COMPLETE QUOTIENT BOOLEAN ALGEBRAS

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ABSTRACT. For I a proper, countably complete ideal on the power set $\mathscr{P}(X)$ for some set X, can the quotient Boolean algebra $\mathscr{P}(X)/I$ be complete? We first show that, if the cardinality of X is at least ω_3, then having completeness implies the existence of an inner model with a measurable cardinal. A well-known situation that entails completeness is when the ideal I is a (nontrivial) ideal over a cardinal κ which is κ^+-saturated. The second author had established the sharp result that it is consistent by forcing to have such an ideal over $\kappa = \omega_1$ relative to the existence of a Woodin cardinal. Augmenting his proof by interlacing forcings that adjoin Boolean suprema, we establish, relative to the same large cardinal hypothesis, the consistency of: $2^{\omega_1} = \omega_3$ and there is an ideal ideal I over ω_1 such that $\mathscr{P}(\omega_1)/I$ is complete. (The cardinality assertion implies that there is no ideal over ω_1 which is ω_2-saturated, and so completeness of the Boolean algebra and saturation of the ideal has been separated.)

For I a proper, countably complete ideal on $\mathscr{P}(X)$ for some set X, can the quotient Boolean algebra $\mathscr{P}(X)/I$ be complete? This question was raised by Sikorski [Si] in 1949. By a simple projection argument as for measurable cardinals, it can be assumed that X is an uncountable cardinal κ, and that I is a κ-complete ideal on $\mathscr{P}(\kappa)$ containing all singletons. In this paper we provide consequences from and consistency results about completeness. Throughout, κ will denote an uncountable cardinal, and by an *ideal over* κ we shall mean a proper, κ-complete ideal on $\mathscr{P}(\kappa)$ containing all singletons.

If κ is a measurable cardinal and I a prime ideal over κ, then of course $\mathscr{P}(\kappa)/I$ is complete, being the two-element Boolean algebra. The following theorem shows that completeness in itself has strong consistency strength:

Theorem A. *If $\kappa \geq \omega_3$ and there is an ideal I over κ such that $\mathscr{P}(\kappa)/I$ is complete, then there is an inner model with a measurable cardinal.*

The restriction $\kappa \geq \omega_3$ is necessary for our proof.

There is a well-known situation in which completeness obtains. An ideal over κ is λ-*saturated* iff whenever $\{X_\alpha | \alpha < \lambda\} \subseteq \mathscr{P}(\kappa) - I$, there are $\alpha < \beta < \lambda$ such that $X_\alpha \cap X_\beta \notin I$. Smith-Tarski [ST] established that if an ideal I over κ is κ^+-saturated, then $\mathscr{P}(\kappa)/I$ is complete. There are several consistency results about the existence of such ideals; modifying a sharp one due to the second author, we establish the following result separating completeness from saturation.

NS denotes the ideal over ω_1 consisting of the nonstationary subsets; for an ideal I over κ and $A \subseteq \kappa$, $I|A$ denotes $\{X \subseteq \kappa | X \cap A \in I\}$, the restriction of I to A; finally, for $S \subseteq \omega_1$, $\widetilde{S}$ denotes $\omega_1 - S$.

Theorem B. *Suppose that κ is a Woodin cardinal, CH holds, and S is a stationary, co-stationary subset of ω_1. Then there is a cardinal-preserving forcing extension with no new reals satisfying*: $2^{\omega_1} = \kappa$, *and $I = \mathrm{NS}|\widetilde{S}$ is an ideal over ω_1 such that $\mathscr{P}(\omega_1)/I$ is a complete Boolean algebra (and I is not ω_2-saturated).*

Of course, starting with a Woodin cardinal we can insure that CH also holds by carrying out a preliminary extension. Also, the statement has the same force if S is replaced by $\widetilde{S}$, but the formulation is notationally convenient for the proof. That the I in the resulting model cannot be ω_2-saturated follows from the following well-known result of Ketonen [Ke]: If CH holds and there is an ω_2-saturated ideal over ω_1 then $2^{\omega_1} = \omega_2$. Adding a further layer of complexity to the proof of Theorem B, we also establish

Theorem C. *Assume the hypotheses of Theorem B. Then there is a forcing extension with no new reals satisfying*: $2^{\omega_1} = \omega_3 = \kappa$, *and $I = \mathrm{NS}|\widetilde{S}$ is an ideal over ω_1 such that $\mathscr{P}(\omega_1)/I$ is a complete Boolean algebra (and I is not ω_2-saturated).*

It follows from 1.1(a) below that I must be ω_3-saturated. This then contributes to the theory of saturated ideals by establishing the relative consistency of $2^\omega = \omega_1$, $2^{\omega_1} = \omega_3$, and the existence of an ω_3-saturated ideal over ω_1.

In §1 we derive some consequences of completeness and establish Theorem A. In §2 we indicate the modifications necessary to a previous proof of the second author to establish Theorem B. Finally in §3 we build on §2 to establish Theorem C, providing iteration lemmas for iterated semiproper forcing with mixed supports. The main mathematical advances in this paper are due to the second author, based on speculations and prodding by the first.

1. CONSEQUENCES OF COMPLETENESS

We first review the various concepts involved to affirm some notation: Let I be an ideal over κ. Then $I^+ = \mathscr{P}(\kappa) - I$, the "positive measure" sets with respect to I. For any $A \subseteq \kappa$, $[A]_I = \{B \subseteq \kappa | A \triangle B \in I\}$. The Boolean algebra $\mathscr{P}(\kappa)/I$ consists of the $[A]_I$'s with the set-theoretic operations modulated by I: $[A]_I \vee [B]_I = [A \cup B]_I$, $[A]_I \wedge [B]_I = [A \cap B]_I$, $-[A]_I = [\kappa - A]_I$, $\mathbf{0}_I = [\varnothing]_I$, and $\mathbf{1}_I = [\kappa]_I$. The subscript I will be suppressed when clear from the context. A Boolean algebra is *complete* iff least upper bounds exist for any collection of its members. $\mathscr{A}$ is an *antichain* with respect to I iff $\mathscr{A} \subseteq I^+$ yet whenever $A, B \in \mathscr{A}$ are distinct, $A \cap B \in I$. Thus, I is λ-saturated iff every antichain with respect to I has cardinality $< \lambda$. Also, $\mathscr{A}$ is a maximal antichain iff $\bigvee\{[A] | A \in \mathscr{A}\} = \mathbf{1}$ yet whenever $A, B \in \mathscr{A}$ are distinct, $[A] \wedge [B] = \mathbf{0}$.

The first significant results bearing on Sikorski's question were derived by Solovay [So], who established the consistency, relative to the existence of a measurable cardinal, of the existence of saturated ideals over accessible cardinals. In passing, he in effect noted the following partial converse to the Smith-Tarski result:

1.1. Lemma. *Suppose that I is an ideal over κ such that $\mathscr{P}(\kappa)/I$ is complete. For any λ, if I is not λ-saturated, then $2^\lambda \leq 2^\kappa$. In particular,*
  (a) *I is 2^κ-saturated.*
  (b) *If $2^\kappa < 2^{\kappa^+}$, then I is κ^+-saturated.*

Proof. Let $\langle A_\alpha | \alpha < \lambda \rangle$ enumerate (without repetitions) an antichain with respect to I. For any $X \subseteq \lambda$, let $a_X = \bigvee \{[A_\alpha] | \alpha \in X\}$. Then $X \neq Y$ implies that $a_X \neq a_Y$. Hence, $2^\lambda \leq |\mathscr{P}(\kappa)/I| \leq 2^\kappa$. □

Kunen established that if there is a κ^+-saturated ideal over κ, then κ is measurable in an inner model. In particular, as Solovay noted, 1.1(b) implies that if there is an ideal I over κ such that $\mathscr{P}(\kappa)/I$ is complete, then $V \neq L$. Kunen asked in the early 1970s whether completeness has strong consistency strength, and Theorem A confirms this, at least if $\kappa \geq \omega_3$.

With our ultimate goal the proof of Theorem A, we now fix an ideal I over κ such that $\mathscr{P}(\kappa)/I$ is complete for the rest of this section. We use the well-known stratagem of considering $\mathscr{P}(\kappa)/I - \{0\}$ as a notion of forcing with $[X] \leq [Y]$ iff $X - Y \in I$, and we denote the corresponding forcing relation simply by $\Vdash$. Note that if $\langle A_\alpha | \alpha < \gamma \rangle$ enumerates without repetitions a maximal antichain with respect to I, then it corresponds to a name $\dot{\tau}$ for an ordinal specified by: $[A_\alpha] \Vdash \dot{\tau} = \beta$ iff $\alpha = \beta$.

The following lemmata derive consequences of completeness using maximal antichains.

1.2. Lemma. *Suppose that $\gamma \leq \kappa$ and $\dot{\tau}$ is a name such that $\Vdash \dot{\tau} \in \gamma$. Then there is a partition $\{B_\xi | \xi < \gamma\}$ of κ such that: if $B_\xi \in I^+$, then $[B_\xi] \Vdash \dot{\tau} = \xi$.*

Proof. Let $\mathscr{A}$ be a maximal antichain with respect to I such that for any $A \in \mathscr{A}$, there is a $\xi_A < \gamma$ such that $[A] \Vdash \dot{\tau} = \xi_A$. By completeness, let $A_\xi \subseteq \kappa$ for $\xi < \gamma$ be such that $[A_\xi] = \bigvee\{[A] | \xi_A = \xi\}$. Then set $B_\xi = A_\xi - \bigcup_{\zeta < \xi} A_\zeta$ for $\xi < \gamma$, so that $[B_\xi] = [A_\xi]$ by κ-completeness. The B_ξ's are as required, once they are slightly modified to constitute a partition of all of κ. □

1.3. Lemma. *Forcing with $\Vdash$ preserves all cardinals $< \kappa$.*

Proof. It suffices to show that if $\gamma < \kappa$ is regular, $\delta < \gamma$, and $\Vdash \dot{\tau}: \delta \to \gamma$, then $\Vdash \exists \eta < \gamma(\dot{\tau}``\delta \subseteq \eta)$. For each $\beta < \delta$, let $\langle B_\xi^\beta | \xi < \gamma \rangle$ satisfy 1.2 for $\dot{\tau}(\beta)$. For each $\alpha < \kappa$, set

$$\eta_\alpha = \sup\{\xi < \gamma | \alpha \in B_\xi^\beta \text{ and } \beta < \delta\},$$

so that $\eta_\alpha < \gamma$ by the regularity of γ. Next, set $E_\eta = \{\alpha < \kappa | \eta_\alpha = \eta\}$ for $\eta < \gamma$. Then $\bigcup_{\eta < \gamma} E_\eta = \kappa$ is a partition. Consequently, for any $X \in I^+$ there is an $\eta < \gamma$ such that $E_\eta \cap X \in I^+$ by κ-completeness and $[E_\eta \cap X] \Vdash \dot{\tau}``\delta \subseteq \eta$. □

The proof of the following proposition is similar; it will not be needed in the rest of the paper. The only early result about complete quotient Boolean algebras other than the Smith-Tarski result, it appeared in terms of distributivity properties in Pierce [P], which also contained a similar formulation of the easy forcing fact that a notion of forcing adjoins a new function: $\lambda \to \lambda$ exactly when it adjoins a new subset of λ.

