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More on the revised GCH and the black box

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Abstract

We strengthen the revised GCH theorem by showing, e.g., that for $\lambda = \mathrm{cf}(\lambda) > \beth_\omega$, for all but finitely many regular $\kappa < \beth_\omega$, it holds that "λ is accessible on cofinality κ" in some weak sense (see below).

As a corollary, $\lambda = 2^\mu = \mu^+ > \beth_\omega$ implies that the diamond holds on λ when restricted to cofinality κ for all but finitely many $\kappa \in \mathrm{Reg} \cap \beth_\omega$.

We strengthen previous results on the black box and the middle diamond: previously it was established that these principles hold on $\{\delta : \delta < \lambda, \mathrm{cf}(\delta) = (\beth_n)^+\}$ for sufficiently large n; here we succeed in replacing a sufficiently large $\beth_n$ with a sufficiently large $\aleph_n$.

The main theorem, concerning the accessibility of λ on cofinality κ, Theorem 3.1, implies as a special case that for every regular $\lambda > \beth_\omega$, for some $\kappa < \beth_\omega$, we can find a sequence $\langle \mathcal{P}_\delta : \delta < \lambda \rangle$ such that $u \in \mathcal{P}_\delta \Longrightarrow \sup u = \delta$ & $|u| < \beth_\omega$, $|\mathcal{P}_\delta| < \lambda$, and we can fix a finite set $\mathfrak{d}$ of "exceptional" regular cardinals $\theta < \beth_\omega$ so that if $A \subseteq \lambda$ satisfies $|A| < \beth_\omega$, there is a pair-coloring $\mathbf{c} : [A]^2 \to \kappa$ so that for every $\mathbf{c}$-monochromatic $B \subseteq A$ with no last element, letting $\delta := \sup B$ it holds that $B \in \mathcal{P}_\delta$—*provided* that $\theta := \mathrm{cf}(\delta)$ is not one of the finitely many "exceptional" members of $\mathfrak{d}$.

Keywords: Revised GCH; Middle diamond; Black box

0. Introduction

The main result of this paper is defining for any cardinal λ a set $\mathfrak{d}_0(\lambda)$ of regular cardinals $< \lambda$ such that for the strong limit $\theta < \lambda$ it holds that $\theta \cap \mathfrak{d}_0(\lambda)$ is finite, and for every $\kappa \in \mathrm{Reg} \cap \theta \backslash \mathfrak{d}_0(\lambda)$, in some sense λ has not too many subsets of cardinality κ. It is our main aim here to use this to show: if $\mathrm{cf}(\lambda) > \mu$ and $\kappa \in \mathrm{Reg} \cap \mu$ satisfies $\lambda = \sup\{\alpha : \kappa \notin \mathfrak{d}_0(|\alpha|)\}$ <u>then</u> λ has a "good" sequence $\langle \mathcal{P}_\alpha : \alpha < \lambda \rangle$, $\mathcal{P}_\alpha \subseteq [\alpha]^{\leq \kappa}$ and if $\lambda = \lambda^\mu$, more (see 3.5, 3.8).

This gives as a main consequence that: if $\mu \geq \theta$, $\lambda = \mathrm{cf}(2^\mu)$ <u>then</u> (λ, κ) has the BB (black box) and (a version of) the middle diamond for all but finitely many $\kappa \in \mathrm{Reg}$ satisfying $\beth_\omega(\kappa) \leq \mu$. Also $\lambda = 2^\mu = \mu^+ > \theta \Rightarrow \lambda$ has the diamond on cofinality κ for all regular κ for which $\beth_\omega(\kappa) < \lambda$ except finitely many.

So this is part of pcf theory [17] continuing in particular [21]. The proof of the main theorem here is adapted to be a shorter proof of the revised GCH theorem from [21] in Section 1 we present a short and self-contained proof of the revised GCH and discuss its potential extensions.

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By pcf theory [17,21] a worthwhile choice of a definition of power for $\kappa < \lambda$ regular is $\lambda^{[\kappa]}$ (or $\lambda^{<\kappa>}$), the minimal cardinality of a family of subsets of λ each of cardinality $\le \kappa$ such that any other subset of λ of cardinality κ is equal to (or is contained in) the union of $< \kappa$ members of the family (see Definition 1.2).

