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Categoricity for abstract classes with amalgamation[1]

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Abstract

Let $\mathfrak{K}$ be an abstract elementary class with amalgamation, and Lowenheim Skolem number LS($\mathfrak{K}$). We prove that for a suitable Hanf number χ_0 if $\chi_0 < \lambda_0 \leqslant \lambda_1$, and $\mathfrak{K}$ is categorical in λ_1^+ then it is categorical in λ_0. © 1999 Elsevier Science B.V. All rights reserved.

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0. Introduction

We try to find something on

$$\text{Cat}_K = \{\lambda: K \text{ categorical in } \lambda\}$$

for K an abstract elementary class with amalgamation (see Definition 0.1 below).

The Los conjecture = Morley theorem deals with the case where K is the class of models of a countable first order theory T. See [14] for more on first order theories. What for T an infinitary language? (For a theory T, K is the class $K_T = \{M: M \models T\}$ we may write Cat_K). Keisler gets what can be gotten from Morley's proof on $\psi \in L_{\aleph_1,\aleph_0}$. Then see [7] on categoricity in $\aleph_1$ for $\psi \in L_{\aleph_1,\aleph_0}$ and even $\psi \in L_{\aleph_1,\aleph_0}(Q)$, and [9, 10] on the behaviour in the $\aleph_n$'s. Makkai Shelah [5] proved, if $T \subseteq L_{\kappa,\aleph_0}$, κ a compact cardinal then $\text{Cat}_K \cap \{\mu^+: \mu \geqslant \beth_{(2^\kappa+|T|)^+}\}$ is empty or is $\{\mu^+: \mu \geqslant \beth_{(2^\kappa+|T|)^+}\}$ (it relies on some developments from [13] but is self-contained).

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It was then reasonable to deal with weakening the requirement on κ to measurability. Kolman Shelah [4] proved that if $\mu \in \mathrm{Cat}_T$, *then* (after cosmetic changes), for the right $\leqslant_T$ the class $\{M: M \models T, \ \|M\| < \lambda\}$ has amalgamation and joint embedding property. This is continued in [19] which gets results on categoricity parallel to the one in [5] for the "downward" implication.

In [11] we deal with abstract elementary classes (they include models of $T \subseteq L_{\kappa,\aleph_0}$, see Definition 0.1), prove a representation theorem (see Claim 0.5 below), and investigate categoricity in $\aleph_1$ (and having models in $\aleph_2$, limits models, realizing and materializing types). Unfortunately, we do not have anything interesting to say here on this context. So we add amalgamation and the joint embedding properties thus getting to the framework of Jonsson [3] (they are the ones needed to construct homogeneous universal models). So this context is more narrow than the ones discussed above, but we do not use large cardinals. We concentrate here, for categoricity on λ, on the case "λ is regular", $\lambda > \beth_{(2^{LS(\mathfrak{K})})^+}$. See later [18, 21, 22] and for more details [15].

We quote the basics from [11] (or [18]).

Definition 0.1. $\mathfrak{K} = (K, \leqslant_{\mathfrak{K}})$ is an abstract elementary class if for some vocabulary $\tau = \tau(K)$, K is a class of $\tau(K)$-models, and the following axioms hold:

AxO: The holding of $M \in K$, $N \leqslant_{\mathfrak{K}} M$ depends on N, M only up to isomorphism i.e. $[M \in K, M \cong N \Rightarrow N \in K]$, and [if $N \leqslant_{\mathfrak{K}} M$ and f is an isomorphism from M onto the τ-model M' mapping N onto N' *then* $N' \leqslant_{\mathfrak{K}} M'$].

AxI: If $M \leqslant_{\mathfrak{K}} N$ then $M \subseteq N$ (i.e. M is a submodel of N).

AxII: $M_0 \leqslant_{\mathfrak{K}} M_1 \leqslant_{\mathfrak{K}} M_2$ implies $M_0 \leqslant_{\mathfrak{K}} M_2$ and $M \leqslant_{\mathfrak{K}} M$ and for $M \in K$.

AxIII: If λ is a regular cardinal, M_i $(i < \lambda)$ is a $\leqslant_{\mathfrak{K}}$-increasing (i.e. $i < j < \lambda$ implies $M_i \leqslant_{\mathfrak{K}} M_j$) and continuous (i.e. for limit ordinal $\delta < \lambda$ we have $M_\delta = \bigcup_{i<\delta} M_i$) *then* $M_0 \leqslant_{\mathfrak{K}} \bigcup_{i<\lambda} M_i \in \mathfrak{K}$.

AxIV: If λ is a regular cardinal, M_i $(i < \lambda)$ is $\leqslant_{\mathfrak{K}}$-increasing continuous, $M_i \leqslant_{\mathfrak{K}} N$ *then* $\bigcup_{i<\lambda} M_i \leqslant_{\mathfrak{K}} N$.

AxV: If $M_0 \subseteq M_1$ and $M_\ell \leqslant_{\mathfrak{K}} N$ for $\ell = 0, 1$, then $M_0 \leqslant_{\mathfrak{K}} M_1$.

ArVI: $LS(\mathfrak{K})$ exists,[2] where $LS(\mathfrak{K})$ is the minimal cardinal λ such that: if $A \subseteq N$ and $|A| \leqslant \lambda$ *then* for some $M \leqslant_{\mathfrak{K}} N$ we have $A \subseteq |M| \leqslant \lambda$ and we demand for simplicity $|\tau| \leqslant \lambda$.

Definition 0.2. (1) $K_\mu =: \{M \in K: \|M\| = \mu\}$.

(2) We say h is a $\leqslant_{\mathfrak{K}}$-embedding of M into N is for some $M' \leqslant_{\mathfrak{K}} N$, h is an isomorphism from M onto M'.

Definition 0.3. (1) We say that μ is a Skolem Lowenheim number of $\mathfrak{K}$ if
$\mu \geqslant |\tau(K)| + \aleph_0$ and
$(*)^\mu_K$ for every $M \in K$, $A \subseteq M$, $|A| \leqslant \mu$ there is M', $A \subseteq M' \leqslant_{\mathfrak{K}} M$ and $\|M'\| \leqslant \mu$.

(2) $LS(\mathfrak{K}) = \mathrm{Min}\{\mu: \mu$ is a Skolem Lowenheim number of $\mathfrak{K}$ and $|\tau(\mathfrak{K})| \leqslant \mu\}$.

[2] We normally assume $M \in \mathfrak{K} \Rightarrow \|M\| \geqslant LS(\mathfrak{K})$, here there is no loss in it. It is also natural to assume $|\tau(\mathfrak{K})| \leqslant LS(\mathfrak{K})$ which just means increasing $LS(\mathfrak{K})$.

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Claim 0.4. (1) *If I is a directed partial order, $M_t \in K$ for $t \in I$ and $s <_I t \Rightarrow M_s \leqslant_{\mathfrak{K}} M_t$* then

(a) $M_s \leqslant_{\mathfrak{K}} \bigcup_{t \in I} M_t \in K$ *for every* $s \in I$,

(b) *if* $(\forall t \in I)$ $[M_t \leqslant_{\mathfrak{K}} N]$ *then* $\bigcup_{t \in I} M_t \leqslant_{\mathfrak{K}} N$.

(2) *If* $A \subseteq M \in K$, $|A| + LS(\mathfrak{K}) \leqslant \mu \leqslant \|M\|$, then *there is* $M_1 \leqslant_{\mathfrak{K}} M$ *such that* $\|M_1\| = \mu$ *and* $A \subseteq M_1$.

(3) *If I is a directed partial order,* $M_t \leqslant N_t \in K$ *for* $t \in I$, *and* $s \leqslant_I t \Rightarrow M_s \leqslant_{\mathfrak{K}} M_t$ & $N_s \leqslant_{\mathfrak{K}} N_t$ *then* $\bigcup_t M_t \leqslant_{\mathfrak{K}} \bigcup_t N_t$.

Claim 0.5. *Let $\mathfrak{K}$ be an abstract elementary class. There are τ^+, Γ such that*

(a) τ^+ *is a vocabulary extending* $\tau(K)$ *of cardinality* $LS(\mathfrak{K})$,

(b) Γ *is a set of quantifier free types in* τ^+ *(each is an m-type for some $m < \omega$)*,

(c) $M \in K$ *iff for some τ^+-model M^+ omitting every $p \in \Gamma$ we have* $M = M^+ \restriction \tau$,

(d) $M \leqslant_{\mathfrak{K}} N$ iff *there are τ^+-models M^+, N^+ omitting every $p \in \Gamma$ such that* $M^+ \subseteq N^+$, $M = M^+ \restriction \tau(K)$, $N = N^+ \restriction \tau(K)$.

(e) *if M^+ is a τ^+-model omitting every $p \in \Gamma$ and* $M \restriction \tau(K) \leqslant_{\mathfrak{K}} N$, *then there is τ^+-model N^+ omitting every $p \in \Gamma$ and* $N^+ \restriction \tau(K) = N$, $M^+ \subseteq N^+$.

Claim 0.6. *Assume K has a member of cardinality* $\geqslant \beth_{(2^{LS(\mathfrak{K})})^+}$ *(here and elsewhere we can weaken this to: has a model of cardinality* $\geqslant \beth_\alpha$ *for every* $\alpha < (2^{LS(\mathfrak{K})})^+$). Then *there is Φ proper for linear orders* (*see* [14, Ch. VII, Section 2]) *such that*

(a) $|\tau(\Phi)| = LS(\mathfrak{K})$,

(b) *for linear orders* $I \subseteq J$ *we have* $EM_\tau(I, \Phi) \leqslant_{\mathfrak{K}} EM(J, \Phi)$ $(\in K)$,

(c) $EM_\tau(I, \Phi)$ *has cardinality* $|I| + LS(K)$ *(so K has a model in every cardinality* $\geqslant LS(K)$).

PART I

1. The framework

Hypothesis 1.1. (a) $\mathfrak{K} = (K, \leqslant_{\mathfrak{K}})$ an abstract elementary class (0.1) so $K_\lambda = \{M \in K: \|M\| = \lambda\}$,

(b) K has amalgamation and the joint embedding property,

(c) K has members of arbitrarily large cardinality, equivalently: K has a member of cardinality at least $\beth_{(2^{LS(\mathfrak{K})})^+}$.

Remark 1.2. (1) So there is a monster $\mathfrak{C}$ (see [8, Ch. I, Section 1]=[14, Ch. I, Section 1]).

Definition 1.3. K is categorical in λ if it has one and only one model of cardinality λ, up to isomorphism.

Definition 1.4. (1) We can define $\mathrm{tp}(\bar{a}, M, N)$ (when $M \leqslant_{\mathfrak{K}} N$, $\bar{a} \subseteq N$), as $(\bar{a}, M, N)/E$ where E is the following equivalence relation: $(\bar{a}^1, M^1, N^1)\ E(\bar{a}^2, M^2, N^2)$ *iff* $M^\ell \leqslant_{\mathfrak{K}} N^\ell$, $\bar{a}^\ell \in {}^\alpha(N^\ell)$ (for some α) and $M^1 = M^2$ and there is $N \in K$ satisfying $M^1 = M^2 \leqslant_{\mathfrak{K}} N$

and $\leqslant_{\mathfrak{K}}$-embedding $f^{\ell}: N^{\ell} \to N$ over M^{ℓ} (i.e. $f \restriction M^{\ell}$ is the identity) and $f^1(\bar{a}^1) = f^2(\bar{a}^2)$. Let $p_1 = p_2 \restriction M_1$ if $M_1 \leqslant_{\mathfrak{K}} M_2 \leqslant_{\mathfrak{K}} M, a \in {}^{\alpha}M$ and $p_{\ell} = \mathrm{tp}(\bar{a}, M_{\ell}, M)$.

(2) We omit N when $N = \mathfrak{C}$ (see later) and may then write $\frac{\bar{a}}{M}$. We can define N is κ-saturated (when $\kappa > LS(\mathfrak{K})$) by: if $M \leqslant_{\mathfrak{K}} N$, $\|M\| < \kappa$ and $p \in \mathscr{S}^{<\omega}(M)$ (see below) then p is realized in M, i.e. for some $\bar{a} \subseteq N$, $p = \mathrm{tp}(\bar{a}, M, N)$.

(3) $\mathscr{S}^{\alpha}(M) = \{\mathrm{tp}(\bar{a}, M, N)$: $\bar{a} \in {}^{\alpha}N$, $M \leqslant_{\mathfrak{K}} N\}$; we define $p \restriction M$ when $M \leqslant_{\mathfrak{K}} N \,\&\, p \in \mathscr{S}(N)$ as $\mathrm{tp}(\bar{a}, M, N_1)$ when $N \leqslant_{\mathfrak{K}} N_1, p = \mathrm{tp}(\bar{a}, N, N_1)$. Let $p \leqslant q$ mean $p \in \mathscr{S}(M)$, $q \in \mathscr{S}(N)$, $p = q \restriction M$; see [13, Ch. II] or [18, Section 0].

(4) $\mathfrak{C}$ is the monster model (as in [14]) so e.g. $\mathrm{tp}(\bar{a}, \phi, M)$ is defined naturally.

(5) $\mathscr{S}(M) = \mathscr{S}^1(M)$ (could just as well use $\mathscr{S}^{<\omega}(M)$).

Definition 1.5. K stable in λ means: $\|M\| \leqslant \lambda \Rightarrow |\mathscr{S}(M)| \leqslant \lambda$ and $\lambda \geqslant LS(\mathfrak{K})$.

Convention 1.6. If not said otherwise, Φ is as in Claim 0.6.

Claim 1.7. *If K is categorical in λ and $\lambda \geqslant LS(\mathfrak{K})$*, then
(a) *K is stable in every μ, $LS(\mathfrak{K}) \leqslant \mu < \lambda$, hence*
(b) *the model $M \in K_{\lambda}$ is $cf(\lambda)$-saturated (if $cf(\lambda) > LS(\mathfrak{K})$).*

Proof. Like [4]. □

Definition 1.8. E_{μ} is the following relation,

$$p \; E_{\mu} \, q \text{ iff for some } M \in K,\ m < \omega \text{ we have}$$
$$p, q \in \mathscr{S}^m(M) \quad \text{and} \quad [N \leqslant_{\mathfrak{K}} M \,\&\, \|N\| \leqslant \mu \Rightarrow p \restriction N = q \restriction N].$$

Obviously, it is an equivalence relation.

Remark 1.9. In previous contexts $E_{LS(\mathfrak{K})}$ is equality, e.g. the axioms of NF in [13, Ch. II, Section 1] show it; but here we do not know – this is the main difficulty. We may look at this as our bad luck, or inversely, a place to encounter some of the difficulty of dealing with $L_{\mu,\omega}$ (in which our context is included).

Claim 1.10. *There is no maximal member in K, in fact for every $M \in K$ there is $N, M \leqslant_{\mathfrak{K}} N \in K$, $\|N\| \leqslant \|M\| + LS(\mathfrak{K})$. $\restriction$ has the obvious properties.*

Proof. Immediate by Hypothesis 1.1.

2. Variant of saturated

Definition 2.1. Assuming K stable in μ and α is an ordinal $< \mu^+, \mu^+ \times \alpha$ means ordinal product.

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(1) $M <^0_{\mu,\alpha} N$ if $M \in K_\mu$, $N \in K_\mu$, $M \leqslant N$ and there is a $\leqslant_{\mathfrak{K}}$-increasing sequence $\bar{M} = \langle M_i\colon i \leqslant \mu \times \alpha\rangle$ which is increasing continuous, $M_0 = M$, $M_{\mu\times\alpha} \leqslant_{\mathfrak{K}} N$ and every $p \in \mathscr{S}^1(M_i)$ is realized in M_{i+1}.

(2) We say $M <^1_{\mu,\alpha} N$ iff $M \in K_\mu$, $N \in K_\mu$, $M \leqslant_{\mathfrak{K}} N$ and there is a $\leqslant_{\mathfrak{K}}$-increasing sequence $\bar{M} = \langle M_i\colon i \leqslant \mu \times \alpha\rangle, M_0 = M$, $M_{\mu\times\alpha} = N$ and every $p \in \mathscr{S}^1(M_i)$ is realized in M_{i+1}.

(3) If $\alpha = 1$, we may omit it.

Lemma 2.2. *Assume K stable in μ and $\alpha < \mu^+$.*

(0) *If $\ell \in \{0,1\}$ and $\alpha_1 < \alpha_2 < \mu^+$ and there is $b \subseteq \alpha_2$ such that $otp(b) = \alpha_1$ and [$\ell = 1 \Rightarrow b$ unbounded in α_2]* then $<^\ell_{\mu,\alpha_2} \subseteq <^\ell_{\mu,\alpha_1}$.

(1) *If $M \in K_\mu$, then for some N we have $M <^0_{\mu,\alpha} N$ and for some $N, M <^1_{\mu,\alpha} N$.*

(2) (a) *If $M \in K_\mu, M \leqslant_{\mathfrak{K}} M' \leqslant^\ell_{\mu,\alpha} N$* then $M \leqslant^\ell_{\mu,\alpha} N$.

(b) *If $M \in K_\mu, M \leqslant_{\mathfrak{K}} M' \leqslant^0_{\mu,\alpha} N' \leqslant_{\mathfrak{K}} N \in K_\mu$ then $M \leqslant^0_{\mu,\alpha} N$.*

(3) *If $\langle M_i\colon i < \alpha\rangle$ is $\leqslant_{\mathfrak{K}}$-increasing sequence in K_μ, $M_i \leqslant^0_\mu M_{i+1}$ and $\alpha < \mu^+$ is a limit ordinal,* then $M_0 \leqslant^1_{\mu,\alpha} \bigcup_{i<\alpha} M_i$.

(4) *If $M \leqslant^0_\mu N$* then:

(a) *any $M' \in K_\mu$ can be $\leqslant_{\mathfrak{K}}$-embedded into N (here we can waive $\|M\| = \mu$).*

(b) *If $M' \leqslant_{\mathfrak{K}} N' \in K_{\leqslant\mu}, h$ is a $\leqslant_{\mathfrak{K}}$-embedding of M' into M then h can be extended to a $\leqslant_{\mathfrak{K}}$-embedding of N' into N.*

(5) *If $M^\ell \leqslant^1_{\mu,\kappa} N^\ell$ for $\ell = 1,2, h$ an isomorphism from M^1 into [onto] M^2 then h can be extended to an isomorphism from N^1 into [onto] N^2.*

(6) *If $M \leqslant^1_{\mu,\kappa} N^\ell$ for $\ell = 1,2$ then $N^1 \cong N^2$ (even over M).*

(7) *If $M \leqslant^0_{\mu,\kappa} N$, $M \leqslant_{\mathfrak{K}} M' \in K_\mu$* then *$M'$ can be $<_{\mathfrak{K}}$-embedded into N over M.*

(8) *If $\mu \geqslant \kappa > LS(\mathfrak{K}), M <^1_{\mu,\kappa} N$ then N is $cf(\kappa)$-saturated.*

Proof. See [13, Ch. II, 3.10, p. 319] and around.