1.4. Proposition. *Suppose that $2^\nu < \kappa$. Then forcing with $\Vdash$ does not adjoin any new functions: $\nu \to 2^\nu$.*

Proof. Suppose $\Vdash \dot{t}\colon \nu \to 2^\nu$. For each $\beta < \nu$, let $\langle B_\xi^\beta | \xi < 2^\nu \rangle$ be as in 1.2 with its $\gamma = 2^\nu$ and its $\dot{t} = \dot{t}(\beta)$. For each $f\colon \nu \to 2^\nu$, let $E_f = \bigcap_{\beta<\nu} B_{f(\beta)}^\beta$. Then $\bigcup_f E_f = \kappa$ is a partition. Consequently, for any $X \in I^+$, there is an f such that $E_f \cap X \in I^+$ by κ-completeness, and $[E_f \cap X] \Vdash \dot{t} = \check{f}$. $\square$

The connection with inner models of measurability is made through the well-known concept of precipitous ideal, due to Jech and Prikry. For an ideal J over λ, if G is generic over V for the corresponding notion of forcing $\mathscr{P}(\lambda)/J - \{0_J\}$, then $\{X \subseteq \lambda | [X] \in G\}$ is an ultrafilter on $\mathscr{P}(\lambda) \cap V$, and for any class A in the sense of V, the ultrapower of A with respect to this ultrafilter using functions in $^\lambda V \cap V$ is called the *generic ultrapower of A by G*. In this situation, if $\dot{t}$ is a name for a function in $^\lambda A \cap V$, we denote by $(\dot{t})$ the name of the equivalence class of $\dot{t}$ in the ultrapower. J is *precipitous* iff for any such G, the generic ultrapower of V by G is well-founded. As Jech and Prikry showed, *if there is a precipitous ideal J over λ, then λ is measurable in an inner model.*

Before we turn to the proof of Theorem A, we establish a partial well-foundedness result about generic ultrapowers. Continuing to work with our fixed ideal I over κ such that $\mathscr{P}(\kappa)/I - \{0\}$ is complete and the corresponding notion for forcing $\Vdash$, we first make an observation related to 1.2.

1.5. Lemma. *Suppose that $\gamma < \kappa$ and $[X] \Vdash \dot{t}\colon \kappa \to \gamma \wedge \dot{t} \in \check{V}$. Then there is an $f\colon \kappa \to \gamma$ such that $[X] \Vdash (\dot{t}) = (\check{f})$; i.e. $[X]$ forces that $\dot{t}$ and $\check{f}$ belong to the same equivalence class in the generic ultrapower.*

Proof. For any set x, let c_x denote the constant function: $\kappa \to \{x\}$. We first note that for any $[Y] \leq [X]$, there is a $[Z] \leq [Y]$ and a $\xi < \gamma$ such that $[Z] \Vdash (\dot{t}) = (\check{c}_\xi)$. (There is a $[\overline{Y}] \leq [Y]$ and a $g\colon \kappa \to \gamma$ such that $[\overline{Y}] \Vdash \dot{t} = \check{g}$. By κ-completeness there is a $\xi < \gamma$ such that $Z = \{i \in \overline{Y} | g(i) = \xi\} \in I^+$. Thus, $[Z] \Vdash (\dot{t}) = (\check{c}_\xi)$ by definition of generic ultrapower.)

Using this, let $\mathscr{A} \subseteq \mathscr{P}(X) \cap I^+$ be a maximal antichain in $\mathscr{P}(X) \cap I^+$ such that for each $A \in \mathscr{A}$, there is a $\xi_A < \gamma$ with $[A] \Vdash (\dot{t}) = (\check{c}_{\xi_A})$. By completeness, let $A_\xi \subseteq X$ for $\xi < \gamma$ be such that $[A_\xi] = \bigvee\{[A] | \xi_A = \xi\}$. Then as in the proof of 1.2, let $B_\xi \subseteq X$ for $\xi < \gamma$ be such that $[B_\xi] = [A_\xi]$ and $\{B_\xi | \xi < \gamma\}$ is a partition of X. Finally, define $f\colon \kappa \to \gamma$ by $f(i) = \xi$ iff $i \in B_\xi$ (and $f(i) = 0$ if $i \notin X$). Then $[X] \Vdash (\dot{t}) = (\check{f})$ by definition of generic ultrapower. $\square$

This leads to

1.6. Proposition. *If κ is a successor cardinal, then $\Vdash$ "the generic ultrapower of κ is well-founded."*

Remark. It follows from the conclusion that e.g. κ is inaccessible in L.

Proof. Suppose that $\kappa = \mu^+$, and assume to the contrary that for some $X \in I^+$, $[X] \Vdash$ "$\langle (\dot{t}_n) | n \in \omega \rangle$ is an infinite descending sequence with $\dot{t}_n \in {}^\kappa\kappa \cap \check{V}$ for $n \in \omega$". We can assume that $[X] \Vdash (\dot{t}_0) = (\check{g}_0)$ for some $g \in {}^\kappa\kappa$, and that $[X] \Vdash \forall n \in \omega \forall \xi < \kappa (\dot{t}_n(\xi) \leq \check{g}_0(\xi))$.

For each $\alpha < \kappa$, let $e_\alpha\colon \alpha + 1 \to \mu$ be an injection. Then let $\dot{\sigma}_n$ for $0 < n < \omega$ be names such that $[X] \Vdash \forall \xi < \kappa (\dot{\sigma}_n(\xi) = \check{e}_{g_0(\xi)}(\dot{t}_n(\xi)))$. Thus,

$[X] \Vdash \dot{\sigma}_n : \kappa \to \mu$. By 1.5, let $f_n : \kappa \to \mu$ for $n \in \omega$ be such that $[X] \Vdash (\dot{\sigma}_n) = (\check{f}_n)$. Next, define $g_n : \kappa \to \kappa$ for $0 < n < \omega$ by: $g_n(i)$ is the unique β such that $f_n(i) = e_{g_0(\xi)}(\beta)$ if $f_n(i)$ is in the range of $e_{g_0(\xi)}$, and $= 0$ otherwise. Then for each $n \in \omega$, $[X] \Vdash (\dot{\tau}_n) = (\check{g}_n)$. Finally, if $T_n = \{i \in \kappa \mid g_{n+1}(i) < g_n(i)\}$ for $n \in \omega$, then $[X] \leq [T_n]$ by our assumption on $[X]$ and the definition of generic ultrapower. Hence, $X \cap \bigcap_n T_n$ is not empty, but any i in this set gives rise to an infinite descending set of ordinals $g_0(i) > g_1(i) > g_2(i) \cdots$, which is a contradiction. $\square$

Turning to the proof of Theorem A, we need another ingredient. The Dodd-Jensen Covering Theorem for their inner model K asserts that if there is no inner model with a measurable cardinal, then for any uncountable set x of ordinals there is a set $y \supseteq x$ such that $y \in K$ and $|y| = |x|$. The definable class K is extensionally preserved in all forcing extensions.

Proof of Theorem A. We take $\kappa \geq \omega_3$. Assume to the contrary that there is no inner model with a measurable cardinal. Suppose briefly that G is generic for $\mathscr{P}(\kappa)/I - \{0\}$. Then $K^V = K^{V[G]}$, and (since forcing does not create inner models of measurability) the conclusion of the Covering Theorem holds in $V[G]$. Since $\kappa \geq \omega_3$ and ω_1 and ω_2 are preserved by the forcing by 1.3, this implies in particular: whenever x is a size ω_1 set of ordinals in $V[G]$, there is a $y \in V$ of size ω_1 such that $y \supseteq x$. (Note that the preservation of ω_2 is needed here.) Recalling a previous remark about maximal antichains corresponding to names for ordinals, this in turn translates in a straightforward manner to the following assertion in the ground model:

(∗)   Whenever $\langle \mathscr{A}_\alpha \mid \alpha < \omega_1 \rangle$ is a sequence of maximal antichains with respect to I, for any $X \in I^+$ there is a $Y \in \mathscr{P}(X) \cap I^+$ such that for any $\alpha < \omega_1$, $|\{A \in \mathscr{A}_\alpha \mid A \cap Y \in I^+\}| \leq \omega_1$.