This gives a good partition of the exponentiation as $\lambda^\kappa = \lambda \Leftrightarrow 2^\kappa \le \lambda \& (\forall \sigma)(\sigma = \mathrm{cf}(\sigma) \le \kappa \Rightarrow \lambda^{<\sigma>} = \lambda)$. So GCH is equivalent to: κ regular $\Rightarrow 2^\kappa = \kappa^+$ and [$\kappa < \lambda$ are regular $\Rightarrow \lambda^{<\kappa>} = \lambda$].

Let $\mathfrak{d}^+(\lambda) = \{\kappa : \kappa$ be regular $< \lambda$ and $\lambda < \lambda^{<\kappa>}\}$. In [21] the revised GCH theorem is proved:

⊛ if $\lambda > \beth_\omega$ then $\mathfrak{d}^+(\lambda) \cap \beth_\omega$ is bounded, i.e., $\lambda = \lambda^{<\kappa>}$ for every sufficiently large regular $\kappa < \beth_\omega$.

We can replace $\beth_\omega$ in the RGCH theorem by any strong limit cardinal θ.

The advances in pcf theory reveal several natural hypotheses. The Strong Hypothesis ($\mathrm{pp}(\mu) = \mu^+$ for every singular μ) is very nice, but it implies the SCH and hence does not follow from ZFC. The status of the Weak Hypothesis (somewhat more than $\{\mu : \mathrm{cf}(\mu) < \mu < \lambda \le \mathrm{pp}(\mu)\}$ is at most countable) is not known but we are sure that its negation is consistent though it has large consistency strength, but not sure about $(\forall \mathfrak{a})(|\mathfrak{a}| \ge |\mathrm{pcf}(\mathfrak{a})|)$. Still better than ⊛ would be the following (which we believe, but do not know, particularly (2)):

Conjecture 0.1. (1) *for every* λ, $\mathfrak{d}^+(\lambda)$ *is finite, or at least*
(2) *for every strong limit* μ, $\lambda \ge \mu \Rightarrow \mathfrak{d}^+(\lambda) \cap \mu$ *is finite.*

Here we define a set $\mathfrak{d}_0(\lambda) \cap \theta$ whose finiteness and other results on it (see 3.1 and consequences) form a step in the right direction and suffice to improve the results of [7]. In particular, the results allow us to use "$\kappa = \aleph_n$ for some n" rather than "some regular $\kappa < \beth_\omega$". This looks like the right direction in infinite abelian group theory (as there are non-free almost κ-free abelian groups of cardinality κ when $\kappa = \aleph_n$). So we can hope to get the right objects in each cardinality $\aleph_n$, whereas consistently they may not exist for arbitrary $\kappa = \mathrm{cf}(\kappa) < \beth_\omega$. However, at the moment the results here do not suffice to get e.g. "there is an $\aleph_n$-free abelian group G for which $\mathrm{Hom}(G, \mathbb{Z}) = \{0\}$"; for this we need $\kappa = \aleph_0 \vee \kappa = \aleph_1$. It is quite "hard" for this to fail for every λ; see [25].

The work here continues also previous work on $I[\lambda]$. By [10], if $\lambda = \mu^+$ and μ is strong limit singular, then for some $A \in I[\lambda]$ and some $\mathbf{c} : [\mu^+]^2 \to \mathrm{cf}(\mu)$, if $B \subseteq \mu$ and $\mathbf{c} \restriction [B]^2$ is constant (or just has bounded range), $\delta = \sup(B)$, $\mathrm{cf}(\delta) \ne \mathrm{cf}(\mu)$, then $\delta \in A$.

By Džamonja and Shelah [2], using [21], if $\lambda = \mu^+$ and μ is strong limit singular, then for some $\kappa < \mu$, for some $A \in I[\lambda]$, if for every $A' \subseteq A$, $|A'| < \theta$ for some $\mathbf{c} : [A'] \to \kappa$, we have: if $B \subseteq A'$, $\mathbf{c} \restriction [B]$ is constant, $\delta = \sup[B]$, $\mathrm{cf}(\delta) > \kappa$, then $\delta \in A$. By [20, 5.20], conditions on T_D help to prove that $I[\lambda]$ is "large".

