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More on simple forcing notions and forcings with ideals

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Abstract

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(1) It is shown that cardinals below a real-valued measurable cardinal can be split into finitely many intervals so that the powers of cardinals from the same interval are the same. This generalizes a theorem of Prikry [9].

(2) Suppose that the forcing with a κ-complete ideal over κ is isomorphic to the forcing of λ-Cohen or random reals. Then for some $\tau < \kappa$, $\lambda^\tau \geqslant 2^\kappa$ and $\lambda \leqslant 2^{<\kappa}$ implies that $2^\kappa = 2^\tau = \operatorname{cov}(\lambda, \kappa, \tau^+, 2)$. In particular, if $2^\kappa < \kappa^{+\omega}$, then $\lambda = 2^\kappa$. This answers a question from [3].

(3) If $A_0, A_1, \ldots, A_n, \ldots$ are sets of reals, then there are disjoint sets $B_0, B_1, \ldots, B_n, \ldots$ such that $B_n \subseteq A_n$ and $\mu^*(B_n) = \mu^*(A_n)$ for every $n > \omega$, where μ^* is the Lebesgue outer measure. For finitely many sets the result is due to N. Lusin.

(4) Let $(P, \leqslant)$ be a σ-centered forcing notion and $\langle A_n \mid n < \omega \rangle$ subsets of P witnessing this. If P, A_n's and the relation of compatibility are Borel, then P adds a Cohen real.

(5) The forcing with a κ-complete ideal over a set X, $|X| \geqslant \kappa$ cannot be isomorphic to a Hechler real forcing. This result was claimed in [3], but the proof given there works only for X of cardinality κ.

In Section 1, we deal with powers of cardinals below a real-valued measurable. The result stated in the abstract and stronger ones in a similar direction are proven.

The rest of the paper may be viewed as a continuation of [3]. In addition to the generic ultrapowers we are using cardinal arithmetics techniques and notions like

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pseudopowers, true cofinalities and covering number. These notions were introduced by the second author, see [10]. We shall state the definitions in appropriate places throughout the paper.

In Section 2, we deal with a question on the number of Cohen or random reals needed forcing with ideal. The principal result is stated in the abstract. Some knowledge of [3] is needed for this section, as well as for the next one.

Section 4 is devoted to the study of the Hechler forcing notion. Statement (4) of the abstract is proved there as well.

The results of Section 4 are due solely to the second author and the rest is mainly joint.

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1. On powers of cardinals below a real-valued measurable

An ideal I is σ-saturated if every pairwise disjoint collection of sets in I^+ is at most countable.

A cardinal κ is real-valued measurable if there exists a σ-additive probability measure μ on κ such that:

(i) all subsets of κ are measurable,

(ii) all singletons $\{\alpha\}$ have measure zero,

(iii) the ideal of null sets is κ-complete (μ is κ-additive),

it is called nowhere prime or atomlessly real-valued measurable if in addition the following holds:

(iv) for every $A \subset \kappa$, if $\mu(A) \neq 0$, then there are disjoint $A_0, A_1 \subset A$ such that $0 < \mu(A_0)$, $\mu(A_1) < \mu(A)$.

Further everywhere we mean by a real-valued measurable an atomlessly real-valued measurable.

If κ is a real-valued measurable cardinal, the ideal of null subsets of κ is a σ-saturated κ-complete ideal. Solovay [19] starting from a measurable cardinal constructed a model with a real-valued measurable cardinal. In fact $2^{\aleph_0}$ can be real-valued measurable. If κ is a real-valued measurable or carries a σ-saturated κ-complete ideal, then $2^{\aleph_0} \geqslant \kappa$.

Prikry [9] showed that the presence of a real-valued measurable cardinal (or σ-saturated ideal) has an influence on powers of cardinals below it:

Theorem 1.1 (Prikry). *If $2^{\aleph_0}$ is real-valued measurable (or carries a σ-saturated $2^{\aleph_0}$-complete ideal), then for every $\lambda < 2^{\aleph_0}$, $2^\lambda = 2^{\aleph_0}$.*

The purpose of this section will be to prove the following:

Theorem 1.2. *Suppose κ is a real-valued measurable (or carries a σ-saturated κ-complete ideal), then the cardinals below κ can be split into finitely many intervals so that the powers of cardinals from the same interval are the same.*

Actually the theorem that we shall prove will give more information and will generalize Prikry's theorem. Using forcing constructions we shall show that any finite number of intervals is possible.

The proof will combine the idea of Prikry with Shelah's cardinal arithmetics.

Let us recall the definitions of the pseudopower and covering numbers which were introduced in [14].

Definition 1.3. (1) For $\mathrm{cof}\,\lambda \leqslant \kappa < \lambda$ the pseudopower $pp_\kappa \lambda$ is the supremum of the cofinalities of the ultraproducts $\prod a/\mathcal{F}$ associated with a set a of at most κ-regular cardinals below λ and an ultrafilter $\mathcal{F}$ on a containing no bounded subsets of λ.

(2) $pp(\lambda)$ is $pp_{\mathrm{cof}\,\lambda}(\lambda)$.

(3) $pp_{\Gamma(\theta,\nu)}(\lambda)$ is the supremum of the cofinalities of the products $\prod a/\mathcal{F}$ associated with a set of cardinality less than θ consisting of regular cardinals below λ and a ν-complete filter $\mathcal{F}$ on a containing no bounded subsets of λ.

(4) The covering number $\mathrm{cov}(\lambda, \kappa, \theta, \sigma)$ is the least μ so that there exists a family $\mathcal{P}$, $|\mathcal{P}| = \mu$, of subsets of λ of cardinality less than κ (i.e. $\mathcal{P} \subseteq \mathcal{P}_\lambda(\lambda)$) so that

$$t \subseteq \lambda \wedge |t| < \theta \rightarrow (\exists \mathcal{P}' \subseteq \mathcal{P})\Big(|\mathcal{P}'| < \sigma \wedge t \subseteq \bigcup_{A \in \mathcal{P}'} A\Big).$$

Note that $\mathrm{cov}(\lambda, \kappa, \kappa, 2)$ is the least size of an unbounded subset of $\mathcal{P}_\kappa(\lambda)$. If $\lambda < \kappa$, then $\mathrm{cov}(\lambda, \kappa, \theta, \sigma) = 1$ for any θ, σ. If $\kappa^+ \leqslant \lambda < \kappa^{+\omega}$ then $\mathrm{cov}(\lambda, \kappa, \theta, 2) = \lambda$ for every $\theta \leqslant \kappa$.

Lemma 1.4. *Suppose that κ carries a κ-complete σ-saturated ideal. Let $\lambda_0 < \lambda < \kappa$. Suppose that for every cardinal λ', $\lambda_0 \leqslant \lambda' < \lambda$, $2^{\lambda'} = 2^{\lambda_0}$. Then $2^\lambda = \mathrm{cov}(2^{\lambda_0}, \kappa, \lambda^+, \aleph_1)$.*

Remark. If $\kappa = 2^{\aleph_0}$, then this holds by Theorem 1.1.

Proof. Suppose otherwise. Let $\lambda_0 < \lambda$ be cardinals witnessing this. Then

$$2^\lambda > \mathrm{cov}(2^{\lambda_0}, \kappa, \lambda^+, \aleph_1).$$

Note that such λ is a regular cardinal. For every $\alpha < \lambda$, $\alpha \geqslant \lambda_0$. Let us fix $f_\alpha : 2^\alpha \leftrightarrow 2^{\lambda_0}$. For $X \subseteq \lambda$ we define $G_X : \lambda \rightarrow 2^{\lambda_0} \times \lambda$ by setting $G_X(\alpha) = \langle f_\alpha(X \cap \alpha), \alpha \rangle$. Then $G''_X(\lambda) \in \mathcal{P}_{\lambda^+}(2^{\lambda_0} \times \lambda)$. By our assumption, $\mathrm{cov}(2^{\lambda_0}, \kappa, \lambda^+, \aleph_1) < 2^\lambda$.

Denote $\text{cov}(2^{\lambda_0}, \kappa, \lambda^+, \aleph_1)$ by δ. There exists a family $\langle N_i \mid i < \delta\rangle$ of elementary submodels of $\langle H_{(2^\lambda)^{++}}, \in, \langle f_\alpha \mid \alpha < \lambda\rangle, \lambda_0, \ldots\rangle$ such that

(a) $|N_i| < \kappa$,

(b) every subset a of $2^{\lambda_0} \times \lambda$ of cardinality $\leqslant\lambda$ is contained in a countable union of elements of $\{N_i \mid i < \delta\}$.

Just pick a family witnessing $\text{cov}(2^{\lambda_0}, \kappa, \lambda^+, \aleph_1) = \delta$ and close each member of it by Skolem functions of the structure.

So for every $X \subset \lambda$, $|X| = \lambda$, there is $i < \delta$ such that $|N_i \cap G''_X(\lambda)| = \lambda$. Hence, for unboundedly many α's below λ, $X \cap \alpha \in N_i$. But $\langle f_\gamma \mid \gamma < \lambda\rangle \in N_i$, so $X \cap \alpha \in N_i$ implies $X \cap \beta \in N_i$ and $f_\beta(X \cap \beta) \in N_i$ for every $\beta < \alpha$. Hence $G''_X(\lambda) \subseteq N_i$. This implies that there are $S \subseteq \mathcal{P}(\lambda)$, $|S| \geqslant \kappa$ and $i^* < \delta$ so that $N_{i^*} \supseteq G''_X(\lambda)$ for every $X \in S$.