We now derive a contradiction by using this to show that I must be precipitous. Suppose then that $[X] \Vdash$ "$\langle (\dot{\tau}_n) \mid n \in \omega \rangle$ is an infinite descending sequence in the generic ultrapower". For each $n \in \omega$, let $\mathscr{A}_n$ be a maximal antichain with respect to I such that whenever $A \in \mathscr{A}_n$ there is an $f_n^A : \kappa \to V$ such that $[A] \Vdash \dot{\tau}_n = \check{f}_n^A$. Let $Y \in \mathscr{P}(X) \cap I^+$ satisfy (∗) for $\langle \mathscr{A}_n \mid n \in \omega \rangle$ and for each $n \in \omega$ let $\langle A_\xi^n \mid \xi < \omega_1 \rangle$ enumerate $\{A \in \mathscr{A}_n \mid A \cap Y \in I^+\}$. Set $B_\xi^n = (A_\xi^n - \bigcup_{\zeta < \xi} A_\zeta^n) \cap Y$ and $B_n = \bigcup_{\xi < \omega_1} B_\xi^n$, and define f_n on B_n by: $f_n = \bigcup_{\xi < \omega_1} f_n^{A_\xi^n} \mid B_\xi^n$. It is easily seen that for $n \in \omega$, $[B_n] = [Y]$ by maximality of $\mathscr{A}_n$ and κ-completeness. Hence $\bigcap_n B_n \neq \varnothing$, again by κ-completeness. But for any i in this set, $f_0(i) > f_1(i) > \cdots$ which is a contradiction. $\square$

Properties of the sort (∗) were first investigated in Baumgartner-Taylor [BT], then in Foreman-Magidor-Shelah [FMS], by Woodin, and extensively in Gitik-Shelah [GS]. A positive answer to the following question would strengthen 1.6 and eliminate the condition $\kappa \geq \omega_3$ from Theorem A.

1.7. **Question.** If κ is a successor cardinal and I is an ideal over κ such that $\mathscr{P}(\kappa)/I$ is complete, is I precipitous?

2. Separating completeness from saturation

We next turn to the proof of Theorem B. We shall build on the proof of the following result of Shelah [Sh4, XVI, Theorem 2.4], which we first describe.

2.1. Theorem. *Suppose that κ satisfies $(*)^a_{ab}$ (see [Sh4, XVI, 2.2]), e.g. is a Woodin cardinal, and S is a stationary, co-stationary subset of ω_1. Then there is a forcing extension with no new reals satisfying: $\kappa = \omega_2$ and $\mathrm{NS}|\widetilde{S}$ is an ideal over ω_1 which is ω_2-saturated.*

This result weakened the large cardinal hypotheses of previous results drawing the same conclusion, and was an outgrowth of Foreman-Magidor-Shelah [FMS]. It was established using concepts and techniques of the second author that we quickly review:

Suppose that $\langle P, \leq \rangle$ is a poset for forcing, λ is regular with $\mathscr{P}(P) \subseteq H_\lambda$, and N is countable with $\langle N, \in \rangle \prec \langle H_\lambda, \in \rangle$. Then $q \in P$ is $\langle N, P \rangle$-*generic* iff for any P-name $\dot{t} \in N$ for an ordinal, $q \Vdash \dot{t} \in \check{N}$. Refining this, $q \in P$ is $\langle N, P \rangle$-*semigeneric* iff for any P-name $\dot{t} \in N$ for a *countable* ordinal, $q \Vdash \ddot{t} \in \check{N}$. $\langle p_n | n \in \omega \rangle$ is a P-*generic sequence for* N iff $p_{n+1} \leq p_n \in N$ for each n and whenever D is a dense set for P with $D \in N$, there is an n such that $p_n \in D$.

$\langle P, \leq \rangle$ is *semiproper* iff for any regular λ such that $\mathscr{P}(P) \subseteq H_\lambda$, there is a closed unbounded subset of $[H_\lambda]^{<\omega_1}$ consisting of N such that $\langle N, \in \rangle \prec \langle H_\lambda, \in \rangle$ satisfying: for any $p \in N$, there is a $q \leq p$ such that q is $\langle N, P \rangle$-semigeneric. For S a stationary subset of ω_1, $\langle P, \leq \rangle$ is S-*closed* iff for any regular λ such that $\mathscr{P}(P) \subseteq H_\lambda$, countable N with $\langle N, \in \rangle \prec (H_\lambda, \in)$ and $N \cap \omega_1 \in S$, and P-generic sequence $\langle p_n | n \in \omega \rangle$ for N, there is a $q \leq p_n$ for every $n \in \omega$. Semiproper is $\{\aleph_1\}$-semiproper and S-closed is $\{S\}$-complete in the sense of Shelah [Sh2]. ω_1-closure readily implies semiproperness. The salient features of these concepts are that if a poset is semiproper, then forcing with it *preserves stationary subsets of* ω_1 (i.e. any stationary subset of ω_1 in the ground model remains stationary in the extension), and if it is S-closed, then it adjoins no new countable sequences of ordinals.

To affirm notation, P is the *countable support iteration* of $\langle P_\alpha, \dot{Q}_\alpha | \alpha < \gamma \rangle$ iff setting $P_\gamma = P$, we have: $P_0 = \{\varnothing\}$; for $\alpha < \gamma$, $\Vdash_{P_\alpha}$ "$\dot{Q}_\alpha$ is a poset" and $P_{\alpha+1} = P_\alpha * \dot{Q}_\alpha$; and for limit $\alpha \leq \gamma$, P_α is the direct limit of $\langle P_\beta | \beta < \alpha \rangle$ in case $\mathrm{cf}(\alpha) > \omega$, and the inverse limit otherwise. Proceeding recursively we can take P_α to consist of functions p (the conditions) with domain α so that for each $\beta < \alpha$, $p(\beta)$ is a P_β-name and $\Vdash_{P_\beta} p(\beta) \in \dot{Q}_\beta$, and supposing that $\Vdash_{P_\beta}$ "$\dot{1}_{Q_\beta}$ is the maximum element of $\dot{Q}_\beta$",

$$\mathrm{supt}(p) = \{\beta < \alpha | \Vdash_{P_\beta} p(\beta) \neq \dot{1}_{Q_\beta}\}$$

is countable, with corresponding partial order on P_α given by: $p \leq q$ iff $\forall \beta < \alpha(p|\beta \Vdash_{P_\beta} p(\beta) \leq q(\beta))$.

Throughout the paper we rely on the following convention: For a notion of forcing P, $\dot{G}_P$ denotes the canonical P-name for its generic object, and if $P = P_\alpha$ in some contextually clear indexing, $\dot{G}_\alpha$ is written for $\dot{G}_{P_\alpha}$.

2.2. Lemma. *Suppose P is the countable support iteration of $\langle P_\alpha, \dot{Q}_\alpha | \alpha < \gamma \rangle$ where for $\alpha < \gamma$, $\Vdash_{P_\alpha}$ "$\dot{Q}_\alpha$ is semiproper and S-closed". Then P is semiproper and S-closed.*

This is a special case of more general iteration lemmas. The appropriate mode of iteration for semiproperness is revised countable support (RCS) iteration, but S-closure at each stage implies that there are no new countable sequences of ordinals, and so RCS iteration reduces to countable support iteration.

In [FMS], 2.1 is established for κ supercompact instead of, e.g. Woodin, by first establishing the consistency with CH of a Martin's Axiom for S-closed notions of forcing that preserve stationary subsets of ω_1 and meeting ω_1 dense sets. Then it suffices to argue with a notion of forcing which in the formulation of Shelah [Sh3] is as follows:

Suppose that $\mathscr{A}$ consists of stationary subsets of ω_1 with $S \in \mathscr{A}$. Then $\mathrm{Seal}_S(\mathscr{A})$ consists of countable sequences $\langle N_\xi | \xi \leq \gamma \rangle$ such that:

(i)   each $N_\xi \in [H_\lambda]^{<\omega_1}$, where $\lambda = (2^{\omega_1})^+$, and $S \in N_0$.
(ii) $\langle N_\xi | \xi \leq \gamma \rangle$ is increasing and continuous.
(iii) for each $\xi \leq \gamma$, $N_\xi \cap \omega_1$ is an ordinal in $\bigcup \{ A | A \in \mathscr{A} \cap N_\xi \}$.

$\mathrm{Seal}_S(\mathscr{A})$ is ordered by: $p \leq q$ iff q is an initial segment of p.

$\mathrm{Seal}_S(\mathscr{A})$ is clearly S-closed, although it may not be semiproper, and forcing with it provides an enumeration of $\mathscr{A}$ in ordertype $\leq \omega_1$ and a closed unbounded subset of the diagonal union of $\mathscr{A}$ according to that enumeration. If $\mathscr{A}$ was a maximal antichain with respect to NS, then it can be shown that forcing with $\mathrm{Seal}_S(\mathscr{A})$ preserves stationary subsets of ω_1. $\mathscr{A}$ is then "sealed": it remains a maximal antichain in any extension that preserves stationary sets, since any stationary set has stationary intersection with the diagonal union of $\mathscr{A}$, and hence with a particular member of $\mathscr{A}$ by Fodor's Lemma. The aforementioned version of Martin's Axiom implied through this means that $\mathrm{NS}|\widetilde{S}$ is ω_2-saturated.

2.1 was established by applying reflection properties directly in [Sh4, XVI]. Let

$$Q_S = \left(\prod \{ \mathrm{Seal}_S(\mathscr{A}) | \mathrm{Seal}_S(\mathscr{A}) \text{ is semiproper} \} \right) * \mathrm{col}(\omega_1, 2^{\omega_1}),$$

the countable support product of all $\mathrm{Seal}_S(\mathscr{A})$'s for $\mathscr{A}$'s that yield semiproper $\mathrm{Seal}_S(\mathscr{A})$, followed by the usual collapse of 2^{ω_1} to ω_1 using countable approximations. Clearly Q_S is S-closed; it is also semiproper (see [Sh4, XIII, 2.8(3)] or [Sh3, 2.8(3), p. 361]). Let P be the countable support iteration of $\langle P_\alpha, \dot{Q}_\alpha | \alpha < \kappa \rangle$ where $\dot{Q}_\alpha$ is a P_α-name for Q_S in the sense of V^{P_α}. Assuming that κ satisfies the large cardinal hypothesis of 2.1, the second author showed that any forcing extension via P satisfies its conclusion.