Discussion 2.3. There (in [13, Ch. II, 3.6]) the main point was that for $\kappa > LS(\mathfrak{K})$, the notions "$\kappa$-homogeneous universal" and κ-saturation (i.e. every "small" 1-type is realized) are equivalent.

Not hard, still [13, Ch. II, 3.6] was a surprise to some. In first order the equivalence saturated $\equiv$ homogeneous universal for $\prec$ seemed a posteriori natural as the homogeneity used was anyhow for sequences of elements realizing the same first order formulas so (forgetting about the models) to some extent this seemed natural; i.e. asking this for any type of 1-element was very natural.

But here, types of 1-element are really meaningful only over a model. So it seems that if over any small submodel every type of 1-element is realized (say in $\mathfrak{A}$) and we want to embed $N \geqslant_{\mathfrak{K}} N_0, N_0 \leqslant_{\mathfrak{K}} \mathfrak{A}$ into $\mathfrak{A}$ over N_0, we encounter the following problem: we cannot continue this as after ω stages we get a set which is not a model (if $LS(\mathfrak{K}) > \aleph_0$ this absolutely necessarily fails; and is if $LS(\mathfrak{K}) = \aleph_0$ at best the situation as in [9]).

This explains a natural preconception making you not believe; i.e. psychological barrier to prove. It *does* not mean that the proof is hard.
In the explanation of Ch. I, *Lemma* 2.2(8), *of course, we assumed* κ *regular.*

Definition 2.4. $M \in K$ is κ-saturated if $\kappa > LS(\mathfrak{K})$ and:
$N \leqslant_{\mathfrak{K}} M, \|N\| < \kappa, p \in \mathscr{S}^1(N) \Rightarrow p$ realized in M.

Remark. Note $\leqslant^1_{\mu,\kappa}, \kappa$ regular are the interesting ones. Still $\leqslant^0_{\mu,\kappa}$ is enough for universality (Lemma 2.2(4)) and is natural, $\leqslant^1_{\mu,\kappa}$ is natural for uniqueness. *But* $<^1_{\mu,\aleph_0} = <^1_{\mu,\aleph_1}$ can be proved only under categoricity (or something like superstability assumptions). *Look* at first order T stable in μ. Then, $M <^1_{\mu,\kappa} N$ is equivalent to

$$\|M\| = \|N\| = \mu, M, N \models T$$

and there is $\langle M_i \colon i \leqslant \kappa \rangle$, $\prec$-increasing continuous such that

$$M_0 = M, \qquad M_\kappa = N,$$

$$(M_{i+1}, c)_{c \in M_i} \quad \text{is saturated.}$$

Question: Now ... is N saturated when $M <^1_{\mu,\kappa} N$?
Answer: It is iff $\mathrm{cf}(\kappa) \geqslant \kappa_r(T)$. See [20, Ch. IV, Section 3].
See on limit and superlimit models in [11].

Proof of Lemma 2.2(8).
Statement: If $M <^1_{\mu,\kappa} N$ (κ regular) then N is κ-saturated.
Note: if $\kappa \leqslant LS(\mathfrak{K})$ the conclusion is essentially empty, but there is no need for the assumption "$\kappa > LS(\mathfrak{K})$".

Proof. Let $\bar{M} = \langle M_i \colon \; i \leqslant \mu \times \kappa \rangle$ witness $M \leqslant^1_{\mu,\kappa} N$ so $M_0 = M, M_{\mu\times\kappa} = N$, $M_i \leqslant_{\mathfrak{K}}$-increasing continuous and every $p \in \mathscr{S}(M_i)$ is realized in M_{i+1}.
Assume

$$(*) \quad N' \leqslant_{\mathfrak{K}} N, \quad \|N'\| < \kappa, \quad p \in \mathscr{S}(N').$$

We should prove that "p is realized in N". But $\langle M_i \colon \; i \leqslant \mu \times \kappa \rangle$ is increasing continuous

$$\mathrm{cf}(\mu \times \kappa) = \kappa > \|N'\|$$

so $N' \leqslant_{\mathfrak{K}} M_{\mu\times\kappa} = \bigcup_{i<\mu\times\kappa} M_i$ implies there is $i(*) < \mu \times \kappa$, such that $N' \subseteq M_{i(*)}$ hence by Axiom V $N' \leqslant_{\mathfrak{K}} M_{i(*)}$. So p has an extension $p^* \in \mathscr{S}(M_{i(*)})$ and p^* is realized in $M_{i(*)+1}$ so in $M_{\mu\times\kappa} = N$. $\square$

Comment: Hence length κ (instead of $\mu \times \kappa$) suffices.
But for the uniqueness it does not. See Lemma 2.2(4) + (5).
Comment: The definition of $\leqslant^0_{\mu,\kappa}, \leqslant^1_{\mu,\kappa}$ is also essentially taken from [13, Ch. II, 3.10]. We need the intermediate steps to construct models so we have to have μ of them in order to deal with all the elements.

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Claim 2.5. *If K is categorical in $\lambda, M \in K_\lambda$ and $cf(\lambda) > \mu$ then: if $N <_{\mathfrak{K}} M \in K_\lambda$, $N \in K_\mu$, $N' <_{\mathfrak{K}} M$, h is an isomorphism from N onto N'*, then *h can be extended to an automorphism of M.*

Proof. We can find $\langle M_i\colon\ i < \lambda\rangle$ which is $<_{\mathfrak{K}}$-increasing continuous, $\|M_i\| = |i| + LS(\mathfrak{K}), M_i <^1_{|i|+LS(\mathfrak{K}), |i|+LS(\mathfrak{K})} M_{i+1}$. By the categoricity assumption without loss of generality $M = \bigcup_{i<\lambda} M_i$. As $\mathrm{cf}(\lambda) > \mu$ for some $i_0 < \lambda$ we have $N, N' \prec M_{i_0}$.

By Lemma 2.2 we can build an automorphism. □

Definition 2.6. For $\mu \geqslant LS(\mathfrak{K})$, we say $N \in K_\mu$ is (μ, κ)-saturated (or (μ, κ)-limit) if for some M we have $M <^1_{\mu,\kappa} N$ (so κ is $\leqslant \mu$, normally regular).

Fact 2.7. (1) *The (μ, κ)-saturated model is unique (even over M) if it exists at all.*
(2) *If M is (μ, κ)-saturated, $\kappa = \mathrm{cf}(\kappa), \mathrm{cf}(\kappa) > LS(\mathfrak{K})$ then M is κ-saturated.*
(3) *If M is (μ, κ)-saturated for every $\kappa = \mathrm{cf}(\kappa) \leqslant \mu$ and $\mu > LS(\mathfrak{K})$ then M is μ-saturated.*

Discussion: It is natural to define saturated as $\|M\|$-saturated. (*I* may have confusions using the other.) This is particularly reasonable when the cardinal is regular, e.g. if K categorical in $\lambda, \lambda = \mathrm{cf}(\lambda)$ the model in K_λ is λ-saturated.

Part of the program is to prove that all the definitions are equivalent.

For now in Definition 2.6 we are not sure that such a model exists.

3. Splitting

Whereas non-forking is very nice in [14], in more general contexts it is not clear whether we have so good a notion, hence we go back to earlier notions from [6], like splitting. It still gives for many cases $p \in \mathscr{S}(M)$, a "definition" of p over some "small" $N \leqslant_{\mathfrak{K}} M$. We need μ-splitting because $E_{LS(\mathfrak{K})}$ is not known to be equality (see 1.8).

Context 3.1. Inside the monster model $\mathfrak{C}$.

Definition 3.2. $p \in \mathscr{S}(M)$ μ-splits over $N \leqslant_{\mathfrak{K}} M$ if

$$\|N\| \leqslant \mu,$$

and there are N_1, N_2, h such that $N \leqslant_{\mathfrak{K}} N_\ell \leqslant_{\mathfrak{K}} M$ for $\ell = 1, 2$ and

$$\|N_1\| = \|N_2\| \leqslant \mu$$

h an elementary mapping from N_1 onto N_2 over N such that the types $p \restriction N_2$ and $h(p \restriction N_1)$ are contradictory and $N \leqslant_{\mathfrak{K}} N_\ell \leqslant_{\mathfrak{K}} M$.

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Claim 3.3. (1) *Assume* $\mathfrak{K}$ *is stable in* $\mu, \mu \geqslant LS(\mathfrak{K})$. *If* $M \in \mathfrak{K}, p \in \mathscr{S}^1(M)$, *then for some* $N_0 \subseteq M, \|N_0\| = \mu, p$ *does not* μ*-split over* N_0 *(see Definition* 3.2*).*

(2) *Moreover, if* $2^\kappa > \mu, \langle M_i\colon\ i \leqslant \kappa+1 \rangle$ *is* $<_\mathfrak{K}$*-increasing,* $\bar{a} \in {}^m(M_{\kappa+1})$, $tp(\bar{a}, M_{i+1}, M_{\kappa+1})$ *does* $(\leqslant \mu)$*-split over* M_i, then K *is not stable in* μ.

Proof. (1) If not, we can choose by induction on $i < \mu$ N_i, N_i^1, N_i^2, h_i such that
(a) $\langle N_i\colon\ i \leqslant \mu \rangle$ is increasing continuous, $N_i <_\mathfrak{K} M, \|N_i\| = \mu$
(b) $N_i \leqslant_\mathfrak{K} N_i^\ell \leqslant_\mathfrak{K} N_{i+1}$
(c) h_i is an elementary mapping from N_i^1 onto N_i^2 over N_i,
(d) $p \restriction N_i^2, h_i(p \restriction N_i^1)$ are contradictory, equivalently distinct (we could have defined them for $i < \mu^+$).

Let $\chi = \mathrm{Min}\{\chi\colon\ 2^\chi > \mu\}$ so $2^{<\chi} \leqslant \mu$. Now contradict stability in μ as in [8, Ch. I, Section 2] or [14, Ch. I, Section 2] (by using models).

(2) Similar proof. □

Conclusion 3.4. If $p \in \mathscr{S}^m(M)$, M is μ^+-saturated, $\kappa = \mathrm{cf}(\kappa) \leqslant \mu$, *then* for some $N_0 <^0_{\mu,\kappa} N_1 \leqslant_\mathfrak{K} M, (\|N_1\| = \mu)$ we have

p is the E_μ-unique extension of $p \restriction N_1$ which does not μ-split over N_0.

4. Indiscernibles and E.M. models

Definition 4.1. Let $h_i : Y \to \mathfrak{C}$ for $i < i^*$.

(1) $\langle h_i\colon\ i < i^* \rangle$ is an indiscernible sequence (of character $<\kappa$) (over A) if *for every* g, a partial one to one order preserving map from i^* to i^* (of cardinality $<\kappa$) *there is* $f \in \mathrm{AUT}(\mathfrak{C})$, such that

$$g(i) = j \Rightarrow h_j \circ h_i^{-1} \subseteq f$$
$$(\text{and } id_A \subseteq f).$$

(2) $\langle h_i\colon\ i < i^* \rangle$ is an indiscernible set (of character κ) (over A) if: for every g partial one to one map from i^* to i^* (with $|\mathrm{Dom}\, g| \leqslant \kappa$) *there is* $f \in \mathrm{AUT}(\mathfrak{C})$, such that

$$g(i) = j \Rightarrow h_j \circ h_i^{-1} \subseteq f$$
$$(\text{and } id_A \subseteq f).$$

(3) $\langle h_i\colon\ i < i^* \rangle$ is a strictly indiscernible sequence, if $i^* \geqslant \omega$ and for some Φ, proper for linear orders (see [8, Ch. VII] or [14, Ch. VII]) in vocabulary $\tau_1 = \tau(\Phi)$ extending $\tau(K)$, there is $M^1 = EM^1(i^*, \Phi)$ such that M^1 is the Skolem Hull of $\{x_i\colon\ i < i^*\}$, and a sequence of unary terms $\langle \sigma_t\colon\ t \in Y \rangle$ such that

$$\sigma_t(x_i) = h_i(t) \quad \text{for } i < i^*, \quad t \in Y$$
$$M^1 \restriction \tau(K) <_\mathfrak{K} \mathfrak{C}.$$

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Notation 4.2. We can replace h_i by the sequence $\langle h_i(t)\colon t \in Y\rangle$.

Definition 4.3. (1) $\mathfrak{K}$ has the (κ, θ)-order property if for every α there are $A \subseteq \mathfrak{C}, \langle \bar{a}_i\colon i < \alpha\rangle$, where $\bar{a}_i \in {}^{\kappa}\mathfrak{C}, |A| \leqslant \theta$ such that

(∗) if $i_0 < j_0 < \alpha, i_1 < j_1 < \alpha$ then for no $f \in \mathrm{AUT}(\mathfrak{C})$ do we have
$f \restriction A = \mathrm{id}_A, f(\bar{a}_{i_0} {}^\frown \bar{a}_{j_0}) = \bar{a}_{j_1} {}^\frown \bar{a}_{i_1}$.

If $A = \emptyset$ i.e. $\theta = 0$, we write "κ-order property".

(2) $\mathfrak{K}$ has the $(\kappa_1, \kappa_2, \theta)$ order property *if* for every α there are $A \subseteq \mathfrak{C}$ such that $|A| \leqslant \theta, \langle \bar{a}_i\colon i < \alpha\rangle$ where $\bar{a}_i \in {}^{\kappa_1}\mathfrak{C}, \langle \bar{b}_i\colon i < \alpha\rangle$ where $\bar{b}_i \in {}^{\kappa_2}\mathfrak{C}$ such that

(∗) if $i_0 < j_0 < \alpha, i_1 < j_1 < \alpha$, *then* for no $f \in \mathrm{AUT}(\mathfrak{C})$ do we have
$f \restriction A = \mathrm{id}_A, f(\bar{a}_{i_0}) = \bar{a}_{j_1}, f(\bar{b}_{j_0}) = \bar{b}_{i_1}$.

Remark 4.4. So we have obvious monotonicity and if $\theta \leqslant \kappa$ we can let $A = \emptyset$; so the (κ, θ)-order property implies the $(\kappa + \theta)$-order property. Also strictly indiscernible sequence is an indiscernible sequence.

Claim 4.5. (1) *If* $\mu \geqslant LS(\mathfrak{K}) + |Y|$ *and* $h_i^0\colon Y \to \mathfrak{C}$, *for* $i < \theta < \beth_{(2^\mu)^+}$ *(e.g.* $h_i^0 = h_i$*)* then *we can find* $\langle h'_j\colon j < i^*\rangle$, *a strictly indiscernible sequence, with* $h'_j : Y \to \mathfrak{C}$ *such that*

(∗) *for every* $n < \omega$, $j_1 < \cdots < j_n < i^*$ *for arbitrarily large* θ *we can find* $i_1 < \cdots < i_n < \theta$ *and* $f \in AUT(\mathfrak{C})$ *such that* $h'_{j_\ell} \circ (h^0_{i_\ell})^{-1} \subseteq f$.

(2) *If in part* (1) *for each* θ, *the sequence* $\langle h_j^0\colon j < \theta\rangle$ *is an indiscernible sequence of character* $\aleph_0$, *in* (∗) *any* $i_1 < \cdots < i_n < i^*$ *will do.*

(3) *In Definition* 4.3 *we can restrict* α *to* $\alpha < \beth_{(2^{\kappa+\theta+LS(\mathfrak{K})})^+}$ *and get an equivalent version.*

(4) *In Definition* 4.3 *we can demand* $\langle \bar{a} {}^\frown \bar{a}_i\colon i < \alpha\rangle$ *is strictly indiscernible (where* $\bar{a}$ *lists* A*) and get an equivalent version.*

(5) *If* $\mu \geqslant LS(\mathfrak{K}) + |Y|$, $N \leqslant_{\mathfrak{K}} \mathfrak{C}$ *and* $h_i^0 : Y \to N$ *for* $i < \theta < \beth_{(2^\mu)^+}$ *and* N^1 *is an expansion of* N *with* $|\tau(N^1)| \leqslant \mu$, then *for some expansion* N^2 *of* N^1 *with* $|\tau(N^2)| \leqslant \mu$ *and* Ψ *we have*

- (a) $\tau(\Psi) = \tau(N^2)$
- (b) *for linear orders* $I \subseteq J$ *we have*
  $EM_\tau(I, \Psi) \leqslant_{\mathfrak{K}} EM(I, \Psi) \in K$
  and the skeleton of $EM_\tau(I, \Psi)$ *is* $\langle \bar{a}_t\colon t \in I\rangle$, $\bar{a}_t = \langle a_{t,y}\colon y \in Y\rangle$
- (c) *for every* $n < \omega$ *for arbitrarily large* $\theta < \beth_{(2^\mu)^+}$ *for some* $i_0 < \cdots i_{n-1} < \theta$, *for every linear order* I *and* $t_0 < \cdots < t_{n-1}$ *in* I, *letting* $J = \{t_0, \ldots, t_{n-1}\}$ *there is an isomorphism* g *from* $EM(J, \Psi) \subseteq EM(I, \Psi)$ *(those are* $\tau(N^2)$*-models) onto the submodel of* N^2 *generated by* $\bigcup_{\ell < n} Rang(h^0_{i_\ell})$ *such that* $h^0_{i_\ell}(y) = g(a_{t_\ell, y})$.

Proof. As in [11].

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Lemma 4.6. *If there is a strictly indiscernible sequence which is not an indiscernible set of character* $\aleph_0$ *called* $\langle \bar{a}^i \colon i<\omega\rangle$, *then* $\mathfrak{K}$ *has the* $|\ell g(a^i)|$*-order property.*
[*Permutation of infinite sets is a more complicated issue.*]

Claim 4.7. (1) *If* $\mathfrak{K}$ *has the* κ*-order property then*

$$I(\chi,\kappa)=2^\chi \text{ for every } \chi>(\kappa+LS(\mathfrak{K}))^+$$

(*and other strong non-structure properties*).

(2) *If* $\mathfrak{K}$ *has the* $(\kappa_1,\kappa_2,\theta)$*-order property and* $\chi \geqslant \kappa=\kappa_1+\kappa_2+\theta$ then *for some* $M\in K_\chi$, *we have* $|\mathcal{S}^{\kappa_2}(M)/E_\kappa|>\chi$.