On pcf theory and versions of the RGCH without the axiom of choice, see [19,9] and more in [24].

We tried to make this paper as self-contained as is reasonably possible.

Definition 0.2. (1) For an ideal J on a set X:

(a) $J^+ = \mathcal{P}(X) \backslash J$; we agree that J determines X so $X = \mathrm{Dom}(J)$ — this is an abuse of notation when $\cup\{A : A \in J\} \subset X$ but usually clear in the context;

(b) for a binary relation R on Y and an ideal J on X and for $f, g \in {}^X Y$, let $f R_J g$ mean $\{t \in X : \neg f(t) R g(t)\} \in J$; the relations we shall use are $=, \ne, <, \le$.

(2) If D is a filter on X, J the dual ideal on X (i.e., $J = \{X \backslash A : A \in D\}$) we may replace J by D in the notation $f R_J g$.

(3) Let $(\forall^J t)\varphi(t)$ mean $\{t : \neg\varphi(t)\} \in J$; similarly $\exists^J, \forall^D, \exists^D$.

(4) Let $S^\lambda_\kappa = \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ and $S^\lambda_{<\kappa} = \{\delta < \lambda : \mathrm{cf}(\delta) < \kappa\}$.

Definition 0.3. (1) Let $\bar{A} = \langle A_i : i \in X \rangle$, D a filter on X, and for simplicity first assume $i \in X \Rightarrow A_i \ne \emptyset$. We let

(a) $T^0_D(\bar{A}) = \sup\{|\mathcal{F}| : \mathcal{F} \subseteq \Pi\bar{A}$ and $f_1 \ne f_2 \in \mathcal{F} \Rightarrow f_1 \ne_D f_2\}$;

(b)

$$T^1_D(\bar{A}) = \mathrm{Min}\{|\mathcal{F}| : \text{(i)} \quad \mathcal{F} \subseteq \Pi(\bar{A}) \tag{1}$$

$$\text{(ii)} \quad f_1 \ne f_2 \in \mathcal{F} \Rightarrow f_1 \ne_D f_2 \tag{2}$$

$$\text{(iii)} \quad \mathcal{F} \text{ maximal under (i) + (ii)}\}; \tag{3}$$

(c) $T^2_D(\bar{A}) = \mathrm{Min}\{|\mathcal{F}| : \mathcal{F} \subseteq \Pi\bar{A}$ and for every $f_1 \in \Pi\bar{A}$, for some $f_2 \in \mathcal{F}$ we have $\neg(f_1 \ne_D f_2)\}$.

(2) If $\{i : A_i = \emptyset\} \in J$ then we let $T^\ell_D(\bar{A}) = T^\ell_{D\restriction Y}(\bar{A} \restriction Y)$ where $Y = \{i : A_i \neq \emptyset\}$; note that if $\{i : A_i \neq \emptyset\} \in J$ then $T^\ell_D(\bar{A}) = 0$.
(3) For $f \in {}^\kappa\mathrm{Ord}$ and $\ell < 3$ let $T^\ell_D(f)$ mean $T^\ell_D(\langle f(\alpha) : \alpha < \kappa\rangle)$.
(4) If $T^0_D(\bar{A}) = T^1_D(\bar{A}) = T^2_D(\bar{A})$ then we let $T_D(\bar{A}) = T^\ell_D(\bar{A})$ for $\ell < 3$; similarly $T_D(f)$; we say that $\mathcal{F}$ witnesses $T_D(\bar{A}) = \lambda$ if it is as in the definition of $T^1_D(\bar{A}) = \lambda$; similarly $T^2_D(f)$.

Remark 0.4. Actually the case $\bar{A} = \bar{\lambda} = \langle \lambda_\alpha : \alpha < \kappa\rangle$ is enough, so we concentrate on it.