Proof of Theorem B. We interlace into the above described proof of 2.1 natural notions of forcing for introducing suprema into $\mathscr{P}(\omega_1)/\mathrm{NS}|\widetilde{S}$: $\langle \mathscr{A}, \mathscr{B} \rangle$ is an *appropriate pair* if $\mathscr{A}$ and $\mathscr{B}$ consist of subsets of ω_1 such that: if $A \in \mathscr{A}$ and $B \in \mathscr{B}$, then $A \cap B \cap \widetilde{S}$ is nonstationary. For such $\langle \mathscr{A}, \mathscr{B} \rangle$, let $\mathrm{Sup}_S(\mathscr{A}, \mathscr{B})$ consist of triples $\langle w, c, d \rangle$ such that:

(i) w is a countable subset of $\mathscr{A} \cup \mathscr{B}$.
(ii) c is a countable, closed set of countable ordinals (so $\bigcup c \in c$).
(iii) $d : c \to 2$.

Order $\mathrm{Sup}_S(\mathscr{A}, \mathscr{B})$ by: $\langle w, c, d \rangle \leq \langle \overline{w}, \overline{c}, \overline{d} \rangle$ iff $\overline{w} \subseteq w$, $\overline{c}$ is an initial

segment of c, $\overline{d} \subseteq d$, and:

if $\alpha \in (c \sim \overline{c}) \cap \widetilde{S}$ and $d(\alpha) = 0$, then $\alpha \notin A$ for any $A \in \overline{w} \cap \mathscr{A}$.

if $\alpha \in (c \sim \overline{c}) \cap \widetilde{S}$ and $d(\alpha) = 1$, then $\alpha \notin B$ for any $B \in \overline{w} \cap \mathscr{B}$.

Suppose that G is $\mathrm{Sup}_S(\mathscr{A}, \mathscr{B})$-generic and set $c_G = \bigcup\{c | \langle w, c, d\rangle \in G\}$ and $d_G = \bigcup\{d | \langle w, c, d\rangle \in G\}$. Then c_G is a closed unbounded subset of ω_1, $d_G^{-1}(\{0\}) \cap A \cap \widetilde{S}$ is countable for every $A \in \mathscr{A}$, and $d_G^{-1}(\{1\}) \cap B \cap \widetilde{S}$ is countable for every $B \in \mathscr{B}$. In particular, if $\mathscr{A} \cup \mathscr{B}$ were a maximal antichain with respect to $\mathrm{NS}|\widetilde{S}$, then $[d_G^{-1}(\{1\})]$ would serve as an upper bound of $\mathscr{A}$ and $[d_G^{-1}(\{0\})]$ of $\mathscr{B}$. These will be *least* upper bounds in the extension if $\mathscr{A} \cup \mathscr{B}$ continues to be maximal there, and this is the only situation that will be germane to the overall argument. Of course, for all this to make sense in the extension we must ascertain that stationary subsets of ω_1 are preserved:

2.3. Lemma.

(a) $\mathrm{Sup}_S(\mathscr{A}, \mathscr{B})$ is ω_1-closed (*and hence semiproper and S-closed*).

(b) *Assuming* CH, $\mathrm{Sup}_S(\mathscr{A}, \mathscr{B})$ is ω_1-linked *with least upper bounds*; i.e. *it is the union of ω_1 sets, each consisting of pairwise compatible elements with least upper bounds.*

Proof. (a) Set $P = \mathrm{Sup}_S(\mathscr{A}, \mathscr{B})$. Suppose that λ is regular with $\mathscr{P}(P) \subseteq H_\lambda$, N is countable with $\langle N, \in\rangle \prec \langle H_\lambda, \in\rangle$ and $\langle p_n | n \in \omega\rangle$ is a P-generic sequence for N. We must find a $q \in P$ such that $q \leq p_n$ for every $n \in \omega$.

For $n \in \omega$ and $p_n = \langle w_n, c_n, d_n\rangle$, since w_n is countable, we can let $C_n \in N$ be a closed unbounded subset of ω_1 such that for any $A \in w_n \cap \mathscr{A}$ and $B \in w_n \cap \mathscr{B}$, $C_n \cap A \cap B \cap \widetilde{S} = \varnothing$. Set $w = \bigcup w_n$, $c = \bigcup c_n$, $d = \bigcup d_n$. Then a simple genericity argument implies that $\bigcup c \in \bigcap C_n$ so that for no $A \in w \cap \mathscr{A}$ and $B \in w \cap \mathscr{B}$ does $\bigcup c \in A \cap B$, and so we can find an $i < 2$ such that $\langle w, c \cup \{\bigcup c\}, d \cup \{\langle \bigcup c, i\rangle\}\rangle$ is in $\mathrm{Sup}_S(\mathscr{A}, \mathscr{B})$ and of course is $\leq p_n$ for every $n \in \omega$.

(b) Note that $\langle w_0, c, d\rangle$ and $\langle w_1, c, d\rangle$ are compatible with least upper bound $\langle w_0 \cup w_1, c, d\rangle$. With CH, there are ω_1 such pairs. □

Let Sup_S be the countable support product of $\mathrm{Sup}_S(\mathscr{A}, \mathscr{B})$ for all appropriate pairs $\langle \mathscr{A}, \mathscr{B}\rangle$. A countable support product of ω_1-closed forcings is readily seen to be ω_1-closed, and ω_1-closed forcings are S-closed and semiproper. Hence by 2.3(a), Sup_S is S-closed and semiproper.

An approach to the proof of Theorem B would be to carry out the countable support iteration of Sup_S through κ stages. Just assuming $\mathrm{cf}(\kappa) > \omega_1$, we would then get the consistency of $2^{\omega_1} = \kappa$ and every appropriate pair $\langle \mathscr{A}, \mathscr{B}\rangle$ with $|\mathscr{A} \cup \mathscr{B}| < \kappa$ can be separated, a consequence in fact of a generalized Martin's Axiom in Baumgartner [B] or in Shelah [Sh1]. It is to ensure κ-saturation, a necessary condition for full completeness by 1.1(a), that we build on the proof of 2.1.

Let P_κ be the countable support iteration of $\langle P_\alpha, \dot{Q}_\alpha | \alpha < \kappa\rangle$, where:

(i) For odd $\alpha < \kappa$, $\dot{Q}_\alpha$ is a P_α-name for Q_S in the sense of V^{P_α}. (Here, Q_S is as in the above outline of the proof of 2.1.)

(ii) For even $\alpha < \kappa$, $\dot{Q}_\alpha$ is a P_α-name for the countable support product of $\mathrm{Sup}_S(\mathscr{A}, \mathscr{B})$'s for all appropriate pairs $\langle \mathscr{A}, \mathscr{B}\rangle$ satisfying $\langle \mathscr{A}, \mathscr{B}\rangle \in$

$V[\dot{G}_\alpha|\{2\gamma|2\gamma < \alpha\}]$. (For $Z \subseteq \alpha$, $\dot{G}_\alpha|Z$ is the P_α-name for $\{p \in \dot{G}_\alpha| \text{supt}(p) \subseteq Z\}$. Note that $\langle \mathscr{A} , \mathscr{B} \rangle$ is to belong to the smaller model, but in the definition of "appropriate pair" the nonstationariness in "$(\forall A \in \mathscr{A})(\forall B \in \mathscr{B})A \cap B \cap \tilde{S}$ is not stationary" is to be in the sense of $V[G_\alpha]$!)

Next, for $\alpha \leq \kappa$ set, by induction on α:

$$P'_\alpha = \{p \in P_\alpha| \text{supt}(p) \subseteq \{2\gamma|2\gamma < \alpha\} \wedge \forall \beta \in \text{supt}(p)(p(\beta) \text{ is a } P'_\beta\text{-name})\}$$

with the inherited order. ($p(\beta)$ being a P'_β-name, it only depends on $\dot{G}_\beta \cap P'_\beta$.) We show that any forcing extension via P'_κ satisfies the conclusion of Theorem B. This follows from the following technical lemma, all of whose parts are established by simultaneous induction; for its (b), note that P'_β-names being P_β-names is justified by an inductive appeal to (a), and Sup_S was defined a few paragraphs ago in the outline of the proof of 2.1.

2.4. Lemma. *For each $\alpha \leq \kappa$:*

(a) $P'_\alpha \lessdot P_\alpha$; *i.e. every maximal antichain of P'_α is a maximal antichain of P_α.*

(b) $P^*_\alpha = \{p \in P_\alpha|\forall \text{even } \beta \in \text{supt}(p)(p(\beta) \text{ is a } P'_\beta\text{-name for a condition in } \text{Sup}_S \text{ in the sense of } V^{P'_\beta}\}$ *is dense in P_α.*

(c) *For any $p, q \in P^*_\alpha$ such that $p|\{2\gamma|2\gamma < \alpha\}$ and $q|\{2\gamma|2\gamma < \alpha\}$ are compatible members of P'_α, there is an automorphism F^α_{pq} such that: $F^\alpha_{pq}(p)$ is compatible with q; F^α_{pq} is the identity on P'_α; and inductively for any $\beta < \alpha$,*

$$F^\beta_{p|\beta, q|\beta} = F^\alpha_{pq}|P_\beta.$$

(d) *If $p \in P^*_\alpha$, then $p|\{2\gamma|2\gamma < \alpha\} \in P'_\alpha$.*

Once this lemma is established, the proof of Theorem B can be completed as follows: Suppose that G' is P'_κ-generic, and by 2.4(a) let G be P_κ-generic such that $G \cap P'_\kappa = G'$. Note first that for any $X \subseteq \omega_1$ with $X \in V[G']$,

$$V[G'] \Vdash X \text{ is stationary iff } V[G] \Vdash X \text{ is stationary.}$$

(If X were nonstationary in $V[G]$, then it would be nonstationary in $V[G \cap P_\alpha]$ for some even $\alpha < \kappa$. But then $X = A \cap B \cap \tilde{S}$ for some $A \in \mathscr{A}$ and $B \in \mathscr{B}$ with $\langle \mathscr{A} , \mathscr{B} \rangle$ an appropriate pair in $V[G \cap P'_\alpha]$, so that $\text{Sup}_S(\mathscr{A} , \mathscr{B})$ at that stage would have adjoined a closed unbounded subset of ω_1 confirming that X is nonstationary in $V[G']$.) The proof of 2.1 still works to show that in $V[G]$, $\kappa = \omega_2$ and $\text{NS}|\tilde{S}$ is κ-saturated. It thus follows that $\text{NS}|\tilde{S}$ is also κ-saturated in $V[G']$.