Define a partition F of S^2

$$F(\{X, Y\}) = \min\{\alpha < \delta \mid X \cap \alpha \neq Y \cap \alpha\}.$$

By Solovay [19], there exists $S^* \subseteq S$, $|S^*| = \kappa$ such that $|F''([S^*]^2)| < \aleph_1$. Set $\alpha^* = \sup\{\alpha \mid \text{for some } X, Y \in S^*, F(X, Y) = \alpha\}$. Then $\alpha^* < \lambda$, since $\text{cf}\,\lambda = \lambda$. But this means that for every $X \in S^*$, $X \cap \alpha^* \in N_{i^*}$ and $|\{X \cap \alpha^* \mid X \in S^*\}| = \kappa$. Which is impossible, since $|N_{i^*}| < \kappa$. Contradiction. □

Proof of 1.2. Suppose now that there are $\lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots < \kappa$ so that $2^{\lambda_0} < 2^{\lambda_1} < \cdots < 2^{\lambda_n} < \cdots$. Suppose also that λ_{n+1} is the least above λ_n such that $2^{\lambda_{n+1}} > 2^{\lambda_n}$. Then by Lemma 1.4

$$\begin{aligned} 2^{\lambda_0} < 2^{\lambda_1} &= \text{cov}(2^{\lambda_0}, \kappa, \lambda_1^+, \aleph_1) < 2^{\lambda_2} \\ &= \text{cov}(2^{\lambda_1}, \kappa, \lambda_2^+, \aleph_1) < \cdots < 2^{\lambda_{n+1}} \\ &= \text{cov}(2^{\lambda_n}, \kappa, \lambda_{n+1}^+, \aleph_1) < \cdots. \end{aligned}$$

We are going to use the following theorems proved in [12, 14, 15].

Theorem A. *Suppose that* $\theta < \chi \leqslant \chi' \leqslant \mu$, $\aleph_0 < \text{cf}\,\chi$, $\text{cf}\,\chi' < \theta$ *and* $pp_{\Gamma(\theta,\aleph_1)}(\chi) \geqslant \mu$. *Then* $pp_{\Gamma(\theta,\aleph_1)}(\chi) \geqslant pp_{\Gamma(\theta,\aleph_1)}(\chi')$.

Theorem B. *Let* $\theta < \kappa \leqslant \chi$, $\aleph_0 < \text{cf}\,\chi < \theta$. *If for every* χ', $\kappa \leqslant \chi' < \chi$, $\aleph_0 < \text{cf}\,\chi' < \theta$, $pp_{\Gamma(\theta,\aleph_1)}(\chi') < pp_{\Gamma,\aleph_1)}(\chi)$ *(or equivalently* $<\chi$), *then* $pp_{\Gamma(\theta,\aleph_1)}(\chi) = pp_{\Gamma((\text{cf}\,\chi)^+,\aleph_1)}(\chi)$.

Theorem C. *Let* $\mu \geqslant \kappa \geqslant \theta$. *Then* $\text{cov}(\nu, \kappa, \theta, \aleph_1) = \sup\{pp_{\Gamma(\theta,\aleph_1)}(\chi) \mid \kappa \leqslant \chi \leqslant \mu, \aleph_0 < \text{cf}\,\chi < \theta\}$.

The next proposition follows from Theorems A, C and Lemma 1.4.

Proposition 1.5.0. *For every* $m < \omega$, $\text{cov}(2^{\lambda_0}, \kappa, \lambda_{m+1}^+, \aleph_1) = \cdots = \text{cov}(2^{\lambda_m}, \kappa, \lambda_{m+1}^+, \aleph_1) = 2^{\lambda_{m+1}} = \text{cov}(2^{\lambda_{m+1}}, \kappa, \lambda_{m+1}^+, \aleph_1)$.

So $\mathrm{cov}(2^{\lambda_0}, \kappa, \lambda_1^+, \aleph_1) < \mathrm{cov}(2^{\lambda_0}, \kappa, \lambda_2^+, \aleph_1) < \cdots < \mathrm{cov}(2^{\lambda_0}, \kappa, \lambda_m^+, \aleph_1) < \cdots$.

The next theorem is a non-GCH analog of the Hajnal–Shelah Theorem [5], [11, 2.12] saying that the set $\{\mu^\theta \mid 2^\theta < \mu\}$ is finite.

Theorem 1.5. *Suppose that $\mu > \kappa$. Then the set $\{\mathrm{cov}(\mu, \kappa, \theta, \aleph_1) \mid \theta < \kappa\}$ is finite.*

Proof. Suppose otherwise. Let $\langle \theta_n \mid n < \omega \rangle$ be an increasing sequence so that for every $n < \omega$

(a) $\theta_n < \kappa$, and

(b) $\mathrm{cov}(\mu, \kappa, \theta_n, \aleph_1) < \mathrm{cov}(\mu, \kappa, \theta_{n+1}, \aleph_1)$.

Since $\mu > \kappa$, by [14, 5.3], $\mathrm{cov}(\mu, \kappa, \theta_n, \aleph_1) \geq \mu$, for every $n < \omega$. Hence, by (b) for all but finitely many n, $\mathrm{cov}(\mu, \kappa, \theta_n, \aleph_1) > \mu$. W.l.o.g. let us assume that this holds for every n. Then, by Theorem C, for every $n < \omega$ there is χ, $\kappa \leq \chi \leq \mu$, $\mathrm{cf}\,\chi < \theta_n$ but $\mu \leq pp_{\Gamma(\theta_n, \aleph_1)}(\chi)$.

Let $n < \omega$. Set

$$\chi_n = \min\{\chi \mid \kappa \leq \chi \leq \mu,\ \mathrm{cf}\,\chi < \theta_n \text{ and } pp_{\Gamma(\theta_n, \aleph_1)}(\chi) \geq \mu\}.$$

Then $\langle \chi_n \mid n < \omega \rangle$ is nonincreasing. So w.l.o.g. $\chi_n = \chi^*$ for some χ^* for every $n < \omega$. Hence, $\kappa \leq \chi < \chi^*$, $\aleph_1 \leq \mathrm{cf}\,\chi < \theta_n$ implies $pp_{\Gamma(\theta_n, \aleph_1)}(\chi) < \chi^*$ for every $n < \omega$. By Theorem B, then $pp_{\Gamma(\theta_n, \aleph_1)}(\chi^*) = pp_{\Gamma((\mathrm{cf}\,\chi^*)^+, \aleph_1)}(\chi^*)$. By Theorem C, $\mathrm{cov}(\mu, \kappa, \theta_n, \aleph_1) = \sup\{pp_{\Gamma(\theta_n, \aleph_1)}(\chi) \mid \kappa \leq \chi \leq \mu,\ \aleph_1 \leq \mathrm{cf}\,\chi < \theta_n\}$ and by Theorem A, for every χ, $\chi^* \leq \chi \leq \mu$, $\aleph_1 \leq \mathrm{cf}\,\chi < \theta_n$, $pp_{\Gamma(\theta_n, \aleph_1)}(\chi^*) \geq pp_{\Gamma(\theta_n, \aleph_1)}(\chi)$. So $\mathrm{cov}(\mu, \kappa, \theta_n, \aleph_1) = \mathrm{cov}(\mu, \kappa, \theta_{n+1}, \aleph_1)$. Contradiction. □

Now, Theorem 1.2 follows from Lemma 1.4 and Theorem 1.5. □ 1.2

Theorem 1.2 and Lemma 1.4 imply:

Corollary 1.6. *Suppose that κ carries a κ-complete σ-saturated ideal. Let $\lambda_0 < \lambda_1 < \kappa$ be such that $2^{\lambda_0} < 2^{\lambda_1}$ and for every α, $\lambda_0 \leq \alpha < \lambda_1$, $2^{\lambda_0} = 2^\alpha$. Then there exists a cardinal δ, $\kappa < \delta \leq 2^{\lambda_0}$ of cofinality λ_1.*

Proof. Suppose otherwise. By Lemma 1.4, $2^{\lambda_1} = \mathrm{cov}(2^{\lambda_0}, \kappa, \lambda_1^+, \aleph_1)$. Let τ, $\kappa \leq \tau \leq 2^{\lambda_0}$ be the least cardinal such that $\mathrm{cov}(\tau, \kappa, \lambda_1^+, \aleph_1) > 2^{\lambda_0}$. Since $\mathrm{cov}(\kappa, \kappa, \lambda_1^+, \aleph_1) = \kappa$, $\tau > \kappa$. By our assumption $\mathrm{cf}\,\tau < \lambda_1$. It is easy to see that $\mathrm{cov}(\tau, \kappa, \lambda_1^+, \aleph_1) \leq \sum_{\alpha<\tau} \mathrm{cov}(|\alpha|, \kappa, \lambda_1^+, \aleph_1)$. Which is impossible. □

Corollary 1.7. *Suppose that κ carries a κ-complete σ-saturated ideal. If $2^{\aleph_0} < \kappa^{+\omega_1}$, then for every $\lambda < \kappa$, $2^\lambda = 2^{\aleph_0}$.*

Corollary 1.8. *Suppose that κ carries a κ-complete σ-saturated ideal. Then there is $\lambda_0 < \kappa$ such that for every λ, $\lambda_0 \leq \lambda < \kappa$, $2^\lambda = 2^{\lambda_0}$.*

It will be interesting to replace $\text{cov}(\mu, \kappa, \theta, \aleph_1)$ by $\text{cov}(\mu, \kappa, \theta, 2)$ in Theorem 1.5. It will provide the 'right' generalization of the Hajnal–Shelah Theorem to non-GCH situations. Unfortunately we do not know whether this is true.

Conjecture. The set $\{\text{cov}(\mu, \kappa, \theta, 2) \mid \theta < \kappa\}$ is always finite.

Using arguments similar to 1.5 it is possible to prove the following.

