_\alpha : \alpha < \delta$  limit  $\rangle$ , for some countable ordinal  $\delta$ , such that each  $c_\alpha$  is a sequence  $\langle a_\beta^\alpha : \beta < \gamma_\alpha \rangle$  (for some countable ordinal  $\gamma_\alpha$ ) of cofinal subsets of  $\alpha$  of order-type  $\omega$ , decreasing by mod finite inclusion (and  $P_0$  is ordered by extension). Suppose that the remainder of  $\bar{P}$  satisfies the following conditions.*

1. *For each nonzero  $\eta < \omega_2$  there is a  $P_\eta$ -name  $\tau_\eta$  for a subset of  $\omega_1$  such that if  $(\tau_\eta)_{G_\eta}$  (where  $G_\eta$  is the restriction of the generic filter to  $P_\eta$ ) is stationary in the  $P_\eta$ -extension and there exists a club  $C \subset \omega_1$  in this extension such that no  $a_\beta^\alpha$  with  $\alpha \in \tau_{G_\eta}$  captures the pair  $\tau_{G_\eta}, C$ , then  $Q_\eta$  is  $\mathbb{C}(\omega_1 \setminus (\tau_\eta)_{G_\eta})$  (and otherwise,  $Q_\eta$  is  $\mathbb{C}(\omega_1)$ ).*

2. *For every pair  $E, C$  of subsets of  $\omega_1$  in any  $P_\eta$ -extension ( $\eta < \omega_2$ ), if  $E$  is stationary in this extension and  $C$  is club and no  $a_\beta^\alpha$  with  $\alpha \in E$  captures  $E, C$ , then there is  $\varrho \in [\eta, \omega_2)$  such that if  $E$  is stationary in the  $P_\varrho$ -extension, then  $Q_\varrho$  is  $\mathbb{C}(\omega_1 \setminus E)$ .*

*Then  $\bar{P}$  is strategically  $\omega$ -closed, and  $\mathcal{C}_s$  holds in the  $\bar{P}$ -extension. Furthermore, in the  $\bar{P}$ -extension,  $\diamond(S)$  holds for every stationary  $S \subset \omega_1$ .*

*Proof.* Let  $X$  be a countable elementary submodel of  $H((2^{|\bar{P}|})^+)$  with  $\bar{P} \in X$ , let  $g$  be an  $X$ -generic filter contained in  $\bar{P} \cap X$ . Let  $\gamma_{X \cap \omega_1}$  be the ordertype of  $X \cap \omega_1$ , and for each  $\beta < \gamma_{X \cap \omega_1}$ , let  $\eta_\beta$  be the  $\beta$ th member of  $X \cap \omega_1$ . For each  $\beta < \gamma_{X \cap \omega_1}$ , let  $a_\beta^{X \cap \omega_1}$  be a cofinal subset of  $X \cap \omega_1$  of ordertype  $\omega$  such that, letting  $g_\eta$  denote the restriction of  $g$  to  $P_\eta$ ,

(a) for all  $\beta' < \beta < \gamma_{X \cap \omega_1}$ ,  $a_\beta^{X \cap \omega_1} \setminus a_{\beta'}^{X \cap \omega_1}$  is finite;

(b)  $a_\beta^\alpha$  is eventually contained in every club subset of  $\omega_1$  in  $X[g_{\eta_\beta}]$  and intersects infinitely every stationary subset of  $\omega_1$  in every  $X[g_{\eta_{\beta'}}]$ ,  $\beta' \in [\beta, \gamma_{X \cap \omega_1})$ .

It remains to see that we can extend  $g$  to a condition whose first coordinate is given by adding

$$c_{X \cap \omega_1} = \langle a_\beta^\alpha : \beta < \gamma_{X \cap \omega_1} \rangle$$

to the union of the first coordinates of the elements of  $g$ , and whose  $\eta$ th coordinate, for each nonzero  $\eta \in X \cap \omega_2$ , is the condition given by the union of  $\{X \cap \omega_1\}$  and the set of realizations of the  $\eta$ th coordinates of the members of  $g$ . We do this by induction on  $\eta$ , letting  $g'_\eta$  be our extended condition in  $P_\eta$ .

<sup>1)</sup> Given sets  $x$  and  $y$ , we say that  $x \subset y$  mod finite if  $x \setminus y$  is finite.