We can conclude that for any maximal antichain $\mathscr{A} \cup \mathscr{B}$ of $\text{NS}|\tilde{S}$ in the sense of $V[G']$ with $\mathscr{A} \cap \mathscr{B} = \varnothing$, $\langle \mathscr{A} , \mathscr{B} \rangle \in V[G' \cap P'_\alpha]$ for some even $\alpha < \kappa$. But then, the forcing with $\text{Sup}_S(\mathscr{A} , \mathscr{B})$ would have adjoined a set $E \subseteq \omega_1$ such that: if $A \in \mathscr{A}$, then $A - (E \cap \tilde{S})$ is not stationary; and if $B \in \mathscr{B}$, then $B \cap E \cap \tilde{S}$ is not stationary. $[E \cap \tilde{S}]$ is thus the supremum for $\mathscr{A}$ in $V[G' \cap P'_{\alpha+1}]$ and hence in $V[G']$. This suffices to show that in $V[G']$, $\mathscr{P}(\omega_1)/\text{NS}|\tilde{S}$ is a complete Boolean algebra.

Finally, P'_κ is S-closed, so that forcing with it adjoins no new countable sequences of ordinals. Consequently, $\{p \in P'_\kappa| \text{ if } \beta \in \text{supt}(p) \text{ and } \langle \dot{w}, \dot{c}, \dot{d} \rangle$

is a component of $p(\beta)$, then $\dot{c} = \check{c}$ and $\dot{d} = \check{d}$ for some $c, d \in V\}$ is dense in P'_κ. Using CH and 2.3(b), a simple Δ-system argument using this dense set then shows that P'_κ has the ω_2-c.c and hence preserves all cardinals, and it is simple to see that it renders $2^{\omega_1} = \kappa$.

Proof of 2.4. Assuming that $\alpha \leq \kappa$ and all four parts hold below α, we verify that they all hold at α.

To first verify that (b) holds at α, suppose that $p \in P_\alpha$ is arbitrary. Let λ be regular and sufficiently large, and N countable such that $\langle N, \in \rangle \prec \langle H_\lambda, \in \rangle$, $p \in N$, and $N \cap \omega_1 \in S$. Let $\langle p_n | n \in \omega \rangle$ be a P_α-generic sequence for N with $p_0 = p$. Incorporating the proof of 2.3(a) into the iteration lemma for S-closed notions of forcing, there is a *least* upper bound $q \in P_\alpha$ for the p_n's specified as follows:

$\operatorname{supt}(q) = \bigcup_n \operatorname{supt}(p_n)$. For even $\beta \in \operatorname{supt}(q)$, $q|\beta$ forces that for component $\langle \dot{w}, \dot{c}, \dot{d} \rangle$ of $q(\beta)$ in some relevant $\operatorname{Sup}_S(\mathscr{A}, \mathscr{B})$ with corresponding $\langle \dot{w}_n, \dot{c}_n, \dot{d}_n \rangle$ in p_n for $n \in \omega$ sufficiently large, $\dot{w}$ is the union of the $\dot{w}_n$'s, $\dot{c}$ is the union of the $\dot{c}_n$'s together with its limit point at the top, and $\dot{d}$ is the union of the $\dot{d}_n$'s together with an arbitrary value for that top limit point. By P_α-genericity of $\langle p_n | n \in \omega \rangle$ and induction it can be assumed that each $\dot{w}_n$ is a P'_β-name and hence that $\dot{w}$ is a P'_β-name, and that $\dot{c} = \check{c}$ for some $c \in V$ and $\dot{d} = \check{d}$ for some $d \in V$.

Continuing to consider that specific component $\langle \dot{w}, \dot{c}, \dot{d} \rangle$ of $q(\beta)$, by definition of such conditions there is a P'_β-name $\dot{w}_0$ such that $q|\beta \Vdash$ "$\langle \dot{w} \cap \dot{w}_0, \dot{w} - \dot{w}_0 \rangle$ is an appropriate pair". By induction, $P'_\beta \lessdot P_\beta$, and the homogeneity property 2.4(c) implies that whenever $r \in P_\beta$, $\dot{t}$ is a P'_β-name, ψ is a one-free variable formula, and $r \Vdash_{P_\beta} \psi(\dot{t})$, then $r|\{2\gamma | 2\gamma < \beta\} \Vdash_{P'_\beta} \psi(\dot{t})$. In particular

$$(q|\beta)|\{2\gamma | 2\gamma < \beta\} \Vdash_{P'_\beta} \text{"}\langle \dot{w} \cap \dot{w}_0, \dot{w} - \dot{w}_0 \rangle \text{ is an appropriate pair"}.$$

This confirms that $q \in P^*_\alpha$ to verify 2.4(b) for α as desired.

At the referee's urging we also elaborate the rest:

To establish (d) at α, let $p \in P^*_\alpha$, β an even ordinal in $\operatorname{supt}(p)$, and set $p^c = p|\{2\gamma | 2\gamma < \beta\}$. It must be shown that $p^c \Vdash_{P_\beta} p(\beta) \in \dot{Q}_\alpha$. We know inductively that $P'_\beta \lessdot P_\beta$, $p^c \in P'_\beta$, and $p^c \Vdash_{P'_\beta} p(\beta) \in \dot{Q}_\beta$. The only problem in trying to replace P'_β by P_β here is that an appropriate pair mentioned in $p(\beta)$ in the sense of $V^{P'_\beta}$ may no longer be one in the sense of V^{P_β}.

Assume to the contrary that for some $q \in P_\beta$ with $q \leq p^c$, $q \Vdash_{P_\beta} \dot{p}(\beta) \notin \dot{Q}_\beta$. By (b) inductively it can be assumed that $q \in P^*_\beta$, and by (d) inductively, $q|\{2\gamma | 2\gamma < \beta\} \in P'_\beta$. Since $q|\{2\gamma | 2\gamma < \beta\} \leq p^c$ in P'_β, there is an automorphism $F^\beta_{q, p|b}$ as in (c) inductively such that $F^\beta_{q, p|\beta}(q)$ is compatible with $p|\beta$. Hence, for some $q^+ \in P_\beta$ with $q^+ \leq q$, $F^\beta_{q, p|\beta}(q^+) \leq p|\beta$. Let $G \subseteq P_\beta$ be P_β-generic over V with $q^+ \in G$. Then $G' = F^\beta_{q, p|\beta}$"$G$ is P_β-generic over V such that $p|\beta \in G'$. But $G \cap P'_\beta = G' \cap P'_\beta$ as $F^\beta_{q, p|\beta}$ is the identity on P'_β, and

so

$$\{\langle \mathscr{A}, \mathscr{B}\rangle | \mathscr{A} \cup \mathscr{B} \subseteq \mathscr{P}(\omega_1) \cap V(G \cap P'_\beta)\}$$
$$= \{\langle \mathscr{A}, \mathscr{B}\rangle | \mathscr{A} \cup \mathscr{B} \subseteq \mathscr{P}(\omega_1) \cap V[G' \cap P'_\beta]\},$$

and so as $V[G] = V[G']$,

$$\{\langle \mathscr{A}, \mathscr{B}\rangle \in V[G \cap P'_\beta] | \langle \mathscr{A}, \mathscr{B}\rangle \text{ is appropriate in } V[G]\}$$
$$= \{\langle \mathscr{A}, \mathscr{B}\rangle \in V[G \cap P'_\beta] | \langle \mathscr{A}, \mathscr{B}\rangle \text{ is appropriate in } V[G'])\}.$$

As also all ω-sequence from V of members of $V[G \cap P'_\beta]$ are in $V[G \cap P'_\beta]$ (as this holds for V and $V[G]$), clearly $(\dot{Q}_\beta)^G = (\dot{Q}_\beta)^{G'}$. This contradicts the choice of q.

Now clause (a) for α follows: For $p \in P_\alpha$, choose q such that $q \leq p \in P_\alpha^*$. Setting $q^c = q|\{2\gamma | 2\gamma < \alpha\}$, $q^c \in P'_\alpha$ by clause (d) and $q^c \Vdash_{P'_\alpha} "p \in P_\alpha / \dot{G}_{P'_\alpha}"$ because if $r \leq q^c \in P'_\alpha$ then $r' = r \cup (q|\{\beta < \alpha | \beta \text{ is odd}\})$ is in P_α^* (check!) and is below r and q hence below p.

We lastly deal with clause (c) for α. If α is a limit, it is immediate: $F^\alpha_{p,q}(r)$ is defined by: $\gamma \in \text{supt}(F^\alpha_{p,q}(r))$ iff for some (equivalently, every) $\beta \in (\gamma, \alpha)$ we have $\gamma \in \text{supt}(F^\beta_{p|\beta, q|\beta}(r))$, and letting $(F^\alpha_{p,q}(r))(\gamma) = (F^\beta_{p|\beta, q|\beta}(r))(\gamma)$ for some (any) such β.

If $\alpha = \beta + 1$, β odd, just note that $\dot{Q}_\beta$ is definable in $V[\dot{G}_{P_\beta}]$ (without parameters). If $\alpha = \beta + 1$, β even, if $G \subseteq P_\beta$ is generic over V, in $V[G \cap P'_\beta]$ there is an automorphism F_β of $\dot{Q}_\beta[G]$ (note: $\dot{Q}_\beta[G] \in V[G \cap P'_\beta]$—see the proof of clause (d)) such that: $F_\beta(p(\beta))$, and $q(\beta)$ are compatible. (The simplest way to see this is to replace in the iteration $\dot{Q}_\alpha$ by its completion.)

3. Collapsing κ to ω_3

In this section we specify the modifications to the proof of Theorem B necessary to establish Theorem C. We define the components of an iteration $\langle P_\alpha, \dot{R}_\alpha | \alpha < \kappa \rangle$ in three cases instead of two:

(i) For $\alpha \equiv 0 \pmod 3$, $\dot{R}_\alpha$ is defined as $\dot{Q}_\alpha$ was before for odd α; i.e. it is $Q_S^{V^{P_\alpha}}$.