Proof. (1) By [17, Ch. III, Section 3] (preliminary version appears in [13, Ch. III, Section 3]) (note the version on e.g. $\triangle(L_{\lambda^+,\omega})$).
(2) Straight. □

Definition 4.8.
(1) Suppose $M\leqslant_{\mathfrak{K}} N$, $p\in\mathcal{S}^m(N)$. Then p divides over M if there are elementary maps $\langle h_i\colon i<\bar{\kappa}\rangle$, $\mathrm{Dom}(h_i)=N$, $h_i \restriction M=id_M$, $\langle h_i\colon i<\bar{\kappa}\rangle$ is a strictly indiscernible sequence and $\{h_i(p)\colon i<\bar{\kappa}\}$ is contradictory i.e. no element (in some $\mathfrak{C}', \mathfrak{C}'<\mathfrak{C}$) realizing all of them.
(2) $\kappa_\mu(\mathfrak{K})$ [or $\kappa^*_\mu(\mathfrak{K})$] is the set of regular κ such that for some $\leqslant_{\mathfrak{K}}$-increasing continuously $\langle M_i\colon i\leqslant\kappa+1\rangle$ in K_μ and $b\in M_{\kappa+1}$ for every $i<\kappa$ we have: $\mathrm{tp}(b,M_\kappa,M_{\kappa+1})$ [or $\mathrm{tp}(b,M_{i+1},M_{\kappa+1})$] divides over M_i.
(3) $\kappa_{\mu,\theta}(\mathfrak{K})$ [or $\kappa^*_{\mu,\theta}(\mathfrak{K})$] is the set of regular κ such that for some $\leqslant_{\mathfrak{K}}$-increasing continuous sequence $\langle M_i\colon i\leqslant\kappa+1\rangle$ in K_θ and $b\in M_{\kappa+1}$ for every $i<\kappa$ we have: $\mathrm{tp}(b,M_\kappa,M_{\kappa+1})$ [or $\mathrm{tp}(b,M_{i+1},M_{\kappa+1})$], μ-divides over M_i (see Definition 4.11 below).

Remark 4.9. (1) Is there a parallel to forking?
(2) Note the difference between $\kappa_\mu(\mathfrak{K})$ and $\kappa^*_\mu(\mathfrak{K})$. Note that now the "local character" is apparently lost.

Fact 4.10.
(1) *In Definition* 4.8(1) *we can equivalently demand*: *no element realizing* $\geqslant\beth_{(2^\chi)^+}$ *of them, where* $\chi=\|N\|$.
(2) *If* $\kappa\in\kappa^*_\mu(\mathfrak{K})$, $\theta=cf(\theta)\leqslant\kappa$ *then* $\theta\in\kappa^*_\mu(\mathfrak{K})$ *and similarly of* $\kappa^*_{\mu,\theta}(\mathfrak{K})$.
(3) $\kappa^*_\mu(\mathfrak{K})\subseteq\kappa_\mu(\mathfrak{K})$ *similarly* $\kappa^*_{\mu,\theta}(\mathfrak{K})\subseteq\kappa_{\mu,\theta}(\mathfrak{K})$.

Definition 4.11. Suppose $M\leqslant_{\mathfrak{K}} N$, $p\in\mathcal{S}(N)$, $M\in K_{\leqslant\mu}$, $\mu\geqslant LS(\mathfrak{K})$.
(1) We say p μ-strongly splits over M, if there are $\langle\bar{a}^i\colon i<\omega\rangle$ such that
 (i) $\bar{a}^i\in{}^{\gamma\geqslant}\mathfrak{C}$ for $i<\omega$, $\gamma<\mu^+$, $\langle\bar{a}^i\colon i<\omega\rangle$ is strictly indiscernible over M
 (ii) for no N^+, $N\leqslant_{\mathfrak{K}} N^+$, $\bar{a}^0\cup\bar{a}^1\subseteq N^+$ and $b\in N^+$ realizing P do we have: $\bar{a}^{0\wedge}\langle b\rangle, \bar{a}^{1\wedge}\langle b\rangle$ realize the same type over M.
(2) We say p explicitly μ-strongly splits over M if in addition $\bar{a}^0\cup\bar{a}^1\subseteq N$.
(3) Omitting μ means any μ (equivalently $\mu=\|N\|$).

Claim 4.12. (1) *Strongly splitting implies dividing with models of cardinality* $\leqslant \mu$ *if* $(*)_\mu$ *holds where* $(*)_\mu = (*)_{\mu,\aleph_0,\aleph_0}$ *and*

$(*)_{\mu,\theta,\sigma}$ *If* $\langle \bar{a}^i : i < i^* \rangle$ *is a strictly indiscernible sequence,* $\bar{a}^i \in {}^\mu\mathfrak{C}$, $\bar{b} \in {}^{\sigma>}\mathfrak{C}$, then *for some* $u \subseteq i^*$, $|u| < \theta$ *and the isomorphism type of* $(\mathfrak{C}, \bar{a}^i {}^\frown \bar{b})$ *for all* $i \in i^* \backslash u$ *is the same.*

Claim 4.13. (1) *Let* $\mu(*) = \mu + \sigma + LS(\mathfrak{K})$. *Assume* $\langle \bar{a}^i : i < i^* \rangle$ *and* $\bar{b}$ *form a counterexample to* $(*)_{\mu,\theta,\sigma}$ *of Claim* 4.12 *and* $\theta > \beth_{(2^{\mu(*)})^+}$ then $\mathfrak{K}$ *has the* $\mu(*)$*-order property.*

(2) *We can also conclude that for* $\chi \geqslant \mu + LS(\mathfrak{K})$, *for some* $M \in K_\chi$ *we have* $|\mathscr{S}^{\ell g(\bar{b})}(M)| > \chi$.

(3) *If we have* "$\theta < \beth_{(2^{\mu(*)})^+}$" *we can still get for every* $\chi \geqslant \mu + \sigma + LS(\mathfrak{K}) + \theta$ *for some* $M \in K_\chi$, *we have* $|\mathscr{S}^{\ell g(\bar{b})}(M)| \geqslant \chi^\theta$.

Proof. Straightforward, using Claim 4.14 below.

Claim 4.14. *Assume* $M = EM(I, \Phi)$, $LS(\mathfrak{K}) + \ell g(\bar{a}_i) \leqslant \mu$, $\mu \geqslant |\alpha| + LS(\mathfrak{K})$ *and* $M \leqslant_{\mathfrak{K}} N$, $\bar{b} \in {}^\alpha N$ *and*

$(*)$ *for no* $J \subseteq I$, $|J| < \beth_{(2^\mu)^+}$ *do we have for all* t, $s \in I \backslash J$, $\mathrm{tp}(\bar{a}_t {}^\frown \bar{b}, \emptyset, N) = \mathrm{tp}(\bar{a}_s {}^\frown \bar{b}, \emptyset, N)$.

Then

(A) *we can find* Φ' *and a formula* φ (*not necessarily first order, but* $\pm\varphi$ *is preserved by* $\leqslant_{\mathfrak{K}}$*-embeddings*) *such that for any linear order* I'
$M = EM(I', \Phi')$, $\bar{a}_t = \bar{a}' {}^\frown \bar{b}_t$, $\ell g(\bar{a}') \leqslant \mu$, $\ell g(\bar{b}_t) = \alpha$,
$\varphi[\bar{a}', \bar{b}_s] \Leftrightarrow t < s$
(*if* $\alpha < \omega$, *this is half the finitary order property*),

(B) *this implies instability in every* $\mu' \geqslant \mu$ *if* $\alpha < \omega$,

(C) *this implies the* $(\mu + |\alpha|)$*-order property,*

(D) *if* $\bar{b} \in {}^\alpha M$ *then* "$|J| < \mu^+$" *in* $(*)$ *suffices,*

(E) *if* $\chi \geqslant \mu$, *for some* $M \in K_\chi$, $|\mathscr{S}^\alpha(M)| > \chi$ *moreover* $|\mathscr{S}^\alpha(M)/E_\mu| > \chi$.

Proof. As we can increase I, without loss of generality the linear order I is dense with no first or last element and is $(\beth_{(2^\mu)^+})^+$-strongly saturated. So for some p and some interval I_0 of I, the set $Y_0 = \{t \in I_0 : \mathrm{tp}(\bar{a}_t {}^\frown \bar{b}, \emptyset, N) = p\}$ is a dense subset of I_0. Also for some $q \neq p$, the set $Y_1 = \{t \in I : \mathrm{tp}(\bar{a}_t {}^\frown \bar{b}, \emptyset, N) = q\}$ has cardinality $\geqslant \beth_{(2^\mu)^+}$ and let $Y_1' \subseteq Y_1$ have cardinality $\beth_{(2^\mu)^+}$. As we can shrink I_0 without loss of generality I_0 is disjoint from Y_1' and as we can shrink Y_1 without loss of generality $(\forall s \in Y_1')(\forall t \in I_0)(s <^I t)$ or $(\forall s \in Y_q')(\forall t \in I_0)(t <^I s)$. By the Erdös–Rado theorem, for every $\theta < \beth_{(2^\mu)^+}$ there are $s_\alpha^\theta \in Y_1'$ for $\alpha < \theta$ such that $\langle s_\alpha^\theta : \alpha < \theta \rangle$ is strictly increasing or strictly decreasing without loss of generality the case does not depend on θ, so as

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we can invert I without loss of generality it is increasing. Let $t^*_\alpha \in Y'_1$ for $\alpha < \beth_{(2^\mu)^+}$ be strictly increasing. Hence (try $(p_1, p_2) = (p, q)$ and $(p_1, p_2) = (q, p)$, one will work)

(∗) we can find $p_1 \neq p_2$ such that

(∗∗) and for every $\theta < \beth_{(2^\mu)^+}$ an increasing sequence $\langle t^\theta_\alpha: \alpha < \theta + \theta \rangle$ of members of I exists such that

$$\alpha < \theta = \mathrm{tp}(\bar{a}_{t^\theta_\alpha} \char`^ \bar{b}, \emptyset, N) = p_0,$$

$$\theta \leqslant \alpha < \theta + \theta \Rightarrow \mathrm{tp}(\bar{a}_{t^\theta_\alpha} \char`^ \bar{b}, \emptyset, N) = p_1.$$

Now we apply Claim 4.5(5) with h^θ_i listing $\bar{a}^\theta_\alpha \char`^ \bar{a}^\theta_{\theta+\alpha} \char`^ \bar{b}$ and letting N^1 be $EM(I, \Phi)$ (so $\tau(N^1) = \tau(\Phi)$) and we get Ψ as there. Now for any linear order I^*, look at $EM(I^*, \Psi)$ and its skeleton $\langle \bar{a}^*_t: t \in I^* \rangle$. Clearly, $\bar{a}^*_t = \bar{a}^1_t \char`^ \bar{a}^2_t \char`^ \bar{b}^*$, and letting M^* be the submodel of $EM_{\tau(\Phi)}(I^*, \Psi)$ generated by $\{a^1_t, a^2_t: t \in I^*\} \cup \bar{b}$, it is isomorphic to $EM(I^* + I^*, \Phi)$, so without loss of generality $M = M^* \restriction \tau(\mathfrak{K}) \leqslant_{\mathfrak{K}} \mathfrak{C}$, so $\mathrm{tp}(\bar{a}^1_t \char`^ \bar{b}, \emptyset, M) = p_1$, $\mathrm{tp}(\bar{a}^2_t \char`^ \bar{b}, \emptyset, M) = p_2$. Now for any χ we can choose $I^* = I^*_\chi$ such that $\mathbf{D} = \{J: J$ an initial segment of I^* and $J \cong I^*$ and $I^* \setminus J$ is isomorphic to $I^*\}$ has cardinality $> \chi$.

So we have proved clause (E) and clause (B), by easy manipulations we get clause (A) and so (C).

We are left with clause (D). Clearly for some $\bar{t} = \langle t_i: i < i^* \rangle$, $i^* < |\alpha|^+ + \aleph_0$ such that $\bar{b} = \langle b_\beta: \beta < \alpha \rangle$, $b_\beta = \tau_\beta(\bar{a}_{t_{i(\beta,0)}}, \ldots, \bar{a}_{t_{i(\beta,n(\beta)-1)}})$ and $i(\beta, \ell) < i^*, \tau_\beta$ a $\tau(\Phi)$-term.

Let $J = \{t_i: i < i^*\}$ so by the version of (∗) used in clause (D), necessarily for some $s_1, s_2 \in I \setminus J$ we have

$$p_1 \neq p_2,$$

where

$$p_1 = \mathrm{tp}(\bar{a}_{s_1} \char`^ \bar{b}, \emptyset, N), \qquad p_2 = \mathrm{tp}(\bar{a}_{s_2} \char`^ \bar{b}, \emptyset, N).$$

Clearly $s_1 \neq s_2$. By renaming without loss of generality $s_1 <^I s_2$ and $0 = i_0 \leqslant i_1 \leqslant i_2 \leqslant i_3 = i^*$ and $t_i <^I s_1 \Leftrightarrow i < i_1$ and $s_1 <^I t_i <^I s_2 \Leftrightarrow i_1 \leqslant i < i_2$ and $s_2 <^I t_i \Leftrightarrow i_2 < i < i_3$.

Renaming without loss of generality $i(\beta, \ell) \notin \{i_1, i_2\}$, and replace t_{i_1}, t_{i_2} by s_1, s_2. So for every linear order I' we can define a linear order I^* with the set of elements

$$\{t_i: i < i_1 \text{ or } i_2 < i < i^*\} \cup \{(s, i): s \in I', i_1 \leqslant i < i_2\}$$

linearly ordered by

$$t_{j_1} < t_{j_2} \quad \textit{if } j_1 < j_2 < i_1,$$

$$t_{j_1} < t_{j_2} \quad \textit{if } i_2 < j_1 < j_2 < i^*,$$

$$t_{j_1} < (s', j') < (s'', j'') < t_{j_2} \quad \begin{array}{l} \textit{if } j_1 < i_1,\ i_2 < j_2 < i^*, \\ s', s'' \in I', j', j'' \in [i_1, i_2] \\ (s' <^{I'} s'') \vee (s' = s'' \,\&\, j' < j''). \end{array}$$

In $M = EM(I^*, \Phi)$ define, for $s \in I'$

$$\bar{c}_{s,i} \text{ is } \bar{a}_{i_l} \text{ if } i \langle i_1 \vee i \rangle i_2,\ \bar{c}_{s,i} = \bar{a}_{(s,i)} \text{ if } i \in [i_1, i_2),$$

$$\bar{b}_s = \langle \tau_\beta(\bar{c}_{s,i(\beta,0)}, \bar{c}_{s,i(\beta,1)}, \ldots, \bar{c}_{s,i(\beta,n(\beta)-1)}: \beta < \alpha \rangle.$$

Easily

$$s' <^{I'} s'' \Rightarrow \text{tp}(\bar{a}_{(s',i_1)} {}^\frown \bar{b}_{s''}, \emptyset, M) = p_1,$$

$$s'' \leqslant^{I'} s' \Rightarrow \text{tp}(\bar{a}_{(s',i_1)} {}^\frown \bar{b}_{s''}, \emptyset, M) = p_2.$$

By easy manipulations we can finish. □

Claim 4.15. *Assume (K is categorical in λ and)*
(a) $1 \leqslant \kappa$, $LS(\mathfrak{K})^\kappa < \theta = cf(\theta) \leqslant \lambda$ *and*
$(\forall \alpha < \theta)(|\alpha|^\kappa < \theta)$
(b) $\bar{a}_i \in {}^\kappa\mathfrak{C}$ *for* $i < \theta$.
Then for some $W \subseteq \theta$ of cardinality θ, the sequence $\langle \bar{a}_i: i \in W \rangle$ is strictly indiscernible.

Proof. Let $M' \prec \mathfrak{C}$, $\|M'\| = \theta$, $\bar{a}_\alpha \subseteq M'$. There is $M'', M' \prec M'' \prec \mathfrak{C}$, $\|M''\| = \lambda$. So $M'' \cong EM(\lambda, \Phi)$ and without loss of generality equality holds. So there is $u \subseteq \lambda$, $|u| \leqslant \theta$, $M' \subseteq EM(u, \Phi)$. So without loss of generality $M' = EM(u, \Phi)$. So $\bar{a}_\alpha \in EM(v_\alpha, \Phi)$, $v_\alpha \subseteq u$, $|v_\alpha| \leqslant \kappa$.

Without loss of generality: otp $v_\alpha = j^*$, so for $\alpha < \beta$, $\text{OP}_{u_\alpha, u_\beta}$ the order preserving map from u_β onto u_α induces $f_{\alpha,\beta}: EM(u_\beta, \Phi) \xrightarrow[\text{onto}]{\text{iso}} EM(u_\alpha, \Phi)$, and without loss of generality $f_{\alpha,\beta}(\bar{a}_\beta) = a_\alpha$.

Now for some $w \in [\theta]^\theta$, $\langle v_\alpha : \alpha \in w \rangle$ *is indiscernible* in the linear order sense (make them a sequence). Now we can create the right Φ.

[Why? Let $u_\alpha = \{\gamma_{\alpha,j}: j < j^*\}$ where $\gamma_{\alpha,j}$ increases with j. For $\alpha < \theta$, $A_\alpha = \{\gamma_{\beta,j}: \beta < \alpha, j < j^*\} \cup \{\bigcup_{\beta < \alpha, j} \gamma_{\beta,j} + 1\}$. Let $\gamma^*_{\beta,j} = \text{Min}\{\gamma \in A_\alpha: \gamma_{\beta,j} \geqslant \gamma\}$ and for each $\alpha \in S^*_0 = \{\delta < \theta: \text{cf}(\delta) > \kappa\}$ let $h(\delta) = \text{Min}\{\beta < \delta: \gamma^*_{\delta,j} \in A_\beta\}$ (defining $\langle A_\beta: \beta \leqslant \delta \rangle$ as increasing continuous, $\text{cf}(\delta) > \kappa \geqslant |j^*|$ and $\gamma^*_{\delta,j} \in A_\delta$ by definition).

By Fodor's lemma for some stationary $S_1 \subseteq S_0$, $h \restriction S_1$ is constantly β^*. As $(\forall \alpha < \theta)$ $(|\alpha|^\kappa < \theta = \text{cf}(\theta))$ for some $S_2 \subseteq S_1$ for each $j < j^*$ and for all $\delta \in S_2$, the truth value of "$\gamma_{\delta,j} \in A_\delta$" (e.g. $\gamma_{\delta,j} = \gamma^*_{\delta,j}$) is the same and $\langle \gamma^*_{\delta,j}: \delta \in S_2 \rangle$ is constant. Now $\langle u_\delta: \delta \in S_2 \rangle$ is as required. See more in [16, Section 7]. □

Definition 4.16. A model M is λ-strongly saturated if
(a) λ-saturated,
(b) strongly λ-homogeneous: if f is a partial elementary mapping from M to M, $|\text{Dom}\, f| < \lambda$ then $(\exists g \in AUT(M))(f \subseteq g)$.

Note: if $\mu = \mu^{<\lambda}$, I a linear order of cardinality $\leqslant \mu$, then there is a λ-strongly saturated dense linear order $J, I \subseteq J$.

Remark. We can even get a uniform bound on $|J|$ (which only depends on μ).

5. Rank and superstability

Definition 5.1. We define for $M \in K_\mu$, $p \in \mathscr{S}^m(M)$, $R(p)$ as follows: $R(p) \geqslant \alpha$ iff for every $\beta < \alpha$ there are $M^+, M \leqslant_{\mathfrak{K}} M^+ \in K_\mu$, $p \subseteq p^+ \in \mathscr{S}^1(M^+), R(p^+) \geqslant \beta \,\&\, [p^+\ \mu$-strongly splits over $M]$. In case of doubt we write R_μ.