For each  $\eta \in \omega_2 \cap X$ , there is a  $P_\eta$ -name  $\sigma \in X$  for a club subset of  $\omega_1$  such that if in the  $P_\eta$ -extension,  $(\tau_\eta)_{G_\eta}$  is stationary and there is a club  $C$  such that  $\tau_{G_\eta}, C$  is not captured by any  $a_\beta^\alpha$  with  $\alpha \in (\tau_\eta)_{G_\eta}$ , then  $\sigma_{G_\eta}$  is such a  $C$ . However, in this case the realizations of  $\tau_\eta$  and  $\sigma$  by  $g$  are captured by  $a_{o.t.(\eta \cap \omega_2)}^{X \cap \omega_1}$ , so  $g'_\eta$  forces that the pair  $\tau_{G_\eta}, \sigma_{G_\eta}$  is captured by  $a_{o.t.(\eta \cap \omega_2)}^{X \cap \omega_1}$ . It follows that  $g'_\eta$  forces that either  $\mathcal{Q}_\eta$  is  $\mathbb{C}(\omega_1)$ , or  $X \cap \omega_1$  is not in  $\tau_{G_\eta}$ . In either case, the union of the members of  $g \cap \mathcal{Q}_\eta$  can be extended to a condition in  $\mathcal{Q}_\eta$  by adding  $\{X \cap \omega_1\}$ .

To see that  $\diamond(S)$  holds for every stationary  $S \subset \omega_1$  in the  $\bar{P}$ -extension, fix such an  $S$  in the  $P_\alpha$ -extension for some  $\alpha < \omega_2$ . Since  $\bar{P}$  is  $(\omega, \infty)$ -distributive, there exists in this extension a sequence  $\langle e_\beta^\delta : \delta, \beta < \omega_1 \rangle$  such that for each  $\delta < \omega_1$  and each  $x \subset \delta$  there are uncountably many  $\beta$  with  $e_\beta^\delta = x$ . Then, letting  $T \in \mathcal{P}(\omega_1)^{V[G_\alpha]}$  be the set such that the realization of  $\mathcal{Q}_\alpha$  is  $\mathbb{C}(T)$ ,  $\mathcal{Q}_\alpha$  adds a  $\diamond$ -sequence  $\langle b_\delta : \delta \in S \rangle$  defined by letting  $b_\delta$  be  $e_\beta^\delta$ , where the  $\beta$ th element of  $T$  above  $\beta$  is the first element of the generic club for  $\mathcal{Q}_\alpha$  above  $\delta$ . That this is a  $\diamond$ -sequence can be seen as follows: note that since  $S$  is stationary in the  $\bar{P}$ -extension, there exist stationarily many elementary submodels  $X$  of any sufficiently large  $H(\theta)^{V[G]}$  in this extension with  $X \cap \omega_1 \in S$ . Then  $X \cap (G/G_\alpha)$  is an  $(X \cap V[G_\alpha], \bar{P}/P_\alpha)$ -generic filter which can be extended to a condition in  $\bar{P}/P_\alpha$  by adding  $X \cap \omega_1$  to each coordinate, and extended again to make any element of  $T \setminus ((X \cap \omega_1) + 1)$  the least element of the generic club for  $\mathcal{Q}_\alpha$  above  $X \cap \omega_1$ . That  $\langle b_\beta : \beta \in S \rangle$  is a  $\diamond$ -sequence then follows by genericity.  $\square$

Section 3 shows that proper forcing does not always preserve the existence of a winning strategy for split.

## 2.2 A strategy for unsplit

In this section we show that it is consistent for unsplit to have a winning strategy in  $\mathcal{SG}$ . We do this via the following guessing principle.

**Definition 2.7** Let  $\mathcal{D}_u$  be the statement that there is a  $\diamond$ -sequence  $\langle \sigma_\alpha : \alpha < \omega_1 \rangle$  such that for every  $E \subset \omega_1$  there is a club  $C \subset \omega_1$  such that

$$\text{either } (\forall \alpha \in C)((E \cap \alpha = \sigma_\alpha) \Rightarrow \alpha \in E) \text{ or } (\forall \alpha \in C)((E \cap \alpha = \sigma_\alpha) \Rightarrow \alpha \notin E).$$