(ii) For $\alpha \equiv 1 \pmod 3$, $\dot{R}_\alpha$ is defined as $\dot{Q}_\alpha$ was before for even α, but for all appropriate pairs $\langle \mathscr{A}, \mathscr{B}\rangle$ satisfying

$$\langle \mathscr{A}, \mathscr{B}\rangle \in V[\dot{G}_\alpha | \{\beta < \alpha | \beta \equiv 1 \pmod 3 \vee \beta \equiv 2 \pmod 3\}]$$

(where $\dot{G}_\alpha | Z$ is as before).

(iii) For $\alpha \equiv 2 \pmod 3$, $\dot{R}_\alpha$ is $\text{col}(\omega_2, 2^{\omega_2})$, the collapse of 2^{ω_2} to ω_2 using ω_1 size approximations, in the sense of $V[\dot{G}_\alpha | \{\beta < \alpha | \beta \equiv 1 \pmod 3 \vee \beta \equiv 2 \pmod 3\}]$.

The latter notion of forcing is semiproper and S-closed, being ω_1-closed. Its introduction necessitates that we define the P_α's with mixed support: Proceeding recursively, for $\alpha \leq \kappa$ let $\overline{P}_\alpha$ consist of functions p with domain α such that for each $\beta < \alpha$, $p(\beta)$ is a $\overline{P}_\beta$-name such that $\Vdash_{\overline{P}_\beta} p(\beta) \in \dot{R}_\beta$, and

$$|\text{supt}(p) \cap \{\beta < \alpha | \beta \equiv 0 \pmod 3 \vee \beta \equiv 1 \pmod 3\}| \leq \aleph_0,$$
$$|\text{supt}(p) \cap \{\beta < \alpha | \beta \equiv 2 \pmod 3\}| \leq \aleph_1.$$

The following lemma will be a consequence of the forthcoming iteration lemmas.

3.1. Lemma. *For each $\alpha \leq \kappa$, $\overline{P}_\alpha$ is S-closed and semiproper.*

Assuming this lemma, the proof of Theorem C can be completed as follows: For $\alpha \leq \kappa$, define

$$\overline{P}'_\alpha = \{p \in \overline{P}_\kappa | \forall \alpha \in \text{supt}(p)(\alpha \equiv 1 \ (\text{mod } 3) \vee \alpha \equiv 2 \ (\text{mod } 3))$$
$$\wedge \forall \beta \in \text{supt}(p)(p(\beta) \text{ is a } P'_\beta\text{-name})\}.$$

($p(\beta)$ being a P'_β-name, it only depends on $\dot{G}_\beta \cap P'_\beta$.) Then any forcing extension via $\overline{P}'_\kappa$ satisfies the conclusion of Theorem C:

Let G' be $\overline{P}'_\kappa$-generic. It can be checked that the analogue of 2.4 holds in the new situation. In particular, there is a $G \overline{P}_\kappa$-generic such that $G \cap \overline{P}'_\kappa = G'$. With 3.1, the proof of 2.1 still works to show that in $V[G]$, $\kappa = \omega_2$ and $\text{NS}|\widetilde{S}$ is κ-saturated. It then follows as before that in $V[G']$, $\text{NS}|\widetilde{S}$ is κ-saturated and $\mathscr{P}(\omega_1)/\text{NS}|\widetilde{S}$ is a complete Boolean algebra.

By standard arguments $\overline{P}'_\kappa$ has the κ-c.c., and the introduction of the collapses $\dot{R}_\alpha$ for $\alpha \equiv 2 \ (\text{mod } 3)$ implies that in $V[G']$, $\kappa = 2^{\omega_1} \leq \omega_3$. But by 3.7 below, ω_2 is preserved so that $\kappa = \omega_3$ in $V[G']$ and so the proof is complete.

The rest of this section is devoted to establishing 3.1 and the forthcoming 3.7. We build on Shelah [Sh2; Sh4, XIV] and refer to them for the more basic details about iterated forcing that are not provided in full.

For $\mu > \omega$, let $\mathscr{K}_\mu$ be the class of $\langle Q, \leq_Q, \leq_Q^0 \rangle$ such that:

(i) $\langle Q, \leq_Q \rangle$ is a semiproper, S-closed notion of forcing, say with maximum element 1_Q.

(ii) $\langle Q, \leq_Q^0 \rangle$ is a poset so that: (a) $\leq_Q$ refines $\leq_Q^0$ (i.e. if $p \leq_Q^0 q$, then $p \leq_Q q$); and (b) $\leq_0$ is μ-closed (i.e. if $\langle p_\alpha | \alpha < \eta \rangle$ is $\leq_Q^0$-decreasing and $\eta < \mu$, then there is a $p \in Q$ such that $p \leq_Q^0 p_\alpha$ for every $\alpha < \eta$).

We often suppress the subscripts Q and furthermore identify $\langle Q, \leq_Q, \leq_Q^0 \rangle$ with its domain Q when there is little possibility of confusion. When we use forcing terminology for such a member of $\mathscr{K}_\mu$, we are referring to the $\langle Q, \leq_Q \rangle$ part.

Next, let $\mathscr{K}_\mu^*$ be the class of $\langle P_\alpha, \dot{Q}_\alpha | \alpha < \gamma \rangle$ where for each $\alpha < \gamma$, P_α is a notion of forcing, $\dot{Q}_\alpha$ is a P_α-name and $\Vdash_{P_\alpha} \dot{Q}_\alpha \in \mathscr{K}_\mu$ and recursively:

(i) P_α consists of functions p with domain α so that for each $\beta < \alpha$, $p(\beta)$ is a P_β-name such that $\Vdash p(\beta) \in \dot{Q}_\beta$, and setting

$$\text{supt}(p) = \{\beta < \alpha | \Vdash p(\beta) \neq 1_{Q_\beta}\}$$

as before,

$$|\text{supt}(p)| \leq \aleph_1 \quad \text{and} \quad |\{\beta < \alpha | \neg(\Vdash p(\beta) \leq_{\dot{Q}_\beta}^0 1_{Q_\beta})\}| \leq \aleph_0.$$

(ii) The ordering on P_α is given by:

$$p \leq q \text{ iff } \forall \beta < \alpha(p|\beta \Vdash p(\beta) \leq_{\dot{Q}_\beta} q(\beta))$$
$$\wedge |\{\beta \in \text{supt}(p)|\neg(p|\beta \Vdash p(\beta) \leq_{\dot{Q}_\beta}^0 q(\beta))\}| \leq \aleph_0.$$

If P_γ is defined by taking $\alpha = \gamma$ in the above, we say that P_γ is the *iteration* of $\langle P_\alpha, \dot{Q}_\alpha | \alpha < \gamma \rangle$. The introduction of the second partial order $\leq^0$ serves less to provide iteration lemmas of potentially wide applicability than to provide a uniform approach to 3.1. For that result, $\leq^0_{\dot{R}_\alpha}$ will coincide with $\leq_{\dot{R}_\alpha}$ when $\alpha \equiv 2 \pmod 3$, i.e. when $\dot{R}_\alpha$ is the Levy collapse $\operatorname{col}(\omega_2, 2^{\omega_2})$; and $\leq^0_{\dot{R}_\alpha}$ will just be equality for $\alpha \equiv 0 \pmod 3$ and $\alpha \equiv 1 \pmod 3$. Note then that by how we defined the $\overline{P}_\alpha$'s from the $\dot{R}_\alpha$'s, $\langle \overline{P}_\alpha, \dot{R}_\alpha | \alpha < \kappa \rangle \in \mathscr{K}_\mu^*$.

The usual iteration facts hold for members of $\mathscr{K}_\mu^*$: e.g., if $\langle P_\alpha, \dot{Q}_\alpha | \alpha < \gamma \rangle \in \mathscr{K}_\mu^*$, for any $\beta < \alpha < \gamma$, $P_\beta \lessdot P_\alpha$, i.e. every maximal antichain of P_β is a maximal antichain of P_α, and for the usual quotient poset P_α / P_β such that $P_\alpha \cong P_\beta * P_\alpha / P_\beta$, P_α / P_β is an iteration of a member of $\mathscr{K}_\mu^*$. To establish 3.1, we must verify that iterations in $\mathscr{K}_\mu^*$ preserve S-closure and semiproperness.

The following lemma provides the main induction step for establishing the preservation of S-closure:

3.2. Lemma. *Suppose that P_γ is the iteration of $\mathscr{P} = \langle P_\alpha, \dot{Q}_\alpha | \alpha < \gamma \rangle \in \mathscr{K}_\mu^*$ and $\delta < \eta \leq \gamma$. Then the following holds for sufficiently large regular λ:*

Assume that N is countable with $\langle N, \in \rangle \prec \langle H_\lambda, \in \rangle$, $N \cap \omega_1 \in S$, $\{\mathscr{P}, \delta, \eta\} \subseteq N$, $\langle p_n | n \in \omega \rangle$ is a P_η-generic sequence for N, and $q \in P_\delta$ satisfies $q \leq p_n | \delta$ for every $n \in \omega$. Then there is a $q^+ \in P_\eta$ such that $q^+ | \delta = q$ and $q \leq p_n$ for every $n \in \omega$.

Proof. By adjusting names, we can assume for convenience in what follows that for each $n \in \omega$ and $\alpha < \eta$:

(i) $p_{n+1} | \alpha \Vdash p_{n+1}(\alpha) \leq^0 p_n(\alpha)$ iff $\Vdash p_{n+1}(\alpha) \leq^0 p_n(\alpha)$, and

(ii) $\Vdash p_{n+1}(\alpha) \leq p_n(\alpha)$.