Definition 5.2. We call K $(\mu, 1)$-superstable if

$$M \in K_\mu \,\&\, p \in \mathscr{S}(M) \Rightarrow R(p) < \infty \quad \text{(equivalently } <(2^\mu)^+).$$

Claim 5.3. *If* $(*)_\mu$ *from Claim* 4.12 *above fails*, then $(\mu, 1)$-*superstability fails.*

Proof. Straightforward.

Claim 5.4. *If K is not $(\mu, 1)$-superstable, then there are a sequence $\langle M_i\colon i \leqslant \omega + 1\rangle$ which is $<_{\mathfrak{K}}$-increasing continuous in K_μ and $\bar{a} \in {}^m(M_\omega + 1)$ such that $(\forall i < \omega)[\frac{\bar{a}}{M_{i+1}}\ \mu$-strongly splits over $M_i]$.*

Proof. As usual.

Claim 5.5. (1) *If K is not $(\mu, 1)$-superstable* then K *is unstable in every χ such that* $\chi^{\aleph_0} > \chi + \mu + 2^{\aleph_0}$.
(2) *If $\kappa \in \kappa^*_\mu(\mathfrak{K})$ and $\chi^\kappa > \chi \geqslant LS(\mathfrak{K})$*, then $\mathfrak{K}$ *is not χ-stable, even modulo E_μ.*

Remark. We intend to deal with the following elsewhere; we need stable amalgamation

$(*)$ if $\kappa \in \kappa_\mu(\mathfrak{K})$, $\operatorname{cf}(\chi) = \kappa$, $\bigwedge_{\lambda < \chi} \lambda^\mu \leqslant \chi$,
then $\mathfrak{K}$ is not χ-stable.

Remark. (1) In (1) this implies, $I(LS(\mathfrak{K})^{+(\omega(\alpha_0+\alpha)+n)}, K) \geqslant |\alpha|$ when $\mu = \aleph_{\alpha_0}$.
Similarly in (2). We conjecture [2] can be generalized to the context of (1).
(2) Note that for FO stable, for κ regular we have $(*)^\kappa_1 \Leftrightarrow (*)^\kappa_2$ where

$(*)^\kappa_1$ an increasing union of a chain $\langle M_i\colon i < \kappa\rangle$ of λ-saturated models of length κ, $\cup_{i<\kappa} M_i$ is λ-saturated,

$(*)^\kappa_2$ $\kappa \in \kappa_r(\mathfrak{K})$.

In [17] $(*)^\kappa_2$ is changed to

$(**)$ $\kappa < \kappa_r(T)$

(really $\kappa_r(\mathfrak{K})$ (i.e. $\kappa_r(T)$) is a *set* of regular cardinals).

From this point of view, T FO is a degenerated case: $\kappa_r(T)$ is an initial segment so naturally we write the first regular not in it. This is a point where [13] opens our eyes.

(3) In fact in Claim 5.5 not only do we get $\|M\| = \chi$, $|\mathscr{S}(M)| > \chi$ but also $|\mathscr{S}(M)/E_\mu| > \chi$.

Proof. (1)

Case I: There are M, N, p, $\langle \bar{a}_i \colon i < i^* \rangle$ as in Claim 4.12 $(*)_\mu$ and $\bar{c}$, (in fact $\ell g(\bar{c}) = 1$) such that $\bar{c}$ realizes $h_i(p)$ for infinitely many i's and fails to realize $h_i(p)$ for infinitely many i's; so without loss of generality $\{i \colon c \text{ realizes } h_i(p)\}$ is countable (even of order type ω), where the h_i are the maps associated with the indiscernible sequence (see Definition 4.1(1)). Let I be a $\beth(\chi + \beth_{(2^\mu)^+})^+$-strongly saturated dense linear order (see Definition 4.16) such that even if we omit $\leqslant \beth_{(2^\mu)^+}$ members, it remains so. By the strict indiscernibility we can find $\langle \bar{a}_t \colon t \in I \rangle, c$ as above.

So there is $u \subseteq I$, $|u| < \beth_{(2^\mu)^+}$ such that $q = \mathrm{tp}(\bar{a}_t {}^\frown \bar{c}, \emptyset, \mathfrak{C})$ is the same for all $t \in I \setminus u$; without loss of generality $q = \mathrm{tp}(\bar{a}_t {}^\frown \bar{c}, \emptyset, \mathfrak{C}) \Leftrightarrow t \in I \setminus u$, so u is infinite. So we can find $i_n \in i^* \cap u$ such that $i_n < i_{n+1}$. Let $I' = I \setminus (u \setminus \{i_n \colon n < \omega\})$, so that I' is still χ^--strongly saturated. Hence for every $J \subseteq I'$ of order type ω for some $c_J (\in \mathfrak{C})$ we have

$$t \in I' \setminus J \Rightarrow \mathrm{tp}(\bar{a}_t {}^\frown \bar{c}_J, \emptyset, \mathfrak{C}) = q,$$

$$t \in J \Rightarrow \mathrm{tp}(\bar{a}_t {}^\frown \bar{c}_J, \emptyset, \mathfrak{C}) \neq q.$$

This clearly suffices.

Case II. Note Case I.

As in [6] (the finitely many finite exceptions do not matter) or see part (2).

(2) Possibly decreasing κ (allowable as $\kappa \in \kappa^*_\mu(\mathfrak{K})$ rather than $\kappa \in \kappa_\mu(\mathfrak{K})$ is assumed) we can find a tree $T \subseteq {}^{\kappa \geqslant}\chi$, so closed under initial segments, $|T \cap {}^{\kappa >}\chi| \leqslant \chi$ but $|T \cap {}^\kappa\chi| > \chi$. (The cardinal arithmetic assumption is needed just for this). Let $\langle M_i \colon i \leqslant \kappa + 1 \rangle$, $c \in M_{\kappa+1}$ exemplify $\kappa \in \kappa^*_\mu(\mathfrak{K})$ and let $T' = T \cup \{\eta {}^\frown \langle 0 \rangle \colon \eta \in {}^\kappa\mathrm{Ord},$ $i < \kappa \rightarrow \eta \restriction i \in T\}$.

Now we can by induction on $i \leqslant \kappa + 1$ choose $\langle h_\eta \colon \eta \in T' \cap {}^i\chi \rangle$, such that

(a) h_η is a $\leqslant_\kappa$-embedding from $M_{\ell g(\eta)}$ into $\mathfrak{C}$

(b) $j < \ell g(\eta) \Rightarrow h_{\eta \restriction j} \subseteq h_\eta$

(c) if $i = j + 1$, $\nu \in T \cap {}^j \chi$, *then* $\langle h_\eta(M_i) \colon \eta \in \mathrm{Suc}_T(\nu) \rangle$ is strictly indiscernible, and can be extended to a sequence of length $\bar{\kappa}$ such that $\langle h_\eta(P \restriction M_i) : \eta = \nu^\wedge \langle \alpha \rangle, \alpha < \bar{\kappa} \rangle$ is contradictory (i.e. as in Definition 4.8(1)).

There is no problem to do this. Let $M \leqslant_{\mathfrak{K}} \mathfrak{C}$ be of cardinality χ and include $\bigcup \{h_\eta(M_i) \colon \ i < \kappa, \ \eta \in T \cap {}^i\chi\}$ hence it includes also $h_\eta(M_\kappa)$ if $\eta \in T \cap {}^\kappa\chi$ as $M_\kappa = \bigcup_{i < \kappa} M_i$.

Let for $\eta \in T \cap {}^\kappa\chi, c_\eta = h_{\eta {}^\frown \langle 0 \rangle}(c)$, so by Claim 4.14 clearly (by clause (C))

(*) if $i < \kappa$, $\eta \in T \cap {}^i\chi$, and $\eta \triangleleft \eta_1 \in T \cap {}^\kappa\chi$, *then*
$\rho \in \mathrm{Suc}_T(\eta)$: for some ρ_1, $\rho \triangleleft \rho_1 \in T \cap {}^\kappa\chi$ and
c_{ρ_1} realizes $\mathrm{tp}(c_{\eta_1}, h_{\eta_1 \restriction (i-1)}(m_{i-1}))\}$
has cardinality $< \beth_{(2^{\mu + LS(\mathfrak{K})})^+}$.

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Now define an equivalence relation $\mathbf{e}$ on $T \cap {}^{\kappa}\chi$:

$$\eta_1 \,\mathbf{e}\, \eta_2 \text{ iff } \mathrm{tp}(c_{\eta_1}, M) = \mathrm{tp}(c_{\eta_2}, M).$$

Now if for some $\eta \in T \cap {}^{\kappa}\chi, |\eta/\mathbf{e}| > \beth_{(2^{\mu+LS(\mathfrak{K})})^+}$ then for some $\eta^* \in T \cap {}^{\kappa>}\chi$, we have

$$\{\nu \restriction (\ell g(\eta^* + 1)) \colon \nu \in \eta/\mathbf{e}\} \text{ has cardinality } > \beth_{(2^{\mu+LS(\mathfrak{K})})^+}$$

which contradicts $(*)$; so if $\chi \geqslant \beth_{(2^{\mu+LS(\mathfrak{K})})^+}$, we are done.

But if for some $\eta \in T \cap {}^{\kappa>}\chi$ the set in $(*)$ has cardinality $\geqslant \kappa$, then we can continue as in case I of the proof of part (1), so assume this never happens. So above if $|\eta/\mathbf{e}| > 2^{\kappa}$, we get again a contradiction. So if $|T \cap {}^{\kappa}\chi| > 2^{\kappa}$, we conclude $|T \cap {}^{\kappa}\chi/\mathbf{e}| = |T \cap {}^{\kappa}\chi|$, so we are done. We are left with the case $\chi < 2^{\kappa}$, covered by 3.3(2) (note that for $\chi < 2^{\kappa}$ the interesting notion is splitting). □

Claim 5.6. *If $\lambda > \mu^+$, $\mu \geqslant LS(\mathfrak{K}, K)$, $\mathfrak{K}$ is categorical in λ then*
(1) K is $(\mu, 1)$-superstable.
(2) $\kappa^*_{\mu}(\mathfrak{K}) \cap \lambda$ is empty.

Proof. (1) If $\lambda > \mu^{+\omega}$, can use Claims 5.5 and 1.7, so $w \log \mathrm{cf}(\lambda) > |LS(\mathfrak{K})|$.

By Claim 1.7 if $M \in K_{\lambda}$ then M is $\mathrm{cf}(\lambda)$-saturated. On the other hand from the Definition of $(\mu, 1)$-superstable we shall get a non-μ^+-saturated model.

Let $\chi = \beth_{(2^{\lambda})^+}$. Assume $\mathfrak{K}$ is not $(\mu, 1)$-superstable so we can find in K_{μ} an increasingly continuous sequence $\langle M_i \colon i \leqslant \kappa + 1\rangle$ and $c \in M_{\omega+1}$ such that $p_{n+1} = \mathrm{tp}(c, M_{n+1}, M_{\omega+1})$ μ-strongly splits over M_n for $n < \omega$. For each $n < \omega$ let $\langle \bar{a}^n_i \colon i < \omega\rangle$ be a strictly indiscernible sequence over M_n exemplifying p_{n+1} μ-strongly splits over M_n (see Definition 4.11). So we can define $\bar{a}^n_i \in \mathfrak{C}$ for $i \in [\omega, \chi)$ such that $\langle \bar{a}^n_i \colon i < \chi\rangle$ is strictly indiscernible over M_n. Let $T_n = \{\eta \in {}^{2n}\chi \colon \eta(2m) < \eta(2m+1) \text{ for } m < n\}$. For $n < \omega$, $i < j < \chi$ let $h^n_{i,j} \in \mathrm{AUT}(\mathfrak{C})$ be such that $h^n_{i,j} \restriction M_n = \mathrm{id}, h^n_{i,j}(\bar{a}^n_0 {}^\frown \bar{a}^n_1) = \bar{a}^n_i {}^\frown \bar{a}^n_j$. Now we choose by induction on $n < \omega$, $\langle f_{\eta} \colon \eta \in T_n\rangle$, $\langle g_{\eta} \colon \eta \in T_n\rangle$, $\langle a^{\eta}_i \colon i < \chi, \eta \in T_n\rangle$ such that
(a) f_{η}, g_{η} are restrictions of automorphisms of $\mathfrak{C}$,
(b) $\mathrm{Dom}(f_{\eta}) = M_n$,
(c) $g_{\eta} \in \mathrm{AUT}(\mathfrak{C})$,
(d) $\bar{a}^{\eta}_i = g_{\eta}(\bar{a}^n_i)$ if $\eta \in T_n$,
(e) $f_{\langle\rangle} = \mathrm{id}_{M_0}$,
(f) $f_{\eta} \subseteq g_{\eta}$,
(g) if $\eta \in {}^{2n}\chi$, $m < n$ then $f_{\eta \restriction (2m)} \subseteq f_{\eta}$,
(h) if $\eta \in {}^{2n}\chi$ and $i < j < \chi$ then $f_{\eta {}^\frown \langle i,j\rangle} \subseteq (g_{\eta} \circ h^n_{i,j}) \restriction M_{n+1}$.
There is no problem to carry the induction. Now choose by induction on $n, M^*_n, \eta_n, i_n, j_n$ such that

(α) $i_n < j_n < \chi$ and $\eta_n = \langle i_0, j_0, \ldots, i_{n-1}, j_{n-1}\rangle$ so $\eta_n \in T_n$,
(β) $M^*_n \in K_{\lambda}$, $M^*_n <^1_{\mu,\omega} M^*_{n+1}$,
(γ) $\mathrm{Rang}(f_{\eta_n}) \subseteq M^*_n$,

(δ) $\bar{a}_i^{\eta_n}, \bar{a}_j^{\eta_n}$ realizes the same type over M_n,
(ε) $\bar{a}_i^{\eta_n}, \bar{a}_j^{\eta_n} \subseteq M^*_{n+1}$.

There is no problem to carry the induction (using the theorem on existence of strictly indiscernibles to choose $i_n < j_n$).

So $\bigcup_{n<\omega} f_{\eta_n}$ can be extended to $f \in \mathrm{AUT}(\mathfrak{C})$. Let $c^* = f(c), M^*_\omega = \bigcup_n M^*_{\eta_n}, M^*_{\omega+1} = f(M_{\omega-1})$. Clearly $\mathrm{tp}(c, M^*_{n+1}, M^*_{\omega+1})$ does μ-split over M_n hence M_ω is not μ^--saturated (as $\mathrm{cf}(\lambda) > \mu$) (see Claim 5.7); contradiction.

(2) Follows. □

Claim 5.7. *If $\mu \geqslant LS(\mathfrak{K})$, $\langle M_i: i \leqslant \delta \rangle$ is $\leqslant_{\mathfrak{K}}$-increasing continuous, $p \in \mathscr{S}^{\leqslant \mu}(M_\delta)$, p μ-strongly splits over M_i for all i (or just μ-splits over M_i)* then *M_δ is not μ^+-saturated.*

Proof. Straightforward.

Claim 5.8. *Assume there is a Ramsey cardinal $> \mu + LS(\mathfrak{K})$. If $\mathfrak{K}$ is not $(\mu, 1)$-superstable*, then *for every $\chi > \mu + LS(\mathfrak{K})$ there are 2^χ pairwise non-isomorphic models in $\mathfrak{K}_\chi$.*

Proof. By [2] for χ regular; together with [17] for all χ.

Lemma 5.9. (1) *If for some M, $|\mathscr{S}(M)/E_\mu| > \chi \geqslant \|M\| + \beth_{(2^\mu)^+}$ and $\mu \geqslant LS(\mathfrak{K})$* then *$\mathfrak{K}$ is not $(\mu, 1)$-superstable.*
(2) *If $\chi^\kappa \geqslant |\mathscr{S}(M)/E_\mu| > \chi^{<\kappa} \geqslant \chi \geqslant \|M\| + \beth_{(2^\mu)^+}$, $\mu \geqslant LS(\mathfrak{K}) + \kappa$* then $\kappa \in \kappa^*_\mu(\mathfrak{K})$.

Proof. No new point when you remember the definition of E_μ (see Definition 1.8).

6. Existence of many non-splitting

Question 6.1. Suppose $\bar{N} = \langle N_i: i \leqslant \delta \rangle$ is $<^1_{\mu,\kappa}$-increasing continuous (we mean for $i < j$, j non-limit $N_i <^1_{\mu,\kappa} N_j$), $\delta < \mu^+$ and $p \in \mathscr{S}^m(N_\delta)$. Is there $\alpha < \delta$ such that for every $M \in \mathfrak{K}_{\leqslant \lambda}, N_\delta \leqslant_{\mathfrak{K}} M$, p has an extension $q \in \mathscr{S}^m(M)$ which does not μ-split over N_α (and so in particular p does not μ-split over N_α).

Remark 6.2. If $p \restriction N_{\alpha+1}$ does not μ-split over N_α, then $p \restriction N_{\alpha+1}$ has at most one extension mod E_μ which does not μ-split over N_α because $N_{\alpha+1} \in K_\mu$ is universal over $N_\alpha, N_{\alpha+1} \leqslant M \in K_\lambda$. Also if $\lambda = \mu$ there is a unique $q \in \mathscr{S}(M)$ which does not μ-split over N_α.

Lemma 6.3. *Suppose K is categorical in λ, $cf(\lambda) > \mu \geqslant LS(\mathfrak{K})$.* Then *the answer to Question* 6.1 *is yes.*

Remark 6.4. We intend later to deal with the case $\lambda > \mu \geqslant \mathrm{cf}(\lambda) + LS(\mathfrak{K})$ as in [4].

Notation. $I \times \alpha$ is $I + I + \cdots$ (α times) (with the obvious meaning).

Proof. Let Φ be proper for linear order, $|\tau(\Phi)| \leqslant LS(\mathfrak{K})$, $EM_\tau(I,\Phi) \in K$ (of power $|I| + LS(\mathfrak{K})$) and $I \subseteq J \Rightarrow EM_\tau(I,\Phi) \leqslant_{\mathfrak{K}} EM_\tau(J,\Phi)$. So $EM_\tau(\lambda,\Phi)$ is μ^+-saturated (by Claim 1.7). Let I^* be a linear order of power $\mu, I^* \times (\alpha+1) \cong I^*$ for $\alpha < \mu^+$ and $I^* \times \omega \cong I^*$ (see [12], App.). By Claim 1.7 we know that $EM_\tau(I^* \times \lambda, \Phi)$ is μ^+-saturated W log $\kappa = \aleph_0$.

Now we choose by induction on i an ordinal $\alpha_i < \mu^+$ and an isomorphism h_i from N_{1+i} onto $EM(I^* \times \alpha_i, \Phi)$, both increasing with i such that for non limit i, $\mathrm{cf}(\alpha_i) = \aleph_0$.