Theorem 1.9. *Let $\mu > \kappa$. Then there is no $\langle \theta_n \mid n < \omega \rangle$ so that for every $n < \omega$*

(a) $\theta_n < \theta_{n+1} < \text{cf}\, \kappa$,

and

(b) $\text{cov}(\mu, \kappa, \theta_n, 2)^{\aleph_0} < \text{cov}(\mu, \kappa, \theta_{n+1}, 2)$,

or an even weaker statement

(b^-) $\text{cov}(\text{cov}(\mu, \kappa, \theta_n, 2), \kappa, \aleph_1, 2) < \text{cov}(\mu, \kappa, \theta_{n+1}, 2)$.

Fremlin [1] asked if the least size of an unbounded subset of $\mathcal{P}_{\aleph_1}(\kappa)$ is always κ, for a real-valued measurable cardinal κ. In our notation the least size of an unbounded subset of $\mathcal{P}_{\aleph_1}(\kappa)$ is $\text{cov}(\kappa, \aleph_1, \aleph_1, 2)$. Since κ is a real-valued measurable it is regular. So

$$\text{cov}(\kappa, \aleph_1, \aleph_1, 2) = \sum_{\lambda<\kappa} \text{cov}(\lambda, \aleph_1, \aleph_1, 2).$$

It is trivial that for $\lambda < \kappa$ $\text{cov}(\lambda, \aleph_1, \aleph_1, 2) \neq \kappa$. So the equivalent formulation of Fremlin's question is as follows:

For a real-valued measurable κ, is there $\lambda < \kappa$ such that $\text{cov}(\lambda, \aleph_1, \aleph_1, 2) > \kappa$?

We have the following partial results related to this problem.

Proposition 1.10. *Suppose that κ carries a σ-saturated ideal. Then for every $\lambda < \kappa$, $pp(\lambda) < \kappa$.*

Proof. Suppose otherwise. Let λ be the least cardinal witnessing this. First note that it is impossible to have $pp(\lambda) > \kappa$. By [14, 1.5.A], there exist $a \subseteq \lambda$ consisting of regular cardinals, $|a| = \text{cf}\, \lambda$ and an ultrafilter $\mathcal{F}$ on a so that $\text{cf}(\prod a/\mathcal{F}) = \kappa$. Now, in a generic ultrapower, $\text{cf}(\prod a/\mathcal{F}) = j(\kappa) > \kappa$, also $\text{cf}\, j(\kappa) > \kappa$. But this is impossible since the ultrapower is formed by c.c.c.-forcing.

So $pp(\lambda) = \kappa$. Work in a generic ultrapower. Then $pp(\lambda) = j(\kappa) > \kappa$ and κ is regular. So for some $a \subseteq \lambda$ consisting of regular cardinals, $|a| = \text{cf}\, \lambda$, $\kappa \in \text{pcf}(a)$ where $\text{pcf}(a)$ is the set of all possible cofinalities for a. Since the forcing used to form the generic ultrapower is c.c.c.-forcing, there is $b \in V$, $b \subseteq \lambda$, consisting of regular cardinals $|b| = \text{cf}\, \lambda$ and $b \supseteq a$. Then $\text{pcf}(b) \supseteq \text{pcf}(a)$. So in the ultrapower $\kappa \in \text{pcf}(b)$. Back in V this means that $\kappa \leqslant \max \text{pcf}(b)$. By [14], then there are $c \subseteq \lambda$ consisting of regular cardinals, $|c| = \text{cf}\, \lambda$ and an ultrafilter $\mathcal{F}$ on a so that $\text{cf}(\prod c/\mathcal{F}) = \kappa$. Which is impossible as is shown above. $\square$

By [16, 3.7(5), 3.8], for $\mu < \lambda$, if there is no fixed point for the $\aleph$-function in the interval (μ, λ), then $pp(\lambda) = \mathrm{cov}(\lambda, (\mu)^+, (\mu^+), 2)$. Hence the following holds.

Corollary 1.11. *Suppose that κ carries a σ-saturated ideal. Then for every λ, below the first fixed point of the $\aleph$-function,* $\mathrm{cov}(\lambda, \aleph_1, \aleph_2, 2) < \kappa$.

Let us show now that it is impossible to improve the conclusion of Theorem 1.2.

Proposition 1.12. *Let $n < \omega$, κ be a measurable cardinal and let there be n strong cardinals above κ. Then there exists a generic extension $V^{\mathscr{P}}$ satisfying the following*:

(a) *κ is a real-valued measurable,*

(b) $2^{\aleph_0} < 2^{\aleph_1} < \cdots < 2^{\aleph_n}$.

Proof. We use the forcing of [4] to construct a generic extension V_1 of V satisfying the following:

(1) κ is a measurable cardinal,

(2) for some λ, $\lambda < \lambda^{\aleph_0} < \lambda^{\aleph_1} < \cdots < \lambda^{\aleph_n}$.

Now, add to V_1 λ-random reals. Then $2^{\aleph_0} = \lambda^{\aleph_0}$, $2^{\aleph_1} = \lambda^{\aleph_1}, \ldots, 2^{\aleph_n} = \lambda^{\aleph_n}$. □

Let us show now that a result similar to 1.2 holds if $\aleph_\omega$ is a nonstrong limit Jonsson cardinal. Recall that a cardinal λ is Jonsson, if every algebra with countably many operations of cardinality λ, has a proper subalgebra of the same cardinality.

Theorem 1.13. *Suppose that $\aleph_\omega$ is a nonstrong limit Jonsson cardinal; then the set $\{2^{\aleph_n} \mid n < \omega\}$ is finite.*

Proof. Pick an elementary submodel M of $H_{(2^{\aleph_\omega})^{++}}$ such that $|M \cap \aleph_\omega| = \aleph_\omega$ and $M \cap \aleph_\omega \neq \aleph_\omega$. Since each $\aleph_n$ $(n < \omega)$ is not Jonsson there is n_0 so that for every $n \geqslant n_0$, $|M \cap \aleph_n| < \aleph_n$.

Suppose now that for $n \geqslant n_0$, $2^{\aleph_n} > \aleph_\omega$ and m is the least number such that $2^{\aleph_m} > 2^{\aleph_n}$. By Theorem 1.3 it is enough to show that $2^{\aleph_m} = \mathrm{cov}(2^{\aleph_n}, \aleph_\omega, \aleph_m^+, \aleph_1)$.

We proceed as in Lemma 1.4.

Suppose that $2^{\aleph_m} > \mathrm{cov}(2^{\aleph_n}, \aleph_\omega, \aleph_m^+, \aleph_1) = \delta$. Let $\langle f_\alpha \mid \alpha < \aleph_m \rangle$, $\langle G_X \mid X \subseteq \aleph_m \rangle$, $\langle N_i \mid i < \delta \rangle$, be as in 1.4 and belong to M. There are $S \subseteq \mathscr{P}(\aleph_m)$, $|S| = \aleph_\omega$ and $i^* < \delta$ so that $N_{i^*} \supseteq G_X''(\aleph_m)$ for every $X \in S$. So there are such sets in M. Since $|M \cap \aleph_\omega| = \aleph_\omega$, $|M \cap S| = \aleph_\omega$. Denote $\alpha^* = \sup(M \cap \aleph_m)$. Then $\alpha^* < \aleph_m$ and for every $X \neq Y \in M \cap S$, $X \cap \alpha^* \neq Y \cap \alpha^*$. By the choice of N_{i^*}, every $X \cap \alpha^* \in N_{i^*}$ for $X \in S$ and $|N_{i^*}| < \aleph_\omega$. But already $M \cap S$ has $\aleph_\omega$ different intersections with α^*. Contradiction. □

2. On the number of Cohen or random reals

It was shown in [3] that the forcing with a κ-complete ideal over a cardinal κ cannot be isomorphic to the forcing of less than κ^+-Cohen or random reals. A natural question is if one can replace κ^+ by 2^κ. It was raised in [3] and also by Fremlin and Kamburelis [8]. Clearly, there are obvious limitations. Thus, start with a model having a measurable cardinal κ and satisfying GCH. Add $\kappa^{+\omega}$-Cohen (or random) reals. Then 2^κ will be $\kappa^{+\omega+1}$ but the forcing with the ideal generated in the natural way by a measure over κ is isomorphic to the forcing for adding $\kappa^{+\omega}$-Cohen (or random) reals. It is possible as well to have the number of Cohen (or random) reals less than 2^κ but still regular. Just start with usual model for $\neg$SCH satisfying the following: "κ is a measurable, $2^\kappa = \kappa^+$, $\kappa^{+\omega}$ is a strong limit and $2^{\kappa^{+\omega}} = \kappa^{+\omega+7}$". Add $\kappa^{+\omega+1}$-Cohen (or random) reals. Then 2^κ will be $\kappa^{+\omega+7}$ but the forcing with the ideal generated by a measure over κ will be isomorphic to the forcing of $\kappa^{+\omega+1}$-Cohen (or random) reals. The above shows that the number of reals needed for the forcing with an ideal over κ is connected with powers of singular cardinals above κ. One can ask what happens if $2^\kappa < \kappa^{+\omega}$, i.e., when such an influence disappears. It will follow from the theorem below that in this case the number of reals is exactly 2^κ.

Throughout this section let us assume that I is a normal ideal over κ and the forcing with it is isomorphic to the forcing of λ-Cohen or random reals. We deal with the Cohen reals case. The reader familiar with [3] will be able to fill in the changes needed for the random reals case.

Let $V[\langle r_\alpha \mid \alpha < \lambda \rangle]$ be a generic extension by forcing with I-positive sets. Let $j: V \to M$ be the corresponding generic elementary embedding.

Lemma 2.1. *For every* $\tau < \kappa$, $\mathrm{cov}(\lambda, \kappa, \tau^+, 2) \geq 2^\tau$.