**Theorem 2.8** *If  $\mathcal{D}_u$  holds, then unsplit has a winning strategy in  $\mathcal{SG}$ .*

**Proof.** Let  $\langle \sigma_\alpha : \alpha < \omega_1 \rangle$  witness  $\mathcal{D}_u$ . Play for unsplit by accepting  $\alpha$  if and only if  $\sigma_\alpha = A_\alpha$ . At the end of the game, the set of  $\alpha$  such that  $\sigma_\alpha = A_\alpha$  is stationary, and there is a club  $C$  such that either for all  $\alpha$  in  $C$ , if  $\sigma_\alpha = A_\alpha$ , then  $\alpha$  is in  $A$ , or for all  $\alpha$  in  $C$ , if  $\sigma_\alpha = A_\alpha$ , then  $\alpha$  is in  $B$ . In either case, split has lost.  $\square$

Our iteration to force  $\mathcal{D}_u$  employs the same strategy as the iteration for  $\mathcal{C}_s$  before. We first force to add a  $\diamond$ -sequence  $\langle \sigma_\alpha : \alpha < \omega_1 \rangle$  by initial segments, and we then iterate to make this sequence witness  $\mathcal{D}_u$ , iteratively forcing a club through the set of those  $\alpha < \omega_1$  for which  $\sigma_\alpha \neq E \cap \alpha$  or  $\alpha \in E$  for each  $E \subset \omega_1$  such that both the sets  $\{\alpha \in E \mid \sigma_\alpha = E \cap \alpha\}$  and  $\{\alpha \in \omega_1 \setminus E \mid \sigma_\alpha = E \cap \alpha\}$  are stationary.

More specifically, we have the following. Given a sequence  $\Sigma = \langle \sigma_\alpha : \alpha < \omega_1 \rangle$  such that each  $\sigma_\alpha$  is a subset of  $\alpha$ , and given  $E \subset \omega_1$ , let  $A(\Sigma, E)$  be the set of  $\alpha \in E$  such that  $\sigma_\alpha = E \cap \alpha$ , and let  $B(\Sigma, E)$  be the set of  $\alpha \in \omega_1 \setminus E$  such that  $\sigma_\alpha = E \cap \alpha$ .

**Theorem 2.9** *Suppose that  $\text{CH} + 2^{\aleph_1} = \aleph_2$  holds, and let  $\bar{P}$  be a countable support iteration*

$$\langle P_\alpha, \mathcal{Q}_\alpha : \alpha < \omega_2 \rangle$$

*such that  $P_0$  is the partial order consisting of sequences  $\langle \sigma_\beta : \beta < \gamma \rangle$ , for some countable ordinal  $\gamma$ , such that each  $\sigma_\beta$  is a subset of  $\beta$ , ordered by extension. Let  $\Sigma$  be the sequence added by  $P_0$  and suppose that the remainder of  $\bar{P}$  satisfies the following conditions.*

1. *Each  $\mathcal{Q}_\alpha$  is either  $\mathbb{C}(\omega_1)$  or  $\mathbb{C}(\omega_1 \setminus B(\Sigma, E))$  for some  $E \subset \omega_1$  such that  $A(\Sigma, E)$  and  $B(\Sigma, E)$  are both stationary.*

2. *For every  $E \subset \omega_1$  in any  $P_\alpha$ -extension ( $\alpha < \omega_2$ ) there is  $\gamma \in [\alpha, \omega_2)$  such that if  $A(\Sigma, E)$  and  $B(\Sigma, E)$  are both stationary in the  $P_\gamma$ -extension, then  $\mathcal{Q}_\gamma$  is  $\mathbb{C}(\omega_1 \setminus B(\Sigma, E))$ .*

*Then  $\bar{P}$  is strategically  $\omega$ -closed, and in the  $\bar{P}$ -extension,  $\mathcal{D}_u$  holds. Furthermore, in the  $\bar{P}$ -extension,  $\diamond(S)$  holds for every stationary  $S \subset \omega_1$ .*

**Proof.** The iteration  $\bar{P}$  is clearly strategically  $\omega$ -closed, since for any countable  $X \prec H((2^{|\bar{P}|})^+)$  and any  $(X, \bar{P})$ -generic filter  $g$  contained in  $X$ , one can extend  $g$  to a condition by making  $\sigma_{X \cap \omega_1}$  unequal to the realization by  $g$  of any name in  $X$  for a subset of  $\omega_1$ , and adding  $X \cap \omega_1$  to all the clubs being added by the  $Q_\alpha$ 's,  $\alpha \in X \cap \omega_2$ . It is clear also that in the  $\bar{P}$ -extension there is no  $E \subset \omega_1$  such that  $A(\Sigma, E)$  and  $B(\Sigma, E)$  are both stationary.