Claim 0.5. (0) *If $D_0 \subseteq D_1$ are filters on κ then $T^\ell_{D_0}(\bar{\lambda}) \le T^\ell_{D_1}(\bar{\lambda})$ for $\ell = 0, 2$.*
(1) *$T^2_D(\bar{\lambda}) \le T^1_D(\bar{\lambda}) \le T^0_D(\bar{\lambda})$; in particular $T^\ell_D(\bar{\lambda})$ is well defined.*
(2) *If $(\forall i)\lambda_i > 2^\kappa$ then $T^0_D(\bar{\lambda}) = T^1_D(\bar{\lambda}) = T^2_D(\bar{\lambda})$, so the supremum in 0.3(a) is obtained (so, e.g., $T^0_D(\bar{\lambda}) > 2^\kappa$ suffice; also $(\forall i)\lambda_i \ge 2^\kappa$ suffice).*

Proof. (0) Check.
(1) $T^1_D(\bar{A})$ is well defined as every family $\mathcal{F}$ satisfying clauses (i) + (ii) there can be extended to one satisfying (i) + (ii) + (iii), so as $\emptyset$ satisfies (i) + (ii) really $T^1_D(\bar{A})$ is well defined. If $\mathcal{F}$ exemplifies the value of $T^1_D(\bar{\lambda})$, it also exemplifies $T^2_D(\bar{\lambda}) \le |\mathcal{F}|$; hence easily $T^2_D(\bar{\lambda}) \le T^1_D(\bar{\lambda})$ and so $T^2_D(\bar{\lambda})$ is well defined. In the definition of $T^0_D(\bar{\lambda})$ the Min is taken over a non-empty set (as maximal such $\mathcal{F}$ exists), so $T^0_D(\bar{\lambda})$ is well defined.

Lastly, if $\mathcal{F}$ exemplifies the value of $T^1_D(\bar{\lambda})$ it also exemplifies $T^0_D(\bar{\lambda}) \ge |\mathcal{F}|$, so $T^1_D(\bar{\lambda}) \le T^0_D(\bar{\lambda})$.
(2) Let μ be 2^κ. Assume that the desired conclusion fails so $T^2_D(\bar{\lambda}) < T^0_D(\bar{\lambda})$, so there is $\mathcal{F}_0 \subseteq \Pi\bar{\lambda}$ such that $[f_1 \neq f_2 \in \mathcal{F}_0 \Rightarrow f_1 \neq_D f_2]$, and $|\mathcal{F}_0| > T^2_D(\bar{\lambda}) + \mu$ (by the definition of $T^0_D(\bar{\lambda})$). Also there is $\mathcal{F}_2 \subseteq \Pi\bar{\lambda}$ exemplifying the value of $T^2_D(\bar{\lambda})$. For every $f \in \mathcal{F}_0$ there is $g_f \in \mathcal{F}_2$ such that $\neg(f \neq_D g_f)$ (by the choice of $\mathcal{F}_2$). As $|\mathcal{F}_0| > T^2_D(\bar{\lambda}) + \mu$, for some $g \in \mathcal{F}_2$ the set $\mathcal{F}^* =: \{f \in \mathcal{F}_0 : g_f = g\}$ has cardinality $> T^2_D(\bar{\lambda}) + \mu$. Now for each $f \in \mathcal{F}^*$ let $A_f = \{i < \kappa : f(i) = g(i)\}$ clearly $A_f \in D^+$. Now $f \mapsto A_f/D$ is a function from $\mathcal{F}^*$ into $\mathcal{P}(\kappa)/D$; hence, as $\mu \ge |\mathcal{P}(\kappa)/D|$, it is not one to one (by cardinality consideration), so for some $f' \neq f''$ from $\mathcal{F}^*$ (hence form $\mathcal{F}_0$) we have $A_{f'}/D = A_{f''}/D$; but so

$$\{i < \kappa : f'(i) = f''(i)\} \supseteq \{i < \kappa : f'(i) = g(i)\} \cap \{i < \kappa : f''(i) = g(i)\} = A_{f'} \mod D$$

and hence is $\neq \emptyset \mod D$, so $\neg(f' \neq_D f'')$, contradicting the choice of $\mathcal{F}_0$. □