Proof. Suppose otherwise. By [3], $\lambda \geq \kappa^+$. So $\mathrm{cov}(\lambda, \kappa, \tau^+, 2) \geq \lambda \geq \kappa^+$. Let $\delta = (\mathrm{cov}(\lambda, \kappa, \tau^+, 2))^+$. By the assumption $\delta \leq 2^\tau$. Pick in V a sequence $\langle x_\beta \mid \beta < \delta \rangle$ of distinct subsets of τ. Apply j to the sequence. Then, in M, $j(\langle x_\beta \mid \beta < \delta \rangle) = \langle x'_\beta \mid \beta < j(\delta) \rangle$. Consider only the δ first members of the extended sequence, i.e. $\langle x'_\beta \mid \beta < \delta \rangle$. Note that for every $\beta < \delta$, $\mathrm{cf}\,\beta = \kappa$, x'_β is a new subset of τ.

For every $\beta < \delta$, $\mathrm{cf}\,\beta = \kappa$, there is $a \in \mathcal{P}_{\tau^+}(\lambda)$ such that $x'_\beta \in V[\langle r_\alpha \mid \alpha \in a \rangle]$. Since $\delta > \mathrm{cov}(\lambda, \kappa, \tau^+, 2)$, there exist $A \subseteq \{\beta < \delta \mid \mathrm{cf}\,\beta = \kappa\}$, $|A| = \delta$ and $b \in \mathcal{P}_\kappa(\lambda)$ such that for every $\beta \in A$, $x'_\beta \in V[\langle r_\alpha \mid \alpha \in b \rangle]$. For simplification of the notation let us assume that $A = \{\beta < \delta \mid \mathrm{cf}\,\beta = \delta\}$. Collasping $|b|$ to $\aleph_0$, we obtain that one Cohen real r adds all of $\langle x'_\beta \mid \beta < \delta, \mathrm{cf}\,\beta = \kappa \rangle$. Once more, in the random reals case one should deal with less than κ random reals.

Let us force now a box sequence $\langle C_\tau \mid \tau < \delta, \mathrm{cf}\,\tau \leq \kappa \rangle$, i.e. a sequence such that

(a) C_τ is a closed unbounded subset of τ of cardinality $\leq \kappa$,
(b) for every $\tau_1 < \tau_2 < \delta$, if τ_1 is a limit point of C_{τ_2} then $C_{\tau_1} = C_{\tau_2} \cap \tau_1$.

The forcing conditions are an approximation to such a sequence of cardinality $<\delta$. Such forcing is (κ^+, ∞)-distributive and it would not effect the ideal I. Namely, the generic embedding j extends easily after the forcing of a box sequence.

Let us assume for simplification of notation that there is a box sequence $\langle C_\tau \mid \tau < \delta, \text{cf}\, \tau \leq \kappa \rangle$ in V. Pick a set of reals $\langle r_\beta \mid \beta < \kappa \rangle$ representing r in M. Let us fix in V a sequence $\langle T_\alpha \mid \alpha < \delta, \text{cf}\, \alpha = \kappa \rangle$ of canonical names of the reals $\langle x_{\bigcup j''(\alpha)} \mid \alpha < \delta, \text{cf}\, \alpha = \kappa \rangle$. For every $\alpha < \delta$, $\text{cf}\, \alpha = \kappa$, let us consider a set $A_\alpha = \{r_\beta \mid \beta < \kappa$, there exists $\gamma \in C_\alpha$ such that $T_\alpha(r_\beta) = x_\gamma\}$.

Claim. *For every* $\alpha < \delta$, $\text{cf}\, \alpha = \kappa$, A_α *is a set of the second category.*

Proof. Note that, in M, $j(T_\alpha) = T_\alpha$, $T_\alpha(r) = x_{\bigcup j''(\alpha)}$ and since C_α is a club in α, $\bigcup j''(\alpha) \in j(C_\alpha)$. $\square$ Claim.

Now, as in [3], there is $\xi(\alpha) < \kappa$ so that the set $A_\alpha \restriction \xi(\alpha) = \{r_\beta \in A_\alpha \mid \beta < \xi(\alpha)\}$ is of the second category. Find a limit point $\rho(\alpha)$ of C_α so that for every $r_\beta \in A_\alpha \restriction \xi(\alpha)$ there is $\gamma \in C_\alpha \cap \rho(\alpha)$ so that $T_\alpha(r_\beta) = x_\gamma$. Note that $C_\alpha \cap \rho(\alpha) = C_{\rho(\alpha)}$. Replacing δ by its stationary subset if necessary, we can assume that there are $\xi^* < \kappa$ and $\rho^* < \delta$ so that for every $\alpha < \delta$, $\text{cf}\, \alpha = \kappa$, $\xi(\alpha) = \xi^*$ and $\rho(\alpha) = \rho^*$. Hence, for every such α the range of $T_\alpha \restriction (A_\alpha \restriction \xi^*)$ is contained in the set $\{x_\gamma \mid \gamma \in C_{\rho^*}\}$. Since $|C_{\rho^*}| \leq \kappa$, $j''(C_{\rho^*})$ is in M. Therefore, the following set belongs to M: $E = \{\alpha \mid \alpha < \delta$, $\text{cf}\, \alpha = j(\kappa)$, $r \in A_\alpha$, $A_\alpha \restriction \xi^*$ is of the second category and for $\beta < \xi^*$ with $r_\beta \in A_\alpha$ there is $\gamma \in j''(C_{\rho^*})$ satisfying $T_\alpha(r_\beta) = x_\gamma\}$.

Notice that $x_{j(\gamma)} = x_\gamma$. Pick some $\alpha^* \in E \setminus j''(\delta)$. As in [3], the values of $\boldsymbol{T}_{\alpha^*}$ can be decided in V on a set of the second category. And the contradiction is derived now exactly as in [3]. $\square$

Lemma 2.2. *If* $j(\kappa) \geq (2^\kappa)^V$, *then* $2^\kappa = \text{cov}(\lambda, \kappa, \aleph_1, 2)$.

Proof. Suppose otherwise. Let $\langle x_\alpha \mid \alpha < \kappa \rangle$ be a sequence of distinct reals in V. Applying j to $\langle x_\alpha \mid \alpha < \kappa \rangle$, we obtain a longer sequence of distinct reals $\langle x_\alpha \mid \alpha < j(\kappa) \rangle$ in M or equivalently in $V[G]$. The reals $\langle x_\alpha \mid \kappa \leq \alpha < j(\kappa) \rangle$ are all new. For every $\alpha < (2^\kappa)^V$ there is a countable $b_\alpha \subseteq \lambda$ so that x_α is added by the Cohen reals with indexes in b_α. Now use $(2^\kappa)^V > \text{cov}(\lambda, \kappa, \aleph_1, 2)$ and $\text{cf}((2^\kappa)^V) > \kappa$. So we are in the situation where less than κ-many Cohen reals are coding at least κ^+-many reals which are in the image of κ-many, i.e., in $j(\langle x_\alpha \mid \alpha < \kappa \rangle)$. Arguments of [3, Theorem 1.2, 2.6] can be applied now in order to derive the contradiction. $\square$

The next proposition characterizes ordinals moved by an elementary embedding.

Proposition 2.3. *Let κ, δ be cardinals and $j: V \to M \approx \{j(f)(\kappa) \mid f \in {}^{\kappa}V \cap V\}$ be an elementary embedding, probably generic but using a forcing satisfying less than κ-c.c, $\mathrm{crit}(j) = \kappa$. Then for any cardinal $\delta \neq j(\kappa)$, $\mathrm{cf}\,\delta \geq \kappa$, $j(\delta) > \delta$ iff for some cardinal $\mu \leq \delta$, $\mathrm{cf}\,\mu = \kappa$, $\delta < j(\mu)$.*

Proof. Let $j(\delta) > \delta$. Pick $f: \kappa \to \delta$ such that $\delta = j(f)(\kappa)$. Since δ is a cardinal, we can assume that $f(\alpha)$ is a cardinal for every $\alpha < \kappa$. Set $\mu = \sup\{f(\alpha) \mid \alpha < \kappa\}$. If f is not almost constant, then $\mathrm{cf}\,\mu = \kappa$ and $j(\mu) > \delta$. Suppose otherwise. Then $\delta = j(\mu)$, for a cardinal $\mu < \delta$. Note that such δ must be a singular cardinal, since $\mathrm{cf}\,\mu > \kappa$ implies $\bigcup j''\mu = j(\mu) = \delta$, i.e. $\mathrm{cf}\,\delta \leq \mu < \delta$, $\mathrm{cf}\,\mu < \kappa$ implies $\mathrm{cf}\,\delta = \mathrm{cf}\,j(\mu) = \mathrm{cf}\,\mu$ and $\mathrm{cf}\,\mu = \kappa$ implies $\delta = \mathrm{cf}\,\delta = \mathrm{cf}\,j(\mu) \leq j(\kappa)$. In the last case the only possibility is $\mu = \kappa$ and $\delta = j(\kappa)$. Consider now $\mu^{+\kappa}$; if $\mu^{+\kappa} < \delta$, then $\mathrm{cf}\,\mu^{+\kappa} = \kappa$ and $j(\mu^{+\kappa}) > \delta$. Otherwise for some $\alpha < \kappa$, $\delta = \mu^{+\alpha}$. But this is impossible since $\mathrm{cf}\,\delta \leq \kappa$. □

We do not know if it is possible to remove the assumptions "$\mathrm{cf}\,\delta \geq \kappa$" and "$j(\kappa) \neq \delta$" from 2.3. Our conjecture is that this is possible. The positive answer to the following question will be sufficient for removing "$\mathrm{cf}\,\delta \geq \kappa$".

Is it true that for every τ, $|j(j(\tau))| = |j(\tau)|$, where j is an embedding of a κ-complete ω_1-saturated ideal over κ or just κ is a measurable and j is the embedding of a measure over κ?