For $i=0$, use the proof of the uniqueness of N_1 over N_0 (see Claim 2.5 and references there); then using the back and forth argument we can find $J_0 \subseteq \lambda$, $|J_0| = \mu$ and isomorphism h_0 from $N_1 = N_{0+1}$ onto $EM(1^* \times J_0, \Phi) \subseteq (I^* \times \lambda, \Phi)$. Now let $J^0 = J_0 \cup \{\alpha < \lambda \colon (\forall \beta \in J_0) \beta < \alpha\}$ so $J^0 \cong \lambda$ (note: J_0 is bounded in λ as $\mathrm{cf}(\lambda) > \mu \geqslant |J_0|$) and also $EM_\tau(I^* \times J^0, \Phi)$ is μ^+-saturated (being isomorphic to $EM_\tau(I^* \times \lambda, \Phi)$), so without loss of generality J_0 is some ordinal $\alpha_0 < \mu^+$.

So we have h_0. The continuation is similar.

Now h_δ is defined $h_\delta : N_\delta \xrightarrow{\text{onto}} EM_\tau(I^* \times \alpha_\delta, \Phi)$, so as $EM_\tau(I^* \times \lambda, \Phi)$ is μ^+-saturated, $h_\delta(p)$ is realized say by $\bar{a}$, so let $\bar{a} = \bar{\sigma}(x_{(t_1,\gamma_1)}, \ldots, x_{(t_n,\gamma_n)})$ where $\bar{\sigma}$ is a sequence of terms in $\tau(\Phi)$ and (t_ℓ, γ_ℓ) is increasing with ℓ (in $I^* \times \lambda$). Let $\beta < \delta$ be such that

$$\{\gamma_1, \ldots, \gamma_n\} \cap \alpha_\delta \subseteq \alpha_\beta.$$

Let

$$\gamma'_\ell = \begin{cases} \gamma_\ell & \text{if } \gamma_\ell < \alpha_\delta, \\ \lambda + \gamma_\ell & \text{if } \gamma_\ell \geqslant \alpha_\delta. \end{cases}$$

Then in the model $N = EM_\tau(I^* \times \lambda + \lambda, \Phi)$, we shall show that the finite sequence $\bar{a}' = \bar{\sigma}(x_{(t_1,\gamma'_1)}, \ldots, x_{(t_n,\gamma'_n)})$ realizes a type as required over $M = EM_\tau(I^* \times \lambda, \Phi)$. Why? Assume toward contradiction that

$(*)$ $\mathrm{tp}(\bar{a}', M, N)$ does μ-split over $M_{\beta+1}$ where over $M_\gamma = EM(I^* \times \alpha_\gamma, \Phi)$, for $\gamma < \delta$.

Let $\bar{\mathbf{c}}, \bar{\mathbf{b}} \in {}^\mu M$ realize the same type over $M_{\beta+1}$ but witness splitting.

We can find $w \subseteq \lambda$, $|w| \leqslant \mu$ such that $\bar{\mathbf{c}}, \bar{\mathbf{b}} \subseteq EM(I^* \times w, \Phi)$. Choose γ such that

$$\sup w < \gamma < \lambda.$$

Let $M^- = EM(I^* \times (\alpha_\delta \cup w \cup [\gamma, \lambda)), \Phi) <_{\mathfrak{K}} M$.
Let $N^- = EM(I^* \times (\alpha_\delta \cup w \cup [\gamma, \lambda) \cup [\lambda, \lambda + \lambda)), \Phi)$.
So still $\bar{\mathbf{c}}, \bar{\mathbf{b}}$ witness that $\mathrm{tp}(\bar{a}', M^-, N^-)$ μ-splits over $M_{\beta+1}$.
There is an automorphism f of the linear order $I^* \times (\alpha_\delta \cup w \cup [\gamma, \lambda)) \cup [\lambda, \lambda + \lambda))$ such that

$$f \restriction (I^* \times \alpha_{\beta+1}) = \text{the identity},$$

$$f \restriction (I^* \times [\gamma + 2, \lambda + \lambda)) = \text{the identity},$$

$$\mathrm{Rang}(\restriction (I^* \times (w \setminus \alpha_{\beta+1}))) \subseteq I^* \times [\alpha_{\beta+1}, \alpha_{\beta+2}).$$

Now f induces an automorphism of N^- naturally called $\hat{f}$.

So

$$\hat{f} \restriction M_{\beta+1} = \text{identity},$$

$$\hat{f}(\bar{a}') = \bar{a}', \qquad \hat{f}(M^-) = M^-.$$

As $\hat{f}$ is an automorphism, $\hat{f}(\bar{\mathbf{c}}), \hat{f}(\bar{\mathbf{b}})$ witness that $\mathrm{tp}(\hat{f}(\bar{a}'), \hat{f}(M^-), \hat{f}(N^-))$ μ-splits over $\hat{f}(M_{\beta-1})$; i.e. $\mathrm{tp}(\bar{a}', M^-, N^-)$ μ-splits over $M_{\beta-1}$. So $\mathrm{tp}(\bar{a}', M_{\beta-2}, N)$ μ-splits over $M_{\beta+1}$.

Now choose $\alpha_\gamma < \mu^+$ for $\gamma \in (\delta, \mu]$, increasing continuous

$$\alpha_{\delta+i} = \alpha_\delta + i, \qquad M_\gamma = EM(I^* \times \alpha_\gamma, \Phi).$$

So $\langle M_\gamma \colon \gamma \leqslant \mu \rangle$ is increasing continuous. Without loss of generality $\delta = \mathrm{cf}(\delta)$. If $\delta = \mu$, we finish by Claim 3.3(2) as categoricity in $\lambda \Rightarrow$ stability in $\chi \in [LS_{\mathfrak{K}}, \lambda)$. So $\delta = \mathrm{cf}(\delta) < \mu$. So for $\gamma_1 \in [\beta, \mu)$ there is an $f \in AUT(I^* \times (\lambda + \lambda))$

$$f \restriction I^* \times \alpha_\beta = \text{identity},$$

$$f \text{ takes } I^* \times [\alpha_\beta, \alpha_{\beta+1}) \text{ onto } I^* \times [\alpha_\beta, \alpha_{\gamma_1 - 1}),$$

$$f \text{ takes } I^* \times [\alpha_{\beta+1}, \alpha_{\beta+2}) \text{ onto } I^* \times [\alpha_{\gamma_1+1}, \alpha_{\gamma_1-2}),$$

$$f \text{ takes } I^* \times [\alpha_{\beta+2}, \alpha_{\gamma_1+3}) \text{ onto } I^* \times [\alpha_{\gamma_1+1}, \alpha_{\gamma_1+2}),$$

$$f \restriction I^* \restriction \times [\alpha_{\gamma_1+3}, \lambda + \lambda) = \text{identity}.$$

As before this shows

$$\mathrm{tp}(\bar{a}^1, M_{\gamma_1+\gamma_2+1}, N) \ \mu\text{-splits over } M_{\gamma_1+\gamma_2}.$$

So $\{\gamma < \mu \colon \mathrm{tp}(\bar{a}^1, M_{\gamma+1}, N)\ \mu\text{-splits over } M_\gamma\}$ has order type μ, so without loss of generality is μ. By Claim 3.3(2) we get a contradiction.

Note: A priori may be $\alpha < \beta < \delta \Rightarrow p \restriction N_\beta$ does not μ-split over N_α but $\alpha < \delta \Rightarrow p$ μ-splits over N_α. The first part of the argument shows in particular p μ-splits over $N_\alpha \Rightarrow p \restriction N_{\alpha+1}$ μ-splits over N_α.

The second part, (as the first part holds for every α) pushes it up to the "new" M_γ's ($\gamma \in (\delta, \mu)$) to enable us to use Claim 3.3(2).

By categoricity $EM(I^* \times \lambda, \Phi)$ is the model in K_λ and by Claim 2.5 it is also unique over $h_i(N_\delta)$. Now having proved the conclusion for $h_\delta(N_\delta), h_\delta(p)$, we can deduce it for $N_{\delta,p}$. $\square$

Theorem 6.5. *Suppose K categorical in λ and model in K_λ is μ^+-saturated (e.g. $cf(\lambda) > \mu$) and $LS(\mathfrak{K}) \leqslant \mu < \lambda$.*

(1) *$M <^1_{\mu,\kappa} N \Rightarrow N$ is saturated.*

(2) *If $\kappa_1, \kappa_2 \leqslant \mu$ and for $\ell = 1, 2$ we have $M_\ell <^1_{\mu,\kappa_\ell} N_\ell$*, then $N_1 \cong N_2$.

(3) *There is $M \in K_\mu$ which is saturated.*

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Remark 6.6. (1) The model we get by (2) we call *the saturated model* of K in μ.
(2) Formally – we do not use Lemma 6.3.
(3) By the same proof $M \leqslant^1_{\mu,\kappa_\ell} N_\ell \Rightarrow N_1 \cong_M N_2$ and we call N *saturated over* M.

Proof. (1) By the uniqueness proofs 2.2 as $M <^1_{\mu,\kappa} N$ there are $\langle M_i: i \leqslant \kappa \rangle$, $M_i <^1_{\lambda,\kappa} M_{i+1}$, $<_{\mathfrak{K}}$-increasing continuous $M_0 = M$, $M_\kappa = N$ and as in the proof of Lemma 6.3 without loss of generality $M_i = EM(\alpha_i, \Phi)$ where $\alpha_i < \mu^+$.

To prove $N = N_\kappa$ is μ-saturated suppose $p \in \mathscr{S}^1(M^*)$, $M^* \leqslant_{\mathfrak{K}} N$, $\|M^*\| < \mu$; as we can extend M^* (as long as its power is $<\mu$ and it is $<_{\mathfrak{K}} N$), without loss of generality $M^* = EM(J, \Phi)$, $J \subseteq \alpha_\kappa, |J| < \mu$.

So for some $\gamma, [\gamma, \gamma + \omega) \cap J = \emptyset$, $\gamma + \omega \leqslant \alpha_\kappa$. We can replace $[\gamma, \gamma + \omega)$ by a copy of λ; this will make the model μ-saturated [alternatively, use $I^* \times$ ordinal as in a previous proof].

But easily this introduces no new types realized over M^*. So p is realized.

(2) Follows by part (1).

By Claim 2.2(4) we finish.

(3) Follows from the proof of part (1).

Remark. In part (1) we have used just $\mathrm{cf}(\lambda) > \mu > LS(\mathfrak{K})$.

Claim 6.7. *Assume K categorical in $\lambda, cf(\lambda) > \mu > LS(\mathfrak{K})$. If $N_i \in K_\mu$ is saturated, increasing with i for $i < \delta, \delta < \mu^+$* then *$N = \bigcup_{i<\delta} N_i \in K_\mu$ is saturated.*

Proof. We prove this by induction on δ, so without loss of generality $\langle N_i: i < \delta \rangle$ is not just $\leqslant_{\mathfrak{K}}$-increasing but also contradicts the conclusion but is increasingly continuous and each N_i saturated. Without loss of generality $\delta = \mathrm{cf}(\delta)$. If $\mathrm{cf}(\delta) = \mu$ the conclusion clearly holds so assume $\mathrm{cf}(\delta) < \mu$. Let $M \leqslant_{\mathfrak{K}} N$, $\|M\| < \mu$ and $p \in \mathscr{S}(M)$ be omitted in N and let $\theta = \delta + \|M\| + LS(\mathfrak{K}) < \mu$, and let $p \leqslant q \in \mathscr{S}(N)$. Now we can choose by induction on $i \leqslant \delta$, $M_i \leqslant N_i$ and $M_i^+ \leqslant_{\mathfrak{K}} N$ such that $M_i \in K_\theta, M_i^+ \in K_\theta$, M_i $\leqslant_{\mathfrak{K}}$-increasing continuous and $M \cap N_i \subseteq M_i$, $j < i \Rightarrow M_j^+ \cap N_i \subseteq M_{i+1}$ and $M_i <^1_{\theta,\omega} M_{i+1}$ and if q does θ-split over M_i then $q \restriction M_i^+$ θ-splits over M_i.

So by Theorem 6.5 we know that M_δ is saturated, and for some $i(*) < \delta$ we have: $q \restriction M_\delta$ does not θ-split over $M_{i(*)}$. But $M^+_{i(*)} \subseteq N = \bigcup_{i<\delta} N_i$, $M^+_{i(*)} \cap N_j \subseteq M_{j+1}$ so $M^+_{i(*)} \subseteq M_\delta$. So necessarily $q \in \mathscr{S}(N)$ does not θ-split over $M_{i(*)}$.

Now we choose by induction on $\alpha < \theta^+, M_{i(*),\alpha}, b_\alpha, f_\alpha$ such that: $M_{i(*),\alpha} \in K_{\theta^+}, M_{i(*)} \leqslant_{\mathfrak{K}} M_{i(*),\alpha} \leqslant_{\mathfrak{K}} N_{i(*)}, M_{i(*),\alpha}$ is $\leqslant_{\mathfrak{K}}$-increasing continuous in $\alpha, b_\alpha \in N_{i(*)}$ realizes $q \restriction M_{i(*),\alpha}$, f_α is a function with domain M_δ and range $\subseteq N_{i(*)}$ such that $\bar{c} = \langle c: c \in M_\delta \rangle, \bar{c}^\alpha =:$ $\langle f_\alpha(c): c \in M_\delta \rangle$ realizes the same type over $M_{i(*),\alpha}$ and $\{b_\alpha\} \cup \mathrm{Rang}(f_\alpha) \subseteq M_{i(*),\alpha+1}$. As $N_{i(*)}$ is saturated we can carry the construction; if some b_α realizes $q \restriction M_\delta$ we are done (as $b_\alpha \in N$ realizes p). Let $d \in \mathfrak{C}$ realize q so

$(*)_1$ $\alpha < \beta < \theta^+ \Rightarrow \bar{c}^\beta{}^\frown\langle b_\alpha \rangle$ does not realize $\mathrm{tp}(\bar{c}^\frown\langle d \rangle, M_{i(*)}, \mathfrak{C})$.
[Why? As $\bar{c}^\frown\langle b_\alpha \rangle$ does not realize $\mathrm{tp}(\bar{c}^\frown\langle d \rangle, M_{i(*)}, \mathfrak{C})$ because d realizes $p = q \restriction \bar{c}$ whereas b_α does not realize $p = q \restriction \bar{c}$.]

On the other hand as q does not θ-split over $M_{i(*)}$ we have $\mathrm{tp}(\bar{c}\hat{}\langle d\rangle, M_{i(*)}, \mathfrak{C}) = \mathrm{tp}(\bar{c}^{\alpha}\hat{}\langle d\rangle, M_{i(*)}, \mathfrak{C})$ so by the choice of b_β:

$(*)_2$ if $\alpha < \beta < \theta^+$ then $\bar{c}^{\alpha}\hat{}\langle b_\beta\rangle$ realizes $\mathrm{tp}(\bar{c}\hat{}\langle d\rangle, M_{i(*)}, \mathfrak{C})$.

We are almost done by Claim 4.14(D). As $N_{i(*)}$ is saturated without loss of generality $N_{i(*)} = EM_{\tau(\mathfrak{K})}(\mu, \Phi)$ and $M_{i(*)} = EM_{\tau(\mathfrak{K})}(\theta, \Phi)$. As before for some $\gamma < \theta^+$ there are sequences $\bar{c}', b'$ in $EM_{\tau(\mathfrak{K})}(\mu + \gamma, \Phi)$ realizing $\mathrm{tp}(\bar{c}, N_{i(*)}, \mathfrak{C}), q \restriction N_{i(*)}$ respectively, here we use $\mathrm{cf}(\lambda) > \mu$ rather than just $\mathrm{cf}(\lambda) \geqslant \mu$. For each $\beta < \theta^+$ there is a canonical isomorphism g_β from $EM_{\tau(\Phi)}(\beta \cup [\mu, \mu + \gamma), \Phi)$ onto $EM_{\tau(\Phi)}(\beta + \gamma, \Phi)$. So without loss of generality $M_{i(*),\alpha} = EM_{\tau(\mathfrak{K})}(\theta + \gamma_\alpha, \Phi), \bar{c}^{\alpha} = g_{\theta-\gamma_\alpha}(\bar{c}'), b_\alpha = g_{\theta-\gamma_\alpha}(b')$. So $(*)_1 + (*)_2$ gives the order property.

[Why only almost from 4.14(D)? We would like to use the "θ-order property fail", now if we could define $\langle \bar{c}^{\beta}\hat{}\langle b_\beta\rangle$: for $\beta < (2^{\theta})^+\rangle$ fine, but we have only $\alpha < \theta^+$, this is too short.] □

We really proved, in Theorem 6.5 (from λ categoricity):

Subfact 6.8. (1) *If $I \subseteq J$ are linear order, of power $< \mathrm{cf}(\lambda)$;*

$(*)$ $t \in J \backslash I \Rightarrow (\exists^{\aleph_0} s \in J)[s \sim_I t]$ *where $s \sim_I t$ means "s, t realize the same Dedekind cut",*

then *every type over $EM_{\tau(K)}(I, \Phi)$ is realized in $EM(J, \Phi)$.*

(2) *Adding more Skolem functions we can omit $(*)$, for a suitable Φ we can make even the extension μ-saturated over $EM_\tau(I, \Phi)$.*

Proof. Why? Use the proof of Theorem 6.5(1).

Replace the cut of t in I by λ: we get $\mathrm{cf}(\lambda)$-saturated model.

7. More on splitting

Hypothesis 7.1. As before + conclusions of Section 6 for $\mu \in [LS(\mathfrak{K}), \mathrm{cf}(\lambda))$, that is

$(*)$ (a) K has a saturated model in μ.

(b) union of increasing chain of saturated models in K_μ of length $\leqslant \mu$ is saturated.

(c) *if* $\langle M_i \colon i \leqslant \delta\rangle$ increasing continuous in K_μ each M_{i+1} saturated over M_i (the previous one), $p \in \mathscr{S}(M_\delta)$ *then* for some $i < \delta$, p does not μ-split over M_i.

Conclusion 7.2. If $p \in \mathscr{S}^m(M)$ and $M \in K_\mu$ is saturated, *then* for some $M^- <^1_{\mu,\omega} M$, $M^- \in K_\mu$ is saturated and p does not μ-split over M^-.

Proof. We can find $\langle M_n \colon n \leqslant \omega\rangle$ in K_μ each M_n saturated $M_n \leqslant^1_{\mu,\omega} M_{n+1}$ and $M_\omega = \bigcup_{n<\omega} M_n$ so M_ω is saturated, without loss of generality $M_\omega = M$. Now using $(*)(c)$ of Hypothesis 7.1 some M_n is O.K. as M^-. □

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Fact 7.3. *If $M_0 \leqslant^1_{\mu,\omega} M_2 \leqslant^1_{\mu,\omega} M_3$, $p \in \mathscr{S}^m(M_3), p$ does not μ-split over M_0,* then *$R(p) = R(p \restriction M_2)$.*

Proof. We can find (by uniqueness) M_1 such that $M_0 \leqslant^1_{\mu,\omega} M_1 \leqslant^1_{\mu,\omega} M_2$.