Claim 0.6. *Let J be a σ-complete ideal on κ.*

(1) *If $\bar{A} = \langle A_i : i < \kappa\rangle, \bar{\lambda} = \langle \lambda_i : i < \kappa\rangle, \lambda_i = |A_i|$ then $T^\ell_J(\bar{A}) = T^\ell_J(\bar{\lambda})$ and if $A \in J, B = \kappa\backslash A$ then $T^\ell_J(\bar{\lambda}) = T^\ell_{J\restriction B}(\bar{\lambda} \restriction B)$.*
(2) *$T_J(\bar{\lambda}) > 2^\kappa$ iff $(\forall^J t)(\lambda_t > 2^\kappa)$; note that $T_J(\bar{\lambda}) > 2^\kappa$ includes its being well defined.*
(3) *$T^\ell_J(\bar{\lambda}^1) \le T^\ell_J(\bar{\lambda}^2)$ if $(\forall^J t)(\lambda^1_t \le \lambda^2_t)$.*
(4) *If $\mathrm{Dom}(J) = \cup\{A_\varepsilon : \varepsilon < \zeta\}, \zeta < \sigma$ and $\lambda_i \ge 2^\kappa$ for $i < \kappa$ then $T^0_J(\bar{\lambda}) = \mathrm{Min}\{T^0_{J\restriction A_\varepsilon}(\bar{\lambda} \restriction A_\varepsilon) : \varepsilon < \zeta$ and $A_\varepsilon \in J^+\}$.*
(5) *In part (4) if $\bar{\lambda} = \langle \lambda_i : i < \kappa\rangle$ then the following are equivalent:*
   (i) *for every $f \in \prod_{i<\kappa} \lambda_i$ we have $T_J(f) < \lambda$;*
   (ii) *for some $\varepsilon < \zeta, A_\varepsilon \notin J$ and for every $f \in \prod_{i<\kappa} \lambda_i$ we have $T_{J\restriction A_\varepsilon}(f \restriction A_\varepsilon) < \lambda$.*

Proof. For example (and the one we use):
(4) Let $A'_\varepsilon = A_\varepsilon \backslash \cup\{A_\xi : \xi < \varepsilon\}$ for $\varepsilon < \zeta$.

First assume that $\mathcal{F} \subseteq \Pi\bar{\lambda}$ and $f_1 \neq f_2 \in \mathcal{F} \Rightarrow f_1 \neq_J f_2$. Then for each $\varepsilon < \zeta$ satisfying $A_\varepsilon \in J^+$, clearly $\mathcal{F}^{[\varepsilon]} = \{f \restriction A_\varepsilon : f \in \mathcal{F}\}$ satisfies $|\mathcal{F}^{[\varepsilon]}| = |\mathcal{F}|$ as $f \mapsto f \restriction A_\varepsilon$ is one to one by the assumption on $\mathcal{F}$ and $\mathcal{F}^{[\varepsilon]} \subseteq \prod_{i\in A_\varepsilon} \lambda_i$; so $|\mathcal{F}| = |\mathcal{F}^{[\varepsilon]}| \le T^0_{J\restriction A_\varepsilon}(\bar{\lambda} \restriction A_\varepsilon)$. As this holds for every $\varepsilon < \zeta$ for which $A_\varepsilon \in J^+$ we get $|\mathcal{F}| \le \mathrm{Min}\{T^0_{J\restriction A_\varepsilon}(\bar{\lambda} \restriction A_\varepsilon) : \varepsilon < \zeta, A_\varepsilon \in J^+\}$. By the demand on $\mathcal{F}$ we get the inequality $\le$ in part (4). Second, assume $\mu < \mathrm{Min}\{T^0_{J\restriction A_\varepsilon}(\bar{\lambda} \restriction A_\varepsilon) : \varepsilon < \zeta, A_\varepsilon \in J^+\}$. So for each such ε there is $\mathcal{F}_\varepsilon \subseteq \prod_{i\in A_\varepsilon} \lambda_i$ such that

$f \neq g \in \mathcal{F}_\varepsilon \Rightarrow f \neq_{J \restriction A_\varepsilon} g, |\mathcal{F}_\varepsilon| \geq \mu^+$. For each $\varepsilon < \zeta$ let $f^\varepsilon_\alpha \in \mathcal{F}_\varepsilon$ be pairwise distinct for $\alpha < \lambda$, and define $f_\alpha \in \Pi\bar{\lambda}$ for $\alpha < \mu^+$ as follows: $f_\alpha \restriction A'_\varepsilon = f^\varepsilon_\alpha$ when $A_\varepsilon \in J^+$; $f_\alpha \restriction A'_\varepsilon$ is zero otherwise.