Is it possible to have a measurable κ and an embedding j of a measure over κ such that for some τ, $j(\tau) > \tau$ and $j(\tau)$ is a cardinal?

Is it possible to have a κ-complete ω_1-saturated ideal over κ such that $\emptyset \Vdash$ "$j(\kappa)$ is a cardinal"? or more specific $\emptyset \Vdash$ "$j(\kappa) = \kappa^{++}$"?

We can deduce now the following:

Theorem 2.4. *Suppose that the forcing with a κ-complete ideal over κ is isomorphic to the forcing of λ-Cohen or random reals. Then*

(1) *If $2^{\kappa} < \kappa^{+\omega}$, then $\lambda = 2^{\kappa}$.*

(2) *If $2^{\kappa} < \kappa^{+\kappa}$, then $2^{\kappa} = \mathrm{cov}(\lambda, \kappa, \aleph_1, 2)$ or for some $\tau < \kappa$, $2^{\tau} = 2^{\kappa}$ and $2^{\kappa} = \mathrm{cov}(\lambda, \kappa, \tau^+, 2)$*

Proof. If $j(\kappa) \geq 2^{\kappa}$, then by Lemma 2.2, $2^{\kappa} = \mathrm{cov}(\lambda, \kappa, \aleph_1, 2) = \lambda$. If in addition $2^{\kappa} < \kappa^{+\omega}$, then also $\lambda < \kappa^{+\omega}$. Hence $\mathrm{cov}(\lambda, \kappa, \aleph_1, 2) = \lambda$. If $j(\kappa) < 2^{\kappa}$, then $j(2^{\kappa}) = (2^{\kappa})^V$ by Proposition 2.3, since $2^{\kappa} < \kappa^{+\kappa}$. Hence, in M, $(2^{\kappa})^V = j(2^{\kappa}) = 2^{j(\kappa)}$. Since $P^V(\kappa) \subseteq M$, $(2^{\kappa})^M \geq (2^{\kappa})^V$. So, in M, $2^{\kappa} = (2^{\kappa})^V = 2^{j(\kappa)}$. By elementarity of j, back in V, for some $\tau < \kappa$, $2^{\tau} = 2^{\kappa}$. This proves (2) of the theorem, since by Lemma 2.1, $2^{\tau} \leq \mathrm{cov}(\lambda, \kappa, \tau^+, 2) \leq \lambda^{\kappa} = 2^{\kappa}$. For (1) note that $2^{\kappa} < \kappa^{+\omega}$ implies $\mathrm{cov}(\lambda, \kappa, \tau^+, 2) = \lambda$. □

The following generalizes a result of Jech–Pikry [6].

Theorem 2.5. *Suppose that the forcing with a κ-complete ideal over κ is isomorphic to the forcing of λ-Cohen or random reals. Let τ be the least $\nu < \kappa$ such that $2^\nu = 2^{<\kappa}$. Then*

(a) $\lambda^\tau = 2^\kappa$,

(b) $\lambda \leq 2^{<\kappa}$ *implies that* $2^\kappa = 2^\tau = \text{cov}(\lambda, \kappa, \tau^+, 2)$.

Remark. By 1.8, τ defined above exsists.

Proof. Force with the ideal. Let $j: V \to M \subseteq V[G]$ be the generic elementary embedding. Since $2^\tau = 2^{<\kappa}$, by elementarity of j, the same is true in M with κ replaced by $j(\kappa)$. Then $(2^\tau)^M = (2^\kappa)^M \geq (2^\kappa)^V$. But we forced with λ-Cohen or random reals, so $(2^\tau)^M = (2^\tau)^{V[G]} \leq (\lambda^\tau)^V$. Hence, $(2^\kappa)^V \leq (\lambda^\tau)^V$, which proves (a). (b) follows also since $2^\tau = 2^{<\kappa}$ and $2^\tau \leq \text{cov}(\lambda, \kappa, \tau^+, 2)$ by 2.1. □

2.6. Some forcing constructions. It is easy to arrange a situation when $2^\kappa = \lambda > 2^{<\kappa}$. Just start with a measurable κ and add κ-Cohen or random reals. As was noted in [3], it is possible to have $\lambda < 2^{\aleph_0}$. Add $\kappa^{+\omega}$-Cohen or random reals to a model with a measurable κ satisfying GCH. Working harder it is possible to construct a model satisfying $\lambda < 2^{\aleph_0} < 2^{\aleph_1} = 2^\kappa$. Thus start with a model having a measurable κ and a strong limit cardinal $\nu > \kappa$ such that $\text{cf}\, \nu = \aleph_1$, $2^\nu = \nu^{+\omega+2}$, $(\nu^{+\omega})^{\aleph_0} = \nu^{+\omega+1}$. Add $\nu^{+\omega}$ Cohen or random reals. Then λ will be $\nu^{+\omega}$, $2^{\aleph_0} = \nu^{+\omega+1}$ and $2^{\aleph_1} = 2^\kappa = \nu^{+\omega+2}$. Using [4], it is possible to arrange also finitely many jumps, for example $\lambda < 2^{\aleph_0} < 2^{\aleph_1} < \cdots < 2^{\aleph_7} = 2^\kappa$. Let us present now another construction, showing that λ may be above $2^{\aleph_0}$, say $2^{\aleph_0} < \lambda < 2^{\aleph_1} = 2^\kappa$. Once more, using [4] it is possible to put λ between $2^{\aleph_5}$ and $2^{\aleph_6}$. Start with a GCH model having a measurable cardinal κ and a supercompact or strong cardinal ν above it. Blow up the power of ν to $\nu^{+\kappa+3}$ and change its cofinality to $\aleph_1$, without adding subsets to κ. Then the elementary embedding of κ of V extends to elementary embedding in the generic extension. Let us denote it by j. In particular, $\nu^{+\kappa+1} < j(\nu^{+\kappa}) < \nu^{+\kappa+2}$. Add now $\nu^{+\kappa}$ Cohen or random reals. Then $2^{\aleph_0} = \nu^{+\kappa}$, $2^{\aleph_1} = 2^\kappa = \nu^{+\kappa+3}$ and λ will be $j(2^{\aleph_0}) = j(\nu^{+\kappa})$.

We do not know whether it is possible to have $2^{<\kappa} < \lambda < 2^\kappa$, more precisely:

Question. Is it consistent that there exists a κ-complete ideal over κ such that the forcing with it is isomorphic to the forcing for adding λ-Cohen or random reals and $2^{<\kappa} < \lambda < 2^\kappa$?

Note that by 2.5(a), $\lambda^\tau = 2^\kappa$ for the least τ with $2^\tau = 2^{<\kappa}$.

3. On splitting into sets of the same Lebesgue outer measure

Peter Komjáth deduced the following from [3]: If $A_1, \ldots, A_n, \ldots$ are subsets of the real line, then there are disjoint $B_n \subseteq A_n$ such that for every interval I, $I \cap A_n$ is of the second category iff $I \cap B_n$ is such.

He suggested that the dual question for measure may follow from [3]. We are going to show here that this is the case. Namely, the following holds:

Theorem 3.1. *If $A_0, A_1, \ldots, A_n, \ldots$ are sets of reals, then there are disjoint sets $B_0, B_1, \ldots, B_n, \ldots$ such that $B_n \subseteq A_n$ and $\mu^*(B_n) = \mu^*(A_n)$ for every $n > \omega$. Here μ^* denotes the Lebesgue outer measure.*

Proof. It is enough to show that always there is $B \subseteq A_0$, $\lambda^*(B) > 0$, so that for every Borel set C and for every $n > 0$, $A_n \cap C$ is not of measure zero iff $(A_n \setminus B) \cap C$ is such.

Suppose otherwise. Then for every nonempty set $B \subseteq A_0$ there is a Borel set C and $n > 0$ so that B almost contains $C \cap A_n$ and $\lambda^*(C \cap A_n) > 0$. Let I be the ideal for all sets of reals having the measure zero intersection with A_0. The forcing with I is isomorphic to the random real forcing (if, for example, all $A_n = A_1$ $(n \geq 1)$) or to the product of random with Cohen forcings. Both cases are impossible by [3, 2.3 and 4.3]. More precisely, the proof there is given for the Cohen$*$random case, but it works for the product and actually for any finite iteration of this forcing notion. □

Problem. Find an elementary proof of Theorem 3.1.

P. Komjáth wrote to us that for finitely many sets $A_1, \ldots, A_n$ the result is due to N. Lusin.

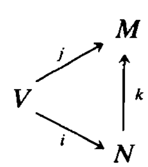
4. On Hechler reals

The main purpose of the present section will be to prove the following:

Theorem 4.1. *The forcing with a κ-complete ideal over a set X, $|X| \geq \kappa$, cannot be isomorphic to a Hechler real forcing.*

Remark. This result was claimed in [3] but the proof given there works only for X of cardinality κ.

Proof. Suppose otherwise. Let I be a κ-complete (and non κ^+-complete) ideal over a set X witnessing this. Force with the Hechler real forcing and let $V[G]$ be a generic extension, $j: V \to M \subseteq V[G]$ the corresponding elementary embedding. Split the embeding in the usual fashion where i is generated by a normal ideal I



over κ. (We are picking a function $f: X \to \kappa$ representing κ in M and projecting I via f). Then $\mathcal{P}(\kappa)/J$ is isomorphic to a complete subalgebra of the Hechler forcing. Let us denote it by Q.

Let us split the result of the proof into a number of lemmas some of them, probably, of independent interest.