We can find an isomorphism h_1 from M_3 onto M_2 over M_1 (by the uniqueness properties $<^1_{\mu,\omega}$). Consider p and $h(p \restriction M_2)$ both from $\mathscr{S}(M_3)$, both do not μ-split over M_0 and have the same restriction to M_1; as $M_0 <^1_{\mu,\omega} M_1$ it follows that $p = h(p \restriction M_2)$. So $R(p \restriction M_2) = R(h(p \restriction M_2)) = R(p)$ as required. □

Claim 7.4 (K categorical in $\lambda, cf(\lambda) > \mu > LS(\mathfrak{K})$). *Suppose $m < \omega, M \in K_\mu$ is saturated, $p \in \mathscr{S}^m(M), M \leqslant_{\mathfrak{K}} N \in K_\mu$, $p \leqslant q \in \mathscr{S}^m(N), N$ saturated over M, q not a stationarization of p (i.e. for no $M^- <^0_{\mu,\omega} M$, q does not μ-split over M^-). Then q μ-divides over M.*

Proof. By Claim 7.5 below and Lemma 6.3 (just p does not μ-split over some N_α).

Claim 7.5. *Assume $M_0 <^1_{\mu,\omega} M_1 <^1_{\mu,\omega} M_2$ all saturated. If $q \in \mathscr{S}(M_2)$ does not μ-split over M_1 and $q \restriction M_1$ does not μ-split over M_0,* then *q does not μ-split over M_0.*

Proof. Let $M_3 \in K_\mu$ be such that $M_2 <^1_{\mu,\omega} M_3$ and $c \in M_3$ realizes q. Choose a linear order I^* such that $I^* \times (\mu + \omega^*) \cong I^* \cong I^* \times \mu$, I^* has no first nor last element (see [12, Appendix]), $|I^*| = \mu$.

Let $I_0 = I^* \times \mu$, $I_1 = I_0 + I^* \times \mathbb{Z}, I_2 = I_1 + I^* \times \mathbb{Z}, I_3 = I_2 + I^* \times \mu$. Clearly without loss of generality $M_\ell = EM_\tau(\Phi, I_\ell)$, let $c = \tau(a_{t_0}, \ldots, a_{t_k}), I_{\ell+1,n} = I_\ell + I^* \times \{m \colon \mathbb{Z} \models m < n\}$ and $I_{0,\alpha} = I^* \times \alpha$. So we can find a (negative) integer $n(*)$ small enough and $m(*) \in \mathbb{Z}$ large enough such that $\{t_0, \ldots, t_n\} \cap I_{2,n(*)} \subseteq I_{1,m(*)}$. Let $M_{1,n} = EM(I_{1,n}, \Phi)$, $M_{2,n} = EM(I_{2,n}, \Phi)$. Clearly $M_0 <^1_{\mu,\omega} M_{1,n} <^1_{\mu,\omega} M_1 <^1_{\mu,\omega} M_{2,n} <^1_{\mu,\omega} M_2$. Clearly (use automorphism of I_3)

$(*)_0$ $q \restriction M_{2,n}$ does not μ-split over $M_{1,m}$ if $\mathbb{Z} \models n < n(*), m(*) \leqslant m \in \mathbb{Z}$.

By Fact 7.3 with $q, M_1, M_{2,n}, M_2, q$ here standing for M_0, M_2, M_3, p there we get

$(*)_1$ $R(q) = R(q \restriction M_{2,n})$ if $n \in \mathbb{Z}$.

Similarly

$(*)_2$ $R(q \restriction M_1) = R(q \restriction M_{1,m})$ if $m \in \mathbb{Z}$.

By $(*)_0$ and Fact 7.3 we have

$(*)_3$ $R(q \restriction M_{2,n(*)}) = R(q \restriction M_{1,m(*)})$.

Similarly we can find $\alpha(*) < \mu$, $\alpha(*)$ successor and $k(*) \in \mathbb{Z}$ such that

$\{t_0, \ldots, t_k\} \cap I_{1,k(*)+1} \subseteq I_{0,\alpha(*)-1}$

and then prove

$(*)_4$ $R(q\restriction M_0)=R(q\restriction M_{0,\alpha})$ if $\alpha(*)\leqslant\alpha<\mu$

$(*)_5$ $R(q\restriction M_{1,k(*)})=R(q\restriction M_{0,\alpha})$ if $\alpha(*)\leqslant\alpha<\mu$.

Together $R(q)=R(q\restriction M_0)$, hence q does not μ-split over M_0 as required. □

Part II

1. Existence of nice Φ

We build EM models, where "equality of types over A in the sense of the existence of automorphisms over A" behaves nicely.

Context 1.1.
(a) $\mathfrak{K}$ is an abstract elementary class with models of cardinality $\geqslant\beth_{(2^{LS(\mathfrak{K})})^+}$; it really suffices to assume:
(a)′ $\mathfrak{K}$ is a class of $\tau(K)$-models, which is $PC_{\kappa^+,\omega}$ with a model of cardinality $\geqslant\beth_{(2^{LS(\mathfrak{K})})^+}$.

Definition 1.2. (1) Let $\kappa\geqslant LS(\mathfrak{K})$, now $\Upsilon^{or}_\kappa=\Upsilon^{or}_{\kappa,\tau}$ is the family of Φ proper for linear orders (see [14, Ch. VII]) such that
 (a) $|\tau(\Phi)|\leqslant\kappa$,
 (b) $EM_\tau(I,\Phi)=EM(I,\Phi)\restriction\tau(K)\in K$,
 (c) $I\subseteq J\Rightarrow EM_\tau(I,\Phi)\leqslant_{\mathfrak{K}}EM_\tau(J,\Phi)$.
(2) Υ^{or} is $\Upsilon^{or}_{LS(\mathfrak{K})}$.

Definition 1.3. We define partial orders $\leqslant^{\oplus}_\kappa$ and $\leqslant^{\otimes}_\kappa$ on Υ^{or}_κ (for $\kappa\geqslant LS(\mathfrak{K})$):
(1) $\Psi_1\leqslant^{\oplus}_\kappa\Psi_2$ *if* $\tau(\Psi_1)\subseteq\tau(\Psi_2), EM(I,\Psi_1)\subseteq EM_{\tau(\Psi_1)}(I,\Psi_2)$ and $EM_\tau(I,\Psi_1)\leqslant_{\mathfrak{K}}EM_\tau(I,\Psi_2)$ for any linear order I.
Again for $\kappa=LS(\mathfrak{K})$ we may drop the κ.
(2) For $\Phi_1,\Phi_2\in\Upsilon^{or}_\kappa$, we say Φ_2 is an inessential extension of Φ_1, $\Phi_1\leqslant^{\mathrm{ie}}_\kappa\Phi_2$ if $\Phi_1\leqslant^{\oplus}_\kappa\Phi_2$ and for every linear order I,

$$EM_\tau(I,\Phi_1)=EM_\tau(I,\Phi_2).$$

(note: there may be more functions in $\tau(\Phi_2)$!)
(3) $\Phi_1\leqslant^{\otimes}_\kappa\Phi_2$ *iff* there is Ψ proper for linear order and producing linear orders such that
 (a) $\tau(\Psi)$ has cardinality $\leqslant\kappa$,
 (b) $EM(I,\Psi)$ is a linear order which is an extension of I: in fact $[t\in I\Rightarrow x_t=t]$,
 (c) $\Phi_2'\leqslant^{\mathrm{ie}}_\kappa\Phi_2$ where $\Phi_2'=\Psi\circ\Phi_1$, i.e.

$$EM(I,\Phi_2')=EM(EM(I,\Psi),\Phi_1).$$

(Though we allow further expansion by functions definable from earlier ones (composition or even definition by cases), as long as the number is $\leqslant \kappa$.)

Claim 1.4. (1) $(\Upsilon^{or}_\kappa, \leqslant^{\otimes}_\kappa)$ *and* $(\Upsilon^{or}_\kappa, \leqslant^{\oplus})$ *are partial orders (and* $\leqslant^{\otimes}_\kappa \subseteq \leqslant^{\oplus}_\kappa$*).*
(2) *Moreover, if* $\langle \Phi_i\colon i<\delta\rangle$ *is a* $\leqslant^{\otimes}_\kappa$*-increasing sequence,* $\delta<\kappa^+$*, then it has a* $<^{\otimes}_\kappa$*-l.u.b.* Φ*;* $EM^1(I,\Phi)=\bigcup_{i<\delta} EM^1(I,\Phi_i)$*. Similarly for* $\leqslant^{\oplus}_\kappa$*.*

Lemma 1.5. *If* $N \leqslant_{\mathfrak{K}} M$*,* $\|M\| \geqslant \beth_{(2^\chi)^+}$*,* $\chi \geqslant \|N\| + LS(\mathfrak{K})$, then *there is* Φ *proper for linear order such that*:
(a) $EM_{\tau(\mathfrak{K})}(\emptyset,\Phi)=N$,
(b) $N \leqslant_{\mathfrak{K}} EM_\tau(I,\Phi)$*, moreover*

$$I \subseteq J \Rightarrow EM_{\tau(\mathfrak{K})}(I,\Phi) \leqslant_{\mathfrak{K}} EM_{\tau(\mathfrak{K})}(I,\Phi).$$

(c) $EM_\tau(I,\Phi)$ *omits every type* $p\in\mathscr{S}(N)$ *which* M *omits, moreover if* I *is finite then* $EM_\tau(I,\Phi)$ *can be* $\leqslant_{\mathfrak{K}}$*-embedded into* M*.*

Proof. Straight by [11, 1.7].

Lemma 1.6. *Assume*
(a) $LS(\mathfrak{K})\leqslant\chi\leqslant\lambda$,
(b) $N_0 \leqslant_{\mathfrak{K}} N_1 \leqslant_{\mathfrak{K}} M$,
(c) $\|N_0\|\leqslant\chi$, $\|N_1\|=\lambda$ *and* $\|M\|\geqslant\beth_{(2^\chi)^+}(\lambda)$,
(d) $\Gamma_1=\{p^1_i\colon i<i^*_1\leqslant\chi\}\subseteq\mathscr{S}(N_1)$*, each* p^1_i/E_χ *omitted by* M*,*
(e) $\Gamma_0=\{p^0_i\colon i<i^*_0\}\subseteq\mathscr{S}(N_0)$*, each* p^0_i *omitted by* M*.*
Then *we can find* $\langle N'_\alpha\colon \alpha\leqslant\omega\rangle$*,* Φ *and* $\langle q^1_i\colon i<i^*_1\rangle$ *such that*
(α) Φ *proper for linear order,*
(β) $N'_\alpha\in\mathfrak{K}_{\leqslant\chi}$ *is* $\leqslant_{\mathfrak{K}}$*-increasing continuous (for* $\alpha\leqslant\omega$*),*
(γ) $N'_0=N_0$ *and* $N'_0\leqslant_{\mathfrak{K}} N_1$*,*
(δ) $q^1_i\in\mathscr{S}(N'_\omega)$*,*
(ε) $EM_{\tau(\mathfrak{K})}(\emptyset,\Phi)$ *is* N'_ω*,*
(ζ) *for linear order* $I\subseteq J$ *we have* $EM_{\tau(\mathfrak{K})}(I,\Phi)\leqslant_{\mathfrak{K}} EM_{\tau(\mathfrak{K})}(J,\Phi)$*,*
(η) *for* n*, there is a* $\leqslant_{\mathfrak{K}}$*-increasing sequence* $\langle N_{n,m}\colon m<\omega\rangle$ *with union* $EM_{\tau(\mathfrak{K})}(n,\Phi)$ *and a* $\leqslant_{\mathfrak{K}}$*-embedding* $f_{n,m}$ *of* $N_{n,m}$ *into* M *such that*
  (i) $N'_m=f_{n,m}N_{0,m}$, $f_{n,m}\subseteq f_{n,m+1}$
  (ii) $f_{n,m}\restriction N_0$ *is the identity,* $Rang(f_{0,m})\subseteq N_1$*,*
  (iii) $f_n(q^1_i\restriction N'_m)=p^1_i\restriction Rang(f_n)$ *for* $i<i^*_1$*,*
(θ) $EM_{\tau(\mathfrak{K})}(I,\Phi)$ *omits every* p^0_i *for* $i<i^*_0$ *and omits every* q^1_i *in a strong sense: for every* $a\in EM_{\tau(\mathfrak{K})}(I,\Phi)$ *for some* n *we have* $q^1_i\restriction N'_n\neq tp(a,N'_n,EM_{\tau(\mathfrak{K})}(I,\Phi))$*.*

Remark. (1) So we really can replace q^1_i by $\langle q^1_i\restriction N'_n\colon n<\omega\rangle$, but for ω-chains by chasing arrows such limit (q^1_i) exists.
(2) Clause (ζ) follows from Clause (η).

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Proof. By [11, 1.7] we can find τ_1, $\tau(\mathfrak{K}) \subseteq \tau_1$, $|\tau_1| \leqslant \chi$ (here we can have $\leqslant LS(\mathfrak{K}) \leqslant \chi$) and an expansion M^+ of M to a τ_1-model and a set Γ of quantifier free types (so $|\Gamma| \leqslant 2^{\aleph_0 + |\tau_1|}$) such that

(A) M^+ omits every $p \in \Gamma$,

(B) for $\bar{a} \in {}^{\omega >}M$ we let $M^-_{\bar{a}} = M^+ \restriction c\ell(\bar{a}, M^+)$ then $M^+_{\bar{a}} \leqslant_{\mathfrak{K}} M^+$, Rang $(\bar{a}) \subseteq$ Rang$(\bar{b})$ $\Rightarrow M^+_{\bar{a}} \restriction \tau(\mathfrak{K}) \leqslant_{\mathfrak{K}} M^+_{\bar{b}} \restriction \tau(\mathfrak{K})$ where $\bar{a} \in {}^{\omega >}(N_\ell) \Rightarrow |M^+_{\bar{a}}| \subseteq N_\ell$.

Note: Further expansion of M^- to M^*, as long as $|\tau(M^*)| \leqslant \chi$ preserves (A) + (B) so we can add

(C) $N_0 \subseteq M^-_{\langle\rangle}$ and letting $M^+_{\bar{a},\ell} = M^+_{\bar{a}} \restriction (|N_\ell| \cap |M^+_{\bar{a}}|)$, $\ell = 1, 2$,

(D) $M^+_{\bar{a},0} \restriction \tau(\mathfrak{K}) \leqslant_{\mathfrak{K}} M^+_{\bar{a},1} \restriction \tau(\mathfrak{K}) \leqslant_{\mathfrak{K}} M^+_{\bar{a}} \restriction \tau(\mathfrak{K})$,

(E) for $i < i^*_1$, the type $p^1_i \restriction (M^+_{\bar{a},1} \restriction \tau(\mathfrak{K}))$ is not realized in $M^-_{\bar{a}} \restriction \tau(\mathfrak{K})$.

Now we choose by induction on n, sequence $\langle f^n_\alpha \colon \alpha < (2^\chi)^+ \rangle$ and N'_m such that

(i) f^n_α is a one-to-one function from $\beth_\alpha(\lambda)$ into M,

(ii) $\langle f^n_\alpha(\zeta) \colon \zeta < \beth_\alpha(\lambda) \rangle$ is n-indiscernible in M^+,

(iii) moreover, if $\alpha < (2^\chi)^+$, and $m \leqslant n$ and $\zeta_1 < \cdots < \zeta_m < \beth_\alpha(\lambda)$ and $\xi_1 < \cdots < \xi_m < \beth_\beta(\lambda)$ *then*: the sequences $\bar{a} = \langle f_\alpha(\zeta_1), \ldots, f_\alpha(\zeta_m) \rangle$, $\bar{b} = \langle f_\beta(\xi_1), \ldots, f_\beta(\xi_m) \rangle$ realize the same quantifier free type in M^- over N^+_1, so there is a natural isomorphism $g_{\bar{b},\bar{a}}$ from $M^+_{\bar{a}}$ onto $M^+_{\bar{b}}$ (mapping $f_\alpha(\zeta_\ell)$ to $f_\beta(\xi_1)$), moreover

$$i < i^*_1 \Rightarrow g_{\bar{b},\bar{a}}(p^1_i \restriction (M^+_{\bar{a},1} \restriction \tau(\mathfrak{K}))) = p^1_i \restriction (M^+_{\bar{b},1} \restriction \tau(\mathfrak{K})) \text{ and } M^+_{\bar{a},1} \restriction \tau(\mathfrak{K}) = N'_m.$$

The rest should be clear. □

Claim 1.7. *Suppose*

(a) $\Phi \in \Upsilon^{or}_\kappa$,

(b) $n < \omega$, u, u_1, u_2 *are subsets of* $\{0, 1, \ldots, n-1\}$ *and* $\sigma_1(\ldots, \bar{x}_\ell, \ldots)_{\ell \in u_1}$, $\sigma_2(\ldots, \bar{x}_\ell, \ldots)_{\ell \in u_2}$, *are* $\tau(\Phi)$*-terms.*

(c) *For every* $\alpha < (2^{LS(\mathfrak{K})})^+$ *(or at least* $\beth_\alpha < \mu(\kappa)$*-see* [14, *Ch.* VII, *Section* 4] *but of this we should be careful as to omit only* $\leqslant LS(\mathfrak{K})$ *types) there are linear orders* $I \subseteq J$, I $\aleph_0$*-homogeneous inside*[3] J, I *of cardinality* $\geqslant \beth_\alpha$, *such that for some (equivalently every)* $t_0 < t_1 < \cdots < t_{n-1}$ *of* I *we have*

$\oplus$ *for some automorphism* f *of* $EM_\tau(J, \Phi)$, $f \restriction EM_\tau(I \setminus \{t_\ell \colon \ell < n,\ \ell \notin u\}, \Phi)$ *is the identity and*

$$f(\sigma_1(\ldots, \bar{a}_{t_\ell}, \ldots)_{\ell \in u_1}) = \sigma_2(\ldots, \bar{a}_{t_\ell}, \ldots)_{\ell \in u_2}.$$

THEN for some Φ', $\Phi \leqslant^{\oplus}_\kappa \Phi'$ *and even* $\Phi \leqslant^{\oplus}_\kappa \Phi'$ *and*

$\otimes$ *for every linear order* I *and* $t_0 < \cdots < t_{n-1}$ *from* I, *there is an automorphism* f *of* $EM_\tau(I, \Phi')$ *such that*

(α) $f \restriction EM(I \setminus \{t_\ell \colon \ell < n,\ \ell \notin u\}, \Phi')$ *is the identity and*

(β) $f(\sigma_1(\ldots, \bar{a}_{t_\ell}, \ldots)_{\ell \in u_1}) = \sigma_2(\ldots, \bar{a}_{t_\ell}, \ldots)_{\ell \in u_2}$.