Now check. $\square$

Definition 0.7. For λ regular uncountable and stationary $S \subseteq \lambda$ let $(D\ell)_{\lambda,S}$ mean that we can find $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha \in S\rangle, \mathcal{P}_\alpha \subseteq \mathcal{P}(\alpha)$ of cardinality $< \lambda$ such that for every $A \subseteq \lambda$ the set $\{\alpha \in S : A \cap \alpha \in \mathcal{P}_\alpha\}$ is stationary.

Definition 0.8. For λ regular uncountable let $I[\lambda]$ be the family of sets $S \subseteq \lambda$ which have a witness $(E, \bar{\mathcal{P}})$ for $S \in I[\lambda]$, which means

$(*)$ E is a club of λ, $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \lambda\rangle, \mathcal{P}_\alpha \subseteq \mathcal{P}(\alpha), |\mathcal{P}_\alpha| < \lambda$, and for every $\delta \in E \cap S$ there is an unbounded subset C of δ of order type $< \delta$ satisfying $\alpha \in C \Rightarrow C \cap \alpha \in \bigcup_{\beta<\delta} \mathcal{P}_\beta$.

Claim 0.9 ([15]). (1) *For λ regular uncountable, $S \in I[\lambda]$ iff there is a pair $(E, \bar{a})$, E a club of λ, $\bar{a} = \langle a_\alpha : \alpha < \lambda\rangle, a_\alpha \subseteq \alpha$ such that $\beta \in a_\alpha \Rightarrow a_\beta = a_\alpha \cap \beta$ and $\delta \in E \cap S \Rightarrow \delta = \sup(a_\delta) > \mathrm{otp}(a_\delta)$ (or even $\delta \in E \cap S \Rightarrow \delta = \sup(a_\delta)), \mathrm{otp}(a_\delta) = \mathrm{cf}(\delta) < \delta$.*
(2) *If $\kappa^+ < \lambda$ and λ, κ are regular, then for some stationary $S \in I[\lambda]$ we have $\delta \in S \Rightarrow \mathrm{cf}(\delta) = \kappa$.*

Claim 0.10. (1) *Assume that $f_\alpha \in {}^\kappa\mathrm{Ord}$ for $\alpha < \lambda, \lambda = (2^\kappa)^+$ or just $\lambda = \mathrm{cf}(\lambda)$ and $(\forall\alpha < \lambda)(|\alpha|^\kappa < \lambda)$ and $S_1 \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) > \kappa\}$ is stationary. Then for some stationary $S_2 \subseteq S_1$ we have: for each $i < \kappa$ the sequence $\langle f_\alpha(i) : \alpha \in S_2\rangle$ is either constant or strictly increasing.*
(2) *If D is a filter on κ and $f_\alpha \in {}^\kappa\mathrm{Ord}$ for $\alpha < \delta$ is $<_D$-increasing and $\mathrm{cf}(\delta) > 2^\kappa$ then $\langle f_\alpha : \alpha < \delta\rangle$ has a $<_D$-eub $f_\delta \in {}^\kappa\mathrm{Ord}$, i.e.,*
(i) $\alpha < \delta \Rightarrow f_\alpha \leq_D f_\delta$,
(ii) $f' \in {}^\kappa\mathrm{Ord}$ & $f' <_D \mathrm{Max}\{f, 1_\kappa\}$*; then* $(\exists\alpha < \delta)(f' <_D f_\alpha)$.

Proof. Part (1) follows easily from the Erdős–Rado partition theorem (see 14.5 in [5]) as follows: color (α, β) for $\alpha < \beta$ in S_1 by the least $i < \kappa$ such that $f_\alpha(i) > f_\beta(i)$ if there is such $i < \kappa$ and color (α, β) by κ otherwise. Since for every color $i < \kappa$ there is no homogeneous set with color i of cardinality ω, there is a homogeneous stationary set $S' \subseteq S_1$ with color κ. Since for each $i < \kappa$ there is club E_i so that either $f_\alpha(i)$ is constant on $S' \cap E_i$ or for every $\alpha < \beta$ in $E_i \cap S'$ it holds that $f_\alpha(i) < f_\beta(i)$, by letting $S_2 = S_1 \cap \bigcap_{i<\kappa} E_i$ we finish the proof of (i).

