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The non-commutative Specker phenomenon in the uncountable case

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An infinitary version of the notion of free products was introduced and investigated by Higman [1]. Let G_i $(i \in I)$ be groups and $*_{i\in X} G_i$ the free product of G_i $(i \in X)$ for $X \Subset I$ and $p_{XY}: *_{i\in Y} G_i \to *_{i\in X} G_i$ the canonical homomorphism for $X \subseteq Y \Subset I$. ($X \Subset I$ denotes that X is a finite subset of I.) Then, the unrestricted free product is the inverse limit $\varprojlim(*_{i\in X} G_i, p_{XY}\colon X \subseteq Y \Subset I)$. We remark $*_{i\in\emptyset} G_i = \{e\}$. Let $\mathbb{Z}_n$ be a copy of the integer group $\mathbb{Z}$ and δ_n be its generator. We use a set theoretic convention $n = \{0, 1, \ldots, n-1\}$ for a natural number $n < \omega$. Since ω is the set of natural numbers and the sets n are cofinal in the family of finite subsets of ω, we have

$$\varprojlim\Bigl(\mathop{*}_{i\in X} \mathbb{Z}_i, p_{XY}\colon X \subseteq Y \Subset \omega\Bigr) \simeq \varprojlim\Bigl(\mathop{*}_{i<m} \mathbb{Z}_i, p_{mn}\colon m \leqslant n < \omega\Bigr).$$

For sake of simplicity, we abbreviate limits $\varprojlim(*_{i\in X} G_i, p_{XY}\colon X \subseteq Y \Subset I)$ and $\varprojlim(*_{i<m} \mathbb{Z}_m, p_{mn}\colon m \leqslant n < \omega)$ by $\varprojlim * G_i$ and $\varprojlim * \mathbb{Z}_n$, respectively, in the sequel. Since $\mathbb{Z}_n$ can be regarded as a subgroup of $\varprojlim * \mathbb{Z}_n$, we regard δ_n

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as an element of $\varprojlim * \mathbb{Z}_n$. An element u of $\varprojlim(*_{i\in X} G_i, p_{XY}\colon X \subseteq Y \Subset I)$ is canonically presented as a map such that $u(X) \in *_{i\in X} G_i$ and $p_{XY}(u(Y)) = u(X)$ for $X \subseteq Y \Subset I$. For $S \subseteq I$, let $p_S : \varprojlim * G_i \to \varprojlim * G_i$ be the canonical projection induced by the homomorphisms obtained by composing the map $p_{X\cap S,X}$ and the inclusion $*_{i\in X\cap S} G_i \hookrightarrow *_{i\in X} G_i$ for $X \Subset I$, i.e., $p_S(x)(X) = x(X \cap S)$ for $X \Subset I$.

An uncountable cardinal κ is measurable if there exists a κ-complete non-principal ultrafilter on κ [2, p. 297]. The following is well known [2, Lemma 27.1]: Let κ be the least cardinal on which there exists a countably complete non-principal ultrafilter. Then κ is a measurable cardinal.

Theorem 1.1. *Let F be a free group. Then for each homomorphism $h : \varprojlim * G_i \to F$ there exist countably complete ultrafilters $u_0, \dots, u_m$ on I such that $h = h \cdot p_{U_0\cup\dots\cup U_m}$ for every $U_0 \in u_0, \dots, U_m \in u_m$. Consequently, if the cardinality of the index set I is less than the least measurable cardinal, then there exists a finite subset X_0 of I and a homomorphism $\bar{h} : *_{i\in X_0} G_i \to F$ such that $h = \bar{h} \cdot p_{X_0}$.*

Previously the second author showed the failure of the Specker phenomenon in the uncountable case in a different situation [3]. (See also [4].) We explain the difference between this result and Theorem 1.1 of the present paper. There is a canonical subgroup of the unrestricted free product, which is called the free complete product and denoted by $\times_{i\in I} G_i$. When an index set I is countable, according to the Higman theorem (Lemma 1.2 and its variant for $\times_{n<\omega} \mathbb{Z}_n$ [1, p. 80]), a homomorphism from $\varprojlim * G_i$ or $\times_{i\in I} G_i$ to a free group factors through a finite free product $*_{i\in F} G_i$. On the other hand, when the index set I is uncountable and each G_i is non-trivial, there exists a free retract of $\times_{i\in I} G_i$ of large cardinality and there are homomorphisms not factoring through any finite free product $*_{i\in F} G_i$, which contrasts with the case when I is countable. This also contrasts with an abelian case, which is known as the Łoś theorem [5]. Theorem 1.1 says that differing from the case of the free complete products the non-commutative Specker phenomenon holds for the unrestricted free products similarly as in the abelian case.

Since the following lemma holds for the free σ-product $\times^{\sigma}_{i\in I} \mathbb{Z}_i$ instead of a free group F [6], Theorem 1.1 also holds for it. (We remark $\times^{\sigma}_{i\in I} \mathbb{Z}_i = \times_{i\in I} \mathbb{Z}_i$, when I is countable.)