To see that at least one of  $A(\Sigma, E)$  and  $B(\Sigma, E)$  is stationary for each  $E \subset \omega_1$ , we first note the following.

**Claim 2.9.1** *Suppose that  $E \subset \omega_1$  is a member of the  $P_\alpha$ -extension, for some  $\alpha < \omega_2$ , and  $A(\Sigma, E)$  is stationary in this extension. Then  $A(\Sigma, E)$  remains stationary in the  $\bar{P}$ -extension.*

Note that  $A(\Sigma, E)$  has countable intersection with  $B(\Sigma, F)$ , for every  $F \subset \omega_1$ . Fix  $X \prec H(((2^{|\bar{P}|})^+)^V)^{V[G_\alpha]}$  (where  $G_\alpha$  is the restriction of the generic filter  $G$  to  $P_\alpha$ ) with  $X \cap \omega_1 \in A(\Sigma, E)$  and  $A(\Sigma, E) \in X$ . Then any  $(X, \bar{P}/P_\alpha)$ -generic filter contained in  $X$  can be extended to a condition by adding  $X \cap \omega_1$  to the clubs being added at every stage of  $\bar{P}$  after the first.

Similar reasoning shows the next two facts, which complete the proof that  $\Sigma$  witnesses  $\mathcal{D}_\omega$  in the  $\bar{P}$ -extension.

**Claim 2.9.2** *Suppose that  $E \subset \omega_1$  is a member of the  $P_\alpha$ -extension, for some  $\alpha < \omega_2$ , and not a member of the  $P_\gamma$ -extension, for any  $\gamma < \alpha$ . Then  $A(\Sigma, E) \cup B(\Sigma, E)$  is stationary in the  $P_\alpha$ -extension.*

To see Claim 2.9.2, let  $\tau$  be a  $P_\alpha$ -name for a subset of  $\omega_1$  which is forced to be unequal to any such subset in any  $P_\gamma$ -extension, for any  $\gamma < \alpha$ . Fix  $X \prec H(((2^{|\bar{P}|})^+)^V)$  with  $\tau \in X$ . Let  $g$  be an  $(X, P_\alpha)$ -generic filter, and note that then the realization of  $\tau \upharpoonright (X \cap \omega_1)$  by  $g$  is different from the realizations of  $\varrho \upharpoonright (X \cap \omega_1)$  by  $g$  for any  $P_\gamma$ -name  $\varrho \in X$  for a subset of  $\omega_1$ , for any  $\gamma \in X \cap \alpha$ . It follows that adding the realization of  $\tau \upharpoonright (X \cap \omega_1)$  by  $g$  to the union of the first coordinate projection of  $g$  gives a condition in  $P_0$  which forces that  $X \cap \omega_1$  is not in any  $\Sigma(B, \varrho_{G_\gamma})$ , for any  $P_\gamma$ -name  $\varrho \in X$  for a subset of  $\omega_1$ , for any  $\gamma \in X \cap \alpha$ . Therefore, we can add  $X \cap \omega_1$  to the clubs being added in every other stage of  $\bar{P}$  in  $X \cap \alpha$ , and get a condition extending every condition in  $g$ .

**Claim 2.9.3** *Suppose that  $E \subset \omega_1$  is a member of the  $P_\alpha$ -extension, for some  $\alpha < \omega_2$ , and  $A(\Sigma, E)$  is non-stationary in this extension. Then  $B(\Sigma, E)$  remains stationary in the  $\bar{P}$ -extension.*

This is similar to the previous claims, noting that every subsequent stage of  $\bar{P}$  forces a club through the complement of a set with countable intersection with  $B(\Sigma, E)$ .

The proof that  $\diamond(S)$  holds for every stationary  $S \subset \omega_1$  in the  $\bar{P}$ -extension is (literally) the same as in the proof of Theorem 2.6.  $\square$

Note that that the iterations  $\bar{P}$  in Theorems 2.6 and 2.9 are strategically  $\omega$ -closed.