Part (2) is Remark 1.2A on page 44, which follows from the pcf Trichotomy Theorem, which is Claim 1.2 on p. 43 of [17]. $\square$

Observation 0.11. *Assume that J, J_1, J_2 are ideals on κ and $J = J_1 \cap J_2$. If $f \in {}^\kappa(\mathrm{Ord} \setminus \omega)$ then $T^\ell_J(f) = \mathrm{Min}\{T^\ell_{J_1}(f), T^\ell_{J_2}(f)\}$.*

Proof. As $J \subseteq J_\ell$ clearly $T_J(f) \leq T_{J_\ell(f)}$ for $\ell = 1, 2$. This proves the inequality $\leq$ in the observation. For the other inequality use pairing functions for each $i < \kappa$. $\square$

1. The revised GCH revisited

Here we give a proof of the RGCH which requires little knowledge; this is the main theorem of [21] — see also [22, Section 1]. The presentation is self-contained; in particular, the pcf theorem is not used (hence proofs of some pcf facts are repeated here in weak forms).

Definition 1.1. (1) For $\lambda \geq \theta \geq \sigma = \mathrm{cf}(\sigma)$ let $\lambda^{[\sigma,\theta]} = \mathrm{Min}\{|\mathcal{P}| : \mathcal{P} \subseteq [\lambda]^{\leq\theta}$; every $u \in [\lambda]^{\leq\theta}$ is the union of $< \sigma$ members of $\mathcal{P}\}$.
(2) Let $\lambda^{[\sigma]} = \lambda^{[\sigma,\sigma]}$.
(3) For $\lambda \geq \theta^{[\sigma,\kappa]}$ let $\lambda^{[\sigma,\kappa,\theta]} = \mathrm{Min}\{|\mathcal{P}| : \mathcal{P} \subseteq [\lambda]^{\leq\kappa}$ such that for every $u \subseteq \lambda$ of cardinality $\leq \theta$ we can find $i^* < \sigma$ and $u_i \subseteq u$ for $i < i^*$ such that $u = \cup\{u_i : i < i^*\}$ and $[u_i]^{\leq\kappa} \subseteq \mathcal{P}\}$.
(4) We may replace θ by $< \theta$ with the obvious meaning (also $< \kappa$).

The following is a relative of Definition 1.1 not used in Section 1 but mentioned in 1.3.

Definition 1.2. (1) For $\lambda \geq \theta \geq \mathrm{cf}(\sigma) = \sigma$ let $\lambda^{<\sigma,\theta>} = \mathrm{Min}\{|\mathcal{P}| : \mathcal{P} \subseteq [\lambda]^{\theta}$; every $u \in [\lambda]^{\leq \theta}$ is included in the union of $< \sigma$ members of $\mathcal{P}\}$.
(2) Let $\lambda^{<\sigma>} = \lambda^{<\sigma,\sigma>}$.
(3) For $\lambda \geq \theta^{<\sigma,\kappa>}$ let $\lambda^{<\sigma,\kappa,\theta>} = \mathrm{Min}\{|\mathcal{P}| : \mathcal{P} \subseteq [\lambda]^{\leq \kappa}$ such that for every $u \subseteq \lambda$ of cardinality $\leq \theta$ we can find $i^* < \sigma$ and $u_i \subseteq u$ for $i < i^*$ such that $u \subseteq \cup\{u_i : i < i^*\}$ and $(\forall v \in [u_i]^{\leq \kappa})(\exists w \in \mathcal{P})(v \subseteq w)\}$.
(4) We may replace θ by $< \theta$ with the obvious meaning (also $< \kappa$).

Observation 1.3. *Let $\lambda > \theta \geq \kappa \geq \sigma = \mathrm{cf}(\sigma)$.*
(1) $\lambda^{<\kappa>} \leq \lambda^{[\kappa]} \leq \lambda^{<\kappa>} + 2^{\kappa}$.
(2) $\lambda^{<\sigma,\theta>} \leq \lambda^{[\sigma,\theta]} \leq \lambda^{<\sigma,\theta>} + 2^{\theta}$ *(but see* (3)*).*
(3) *If* $\mathrm{cf}(\theta) < \sigma$ *then* $\lambda^{<\sigma,\theta>} = \Sigma\{\lambda^{<\sigma,\theta'>} : \sigma \leq \theta' < \theta\}$ *and* $\lambda^{[\sigma,\theta]} = \Sigma\{\lambda^{[\sigma,\theta']} : \sigma \leq \theta' < \theta\}$.
(4) $\lambda^{<\sigma,\kappa,\theta>} \leq \lambda^{[\sigma,\kappa,\theta]} \leq \lambda^{<\sigma,\kappa,\theta>} + 2^{\kappa}$.