Lemma 4.2. *Let D be a filter on ω in V which is $L(\mathbb{R})$-definable. Then there are $2^{\aleph_0}$ almost disjoint D-positive sets.*

Proof. Let φ be a formula defining D in $L(\mathbb{R})$. Force a Cohen real $r \subseteq \omega$. Then, if $\varphi(r)$ (or $\neg\varphi(r)$) is true, i.e., $r \in D$ in the extension, then also $\varphi(\omega \setminus r)$ is true, since for every n, $(\omega \setminus r \setminus n) \cup (r \cap n)$ is Cohen as well over V. This is impossible. So both r and $\omega \setminus r$ are D-positive. Since one Cohen real produces $2^{\aleph_0}$ almost disjoint Cohen reals over V, we obtain $2^{\aleph_0}$-almost disjoint D-positive sets in a Cohen extension of V.

Now, a Hechler real-forcing adds a Cohen real (see Truss [20]). So in M, using homogeneity of the forcing, $D^M = \{x \subseteq \omega \mid L(\mathbb{R}) \models \varphi(x)\}$ has $2^{\aleph_0}$-almost disjoint positive sets. By the elementarity of j, the same is true in V. $\square$

Remark 4.2A. (1) The '$L(\mathbb{R})$-definability' can be replaced by any absolute enough definition.

(2) Without using the hypothesis of 4.1 still similar lemmas hold:

(a) If r, $\omega \setminus r$ satisfy φ, then in $V[r] \models (\exists x)(\varphi(x) \wedge \varphi(\omega \setminus x))$. Hence, this holds in V, if φ is Σ^1_2.

(b) By (a), if φ is $\boldsymbol{\Sigma}^1_2$ we can find in $V[r]$ a perfect set P of pairwise almost disjoint $x \subseteq \omega$ such that $\neg\varphi(\omega \setminus x)$. Hence $V[r] \models \exists P\, \forall x \in P\, (\varphi(x) \wedge \neg\varphi(\omega \setminus x))$, which is a Σ^1_3-statement. If φ was $\boldsymbol{\Sigma}^1_1$, then it is a Σ^1_2-statement and so is true in V.

(3) If in (b) we like to have only $\aleph_0$ pairwise disjoint sets, then we need absoluteness of $(\exists h)(h: \omega \to \omega \wedge \bigwedge_n \neg\varphi(\omega \setminus h^{-1}(\{n\}))$. Which is OK if φ is Π^1_2. So φ being $\boldsymbol{\Delta}^1_2$ is enough for (a) + (b), i.e. for the lemma.

(4) If $(\forall r \in \mathbb{R}^V)$ ($r^\#$ exists), then we do have more absoluteness.

Now we would like to show that the forcing with Q (the subalgebra of Hechler forcing defined above) adds a Cohen real. Let us prove a more general result.

Proposition 4.3. *Let $\langle P, \leqslant\rangle$ be a σ-centered forcing notion and $\langle A_n \mid n < \omega\rangle$ subsets of P witnessing this. Suppose that both $\langle P, \leqslant\rangle$ and $\langle A_n \mid n < \omega\rangle$ have 'simple' definitions. Then P adds a Cohen real.*

Remark. (1) The exact meaning of 'simple' will be clear from the proof; see also Remark 4.3A for precise computation. It will include Borel forcing and the forcing notion Q of our prime interest. Note that in our case, for every real $r \in V$

there is $r^{\#} \in V$, so we have more absoluteness. A general class of 'simple' forcing notions was worked out in Judah–Shelah [7].

(2) The Mathias forcing with ultrafilter is σ-centered but does not add a Cohen real.

Proof. It is well known that a σ-centered forcing adds a new real.

Let $\boldsymbol{r}$ be a name of a new real added by P. For every $p \in P$ and $n < \omega$ let us consider $T_p = \{\eta \in {}^{\omega>}2 \mid p \nVdash (\eta \text{ is not an initial segment of } \boldsymbol{r}\}$ and $T_n = \bigcap \{T_p \mid p \in A_n\}$.

Claim 1. *T_n has an infinite branch.*

Proof. Suppose otherwise. Then $T_n \subseteq {}^{m>}2$ for some $m < \omega$. For every $\eta \in {}^{m}2$ there is $p_\eta \in A_n$ so that $p_\eta \Vdash (\eta$ is not an initial segment of $\boldsymbol{r})$. Find some $p \in A_n$ above all $\{p_\eta \mid \eta \in {}^{m}2\}$. Then $p \Vdash (\boldsymbol{r}$ has not initial segment in ${}^{m}2)$ which is impossible. Contradiction. □ Claim 1

Let η_n be an ω-branch of T_n. For every $p \in A_n$ define

$$b(p, \eta_n) = \{k < \omega \mid \text{for some } q \geq p,\ q \Vdash (\boldsymbol{r} \cap \eta_n = \eta_n \restriction k)\}.$$

Notice that $q_1 \leq q_2$ implies $b(q_1, \eta_n) \supseteq b(q_2, \eta_n)$.

Claim 2. *For each n, $\{b(p, \eta_n) \mid p \in A_n\} \cup \{\omega - k \mid k \in \omega\}$ generates a filter over ω.*

Proof. For every $q_1, \ldots, q_l \in A_n$ $(l < \omega)$ and $m < \omega$ pick $p \in A_n$ such that $p \geq q_1, \ldots, q_l$. Since $\eta_n \restriction m \in T_p$, there is $q \geq p$ such that $q \Vdash \boldsymbol{r} \restriction m = \eta_n \restriction m$. But $\emptyset \Vdash \boldsymbol{r} \neq \eta_n$. Hence there are $q' \geq q$ and $k > m$ such that $q' \Vdash (\boldsymbol{r} \cap \eta_n = \eta_n \restriction k)$. □ Claim 2

Let η_n be an ω-branch of T_n. For every $p \in A_n$ define

$$b(p, \eta_n) = \{k < \omega \mid \text{for some } q \geq p,\ q \Vdash (\boldsymbol{r} \cap \eta_n = \eta_n \restriction k)\}.$$

Notice that $q_1 \leq q_2$ implies $b(q_1, \eta_n) \supseteq b(q_2, \eta_n)$.

Claim 3. *For each n, $\{b(p, \eta_n) \mid p \in A_n\} \cup \{\omega - k \mid k \in \omega\}$ generates a filter over ω.*

Proof. For every $q_1, \ldots, q_l \in A_n$ $(l < \omega)$ and $m < \omega$ pick $p \in A_n$ such that $p \geq q_1, \ldots, q_l$. Since $\eta_n \restriction m \in T_p$, there is $q \geq p$ *such that* $q \Vdash \boldsymbol{r} \restriction m = \eta_n \restriction m$. But $\emptyset \Vdash \boldsymbol{r} \neq \eta_n$. Hence there are $q' \geq q$ and $k > m$ such that $q' \Vdash (\boldsymbol{r} \cap \eta_n = \eta_n \restriction k)$. □ Claim 3

Let us denote this filter by D_n.

Now using the simplicity of $\langle P, \leqslant\rangle$ and $\langle A_n \mid n<\omega\rangle$, it is possible to find $2^{\aleph_0}$ almost disjoint D_n-positive sets. Then it is not hard to split ω into disjoint subsets $\langle a_i \mid i<\omega\rangle$ which will be D_n-positive for every $n<\omega$. (First find disjoint $\langle b_n \mid n<\omega\rangle$ such that b_n is D_n-positive and then split each of b_i's.)

Define by induction a sequence $\langle k_n \mid n<\omega\rangle$ so that the sets

$$B_n=\{\eta_n \restriction l \mid k_n \leqslant l<\omega\}$$

will be disjoint if $\eta_n \neq \eta_m$ and equal otherwise. Let $B=\bigcup_{n<\omega} B_n$. Then $\emptyset \Vdash \forall l\, \exists m \geqslant l\, \boldsymbol{r} \restriction m \in B$. We shall use this below in order to define a name $\boldsymbol{v}$ of a Cohen real. Let $\langle \rho_i \mid i<\omega\rangle$ be an enumeration of ${}^{\omega>}2$ so that each sequence appears infinitely many times.

Let r be a generic real. Define in $V[r]$ a real v. Pick the first $l_0<\omega$ so that $r \restriction l_0 \in B$. Let n_0 be the least n so that $r \restriction l_0 \in B_n$. Pick the first $m_0>l_0$ so that $r \restriction m_0 \notin B_{n_0}$. There exists $k_0<\omega$ so that $m_0 \in a_{k_0}$. Continue, pick the first $l_1 \geqslant m$ so that $r \restriction l_1 \in B$. Find n_1 so that $r \restriction l_1 \in B_{n_1}$. Define m_1 and k_1 as above. Continue in the same fashion. This process defines a sequence $\langle k_i \mid i<\omega\rangle$. Set v to be

$$\rho_{k_0}{}^\frown\rho_{k_1}{}^\frown\rho_{k_2}{}^\frown\cdots.$$

Let us show that v is a Cohen real over V. Work in V. Let $\boldsymbol{v}$ be a name of v. Let $T \subseteq {}^{\omega>}2$ be a nowhere dense set. For every $p \in P$ we shall find $q \geqslant p$ such that $q \Vdash \boldsymbol{v}$ is not a branch of T. Let $p \in P$. Find $n<\omega$ so that $p \in A_n$. Let us try to interpret $\boldsymbol{v}$ using η_n instead of r. The process will work up to some stage $j<\omega$, so that $\eta_n \restriction l_n \in B_n$. Denote $\rho_{k_0}{}^\frown\rho_{k_1}{}^\frown\rho_{k_{j-1}}$ by ρ^*. There is ρ so that $\rho^{*\frown}\rho \in {}^{\omega>}2\setminus T$. Find also $k<\omega$ so that $\rho=\rho_k$. Consider the set $E=\{m<\omega \mid m>l_j$ and for some $q \geqslant p,\ q \Vdash \boldsymbol{r} \cap \eta_n=\eta_n \restriction m\}$. It belongs to D_n. So there is $m \in E \cap a_k$. Pick $q \geqslant p$ such that $q \Vdash \boldsymbol{r} \cap \eta_n=\eta_n \restriction m$. Then $q \Vdash \rho^{*\frown}\rho_k$ is an initial segment of $\boldsymbol{v}$. But this means that $q \Vdash \boldsymbol{v}$ is not a branch of T. $\square$ 4.3

Remark 4.3.A. What should be the definition of $\langle P, \leqslant\rangle$ and A_n's for the proof to work? For applying 4.2 in just ZFC we need "D_n is $\boldsymbol{\Sigma}_1^1$".