We now define a function q^+ with domain η as follows: Fix a well-ordering W of sufficiently large V_ρ. Set $q^+ | \delta = q$. For $\delta \leq \alpha < \eta$, having defined $q^+ | \alpha$ for $\alpha < \eta$ so that recursively $q^+ | \alpha \in P_\alpha$, define $q^+(\alpha)$ as follows:

(a) If for some $k \in \omega$, $q^+ | \alpha \Vdash$ "$\langle p_n(\alpha) | k < n < \omega \rangle$ is $\leq^0$-descending in $\dot{Q}_\alpha - \{\dot{1}_{Q_\alpha}\}$", then since by definitions of $\mathscr{K}_\mu$ and $\mathscr{K}_\mu^*$, $\Vdash_{P_\alpha}$ "$\leq^0_{\dot{Q}_\alpha}$ is μ-closed" and $\mu > \omega$, there is a P_α-name τ so that $q^+ | \alpha \Vdash$ "τ is a $\leq^0$-lower bound for $\langle p_n(\alpha) | k < n < \omega \rangle$". Let $q^+(\alpha)$ be the W-least such τ. Else:

(b) If for some $k \in \omega$ and P_α-name τ, $q^+ | \alpha \Vdash$ "$\langle p_n(\alpha) | k < n < \omega \rangle$ is $\leq_{Q_\alpha}$-descending in $\dot{Q}_\alpha - \{1_{Q_\alpha}\}$ with τ a $\leq_{Q_\alpha}$-lower bound", then let $q^+(\alpha)$ be the W-least such τ. Otherwise:

(c) Set $q^+(\alpha) = \dot{1}_{Q_\alpha}$.

This definition perpetuates $q^+ | \alpha \in P_\alpha$ for every $\alpha \leq \eta$: Clearly,

$$\left| \operatorname{supt}(q^+ | \alpha) \right| \leq \left| \bigcup_n \operatorname{supt}(p_n | \alpha) \right| \leq \aleph_1.$$

Also, conditions on $\leq^0$ and (i) above imply that there is a countable set E such that for $\beta \in \alpha - E$, $\Vdash$ "$p_{n+1}(\beta) \leq^0 p_n(\beta) \leq^0 \dot{1}_{Q_\beta}$" for every $n \in \omega$. For such β, $q^+(\beta)$ was defined either through clause (a), or if not, neither through clause (b) as $\leq$ refines $\leq^0$, but through clause (c). But both (a) and (c) lead to $\Vdash q^+(\beta) \leq^0 \dot{1}_{Q_\zeta}$, and so

$$\{ \beta < \alpha | \neg(\Vdash q^+(\beta) \leq^0 \dot{1}_{Q_\beta}) \} \subseteq E,$$

confirming that $q^+ | \alpha \in P_\alpha$.

We next establish by induction on $\alpha \leq \eta$ that

(1) for every $n \in \omega$, $q^+|\alpha \Vdash q^+(\alpha) \leq p_n(\alpha)$.

(2) If $\alpha \notin N - \delta$, then for every $n \in \omega$, $q^+|\alpha \Vdash q^+(\alpha) \leq^0 p_n(\alpha)$.

As N is countable, this suffices to verify that $q^+ \leq p_n$ for every $n \in \omega$ as desired. For $\alpha < \delta$, the results are immediate; what remains splits into two cases:

Case 1. $\delta \leq \alpha < \eta$ and $\alpha \in N$. Then P_α, $\dot{Q}_\alpha \in N$, and we have $q|\alpha \leq p_n|\alpha$ for every $n \in \omega$ by induction. Since $\langle p_n|n \in \omega\rangle$ is a P_η-generic sequence for N, it follows first that $\langle p_n|\alpha \mid n \in \omega\rangle$ is a P_α-generic sequence for N, and second that $q^+|\alpha \Vdash$ "$\langle p_n(\alpha)|n \in \omega\rangle$ is a $\dot{Q}_\alpha$-generic sequence for $\check{N}[\dot{G}_\alpha]$". But by definitions of $\mathscr{K}_\mu$ and $\mathscr{K}_\mu^*$, $\Vdash_{P_\alpha}$ "$\dot{Q}_\alpha$ is S-closed", and since $N \cap \omega_1 \in S$, (a) or (b) of the definition of q^+ applies at α and (1) follows. (2) holds vacuously.

Case 2. $\delta \leq \alpha < \eta$ and $\alpha \notin N$. The $\leq^0$ conditions on the p_n's imply that for each $n \in \omega$,

$$E_n = \{\beta < \eta | \neg (\Vdash p_{n+1}(\beta) \leq^0 p_n(\beta) \leq^0 \dot{1}_{Q_\beta})\}$$

is countable, and clearly $E_n \in N$, so that E_n is countable in N. Hence, $\bigcup_n E_n \subseteq N$, so that $\alpha \notin \bigcup_n E_n$. Hence, $q^+(\alpha)$ was defined either through clause (a), or if not, neither through clause (b) as $\leq$ refines $\leq^0$, but through clause (c). Both (1) and (2) now follow in this case also. $\square$

The following preservation result now follows in straightforward fashion:

3.3. Proposition. *Suppose that P_γ is the iteration of $\langle P_\alpha, \dot{Q}_\alpha | \alpha < \gamma\rangle \in \mathscr{K}_\mu^*$ and $\delta < \eta \leq \gamma$. Then P_η/P_δ is S-closed.*

The following lemma provides the main induction step for establishing the preservation of semiproperness. Again, for a notion of forcing P, $\dot{G}_P$ denotes the canonical P-name for its generic object. For a set M and $q \in P$, to say that q *decides* $\dot{G}_P \cap \check{M}$ means of course that there is a set $A \in V$ such that $q \Vdash \check{A} = \dot{G}_P \cap \check{M}$. This happens for example if M is countable and q is a lower bound to a P-generic sequence for M, with $A = \{r \in P \cap M | q \leq r\}$. Finally, for $\lambda > \omega$ regular and $\langle N, \in\rangle \prec \langle H_\lambda, \in\rangle$ with $P \in N$, if G is P-generic over V, then let $N[G]$ be the set of interpretations $\{\tau[G] | \tau \in N$ is a P-name$\}$. By Shelah [Sh2, p. 88],

$$(*) \qquad\qquad \langle N[G], \in\rangle \prec \langle H_\lambda^{V[G]}, \in\rangle.$$

We let $N[G_P]$ be a canonical P-name for $N[G]$.

3.4. Lemma. *Suppose that P_γ is the iteration of $\mathscr{P} = \langle P_\alpha, \dot{Q}_\alpha | \alpha < \gamma\rangle \in \mathscr{K}_\mu^*$ and $\delta < \eta \leq \gamma$. Then the following holds for sufficiently large regular λ:*

Assume that N and M are countable with $\langle N, \in\rangle \prec \langle M, \in\rangle \prec \langle H_\lambda, \in\rangle$, $\{\mathscr{P}, \delta, \eta\} \subseteq N$, $N \in M$, and $M \cap \omega_1 \in S$. Assume also that $q \in P_\delta$ is $\langle N, P_\delta\rangle$-semigeneric, q is $\langle M, P_\delta\rangle$-generic and decides $\dot{G}_{P_\delta} \cap \check{M}$, and $p \in P_\eta \cap N$ with $q \leq p|\delta$.

Then there is a $q^+ \in P_\eta$ with $q^+ \leq p$ and $q^+|\delta = q$ such that q^+ is $\langle N, P_\eta\rangle$-semigeneric and q^+ is $\langle M, P_\eta\rangle$-generic and decides $\dot{G}_{P_\eta} \cap \check{M}$.

Proof. We establish this by induction on η, for all δ, N, M, p, and q.

For η a successor, we can clearly assume that $\eta = \delta + 1$. By definitions of K_μ and $\mathscr{K}_\mu^*$, $\Vdash_{P_\delta}$ "$\dot{Q}_\delta$ is semiproper", and so with $(*)$ just before 3.1 in mind, we have

$(**)$ $\qquad \Vdash_{P_\delta} \exists r \in \dot{Q}_\delta (r$ is $\langle N[G_{P_\delta}], P_\delta\rangle$-semigeneric $\wedge\ r(\delta) \leq p(\delta))$.

Since $N \in M$ and q is $\langle M, P_\delta\rangle$-generic and decides $\dot{G}_{P_\delta} \cap \check{M}$, by $(*)$ applied syntactically there is a $q' \in P_\delta \cap M$ with $q \leq q'$ and a P_δ-name $r' \in M$ such that q' forces the assertion of $(**)$ with r'. Since $M \cap \omega_1 \in S$, $\Vdash_{P_\delta}$ "$\dot{Q}_\delta$ is S-closed" by the definitions, and with $(*)$ in mind for M, it is now straightforward to find a $q^+ \in P_\eta$ as desired, noting that in the sense of $\Vdash_{P_\delta}$, any lower bound to a $\dot{Q}_\delta$-generic sequence for $\check{M}$ decides $\dot{G}_{Q_\delta} \cap \check{M}$.

For η a limit, let $\langle \beta_n | n \in \omega \rangle$ enumerate $M \cap \{\beta | \delta \leq \beta < \eta\}$ with $\beta_0 = \delta$. Let $\{\dot{\tau}_n^l : n < \omega\}$ enumerate the P_η-names of ordinals belonging to N if $l = 1$ and M if $l = 2$. Define α_n, q_n, p_n, and N_n by induction on $n \in \omega$ so that henceforth writing $\dot{G}_n$ for $\dot{G}_{P_{\alpha_n}}$, we have:

(a) $\alpha_0 = \delta$, $q_0 = q$, $p_0 = p$ and generally $q_n \in P_{\alpha_n}$ with $q_n \leq p|\alpha_n$ and $q_{n+1}|\alpha_n = q_n$, $p_n \in N[\dot{G}_n] \cap P_{\alpha_n}$ ($\subseteq M \cap P_\alpha$), $p_{n+1} \leq p_n$, and p_{n+1} decides a value for $\dot{\tau}_n$.

(b) q_n is $\langle N, P_{\alpha_n}\rangle$-semigeneric, and q_n is $\langle M, P_{\alpha_n}\rangle$-generic and decides $\dot{G}_n \cap \check{M}$.

(c) $q_n \Vdash \alpha_{n+1} = \max\{\{\beta_k | k \leq n\} \cap N[\dot{G}_n]\}$.
(Note for (c) that if q_n decides $\dot{G}_n \cap \check{M}$ by (b), then $N \in M$ implies that q_n decides $\dot{G}_n \cap \check{N}$ and hence $N[\dot{G}_n]$.)