### 2.3 $\Sigma_2^2$ maximality

The statements that split and unsplit have winning strategies in  $\mathcal{SG}$  are each  $\Sigma_2^2$  in a predicate for  $NS_{\omega_1}$ , and they are obviously not consistent with each other. Woodin (see [6]) has shown that if there is a proper class of measurable Woodin cardinals, then there exists in a forcing extension a transitive class model of ZFC satisfying all  $\Sigma_2^2$  sentences  $\varphi$  such that  $\varphi + \text{CH}$  can be forced over the ground model. The results here show that this result cannot be extended to include a predicate for  $NS_{\omega_1}$ . This was known already, in that  $\diamond^*$  (in the sense of [7]) and “the restriction of  $NS_{\omega_1}$  to some stationary set is  $\aleph_1$  dense” were both known to be consistent with  $\diamond$  (the second of these is due to Woodin, uses large cardinals, and is unpublished, though a related proof, also due to Woodin, appears in [3]). Our example is simpler and does not use large cardinals; it also gives (we believe, for the first time) a counterexample consisting of two sentences each consistent with “ $\diamond(S)$  holds for every stationary set  $S \subset \omega_1$ ”.

### 2.4 A determined variation

There are many natural variations of  $\mathcal{SG}$ . We show that one such variation is determined.

**Theorem 2.10** *Let  $\mathcal{G}$  be the following game of length  $\omega_1$ . In round  $\alpha$ , player I puts  $\alpha$  into one of two sets  $E_0$  and  $E_1$ , and player II puts  $\alpha$  into one of two sets  $A_0$  and  $A_1$ . After all  $\omega_1$  rounds have been played, II wins if one of the following pairs of set are both stationary:*

1.  $E_0 \cap A_0$  and  $E_0 \cap A_1$ ,
2.  $E_1 \cap A_0$  and  $E_1 \cap A_1$ .

*Then II has a winning strategy in  $\mathcal{G}$ .*

*Proof.* Let  $B_{00}, B_{01}, B_{10}$ , and  $B_{11}$  be pairwise disjoint stationary subsets of  $\omega_1$ . In round  $\alpha$ , if  $\alpha$  is in  $B_{ij}$ , let  $\text{II}$  put  $\alpha$  in  $A_i$  if  $\text{I}$  put  $\alpha$  in  $E_0$  and in  $A_j$  otherwise. Then after all  $\omega_1$  many rounds have been played, suppose that  $A_i \cap E_0$  is nonstationary. Then  $B_{i0}, B_{i1}$  are both contained in  $E_1$  modulo  $NS_{\omega_1}$ , which means that  $E_1 \cap A_0$  and  $E_1 \cap A_1$  are both stationary. Similarly, if  $A_i \cap E_1$  is nonstationary, then  $B_{0i}, B_{1i}$  are both contained in  $E_0$  modulo  $NS_{\omega_1}$ , which means that  $E_0 \cap A_0$  and  $E_0 \cap A_1$  are both stationary.  $\square$

### 3 Indeterminacy from forcing axioms

The axiom  $\text{PFA}^{+2}$  says that whenever  $P$  is a proper partial order,  $D_\alpha$  ( $\alpha < \omega_1$ ) are dense subsets of  $P$ , and  $\sigma_1, \sigma_2$  are  $P$ -names for stationary subsets of  $\omega_1$ , there is a filter  $G \subset P$  such that  $G \cap D_\alpha \neq \emptyset$  for each  $\alpha < \omega_1$ , and such that  $\{\alpha < \omega_1 \mid (\exists p \in G)(p \Vdash \check{\alpha} \in \sigma_i)\}$  is stationary for every  $i \in \{1, 2\}$ . Theorems 2.6 and 2.9 together show that  $\text{PFA}^{+2}$  implies the indeterminacy of  $\mathcal{SG}$ . Furthermore, a straightforward argument shows that the following statement implies the nonexistence of a winning strategy for unsplit in  $\mathcal{SG}$ , where  $\text{Add}(1, \omega_1)$  is the partial order that adds a subset of  $\omega_1$  by initial segments: for any pair  $\sigma_1, \sigma_2$  of  $\text{Add}(1, \omega_1)$ -names for stationary subsets of  $\omega_1$ , there is a filter  $G \subset \text{Add}(1, \omega_1)$  realizing both  $\sigma_1$  and  $\sigma_2$  as stationary sets. This statement is trivially subsumed by  $\text{PFA}^{+2}$ , but also holds in the collapse of a sufficiently large cardinal to be  $\omega_2$ , and thus is consistent with CH.