Let $I^l=\langle r_i^l \mid i<\omega\rangle$ be the maximal antichain of P, $r_i^l \Vdash \boldsymbol{r}(l)=h^l(i)$, $h^l:\omega \to \{0,1\}$ for every $l<\omega$. Now $y \in D_n$ iff

$$\begin{aligned}\exists\langle p_1,\ldots,p_k\rangle\Big[\bigwedge_{i=1}^{k} p_i \in A_n \wedge (\forall m \in \omega\setminus y)(\forall_{i_0},\ldots,i_m<\omega)\\ \neg(h^0(i_0)=\eta_n(0) \wedge \cdots \wedge h^{m-1}(i_{m-1})=\eta_n(m-1) \wedge h^m(i_m) \neq \eta_n(m)\\ \wedge \operatorname{comp}\{p_1,\ldots,p_k,r_{i_0}^0,\ldots,r_{i_m}^m\})\Big]\end{aligned}$$

where $\operatorname{comp}\{q_1,\ldots,q_n\}$ means that this set has an upper bound in P.

So for proving 4.3 in ZFC the following is enough: "The sets P, A_n $(n<\omega)$ are $\boldsymbol{\Sigma}_1^1$ and comp is $\boldsymbol{\Pi}_1^1$".

The next lemma which deals purely with Hechler forcing is sufficient in order to complete the proof of 4.1.

Lemma 4.4. *There exists a name $\boldsymbol{\tau}$ in Cohen forcing such that:*

(a) $\Vdash_{\text{Cohen}}$ "*$\boldsymbol{\tau}$ is a name in Hechler forcing of a member of $^{\omega}\omega$*".

(b) *If $\boldsymbol{e} \in V$ is a Hechler name of a member of $^{\omega}\omega$, V^* is a countable submodel of a large enough portion of V such that $\boldsymbol{e} \in V^*$ and r is a Cohen real over V^*, then*

$$\Vdash_{\text{Hechler}} \text{“}\boldsymbol{\tau}(r) \text{ is not dominated by } \boldsymbol{e}\text{”}.$$

Proof of 4.1 from Lemma 4.4. By Lemma 4.2 and Proposition 4.3, Q adds a Cohen real r over V. Since the projection J of I concentrates over κ, we can pick a sequence $\langle r_\alpha \mid \alpha < \kappa \rangle \in V$ representing r in the generic ultrapower.

By elementarity of i, for every countable elementary submodel V^* of a large enough portion of V, $\{\alpha < \kappa \mid r_\alpha \text{ is not Cohen over } V^*\} \in J$.

Since there is a Hechler over V real in M and j is an elementary embedding, for every $R \subseteq (^{\omega}\omega)^V$ of cardinality less than κ there exists $f \in (^{\omega}\omega)^V$ dominating every function of R. Then in M, for every $R \subseteq {}^{\omega}\omega$, $|R| = \kappa$, there exists $f \in {}^{\omega}\omega$ dominating every function of R.

Let $\boldsymbol{\tau}$ be as in Lemma 4.4 and let $h \in M$ be the Hechler real over V generating everything. Consider, in M, $R = \{\boldsymbol{\tau}(r_\alpha)(h) \mid \alpha < \kappa\}$. Then, there exists $e \in {}^{\omega}\omega \cap M$ which dominates R. Let $\boldsymbol{e}$ be a Hechler name of e in V. Pick in V a countable submodel V^* such that $\boldsymbol{e} \in V^*$. By the above, almost all (mod J) r_α's are Cohen over V^*. But this is impossible by 4.4(b). Contradiction. $\square$

Let us turn to the proof of Lemma 4.4. It will be more convenient to deal only with strictly increasing sequences in the definition of Hechler forcing further. Also $^{\omega>}\omega$ will be understood in the same way. We shall interpret the Cohen forcing as Cohen$' = \{(h, g) \mid$ (a) h is a finite function from $^{\omega>}\omega$ to $\{0, 1\}$; (b) $\operatorname{dom} h$ is closed under initial segments; (c) $\operatorname{dom} g = \{t^\frown\langle k\rangle \mid t^\frown\langle k\rangle \in \operatorname{dom} h$ and $h(t) = 0$; (d) $\operatorname{rng} g \subseteq \omega\}$. Let $\langle \bar{h}, \bar{g}\rangle$ be a generic object of Cohen$'$. Then $\bar{h}: {}^{\omega>}\omega \to \{0, 1\}$, $g: \{t^\frown\langle k\rangle \mid t^\frown\langle k\rangle \in {}^{\omega>}\omega,\ h(t) = 0\} \to \omega$. Define now a Hechler name σ. For a Hechler real $f \in {}^{\omega}\omega$ let k_n be the nth member of $\{k \mid h(f \restriction k) = 0\}$. Set $\sigma[f](n) = g(f \restriction (k_n + 1))$.

Suppose that there is a condition $((h, g), (\boldsymbol{t}, \boldsymbol{f})) \in$ Cohen$' *$ Hechler and $\boldsymbol{e} \in V$ such that $((h, g), (\boldsymbol{t}, \boldsymbol{f})) \Vdash$ "there is $\boldsymbol{n}^*$ such that for every $n \geq \boldsymbol{n}^*$, $\boldsymbol{\sigma}$ [Hechler real] $(n) < \boldsymbol{e}$ [Hechler real] (n)".

Extending (h, g) if necessary, we may assume that for some t^* and n^*, $\boldsymbol{t} = \check{t}^*$ and $\boldsymbol{n}^* = \check{n}^*$.

Let us pick an elementary submodel N of a large enough fragment of V such that $\boldsymbol{e} \in N$. Then for every n there is a predense set $\tau_n \in N$ such that $N \models \tau_n$ dense open, and every $p \in \tau_n$ decides the value of $\boldsymbol{e}(n)$. Let $f^* \in V$ be Hechler over N. Define a set $\tau_n^* \in N[f^*]$ as in [3, 3.2], i.e., for every $\langle t, g\rangle \in \tau_n$ find $k < \omega$ so that $k \geq \operatorname{length} t$ and $f^*(m) > g(m)$ for every $m \geq k$. Take all possible extensions of $\langle t, g\rangle$ to conditions of the form $\langle t', g\rangle$ with t' of length k. Change g by f^* in each of them. Let τ_n^* be a set consisting of such conditions. Repeating the proof of Claim 3.2.1 from [3] we obtain the following:

Claim 4.4.1. *For every* $n < \omega$ *and* $\langle t_1, f_1 \rangle \in \text{Hechler}$ *there is* $s \geq t_1$ *such that* $\langle s, f^* \rangle \in \tau_n^*$ *and* $\langle s, f^* \rangle$, $\langle t_1, h_1 \rangle$ *are compatible.*

W.l.o.g. we can assume that $(h, g) \Vdash_{\text{Cohen}'} f(n) \geq \check{f}^*(n)$ for every $n \geq \text{length}(t^*)$. Let $I_n = \{\langle t, f^* \rangle \in \tau_n^* \mid t \geq t^*\}$.

Let n be above $\max(\text{length}(t^*), n^*) + 1$. We extend $\langle h, g \rangle$ and t^* to $\langle h_0, g_0 \rangle$ and t_0 so that $\langle h_0, g_0 \rangle \geq \langle h, g \rangle$, $\langle h_0, g_0 \rangle \Vdash \langle t_0, f \rangle \geq \langle t^*, f \rangle$ and t_0 determines *exactly* the first $n-1$ members of $\{k \mid h_0(\text{Hechler} \restriction k) = 0\}$. Let k_{n-1} be the nth member of this set. The exactness (or t_0 is such of the minimal possible length) implies $\text{length}(t_0) = k_{n-1}$. Let us explain the idea of the rest of the proof by dealing with two particular cases.

Case 1. $t_0 \in I_n$.

Extend $\langle h_0, g_0 \rangle$ to a condition $\langle h_0', g_0' \rangle$ determining the value $f(k_{n-1})$. We choose some i above all the elements of the set $\text{rng}\, t_0 \cup \{f(k_{n-1})\} \cup \bigcup \{\text{rng}\, q \mid q \in \text{dom}\, h_0'\}$. Let $t_1 = t_0 \cup \{\langle k_{n-1}, i \rangle, \langle k_{n-1}+1, i+1 \rangle\}$. Extend h_0' to h_1 by adding $t_1 \restriction k_{n-1}+1$ and t_1 to its domain, and set $h_1(t_1 \restriction k_{n-1}+1) = 0$. Finally, we extend g_0' to g_1 by adding $t_1 \restriction k_{n-1}+1$ and t_1 to its domain and setting $g_1(t_1) =$ the value $\langle t_0, f^* \rangle$ forces on $\boldsymbol{e}(n)$. Then the condition $\langle \langle h_1, g_1 \rangle, \langle t_1, f \rangle \rangle$ is stronger than $\langle \langle h_0, g_0 \rangle, \langle t_0, f \rangle \rangle$ and it forces "$\boldsymbol{\sigma}[\text{Hechler}](n) = \boldsymbol{e}[\text{Hechler}](n)$" which is impossible.

Case 2. *There are* $m > k_{n-1}$ *and* $\langle t_0^i \mid t_0^i \in {}^m\omega,\ i < \omega \rangle$ *such that*

(a) $t_0^i \geq t_0$,

(b) $t_0^i(\text{length}\, t_0) \geq i$,

(c) $\langle t_0^i, f^* \rangle \in I_n$.