Lemma 1.2 (Higman [1]). *For each homomorphism $h : \varprojlim * \mathbb{Z}_n \to F$ there exists $m < \omega$ and a homomorphism $\bar{h} : *_{k<m} \mathbb{Z}_k \to F$ such that $h = \bar{h} \cdot p_m$, where $p_m : \varprojlim * \mathbb{Z}_n \to *_{n<m} \mathbb{Z}_n$ is the canonical projection.*

Lemma 1.3. *Let $I = \bigcup\{I_n\colon n < \omega\}$ with $I_n \subseteq I_{n+1}$ and $x_n \in \varprojlim * G_i$ be such that $p_{I_n}(x_n) = e$. Then there exists a homomorphism $\varphi : \varprojlim * \mathbb{Z}_n \to \varprojlim * G_i$ such that $\varphi(\delta_n) = x_n$ for each $n < \omega$.*

Proof. We define $\psi_{mX} : *_{k<m} \mathbb{Z}_k \to *_{i\in X} G_i$ by $\psi_{mX}(\delta_k) = p_X(x_k)$. Let $X \Subset I_m$. Since $p_X(x_k) = e$ for $k \geqslant m$, we have $\psi_{mX} \cdot p_{mn} = \psi_{nX}$ for $n \geqslant m$ and consequently $\psi_{mX} \cdot p_m = \psi_{nX} \cdot p_n$. Define $\varphi_X : \varprojlim * \mathbb{Z}_n \to *_{i\in X} G_i$ as $\psi_{mX} \cdot p_m$ and let $X \subseteq Y \Subset I_n$. Since $p_{XY} \cdot \psi_{nY} = \psi_{nX}$, we have $p_{XY} \cdot \varphi_Y = p_{XY} \cdot \psi_{nY} \cdot p_n = \psi_{nX} \cdot p_n = \varphi_X$. By the universal property of the inverse limit we have $\varphi : \varprojlim * \mathbb{Z}_n \to \varprojlim * G_i$ such that $p_X \cdot \varphi = \varphi_X$ for $X \Subset I$ and hence $\varphi(\delta_n) = x_n$ for each $n < \omega$. $\square$

For a homomorphism $h : \varprojlim * G_i \to F$, let

$$\mathrm{supp}(h) = \big\{X \subseteq I:\ p_X(g) = e \text{ implies } h(g) = 0 \text{ for each } g\big\}.$$

In the sequel we assume that h is non-trivial. We remark the following facts:

(1) $p_X \cdot p_Y = p_{X\cap Y}$ for $X, Y \subseteq I$;
(2) $\mathrm{supp}(h) = \{X \subseteq I:\ h(g) = h(p_X(g))$ for each $g\}$;
(3) $\mathrm{supp}(h)$ is a filter on I.

Lemma 1.4. *Let $A_n \subseteq A_{n+1} \subseteq I$ and $A = \bigcup\{A_n:\ n < \omega\}$ and $B_{n+1} \subseteq B_n \subseteq I$ and $B = \bigcap\{B_n:\ n < \omega\}$. If $A_n \notin \mathrm{supp}(h)$ for each n, then $A \notin \mathrm{supp}(h)$ and if $B_n \in \mathrm{supp}(h)$ for each n, then $B \in \mathrm{supp}(h)$.*

Proof. Suppose that $A \in \mathrm{supp}(h)$. Take g_n so that $h(g_n) \neq 0$ and $p_{A_n}(g_n) = e$ for each n and let $u_n = p_A(g_n)$. Since $I = \bigcup\{A_n \cup (I \setminus A):\ n < \omega\}$ and $p_{A_n\cup(I\setminus A)}(u_n) = p_{A_n}(g_n) = e$, by Lemma 1.3 we have a homomorphism $\varphi : \varprojlim * \mathbb{Z}_n \to \varprojlim * G_i$ such that $\varphi(\delta_n) = u_n$ for each $n < \omega$. Then $h \cdot \varphi(\delta_n) \neq 0$ for each n, which contradicts Lemma 1.2.

To show the second proposition by contradiction, suppose that $B \notin \mathrm{supp}(h)$. Then we have $g \in \varprojlim * G_i$ such that $p_B(g) = e$ but $h(g) \neq 0$. Let $v_n = p_{B_n}(g)$. Since $I = \bigcup\{B \cup (I \setminus B_n):\ n < \omega\}$ and $p_{B\cup(I\setminus B_n)}(v_n) = p_B \cdot p_{B_n}(g) = e$, we apply Lemma 1.3 and have a homomorphism $\varphi : \varprojlim * \mathbb{Z}_n \to \varprojlim * G_i$ such that $\varphi(\delta_n) = v_n$ for each $n < \omega$. Then we have a contradiction similarly as the above. $\square$

Lemma 1.5. *Let $A_0 \notin \mathrm{supp}(h)$. Then there exist A satisfying the following*:

(1) $A_0 \subseteq A \notin \mathrm{supp}(h)$;
(2) *for $X \subseteq I$, $A \cup X \notin \mathrm{supp}(h)$ imply $(I \setminus X) \cup A \in \mathrm{supp}(h)$.*

Proof. We construct $A_n \notin \mathrm{supp}(h)$ by induction as follows. Suppose that we have constructed $A_n \notin \mathrm{supp}(h)$. If A_n satisfies the required properties of A, we have finished the proof. Otherwise, there exist $A_n \subseteq A_{n+1} \subseteq I$ such that $A_{n+1} \notin \mathrm{supp}(h)$ and $(I \setminus A_{n+1}) \cup A_n \notin \mathrm{supp}(h)$. We claim that this process finishes in a finite step. Suppose that the process does not stop in a finite step.