The axiom Martin's Maximum [4] says that whenever  $P$  is a partial order such that forcing with  $P$  preserves stationary subsets of  $\omega_1$  and  $D_\alpha$  ( $\alpha < \omega_1$ ) are dense subsets of  $P$ , there is a filter  $G \subset P$  such that  $G \cap D_\alpha \neq \emptyset$  for each  $\alpha < \omega_1$ .

**Theorem 3.1** *Martin's Maximum implies that  $\mathcal{SG}$  is undetermined.*

*Proof.* Fix a strategy  $\Sigma$  for unsplit in  $\mathcal{SG}$ , and let  $E, A$ , and  $B$  be the result of a generic run of  $\mathcal{SG}$  where unsplit plays by  $\Sigma$  (the partial order consists of countable partial plays where unsplit plays by  $\Sigma$ , ordered by extension). If the complement of  $E$  has stationary intersection with every stationary subset of  $\omega_1$  in the ground model, one can force to kill the stationarity of  $E$  in such a way that the induced two step forcing preserves stationary subsets of  $\omega_1$  and produces a run of  $\mathcal{SG}$  where unsplit plays by  $\Sigma$  and loses. If the complement of  $E$  does not have stationary intersection with some stationary  $F \subset \omega_1$  in the ground model, then there is a partial run of the game  $p$  and a name  $\tau$  for a club such that  $p$  forces that  $E$  will contain  $F \cap \tau_G$ . Then there exists in the ground model a run of  $\mathcal{SG}$  extending  $p$  in which unsplit plays by  $\Sigma$  and loses: split picks a pair of disjoint stationary subsets  $F_0, F_1$  of  $F$ , and plays so that

1. for every  $\alpha < \omega_1$ , some initial segment of the play forces some ordinal greater than  $\alpha$  to be in  $\tau$ ;
2. whenever unsplit accepts  $\alpha \in F$ , split puts  $\alpha$  in  $A$  if  $\alpha \in F_0$  and puts  $\alpha \in B$  if  $\alpha \in F_1$ .

Now fix a strategy  $\Sigma$  for split in  $\mathcal{SG}$ , and generically add a regressive function  $f$  on  $\omega_1$  by initial segments. Let  $E^\alpha = f^{-1}(\alpha)$  and let  $A^\alpha, B^\alpha$  be the responses given by  $\Sigma$  to a play of  $E^\alpha$  by unsplit. Note that each  $E^\alpha$  will be stationary.

Suppose that there exist  $\alpha < \omega_1$  and stationary sets  $S, T$  in the ground model such that both  $(S \cap E^\alpha) \setminus A^\alpha$  and  $(T \cap E^\alpha) \setminus B^\alpha$  are nonstationary. Then there is a condition  $p$  in our forcing (i. e., a regressive function on some countable ordinal) such that  $p$  forces  $(S \cap E^\alpha) \subset A^\alpha$  and  $(T \cap E^\alpha) \subset B^\alpha$ , modulo nonstationarity (and so in particular  $S$  and  $T$  have nonstationary intersection). Let  $\tau$  be a name for a club disjoint from  $(S \cap E^\alpha) \setminus A^\alpha$  and  $(T \cap E^\alpha) \setminus B^\alpha$ . Extend  $p$  to a filter  $f$  (identified with the corresponding function) realizing  $\tau$  as a club subset of  $\omega_1$ , at successor stages extending to add a new element to the realization of  $\tau$ , and at limit stages (when for some  $\beta < \omega$ ,  $f \upharpoonright \beta$  has been decided and  $f(\beta)$  has not, and  $\beta$  is forced by  $f \upharpoonright \beta$  to be a limit member of the realization of  $\tau$ ) extending so that  $f(\beta) = \alpha$  if and only if  $\beta \in S$ . Then the run of  $\mathcal{SG}$  corresponding to  $f^{-1}(\alpha)$  is winning for unsplit, since the corresponding set  $B^\alpha$  is nonstationary.