We extend first $\langle h_0, g_0 \rangle$ to a condition $\langle h_0', g_0 \rangle$ determining $f \restriction m+1$. Then we shall pick $i < \omega$ above all the elements of the set $\text{rng}\, t_0 \cup \{f(m)\} \cup \bigcup \{\text{rng}\, q \mid q \in h_0'\}$. Extend h_0' to h_1 by adding all the initial segments s of t_0^i such that $t_0 < s \leq t_0^i$ and setting $h_1(s) = 1$. Extend g_0' to g_1 by adding $t_0^i \restriction (k_{n+1}+1)$ to its domain. We already may reach the contradiction by setting $g_i(t_0^i \restriction (k_{n-1}+1))$ to be the value $\langle t_0^i, f^* \rangle$ forces on $\boldsymbol{e}(n-1)$, but let us instead proceed on one more step and deal with the situation which will arise in the general case. So we extend t_0^i to t_1 by adding $\{(m, \max(\text{rng}\, t_0^i)+1\rangle, \langle m+1, \max(\text{rng}\, t_0^i)+2\rangle\}$. Extend h_1 to h_1^* by adding $t_1 \restriction m+1$ and t_1 to its domain and setting $h_1^*(t_1 \restriction m+1) = 0$. We extend g_1 to g_1^* as follows: add t_1 to its domain and set $g_1^*(t_1)$ to be the value $\langle t_0^i, f^* \rangle$ forces on $\boldsymbol{e}(n)$. As in Case 1, we obtain now a contradiction. In order to deal with the general case we introduce a notion of rank.

Definition 4.4.2. Let $t \in {}^{\omega>}\omega$ and $A \subseteq {}^{\omega>}\omega$. Define by induction $\mathrm{rk}(t, A)$.

(a) $\mathrm{rk}(t, A) = 0$ iff for some $k < \mathrm{length}(t)$, $t \restriction k \in A$.

(b) $\mathrm{rk}(t, A) = \alpha$ iff there is no $\beta < \alpha$ such that $\mathrm{rk}(t, A) = \beta$, but there are $m < \omega$ and $\langle t_k \mid k < \omega \rangle$ such that for every $k < \omega$: $t \leqslant t_k$, $t_k \in {}^m\omega$, $t_k(\mathrm{length}(t)) \geqslant k$, and for some $\beta_k < \alpha$, $\mathrm{rk}(t_k, A) = \beta_k$.

Claim 4.4.3. *Let $(t^*, f^*) \in$ Hechler, $I \subseteq \{(t, f^*) \mid (t, f^*) \in \text{Hechler}\}$ be a predense set in the Hechler forcing and $A = \{t \mid (t, f^*) \in I\}$. Then $\mathrm{rk}(t^*, A) < \omega_1$.*

Proof. Suppose otherwise. Let $A^* = \{t \mid t \geqslant t^* \text{ and } \mathrm{rk}(t, A) < \omega_1\}$. Consider a set $S = \{q \mid q \in {}^{\omega>}\omega, \langle q, f^* \rangle \geqslant \langle t^*, f^* \rangle$ and there is no $t \in A^*$ such that $\mathrm{length}(t) \leqslant \mathrm{length}(q)$ and for every $i < \mathrm{length}(t)$, $t(i) < q(i)\}$. Then $t^* \in S$ and S is closed under initial segments. Notice that S has no ω-branches. Since, if $\langle q_i \mid i < \omega \rangle$ is such a branch, then $\langle t^*, \bigcup_{i<\omega} q_i \rangle \geqslant \langle t^*, f^* \rangle$ so for some $t \in A$, $\langle t, \bigcup_{i<\omega} q_i \rangle \geqslant \langle t^*, \bigcup_{i<\omega} q_i \rangle$ (it is impossible to have $t \leqslant t^*$, since then $\mathrm{rk}(t^*, A) = 0$). But then there is $i_0 < \omega$ such that $i_0 = \mathrm{length}\, t$; $l \leqslant \mathrm{length}\, t$ implies $t(l) \geqslant q_{i_0}(l)$ which contradicts the definition of S. Hence S has a maximal element. Let q be such an element. Then $q^\frown\langle k \rangle \notin S$ for every k, actually only k's above $\max(\mathrm{rng}\, q)$ are relevant, but for simplification of the notation we shall ignore these. So there is $t_k \in A^*$ witnessing $q^\frown\langle k \rangle \notin S$. Then $\mathrm{length}(t_k) \leqslant \mathrm{length}(q) + 1$. It is impossible to have strict inequality, since then t_k will witness $q \notin S$. So $\mathrm{length}(t_k) = \mathrm{length}(q) + 1$. Now, there are $\bar{t} \in {}^m\omega$, $\mathrm{length}\, t^* \leqslant m \leqslant \mathrm{length}\, q$ and infinite set $B \subseteq \omega$ such that for every $k \in B$, $t_k \restriction m = \bar{t}$ and $\{t_k(m) \mid k \in B\}$ are pairwise distinct. It is easy to rename B now and $\langle t_k \mid k \in B \rangle$ in order to obtain a sequence $\langle t'_k \mid k < \omega \rangle$ witnessing $\mathrm{rk}(\bar{t}) < \omega_1$. But then $\bar{t}$ will witness $q \notin S$. Contradiction. □ 4.4.3

Let A be $\{t \mid (t, f^*) \in I_n\}$. By Claim 4.4.3, then $\mathrm{rk}(t_0, A) < \omega_1$. The cases $\mathrm{rk}(t_0, A) = 0$ or 1 correspond to Cases 1 and 2 above. We shall define by induction a sequence $t_1, t_2, \ldots, t_m, \ldots$ such that $t_0 < t_1 < t_2 < \cdots < t_m < \cdots$ and $\mathrm{rk}(t_0, A) > \mathrm{rk}(t_1, A) > \cdots$. Since ordinals are well-founded, after finitely many stages the process will terminate. Let us describe the construction of t_1. The rest is similar. Let $\mathrm{rk}(t_0, A) = \alpha$. By the definition of the rank, there are $m > \mathrm{length}(t_0)$ and $\langle t_0^i \mid i < \omega, t_0^i \in {}^m\omega \rangle$ such that

(1) $t_0^i \geqslant t_0$,

(2) $t_0^i(\mathrm{length}(t_0)) \geqslant i$, and

(3) $\mathrm{rk}(t_0^i) < \alpha$.

Extend now $\langle h_0, g_0 \rangle$ to a stronger condition $\langle h'_0, g'_0 \rangle$ determining $f \restriction m$. Pick $i < \omega$ to be above all the elements of the set $\mathrm{rng}\, t_0 \cup \mathrm{rng} f \restriction m \cup \bigcup \{q \mid q \in \mathrm{dom}\, h'_0\}$. Let t_1 be t_0^i. Extend $\langle h'_0, g'_0 \rangle$ to $\langle h_1, g_1 \rangle$ as follows. Add each s, $t_0 < s \leqslant t_1$, to the domain of h'_0 and set $h_1(s) = 1$. Add $t_1 \restriction \mathrm{length}(t_0) + 1$ to the domain of g_1 and give to $g_1(t_1 \restriction \mathrm{length}(t_0) + 1)$ any value.

Now we continue and using t_1, $\langle h_1, g_1 \rangle$ define t_2, $\langle h_2, g_2 \rangle$ and so on. After finitely many stages rank 1 or rank 0 is reached. Then the contradiction is derived as in Cases 1 or 2. □

5. Open problems

Let us list here the problems mentioned in this paper.

Problem 1. Is the set $\{\mathrm{cov}(\mu, \kappa, \theta, 2) \mid \theta < \kappa\}$ always finite?

It is related to the Hajnal–Shelah Theorem saying that the set $\{\kappa^\theta \mid 2^\theta < \kappa\}$ is always finite. We conjecture that the answer is affirmative.

Problem 2 (Fremlin). Let κ be a real-valued measurable. Is there $\lambda < \kappa$ such that $\mathrm{cov}(\lambda, \aleph_1, \aleph_1, 2) > \kappa$?[1]

By 1.11, $\mathrm{cov}(\lambda, \aleph_1, \aleph_1, 2) < \kappa$ for every λ below the first fixed point of the $\aleph$-function.

Problem 3. Suppose that the forcing with a κ-complete ideal over κ is isomorphic to the forcing for adding λ-Cohen or random reals. Is it possible that $2^{<\kappa} < \lambda < 2^\kappa$?

By Theorem 2.5, $\lambda^\tau = 2^\kappa$ for the least τ such that $2^\tau = 2^{<\kappa}$. But we do not know even the simplest case: $2^{\aleph_0} = 2^{<\kappa} = \kappa$, $\lambda = \kappa^{+\omega}$ and $2^\kappa = \kappa^{+\omega+1}$.

Let κ be a measurable cardinal and j be an elementary embedding of a measure over κ.

Problem 4. Is $|j(j(\tau))| = |j(\tau)|$ for every ordinal τ?

Problem 5. Is there an ordinal τ such that $j(\tau) > \tau$ and $j(\tau)$ is a cardinal?

Problem 6. It is possible to have a κ-complete ω_1-saturated ideal over κ such that

$$\emptyset \Vdash \text{“}j(\kappa) \text{ is a cardinal”?}$$

or even $\emptyset \Vdash$ “$j(\kappa) = \kappa^{++}$”?, where j is the generic embedding.

If we replace “ω_1-saturated” by “precipitous” then [2] provides the affirmative answer.

Problem 7. Can the forcing with a σ-ideal be isomorphic to a Borel forcing notion, or to one having a simple absolute definition?

This question was raised in [3]. Note that the proof for the Hechler real in Section 4 may be used for every simple forcing for which it is possible to define the notion of rank.

[1] “No”, to appear in Shelah [18].

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