The case $n = 0$ follows from our initial assumption. Suppose now that q_n has already been defined. By $(*)$ just before 3.4 applied syntactically, $\Vdash$ "$N[\dot{G}_n] \prec M[\dot{G}_n] \prec \langle H_\lambda^{V[\dot{G}_n]}, \in \rangle$". Moreover, since q_n is $\langle M, P_{\alpha_n}\rangle$-generic, we have $q_n \Vdash M[\dot{G}_n] \cap \omega_1 = \check{M} \cap \omega_1 \in \check{S}$. Hence, with α_{n+1} as stipulated by (c) it is straightforward to apply the induction hypothesis in the sense of $q_n \Vdash$ and then to find an appropriate $q_{n+1} \in P_{\alpha_{n+1}}$ as desired. There is no problem in defining p_{n+1}.

We can now define a $q^+ \in P_\eta$ so that $\mathrm{supt}(q^+) = \bigcup_n \mathrm{supt}(q_n)$, and for any β in this set, $q^+(\beta) = q_n(\beta)$ for some (any) n such that $\beta \in \mathrm{supt}(q_n)$. As in [Sh2, Sh4], $q \leq p_n$ so q^+ is $\langle N, P_\alpha\rangle$-semigeneric. q^+ is not necessarily $\langle M, P_\alpha\rangle$-generic, but its existence shows that there is $q' \in P_\eta \cap M$, $q'|\alpha \leq q^+$, q' is $\langle N, P_\alpha\rangle$-semigeneric. Now we can find a q^+ really as required. $\square$

The following preservation result is now clear, since for $\langle N, \in \rangle \prec \langle H_\lambda, \in \rangle$ as in 3.4, we can always find a countable M such that $N \in M$, $\langle M, \in \rangle \prec \langle H_\lambda, \in \rangle$, and $M \cap \omega_1 \in S$ by the stationariness of S.

3.5. Proposition. *Suppose that P_γ is the iteration of $\langle P_\alpha, \dot{Q}_\alpha | \alpha < \gamma \rangle \in \mathscr{K}_\mu^*$ and $\delta < \eta \leq \gamma$. Then P_η/P_δ is semiproper.*

Proof of 3.1. By 3.3 and 3.5 we get S-completeness and semiproperness. $\square$

Finally, we establish the preservation of ω_2 in a special case; a similar result appears in Shelah [Sh2, VIII, §1].

3.6. Proposition. *Suppose that* CH *holds, and P is the iteration of $\mathscr{P} = \langle P_\alpha, \dot{Q}_\alpha | \alpha < \gamma \rangle \in \mathscr{K}_{\omega_2}^*$ where for each $\alpha < \gamma$, $\Vdash_{P_\alpha}$ "$\leq_{\dot{Q}_\alpha}^0 = \leq_{\dot{Q}_\alpha}$ and Q_α sat-*

isfies $\aleph_2$-*pic*" (*see* [Sh2, VII, *Definition* 2]) *or* $\Vdash_{P_\alpha}$ " $\leq^0_{\dot{Q}_\alpha}$ *is the equality relation on* $\dot{Q}_\alpha$." *Then forcing with* P *preserves* ω_2 .

Proof. Assume to the contrary that for some $p \in P$ and P-name τ, $p \Vdash$ " $\tau: \check{\omega}_1 \to \check{\omega}_2$ is a bijection". Taking a regular λ sufficiently large, we proceed by induction on $\xi < \omega_2$ to define $p_\xi \in P$ so that $\xi < \zeta < \omega_2$ implies that for every $\alpha < \gamma$, $\Vdash p_\zeta(\alpha) \leq^0 p_\xi(\alpha)$, and accompanying $q_\xi \leq p_\xi$, N_ξ, and A_ξ as follows:

Set $p_0 = p$. At limits $\zeta < \omega_2$ with p_ξ having been defined for $\xi < \zeta$, by the ω_2-closure of the $\leq^0_{\dot{Q}_\alpha}$'s, let $p_\zeta \in P$ be such that for every $\xi < \zeta$ and $\alpha < \gamma$, $\Vdash p_\zeta(\alpha) \leq^0 p_\xi(\alpha)$.

To handle the successor stage, suppose that p_ξ is given. First let N_ξ be countable with $\langle N_\xi, \in \rangle \prec \langle H_\lambda, \in \rangle$, $\{\mathscr{P}, \xi, p_\xi\} \subseteq N_\xi$, and $N_\xi \cap \omega_1 \in S$. Then define $q_\xi \leq p_\xi$ by first choosing a P-generic sequence for N_ξ starting with p_ξ and then using the clauses (a), (b), and (c) as in the proof of 3.2. Hence q_ξ is $\langle N_\xi, P \rangle$-generic and setting $A_\xi = \{r \in P \cap N_\xi | q_\xi \leq r\}$, we have $q_\xi \Vdash \check{A}_\xi = \dot{G}_P \cap \check{N}_\xi$. We can assume that $\Vdash q_\xi(\alpha) \leq^0 p_\xi(\alpha)$ iff $q_\xi|\alpha \Vdash q_\xi(\alpha) \leq^0 p_\xi(\alpha)$; similarly for $\leq$; and also that each $q_\xi(\alpha)$ depends only on $p_\xi(\alpha)$ and not on ξ (by carrying out the entire procedure canonically according to some well-ordering of a sufficiently large V_ρ). Finally, define $p_{\xi+1}$ as follows:

$$p_{\xi+1}(\alpha) = \begin{cases} q_\xi(\alpha) & \text{if } \Vdash q_\xi(\alpha) \leq^0 p_\xi(\alpha), \\ p_\xi(\alpha) & \text{otherwise.} \end{cases}$$

Proceeding with the proof, a straightforward Δ-system argument with CH shows that there are $\xi_0 < \xi_1 < \omega_2$ such that: $N_{\xi_0} \cap \omega_1 = N_{\xi_1} \cap \omega_1$, and there is an isomorphism $h: \langle N_{\xi_0}, \in, A_{\xi_0} \rangle \to \langle N_{\xi_1}, \in, A_{\xi_1} \rangle$ with $h|(N_{\xi_0} \cap N_{\xi_1})$ the identity, $h(\mathscr{P}) = \mathscr{P}, h(\xi_0) = \xi_1$, and $h(p_{\xi_0}) = p_{\xi_1}$. By assumption, there is an $\eta \in N_{\xi_0} \cap \omega_1$ such that $q_{\xi_0} \Vdash \dot{\tau}(\eta) = \xi_0$. Hence, $\exists r \in A_{\xi_0}(r \Vdash \dot{\tau}(\eta) = \xi_0)$. Applying h, we have $\exists r \in A_{\xi_1}(r \Vdash \dot{\tau}(\eta) = \xi_1)$ since h is the identity on $N_{\xi_0} \cap \omega_1$. Consequently, $q_{\xi_1} \Vdash \dot{\tau}(\eta) = \xi_1$. However, it is not difficult to check that q_{ξ_0} and q_{ξ_1} are compatible, reaching a contradiction:

The definitions of q_{ξ_0} and q_{ξ_1} as in the proof of 3.2 show that the countable sets

$$E_i = \{\alpha \in \text{supt}(q_{\xi_i}) | \neg (\Vdash q_{\xi_i}(\alpha) \leq^0 p_{\xi_i}(\alpha))\}$$

for $i < 2$ are such that $E_i \in N_{\xi_i}$ and E_i is countable in N_{ξ_i} . We now argue that $\Vdash$ " $q_{\xi_0}(\alpha)$ and $q_{\xi_1}(\alpha)$ are compatible" by cases, depending on whether $\alpha \in \gamma - E_0$, $\alpha \in E_0 - E_1$, or $\alpha \in E_0 \cap E_1$: If $\alpha \in \gamma - E_0$, then $\Vdash q_{\xi_1}(\alpha) \leq p_{\xi_1}(\alpha) \leq^0 p_{\xi_0+1}(\alpha) \leq^0 q_{\xi_0}(\alpha)$. If $\alpha \in E_0 - E_1$, then $\Vdash q_{\xi_0}(\alpha) \leq p_{\xi_0}(\alpha)$ and $\alpha \in E_0$ implies that we are in the case $\Vdash_{P_\alpha}$ " $\leq^0_{\dot{Q}_\alpha}$ is equality" of our assumption about P . Consequently, $\alpha \notin E_1$ implies that $\Vdash q_{\xi_1}(\alpha) = p_{\xi_1}(\alpha) = p_{\xi_0}(\alpha)$, and so we have $\Vdash q_{\xi_0}(\alpha) \leq q_{\xi_1}(\alpha)$. Finally, if $\alpha \in E_0 \cap E_1$, then $\alpha \in N_{\xi_0} \cap N_{\xi_1}$ so the first case in the proposition occurs and we apply " $\aleph_2$-pic". This completes the proof. $\square$

In 3.6 we can combine the two possibilities to one as implicit in the proof.

3.7. Proposition. P'_κ *(from the proof of Theorem* C) *preserves* ω_2 .

Proof. Let $\langle P^1_\alpha, \dot{Q}^1_\alpha: \alpha < \kappa \rangle \in \mathscr{K}_\mu$ be as above except that for $\alpha \equiv 0 \pmod 3$, $\dot{Q}^1_\alpha$ is the trivial forcing (and if $\alpha \not\equiv 0 \pmod 3$, $\dot{Q}^1_\alpha = \dot{R}_\alpha$). Essentially, $P^1_\alpha =$

P'_α, and hence it suffices to prove that P^1_α preserves ω_2. Now the assumption of 3.6 clearly holds for $\langle P^1_\alpha, \dot{Q}^1_\alpha : \alpha < \kappa \rangle$ (for $\aleph_2$-pic—see the discussion of Q_S for more). $\square$

This finally completes the proof of Theorem C.

Instead of $2^{\omega_1} = \omega_3$, for any regular ν such that $\omega_2 \leq \nu < \kappa$, it is possible to arrange $2^{\omega_1} = \nu^+ = \kappa$ using $\mathscr{K}^*_\nu$ with $|\mathrm{supt}(p)| \leq \nu$ is place of $|\mathrm{supt}(p)| \leq \omega_1$ in its definition.

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