Then we have A_n's and so let $A = \bigcup\{A_n\colon\ n < \omega\}$. Then $A \notin \mathrm{supp}(h)$ by Lemma 1.4. Since $I \setminus A \subseteq I \setminus A_{n+1}$, $(I \setminus A) \cup A_n \notin \mathrm{supp}(h)$ for each $n < \omega$. Now $I = \bigcup\{(I \setminus A) \cup A_n\colon\ n < \omega\}$ and hence $I \notin \mathrm{supp}(h)$ by Lemma 1.4, which is a contradiction. □

Proof of Theorem 1.1. Let $h: \varprojlim * G_i \to F$ be a non-trivial homomorphism. Apply Lemma 1.5 to $A_0 = \emptyset$, then we have A. We define u_0 as follows.

$X \in u_0$ if and only if $A \cup X \in \mathrm{supp}(h)$ for $X \subseteq I$. Then u_0 is a countably complete ultrafilter on I by Lemma 1.4. We let $I_0 = I \setminus A$, then obviously $I \setminus I_0 \notin \mathrm{supp}(h)$.

When $I_0 \in \mathrm{supp}(h)$, then $h = h \cdot p_{U_0}$ for every $U_0 \in u_0$ and we have finished the proof. Otherwise, we construct $I_n \notin \mathrm{supp}(h)$ and countably complete ultrafilters u_n on I with $I_n \in u_n$ by induction as follows. Suppose that $\bigcup_{i=0}^n I_i \notin \mathrm{supp}(h)$, we apply Lemma 1.5 to $A_0 = \bigcup_{i=0}^n I_i \notin \mathrm{supp}(h)$ and get a countably complete ultrafilter u_{n+1} on I with $I_{n+1} \in u_{n+1}$ so that $I \setminus I_{n+1} \notin \mathrm{supp}(h)$.

To show that this procedure stops in a finite step, suppose the negation. Since $(I \setminus \bigcup_{k=0}^\infty I_k) \cup \bigcup_{k=0}^n I_k$ is disjoint from I_{n+1}, $(I \setminus \bigcup_{k=0}^\infty I_k) \cup \bigcup_{k=0}^n I_k \notin \mathrm{supp}(h)$ for each n. Then we have $I \notin \mathrm{supp}(h)$ by Lemma 1.4, which is a contradiction.

Now we have pair-wise disjoint subsets $I_0, \dots, I_n$ of I such that $I_0 \cup \cdots \cup I_n \in \mathrm{supp}(h)$. By the construction, $X \in u_k$ if and only if $\bigcup_{i \neq k} I_i \cup X \in \mathrm{supp}(h)$ and hence $\bigcup_{i \neq k} I_i \cup U_k \in \mathrm{supp}(h)$ for $U_k \in u_k$ $(0 \leqslant k \leqslant n)$. Since $\mathrm{supp}(h)$ is a filter, $\bigcup_{k=0}^n U_k \in \mathrm{supp}(h)$ and we have the first proposition.

If each u_k contains a singleton $\{i_k\}$, we have $X_0 = \{i_0, \dots, i_n\} \in \mathrm{supp}(h)$. Then we have a homomorphism $\bar{h}: *_{i \in X_0} G_i \to F$ such that $h = \bar{h} \cdot p_{X_0}$. When the cardinality of I is less than the least measurable cardinal, every countably complete ultrafilter u on I is principal by the well-known fact mentioned just before Theorem 1.1 and, hence, u contains a singleton $\{i\}$ for some $i \in I$. Consequently we have the second proposition. □

Remark 1.6. (1) As explained before, when index sets are uncountable, the unrestricted free products and the free complete products behave differently with respect to the Specker phenomenon. The parts of the proof of Theorem 1.1 that do not generalize are applications of of Lemma 1.3. Proposition 1.9 of [7] is an analogue of Lemma 1.3 for the free complete products, but has some restricting hypotheses, which prevent its use.

(2) In [6, Theorem 1.2] we showed the Specker phenomenon holds for general inverse limits over a countable index set I and in [6, Remark 2, p. 102] we demonstrated by an example that this is not true if I is uncountable.

(3) In the abelian case $\varprojlim(\bigoplus_{i \in X} A_i,\ p_{XY}\colon\ X \subseteq Y \Subset I)$ is isomorphic to the direct product $\prod_{i \in I} A_i$ and we can analyze homomorphisms from $\prod_{i \in I} A_i$ to $\mathbb{Z}$ using ultraproducts when the cardinality of the index set I is greater than the least measurable cardinal [5]. We have not found a way to analyze homomorphisms in Theorem 1.1 so far.

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