If there exist no such  $\alpha, S, T$ , there is a function  $h$  on  $\omega_1$  such that each  $h(\alpha) \in \{A^\alpha, B^\alpha\}$  and the forcing to shoot a club through the set of  $\beta$  such that  $f(\beta) = \alpha \Rightarrow \beta \in h(\alpha)$  preserves stationary subsets of the ground model. Then Martin's Maximum applied to the corresponding two step forcing produces a run of  $\mathcal{SG}$  (the run for any  $f^{-1}(\alpha)$  which is stationary) where split plays by  $\Sigma$  and loses.  $\square$

Theorem 3.1 leads to the following question.

**Question 3.2** *Does the Proper Forcing Axiom imply that  $\mathcal{SG}$  is not determined?*

The following question is also interesting. The consistency of the  $\aleph_1$ -density of  $NS_{\omega_1}$  (relative to the consistency of  $AD^{L(\mathbb{R})}$ ) is shown in [13].

**Question 3.3** *Does the  $\aleph_1$ -density of  $NS_{\omega_1}$  decide the determinacy of  $\mathcal{SG}$ ?*

## 4 MLO games

The second-order Monadic Logic of Order (MLO) is an extension of first-order logic with logical constants  $=$ ,  $\in$ , and  $\subset$  and a binary symbol  $<$  as the only non-logical constant, allowing quantification over subsets of the domain. Every ordinal is a model for MLO, interpreting  $<$  as  $\in$ .

Given an ordinal  $\alpha$ , an MLO game of length  $\alpha$  is determined by an MLO formula  $\varphi$  with two free variables for subsets of the domain. In such a game, two players each build a subset of  $\alpha$ , and the winner is determined by whether these two sets satisfy the formula in  $\alpha$ .

Büchi and Landweber [1] proved the determinacy of all MLO games of length  $\omega$ . Recently, Shomrat [12] extended this result to games of length less than  $\omega^\omega$ , and Rabinovich [10] extended it further to all MLO games of countable length. The stationary set splitting game is an example of an MLO game of length  $\omega_1$  whose determinacy is independent of ZFC.

We thank Assaf Rinot for pointing out to us the connection between  $\mathcal{SG}$  and MLO games.

**Acknowledgements** The work in this paper began during the Set Theory and Analysis program at the Fields Institute in the Fall of 2002. The first author is supported in part by NSF grant DMS-0401603, and thanks Juris Steprāns, Paul Szeptycki, and Tetsuya Ishii for helpful conversations on this topic. The research of the second author is supported by the United States-Israel Binational Science Foundation. This is the second author's publication 902. Some of the research in this paper was conducted during a visit by the first author to Rutgers University, supported by NSF grant DMS-0600940.

## References

- [1] J. R. Büchi and L. H. Landweber, Solving sequential conditions by finite-state strategies. *Trans. Amer. Math. Soc.* **138**, 295 – 311 (1969).
- [2] Q. Feng and T. Jech, Projective stationary sets and a strong reflection principle. *J. London Math. Soc.* **58**, 271 – 283 (1998).
- [3] M. Foreman, Ideals and generic elementary embeddings. To appear in *Handbook of Set Theory*.
- [4] M. Foreman, M. Magidor, and S. Shelah, Martin's Maximum, saturated ideals, and non-regular ultrafilters. Part I. *Annals of Mathematics* **127**, 1 – 47 (1988).
- [5] T. Jech, *Set Theory. The Third Millennium Edition, revised and expanded*. Springer Monographs in Mathematics (Springer-Verlag, 2003).
- [6] R. Ketchersid, P. Larson, and J. Zapletal,  $\Sigma_2^2$ -maximality, homogeneity and the stationary tower. Preprint.
- [7] K. Kunen, *Set Theory* (North-Holland, 1980).
- [8] P. Larson, The canonical function game. *Arch. Math. Logic* **44**, 817 – 827 (2005).
- [9] J. Moore, Set Mapping Reflection. *J. Math. Logic* **5**, 87 – 98 (2005).
- [10] A. Rabinovich, The Church problem for countable ordinals. Preprint, 2007.
- [11] S. Shelah, *Proper and Improper Forcing. Perspectives in Mathematical Logic* (Springer-Verlag, 1998).
- [12] A. Shomrat, Uniformization problems in the monadic theory of countable ordinals. Manuscript, 2006.
- [13] W. H. Woodin, The axiom of determinacy, forcing axioms and the nonstationary ideal. *De Gruyter Series in Logic and Its Applications 1* (Walter de Gruyter & Co., 1999).