(γ) $f = F(-, t_0, \ldots, t_{n-1})$ *for some* $F \in \tau(\Phi')$.

[3] This means that every partial order preserving function h from I to I can be extended to an automorphism of J.

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Proof. Expand $M = EM(J,\Phi)$, by the predicates $Q_1 = \{\bar{a}_t\colon t \in I\}, Q_2 = \{\bar{a}_t\colon t\in J\}$. For $t_0 < \cdots < t_{n-1} \in I$, let $f_{t_0,\ldots,t_{n-1}} \in AUT(EM_\tau(J,\Phi))$ as in ($\oplus$) and let g_ℓ (for $\ell < \omega$) be functions from M into $\{\bar{a}_t\colon t\in J\}$ such that $\forall \bar{x}\in M$, $x = \tau_x(g_0(x),\ldots,g_{n-1}(x))$ such that $g_\ell(x) = a_{t_{x,\ell}}$, $t_{x,\ell} <^J t_{x,\ell+1}$, let $P_\tau = \{x\colon \tau_x = \tau\}$.

Let F be an $(n+1)$-ary function, $F(\bar{a}_{t_0},\ldots,\bar{a}_{t_{n-1}}, b) = f_{t_0\ldots t_{n-1}}(b)$ when defined. The model we get we call M^+. Now use the omitting types theorem, i.e. Claim 4.5(5). So there is a model N^+ and $\langle \bar{b}_n\colon n<\omega\rangle$ indiscernible in it such that $N^+ \equiv M^+, N^+$ omits all types which M^+ omits, for every $m<\omega$ for some $s_0 < \cdots < s_{n-1}$ from I the type of $\bar{b}_0{}^\frown \cdots \bar{b}_{n-1}$ in N^+ is equal to the type of $a_{s_0}{}^\frown\cdots{}^\frown a_{s_{n-1}}$ in M^+. Define Φ' such that $EM(I^*,\Phi')$ is a $\tau(N^+)$-model generated by $\{\bar{a}_t\colon t\in I^*\}$ such that $t_0 < \cdots < t_{n-1} \in I^* \Rightarrow$ type of $\bar{a}_{t_0}{}^\frown\cdots{}^\frown\bar{a}_{t_{n-1}}$ in $EM(I^*,\Phi')$ is equal to type of $\bar{b}_0{}^\frown\cdots{}^\frown\bar{b}_{n-1}$ in N^+.
Why is $\Phi <^{\otimes} \Phi'$ and not just $\Phi <^{\oplus} \Phi'$?
Here we use Q_1, Q_2 in M^+ we have

(*) every $c\in M^+$ is in the τ_Φ-Skolem Hull of $Q_2^{M^+} = \{\bar{a}_t\colon t\in J\}$.

So

(*)′ M^+ omits the type

$$p(x) = \left\{ \neg(\exists \bar{y}_0,\ldots,\bar{y}_{n-1}) \left(\bigwedge_{\ell<n} Q_2(\bar{y}_\ell) \,\&\, x = \sigma(y_0,\ldots,y_n)\colon \sigma\in\tau_\Phi \right) \right\}. \qquad \square$$

Conclusion 1.8. For $\kappa \geqslant LS(\mathfrak{K})$ there is $\Phi^* \in \Upsilon_\kappa^{or}$ (in fact for every $\Phi\in\Upsilon_\kappa^{or}$ there is Φ^*, $\Phi \leqslant_\kappa^\otimes \Phi^* \in \Upsilon_\kappa^{or}$) satisfying

(a) if Φ^* satisfies the assumptions of 1.7 for some I, J *then* it satisfies its conclusion,
(b) moreover if $\kappa \geqslant 2^{LS(\mathfrak{K})}$, for some $\chi(\Phi^*) < \mu(\kappa)$ (see [14, Ch. VII, Section 4]), we can weaken the assumption $\alpha < \beth_{(2^\kappa)^+}$ to $\beth_\alpha \leqslant \chi(\Phi^*)$,
(c) moreover, in Claim 1.7 we can omit "I is $\aleph_0$-homogeneous inside J",
(d) also we can demand the automorphism to be $F(-,\bar{a}_{t_0},\ldots,\bar{a}_{t_{n-1}})$ for some function symbol $F\in\tau(\Phi^*)$,
(e) also we can replace clause (α) of $\otimes$ (of Claim 1.7) by: f extends some automorphism of $EM(I\backslash\{t_\ell\colon \ell<n,\ \ell\notin u\},\Phi^*)$ definable as in clause (γ) or $\oplus$ of 1.7.
(f) we can deal similarly with automorphisms extending a given $f \restriction EM(I\restriction \{t_\ell\colon \ell < n\})$ and having finitely many demands.

Proof. For (a) iterate Claim 1.7, by bookkeeping looking at all $\langle \sigma_1,\sigma_2,u,u_1,u_2\rangle$ and use Claim 1.4 for noting that the iteration is possible. Now (b) holds as $\mathrm{cf}(\mu(\kappa)) > \kappa$, and the number of terms is $\leqslant\kappa$. For (c) we can let Ψ be such that $EM(I,\Psi)$ is an $\aleph_0$-homogeneous linear order, $|\tau(\Psi)| = \aleph_0$ and use $\Psi\circ\Phi^*$. The rest are easy, too. $\square$

Lemma 1.9. *Let Φ^* be as in Conclusion* 1.8, *and I be a linear order of cardinality $\chi(\Phi^*)$ (where $\chi(\Phi^*)$ is from Conclusion* 1.8). *Assume $\sigma(\bar{x}_0,\ldots,\bar{x}_{n-1})$ is a term in $\tau(\Phi^*)$, for $\ell = 1,2$ we have $t_0^\ell < \cdots < t_{n-1}^\ell$ and $u \subseteq \{\ell\colon t_\ell^1 = t_\ell^2\}$, and there is no* *automorphism* f *of* $EM(I,\Phi^*)$ *such that* $f \restriction EM(I\setminus\{t^1_\ell,t^2_\ell\colon \ell<n,\ \ell\notin u\},\Phi^*)$ *is the identity, and* $f(\sigma(\bar{a}_{t^1_0},\ldots,))=\sigma(\bar{a}_{t^2_0},\ldots,)$.

Then

(1) *for* $\chi>\chi(\Phi^*)^-$ *we have* $I(\chi,K)=2^\chi$.

(2) *We have the* $\chi(\Phi^*)$*-order property in the sense of* (*see more* [13, *Ch.* III, *Section* 3] *or better* [17, *Ch.* III, *Section* 3]).

Proof. Without loss of generality I is dense.

We can find $t^3_0<\cdots<t^3_{n-1}$ such that

$$\ell\in u\Rightarrow t^3_\ell=t^1_\ell,$$

$$\ell\notin u\Rightarrow t^3_\ell\notin\{t^1_m,t^2_m\colon m<n\}.$$

$\otimes_1$ there is no automorphism f of $EM(I,\Phi^*)$ such that $f\restriction EM_\tau(I\setminus\{t^1_\ell,t^2_\ell,t^3_\ell\colon \ell<n,\ \ell\notin u\},\Phi)$ is the identity and $f(\sigma(\bar{a}_{t^1_0},\ldots))=\sigma(\bar{a}_{t^2_0},\ldots)$
[Why? If there is easily some Φ contradicts Conclusion 1.8(a)]

$\otimes_2$ for some $k\in\{1,2\}$, there is no automorphism f of $EM(I,\Phi)$ which is the identity of $EM(I\setminus\{t^1_\ell,t^2_\ell,t^3_\ell\colon \ell<n,\ \ell\notin u\},\Phi)$ and $f(\sigma(\bar{a}_{t^k_0},\ldots))=\sigma(\bar{a}_{t^3_0},\ldots)$
[Why? If not such f_1,f_2 exists and $f_2^{-1}\circ f_1$ contradict $(*)_2$].

$\otimes_3$ For some $k\in\{1,2\}$ there is no automorphism f of $EM(I,\Phi)$ which is the identity on $EM(I\setminus\{t^k_\ell,t^3_\ell\colon \ell<n,\ \ell\notin u\},\Phi)$ and $f(\sigma(\bar{a}_{t^k_0},\ldots))=\sigma(a_{t^3_0},\ldots)$ [Why? We negate a stronger demand than in $(*)_2$].

By renaming we get that without loss of generality

$$t^1_\ell=t^2_k\Rightarrow\ell=k\in u.$$

By the transitivity of "there is an automorphism" we can assume that just for a singleton $\ell(*)$, $t^1_{\ell(*)}\neq t^2_{\ell(*)}$. Now if we increase u, surely such isomorphism does not exist so without loss of generality $u=\{\ell<n\colon \ell\neq\ell(*)\}$ and $t^1_{\ell(*)}<_I t^2_{\ell(*)}$. Let $I^0=\{t\in I\colon t<_I t^1_{\ell(*)}\}$, $I^1=\{t\in I\colon t^1_{\ell(*)}\leqslant_I t<_I t^2_{\ell(*)}\}$, $I^2=\{t\in I\colon t^2_{\ell(*)}<_I t\}$ (yes: $<_I$ not $\leqslant_I$).

Now for every linear order J we can define $I(J)$ as follows: $I(J)$ is a linear order which is the sum $I^0+\sum_{t\in J}I^1_t(J)+I^2$, $I^1_t(J)$ is isomorphic to I^1, so let $f_t:I^1\to I^1_t(J)$ be such an isomorphism. Let $\bar{\mathbf{b}}^t$ list $EM(I^0+I^1_t(J)+I^2)$ (such that for t,s, $(id_{I^0}+f_sf_t^{-1}+id_{I^2})$ induces a mapping from $\bar{\mathbf{b}}^t$ onto $\bar{\mathbf{b}}^s$). Let $\bar{c}^t=f_t(\sigma(t^1_0,\ldots,t^1_{n-1}))$. Now

$(*)_1$ if $s_0<_J r<_J s_1$ then there is no automorphism f of $EM_\tau(I(J),\Phi^*)$ over $\bar{\mathbf{b}}^r$ mapping $\bar{c}^{s_0}$ to $\bar{c}^{s_1}$,

$(*)_2$ if J is $\aleph_0$-homogeneous (or just 2-transitive) and $r<_J s_0$ & $r<_J s_1$ or $s_0<_J r$ & $s_1<_J r$ *then* there is an automorphism f of $EM_\tau(I(J),\Phi^*)$ over $\mathbf{b}^r$ mapping $\bar{c}^{s_0}$ to $\bar{c}^{s_1}$.

So by [17, Ch. III, Section 3] (or earlier version [13, Ch. III, Section 3]), we have the order property for sequences of length $\chi(\Phi^*)$; the formula appearing in the definition of the order is preserved by automorphisms of the model; though it looks as second order, it does not matter. So conclusion (2) holds and (1) follows. □

Claim 1.10. *Assume*
(a) *K is categorical in λ,*
(b) *the $M \in K_\lambda$ is χ^+-saturated (holds if $cf(\lambda) > \chi$),*
(c) *$\chi \geqslant LS(\mathfrak{K})$.*
Then *every $M \in K$ of cardinality $\geqslant \beth_{(2^\chi)^+}$ (or just $\geqslant \beth_{\mu(\chi)}$ if $\chi \geqslant 2^{LS(\mathfrak{K})}$) is χ^+-saturated.*

Proof. If M is a counterexample, let $N \leqslant_{\mathfrak{K}} M$, $\|N\| \leqslant \chi$ and $p \in \mathscr{S}(N)$ be omitted by N. By the omitting type theorem for abstract elementary classes (see Lemma 1.5, i.e. [11]), we get $M' \in K_\lambda$, $N \leqslant_K M'$, M' omitting p a contradiction.

Claim 1.11. *Assume*
(a) *$LS(\mathfrak{K}) \leqslant \chi$,*
(b) *for every $\alpha < (2^\chi)^+$ there are $M_\alpha <_{\mathfrak{K}} N_\alpha$ (so $M_\alpha \neq N_\alpha$), $\|M_\alpha\| \geqslant \beth_\alpha$ and $p \in \mathscr{S}(M_\alpha)$ such that $c \in N_\alpha \Rightarrow \neg pE_\chi\ tp(c, M_\alpha, \mathfrak{C})$.*
(1) *For every $\theta > \chi$ there are $M <_{\mathfrak{K}} N$ in K_θ and $p \in \mathscr{S}(M_\alpha)$ as in clause* (b).
(2) *Moreover, if Φ is proper for orders as usual, $|\tau(\Phi)| \leqslant \chi$, $\beth_{(2^\chi)^+} \leqslant \lambda$, K categorical in λ and I a linear order of cardinality θ,* then *we can demand $M = EM_\tau(I, \Phi)$.*

Proof. Straightforward.

2. Small pieces are enough and categoricity

Context 2.1.
(i) $\mathfrak{K}$ an abstract elementary class,
(ii) $\mathfrak{K}$ categorical in λ, $\lambda > LS(\mathfrak{K})$,
(iii) some ($\equiv$ any) $M \in K_\lambda$ is saturated (if λ is regular this holds),
(iv) Φ^* is as in Conclusion 1.8.

Hence

Fact 2.2. *For $\mu \in [LS(\mathfrak{K}), \lambda)$, there is a saturated model of cardinality μ, (why? by part* I 6.5(3A)) *and there is also $\Phi^* \in \Upsilon_\mu^{or}$ as in Conclusion* 1.8.

Main Claim 2.3. *If $M \in K$ is a saturated model of cardinality χ, $\chi(\Phi^*) < \chi < cf(\lambda) \leqslant \lambda$ then $\mathscr{S}(M)$ has character $\leqslant \chi(\Phi^*)$, i.e. if $p_1 \neq p_2$ are in $\mathscr{S}(M)$ then for some $N \leqslant_{\mathfrak{K}} M$, $N \in K_{\chi(\Phi^*)}$ we have $p_1 \restriction N \neq p_2 \restriction N$.*

Proof. We can find $I \subseteq J$, $|I| = \chi$, $|J| = \lambda$, $M = EM_\tau(I, \Phi^*) \leqslant_{\mathfrak{K}} N^* = EM_\tau(J, \Phi^*)$ and I, J are $\aleph_0$-homogeneous. So by Claim I 6.7: every $p \in \mathscr{S}(M)$ is realized in N^* and

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say p is realized by $\sigma_p(\bar{a}_{t_{p,0}}, \bar{a}_{t_{p,1}}, \ldots, \bar{a}_{t_{p,n_p-1}})$ where $t_{p,0} < t_{p,1} < \cdots < t_{p,n_p-1}$. If the conclusion fails, then we can find $q \neq p$ in $\mathscr{S}(M)$ such that

$$(*) \quad N \leqslant_{\mathfrak{K}} M, \quad \|N\| \leqslant \chi(\Phi^*) \Rightarrow p \restriction N = q \restriction N.$$

Choose $I' \subseteq J$, dense, homogeneous inside $J, |I'| = \chi(\Phi^*)$ such that $\{t_{p,\ell}: \ell < n_p\} \subseteq I'$ and $\{t_{q,\ell}: \ell < n_q\} \subseteq I'$ and let $M' = EM_\tau(I' \cap I, \Phi^*) \leqslant_{\mathfrak{K}} M, I' \backslash \{t_{p,e}, t_{q,h}: l < n_p,\ h < n_q\} \subseteq I$.

So $p \restriction M' = q \restriction M'$; so Claim 1.7 becomes relevant (i.e. Conclusion 1.8(b)) considering the $\aleph_0$-homogeneity of J) hence by the choice of Φ^*, $p = q$ contradiction. □

Conclusion 2.4. Assume $M \in K_\chi$ is saturated, $\chi(\Phi^*) \leqslant \chi < \lambda$, I directed, $\langle M_t: t \in I \rangle$ is a $\leqslant_{\mathfrak{K}}$-increasing family of $\leqslant_{\mathfrak{K}}$-submodels of M, each saturated and $[t < s \Rightarrow M_t$ saturated over $M_s]$ and $\|M_t\| \leqslant \chi(\Phi^*)$, *then* for every $p \in \mathscr{S}(\bigcup_{t \in I} M_t)$ for some $t^* \in I$:

$(*)$ p does not μ-split over M_{t^*}

(and even does not χ-split over M_{t^*}).

Proof. Clearly by the proof of 2.3.

Claim 2.5. *If T is categorical in λ, the $M \in K_\lambda$ is μ^--saturated, $LS(\mathfrak{K}) \leqslant \chi(\Phi^*) \leqslant \mu < \lambda$, $\langle M_i: i < \delta \rangle$ an $<_{\mathfrak{K}}$-increasing sequence of μ^+-saturated models then $\bigcup_{i<\delta} M_i$ is μ^+-saturated.*

Remark. (1) Hence this holds for limit cardinals $> LS(\mathfrak{K})$.

(2) The addition compared to Claim I 6.7 is the case $\mathrm{cf}(\lambda) = \mu^+, \mu^{++}$ e.g. $\lambda = \mu^-$.

Proof. Let $M_\delta = \bigcup_{i<\delta} M_i$ and assume M_δ is not μ^+-saturated. So there are $N \leqslant_{\mathfrak{K}} M_\delta$ of cardinality $\leqslant \mu$ and $p \in \mathscr{S}(N)$ which is not realized in M_δ. Choose $p_1 \in \mathscr{S}(M_\delta)$ such that $p_1 \restriction N = p$.

Without loss of generality N is saturated (by I 6.7 or induction on μ).

Let $\chi = \chi(\Phi^*)$, without loss of generality $\delta = \mathrm{cf}(\delta)$.

We claim

⊗ there are $i(*) < \delta$ and $N^* \leqslant_{\mathfrak{K}} M_{i(*)}$ of cardinality χ such that p does not χ-split over N^*.

Why? Assume toward contradiction that this fails. The proof of ⊗ splits to two cases.

Case I: $\mathrm{cf}(\delta) \leqslant \chi$.

We can choose by induction on $i < \delta$, N_i^0, N_i^1 such that

(a) $N_i^0 \leqslant_{\mathfrak{K}} M_i$ has cardinality χ,

(b) N_i^0 is increasingly continuous,

(c) $N_i^0 <^1_{\chi,\omega} N_{i+1}^0$,

(d) $N_i^0 \leqslant_{\mathfrak{K}} N_i^1 \leqslant_{\mathfrak{K}} M_\delta$,

(e) N_i^1 has cardinality $\leqslant \chi$,

(f) $p_1 \restriction N_i^1$ does χ-split over N_i^0,

(g) for ε, $\zeta < i$, $N_\varepsilon^1 \cap M_\zeta \subseteq N_i^0$.

There is no problem to carry the definition and then $N_i^1 \subseteq \bigcup_{j<\delta} N_j^0$ and $\langle N_i^0 : i<\delta \rangle, p_1 \restriction \bigcup_{i<\delta} N_i^0$ contradicts Lemma 6.3.

Case II: $\mathrm{cf}(\delta) > \chi$.

Now by Claim 3.3

(∗) for some $N^* \leqslant_{\mathfrak{K}} M_\delta$ of cardinality χ we have p_1 does not χ-split over N^*.

As $\delta = \mathrm{cf}(\delta) \geqslant \mu > \chi$, for some $i(*) < \delta$ we have $N^* \leqslant_{\mathfrak{K}} M_{i(*)}$. This ends the proof of ⊗.

So $i(*)$, N^* are well defined $w\log N^*$ is saturated. Let $c \in \mathfrak{C}$ realize p_1. We can find a $\leqslant_{\mathfrak{K}}$-embedding h of $EM_\tau(\mu^+ + \mu^+, \Phi)$ into $\mathfrak{C}$ such that

(a) N^* is the h-image of $EM_\tau(\chi, \Phi)$,
(b) h maps $EM_\tau(\mu^+, \Phi)$ onto $M' \leqslant_K M_{i(*)}$,
(c) every $d \in N$ and c belong to the range of h.

By renaming, h is the identity, clearly for some $\alpha < \mu^+$ we have $N \cup \{c\} \subseteq EM_\tau(\alpha \cup [\mu^+, \mu^+ + \alpha))$, so some list $\bar{b}$ of the members of N is $\bar{\sigma}(\ldots, \bar{a}_i, \ldots, a_{\mu^+ + j})_{i<\alpha, j<\alpha}$ (assume $\alpha > \mu$ for simplicity) and $c = \sigma^*(\ldots, \bar{a}_i, \ldots, a_{\mu^+ + j}, \ldots)_{i \in u, j \in w}$ ($u, w \subseteq \mu^+$ finite as, of course, only finitely many $\bar{a}_i$'s are needed for the term σ^*).

For each $\gamma < \mu^+$ we define $\bar{b}^\gamma = \bar{\sigma}(\ldots, \bar{a}_i, \ldots, a_{(1+\gamma)\alpha + j}, \ldots)_{i<\alpha, j<\alpha}$ and $c^\gamma = \sigma^*(\ldots, \bar{a}_i, \ldots, a_{(1+\gamma)\alpha + j}, \ldots)_{i,j}$ and stipulate $\bar{b}^{\mu^+} = \bar{b}$, $c^{\mu^+} = c$ and let $q = \mathrm{tp}(\bar{b}\,\hat{}\,c, N^*, \mathfrak{C})$. Clearly $\langle \bar{b}^\gamma\,\hat{}\,c^\gamma : \gamma < \mu^+ \rangle\hat{}\,\bar{b}\,\hat{}\,c$ is a strictly indiscernible sequence over N^* and $\subseteq M_\delta \cup \{c\}$, so also $\{\bar{b}^\gamma : \gamma \leqslant \mu\} \subseteq M_\delta$ is strictly indiscernible over N. [Why? Use $I \supseteq \mu^+ + \mu^+$ which is a strongly μ^{++} saturated dense linear order and use automorphisms of $\mathrm{EM}(I, \Phi)$ induced by an automorphism of I.]

As c realizes p_1 clearly $\mathrm{tp}(c, M_\delta)$ does not χ-split over N^* hence by Main claim 2.3 necessarily $\mathrm{tp}(\bar{b}^\gamma\hat{}\,c, N^*, \mathfrak{C})$ is the same for all $\gamma \leqslant \mu^+$, hence $\gamma < \mu^+ \Rightarrow \mathrm{tp}(\bar{b}^\gamma\hat{}\,c^{\mu^+}, N^*, \mathfrak{C}) = q$, so by the indiscernibility $\beta < \gamma \leqslant \mu^+ \Rightarrow \mathrm{tp}(\bar{b}^\beta\hat{}\,c^\gamma, N^*, \mathfrak{C}) = q$.

By the indiscernibility for some q_1,

$$\beta < \gamma < \mu^+ \Rightarrow \mathrm{tp}(\bar{b}^\gamma\hat{}\,c^\beta, N^*, \mathfrak{C}) = q_1.$$

If $q \neq q_1$, then $\mathrm{tp}(c_0, \bar{b}^1, \mathfrak{C}) \neq \mathrm{tp}(c_2, \bar{b}^1, \mathfrak{C})$, but $\mathrm{Rang}(\bar{b}^\gamma)$ is a model $N_\gamma^* \leqslant_{\mathfrak{K}} M_{i(*)}$, $N^* \leqslant_{\mathfrak{K}} N_\gamma^*$, so by Lemma 1.9, for some $v \subseteq \ell g(\bar{b}^\gamma)$ of cardinality χ, $\mathrm{tp}(c_0, \bar{b}^1 \restriction v, \mathfrak{C}) \neq \mathrm{tp}(c_2, \bar{b}^1 \restriction v, \mathfrak{C})$. So clearly we get the $(\chi, \chi, 1)$-order property contradiction to Claim 4.14.

Hence necessarily $\beta \leqslant \mu^+$ & $\gamma \leqslant \mu^+$ & $\beta \neq \gamma \Rightarrow \mathrm{tp}(\bar{b}^\beta\hat{}\,c^\gamma, N^*, \mathfrak{C}) = q$. For $\beta = \mu^+$, $\gamma = 0$ we get that $c^\gamma \in M_{i(*)} \leqslant_K M_\delta$ realizes $\mathrm{tp}(c^\gamma, N, \mathfrak{C}) = p_1 \restriction N$ as desired. □

We could have proved earlier

Observation 2.6.

(1) If M is θ-saturated, $\lambda > \theta > LS(\mathfrak{K})$ and $N \leqslant_{\mathfrak{K}} M$, $N \in K_{\leqslant \theta}$ then there is N', $N \leqslant_{\mathfrak{K}} N' \leqslant_{\mathfrak{K}} M$, $N' \in K_\theta$ and every $p \in \mathscr{S}(N)$ realized in M is realized in N'.
(2) Assume $\langle N_i : i \leqslant \delta \rangle$ is $\leqslant_{\mathfrak{K}}$-increasingly continuous, $N_i \in K$, $N_i \leqslant M$, M is θ-saturated, and every $p \in \mathscr{S}(N_i)$ realized in M is realized in N_{i+1} then
    (a) if $\delta = \theta \times \sigma$, $\mathrm{LS}(\mathfrak{K}) < \sigma = \mathrm{cf}(\sigma) \leqslant \theta$, then N_δ is σ-saturated
    (b) if $\delta = \theta \times \theta$, $\theta > \mathrm{LS}(\mathfrak{K})$, *then* N_δ is saturated.

Theorem 2.7 (The Downward categoricity theorem for λ successors). *If λ is successor $\geqslant \mu(\chi(\Phi^*)) = \mu < \chi < \lambda$, then K is categorical in χ.*

Remark. (1) We intend to try to prove in future work that also in $K_{>\lambda}$ we have categoricity and deal with λ not successor. This calls for using $\mathscr{P}^-(n)$-diagram as in [9], etc.
(2) We need some theory of orthogonality and regular types parallel to [8, Ch. V] = [14, Ch. V], as done in [20] and then [5] (which appeared) and then [4, 19]. Then the categoricity can be proved as in those papers.

Proof. Let $K' = \{M \in K: M$ is $\chi(\Phi^*)$-saturated of cardinality $\geqslant \chi(\Phi^*)\}$. So

$(*)_1$ K' is closed under $\leqslant_{\mathfrak{K}}$-increasing unions, and there is a saturated $M \in K_\lambda$

$(*)_2$ if $\chi \geqslant \beth_{(2^{LS(\mathfrak{K})})^+}(\chi(\Phi^*))$ and $M \in K_\chi$ then $M \in K'_\chi$
[Why? Otherwise by Claim 1.5 there is a non $LS(\mathfrak{K})^+$-saturated $M \in K_\lambda$ contradicting Lemma 2.2],

$(*)_3$ if $M \in K'$, $p \in \mathscr{S}(M)$ *then* for some $M_0 \leqslant_{\mathfrak{K}} M$, $M_0 \in K'$ and p does not $\chi(\Phi^*)$-split over M_0 so $\|M_0\| = \chi(\Phi^*)$.

$(*)_4$ *Definition*: for $\chi \in [\chi(\Phi^*), \lambda)$ and $M \in K'_\chi$ and $p \in \mathscr{S}(M)$ we say p is minimal if
(a) p is not algebraic which means p is not realized by any $c \in M$,
(b) if $M \leqslant_{\mathfrak{K}} M' \in K'_\chi$ and $p_1, p_2 \in \mathscr{S}(M')$ are non-algebraic extending p, *then* $p_1 = p_2$.

$(*)_5$ *Fact*: if $M \in K'_\chi$ is saturated, $\chi \in [\chi(\Phi^*), \lambda)$, *then* some $p \in \mathscr{S}(M)$ is minimal
[Why? If not, we choose by induction on $\alpha < \chi$ for every $\eta \in {}^\alpha 2$ and triple (M_η, N_η, a_η) and $h_{\eta,\eta \restriction \beta}$ for $\beta \leqslant \alpha$ such that
(a) $M_\eta <^*_{\mathfrak{K}} N_\eta$ and $a_\eta \in N_\eta \backslash M_\eta$,
(b) $\langle M_{\eta \restriction \beta}: \beta \leqslant \alpha \rangle$ is $\leqslant_{\mathfrak{K}}$-increasingly continuous,
(c) $M_{\eta \restriction \beta} <^1_{\mu,\omega} M_{\eta \restriction (\beta+1)}$,
(d) $h_{\eta,\eta \restriction \beta}$ is a $\leqslant_{\mathfrak{K}}$-embedding of $N_{\eta \restriction \beta}$ into N_η which is the identity on $M_{\eta \restriction \beta}$ and maps $a_{\eta \restriction \beta}$ to a_η,
(e) if $\gamma \leqslant \beta \leqslant \alpha$, $\eta \in {}^\alpha 2$, then $h_{\eta,\eta \restriction \gamma} = h_{\eta,\eta \restriction \beta} \circ h_{\eta \restriction \beta, \eta \restriction \gamma}$,
(f) $M_{\eta \frown \langle 0 \rangle} = M_{\eta \frown \langle 1 \rangle}$ but
$\mathrm{tp}(a_{\eta \frown \langle 0 \rangle}, M_{\eta \frown \langle 0 \rangle}, N_{\eta \frown \langle 0 \rangle}) \neq \mathrm{tp}(a_{\eta \frown \langle 1 \rangle}, M_{\eta \frown \langle 1 \rangle}, N_{\eta \frown \langle 1 \rangle})$.
No problem to carry the definition and then we can get a contradiction to stability in χ; for successor use I6.3].

$(*)_6$ Fix $M^* \in K'_{\chi(\Phi^*)}$ and minimal $p^* \in \mathscr{S}(M^*)$.

$(*)_7$ If $M^* \leqslant_{\mathfrak{K}} M \in K'_{<\lambda}$, *then* p^* has a non-algebraic extension of $p \in \mathscr{S}(M)$, moreover it is unique and also p is minimal if M saturated.
[Why? Existence by Lemma 6.3, uniqueness modulo $E_{\chi(\Phi^*)}$ follows hence uniqueness. Applying this to an extension M' of M of cardinality $< \lambda$ we get p is minimal.]

$(*)_8$ There are no $M_1, M_2 \in K'$ such that
(a) $M^* \leqslant_{\mathfrak{K}} M_1 \leqslant_{\mathfrak{K}} M_2$,
(b) M_1, M_2 are saturated of cardinality λ_1, $\lambda_1^+ = \lambda$,

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(c) $M_1 \neq M_2$,

(d) no $c \in M_2 \backslash M_1$ realizes p^*.

[Why? If there are, we choose by induction on $\zeta < \lambda$, $N_\zeta \in K_{\lambda_1}$ is $\leqslant_{\mathfrak{K}}$-increasingly continuous, each N_ζ is saturated, $N_0 = M_1$, $N_\zeta \neq N_{\zeta+1}$ and no $c \in N_{\zeta+1} \backslash N_\zeta$ realizes p^*. If we succeed, then $N = \bigcup_{\zeta<\lambda} N_\zeta$ is in K_λ (as $N_\zeta \neq N_{\zeta+1}$!) but no $c \in N \backslash N_0$ realizes p^* so it is not saturated (as $\{\zeta\colon c \notin N_\zeta\}$ is an initial segment of λ, non-empty (0 is in)) so it has a last element ζ, so $c \in N_{\zeta+1} \backslash N_\zeta$ as c realizes P^* cont.; so N is not saturated, contradiction. For $\zeta = 0$, $N_0 = M_1$ is okay by clause (b). If ζ is limit $< \lambda$, let $N_\zeta = \bigcup_{\varepsilon<\zeta} N_\varepsilon$, clearly $N_\zeta \in K_{\lambda_1}$ and it is saturated by 2.5. If $\zeta = \varepsilon + 1$, note that as N_ζ, M_1 are saturated in K_{λ_1} and $\leqslant_{\mathfrak{K}}$-extends M^* which has smaller cardinality, there is an isomorphism f_ζ from M_1 onto N_ε which is the identity on M^*. We define N_ζ such that there is an isomorphism f_ζ^+ from M_2 onto N_ζ extending f. By assumption (b), $N_\zeta \in K_{\lambda_1}$ is saturated by assumption (c), $N_\zeta \neq N_{\zeta+1}$, and by assumption (d), no $c \in N_{\zeta+1} \backslash N_\zeta$ realizes p^* (as $f_\zeta \restriction M^* =$ the identity). So as said above, we have derived the desired contradiction.]

$(*)_9$ If $M \in K'_{<\lambda}$ and $M^* \leqslant M <_{\mathfrak{K}} N$, M has cardinality $\geqslant \theta^* = \beth_{(2^{\chi(\Phi^*)})^+}$ (or just $\geqslant \theta^*$ $=: \mu(\chi(\Phi^*))$), *then* some $c \in N \backslash M$ realizes p^*.

[Why? By $(*)_2, M, N$ are θ^*-saturated. So we can find saturated $M' \leqslant_{\mathfrak{K}} M$, $N' \leqslant_{\mathfrak{K}} N$ of cardinality θ^* such that $M' = N' \cap M$, $M^* \neq N'$ (why? by Observation 2.6). So still no $c \in N' \backslash M'$ realizes p^*. We would like to transfer the appropriate omitting type theorem of this situation from θ^* to λ_1; the least trivial point is preserving the saturation. But this can be expressed as: "is isomorphic to $EM(I, \Phi)$ for some linear order I" for appropriate Φ, and this is easily transferred.]

$(*)_{10}$ If $M \in K'_{\leqslant\lambda}$ has cardinality $\geqslant \theta^* = \beth_{(2^{\chi(\Phi^*)})^+}$ then it is θ^*-saturated (so $\in K'_{\leqslant\lambda}$).

[why? included in the proof of $(*)_9$].

$(*)_{11}$ If $M \in K'_{\leqslant\lambda}$ has cardinality $\geqslant \theta^*$, *then* M is saturated

[why? Assume not by $(*)_{10}$, M is θ^*-saturated let M be θ-saturated not θ^+-saturated by $(*)_{10}$, $\theta \geqslant \theta^*$, without loss of generality $M^* \leqslant_{\mathfrak{K}} M$. Let $M_0 \leqslant_{\mathfrak{K}} M$ be such that $M^* \leqslant_{\mathfrak{K}} M_0 \in K_\theta$ and some $q \in \mathscr{S}(M_0)$ is omitted by M. Now choose by induction on $i < \theta^+$ a triple (N_i^0, N_i^1, f_i) such that

(a) $N_i^0 \leqslant_{\mathfrak{K}} N_i^1$ belong to K_θ and are saturated,

(b) N_i^0 is $\leqslant_{\mathfrak{K}}$-increasingly continuous,

(c) N_i^1 is $\leqslant_{\mathfrak{K}}$-increasingly continuous,

(d) $N_0^0 = M_0$ and $d \in N_0^1$ realizes q,

(e) f_i is a $\leqslant_{\mathfrak{K}}$-embedding of N_i^0 into M, increasing continuous, $f_0 = \text{id}$,

(f) for each i, for some $c_i \in N_i^1 \backslash N_i^0$ we have $c_i \in N_{i+1}^0$.

If we succeed, let $E = \{\delta < \theta^+\colon \delta$ limit and for every $i < \delta$ and $c \in N_i^1$ we have $(\exists j < \theta^+)(c_j = c) \rightarrow (\exists j < \delta)(c_j = c)\}$. Clearly E is a club of θ^+, and for each $\delta \in E$, c_δ belongs to $N_\delta^1 = \bigcup_{i<\delta} N_i^1$ so there is $i < \delta$ such that $c_\delta \in N_i^1$, so for some $j < \delta$, $c = c_j$ so $c_\delta = c_j \in N_{j+1}^0 \leqslant_{\mathfrak{K}} N_\delta^0$, contradiction to clause (f).

So we are stuck for some ζ, now $\zeta \neq 0$ trivially. ζ not limit by Claim 2.5, so $\zeta = \varepsilon + 1$. Now if $N_\varepsilon^0 = N_\varepsilon^1$, then $f_\varepsilon(d) \in M$ realizes q a contradiction, so $N_\varepsilon^0 <_{\mathfrak{K}} N_\varepsilon^1$.

Also $f_\varepsilon(N^0_\varepsilon) <_{\mathfrak{K}} M$ by cardinality consideration. Now by $(*)_9$ some $c_\varepsilon \in N^1_\varepsilon \setminus N^0_\varepsilon$ realizes p^*.

We can find $N^\prime_\zeta \leqslant_{\mathfrak{K}} M$ such that $f_\varepsilon(N^0_\varepsilon) <_{\mathfrak{K}} N^\prime_\zeta \in K_0$, $N^\prime_\zeta$ saturated (why? by Observation 2.6).

Again by $(*)_9$ we can find $c^\prime_\zeta \in N^\prime_\zeta \setminus f_\varepsilon(N^0_\varepsilon)$ realizing p^*. By $(*)_5$ clearly $\mathrm{tp}(c^\prime_\zeta, f_\varepsilon(N^0_\varepsilon), M) = f_\varepsilon(\mathrm{tp}(c_\zeta, N^0_\varepsilon, N^1_\varepsilon))$ so we can find saturated, $N^1_\zeta \in K_0$ which $\leqslant_{\mathfrak{K}}$-extended N^1_ε and $g^\prime_\varepsilon$ is a $\leqslant_{\mathfrak{K}}$-embedding of $N^\prime_\zeta$ into N^1_ζ extending f^{-1}_ε and $g^\prime_\varepsilon(c^\prime_\zeta) = c_\zeta$. Let $N^0_\zeta = N^0_{\varepsilon+1} = g_\varepsilon(N^\prime_\zeta)$.

So we can carry the construction, contradiction, so $(*)_{11}$ holds.]

$(*)_{12}$ K_λ is categorical in every $\chi \in [\beth_{(2^{\mathcal{A}(\Phi^*)})^+}, \lambda)$

[why? by $(*)_{11}$ every model is saturated and the saturated model is unique]. □

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