Proof. Easy. □

The main claim of this section is

Claim 1.4. *Assume*

(a) $\aleph_0 < \sigma = \mathrm{cf}(\sigma) \leq \kappa < \partial \leq \theta$,
(b) *J is a σ-complete ideal on κ,*
(c) $\bar{\lambda} = \langle \lambda_i : i < \kappa \rangle$ *and* $\partial < \lambda_i$ *for any* $i < \kappa$,
(d) $T_J(\bar{\lambda}) = \lambda$,
(e) $\lambda_i^{[\partial,\theta]} = \lambda_i$ *for* $i < \kappa$ *(yes ∂ not ∂_i!),*
(f) *if* $\partial_i < \partial$ *for* $i < \kappa$ *then* $\prod_{i<\kappa} \partial_i < \partial$,
(g) $\theta = \theta^{\kappa}$ *and* $2^{\theta} \leq \lambda$.

<u>Then</u> $\lambda^{[\partial,\theta]} = \lambda$.

Remark 1.5. (1) We may consider using a μ^+-free family $\bar{f}$ (see Section 2).
(2) Actually we use less than $T_J^1(\bar{\lambda}) = \lambda$; we just use
  (a) there are $f_\alpha \in \prod_{i<\kappa} \lambda_i$ for $\alpha < \lambda$ such that $\alpha < \beta \Rightarrow f_\alpha \neq_J f_\beta$,
  (b) there are $f_\alpha \in \prod_{i<\kappa} \lambda_i$ for $\alpha < \lambda$ such that for every $f \in \prod_{i<\kappa} \lambda_i$ for some α, $\neg(f \neq_J f_\alpha)$.
(3) Actually, "$\aleph_0 < \sigma$" is not used here.

Proof. Let $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$ be pairwise J-different, $f_\alpha \in \prod_{i<\kappa} \lambda_i$ (i.e. $\alpha \neq \beta \Rightarrow \{i : f_\alpha(i) = f_\beta(i)\} \in J$).

For each $i < \kappa$ let $\mathcal{P}_i \subseteq [\lambda_i]^{\leq\theta}$ be of cardinality λ_i and witness $\lambda_i^{[\partial,\theta]} = \lambda_i$; that is: every $u \in [\lambda_i]^{\leq\theta}$ is the union of $< \partial$ members of $\mathcal{P}_i$; such a family exists by assumption (e). Let $M \prec (\mathcal{H}(\chi), \in)$ be of cardinality λ such that $\lambda + 1 \subseteq M$ and $\bar{f}$, $\langle \lambda_i : i < \kappa \rangle$, $\langle \mathcal{P}_i : i < \kappa \rangle$, J, $\mathcal{P}(\kappa)$ belong to M.

Let $\mathcal{P} = M \cap [\lambda]^{\leq\theta}$. We shall show that $\mathcal{P}$ exemplifies the desired conclusion. Now $\mathcal{P}$ is a family of $\leq \|M\| = \lambda$ subsets of λ each of cardinality $\leq \theta$; hence it is enough to show

(∗) if $u \in [\lambda]^{\leq\theta}$ then u is included in the union of $< \partial$ members of $\mathcal{P}$ (or equal to; equivalent here as $2^{\theta} \leq \lambda$ hence $u_1 \subseteq u_2 \in \mathcal{P} \Rightarrow u_1 \in \mathcal{P}$).

Proof of (∗): For every $i < \kappa$ let $u_i = \{f_\alpha(i) : \alpha \in u\}$; so $u_i \in [\lambda_i]^{\leq\theta}$, and hence we can find $\langle v_{i,j} : j < j_i \rangle$ such that $v_{i,j} \in \mathcal{P}_i$ and $u_i = \cup\{v_{i,j} : j < j_i\}$ and $0 < j_i < \partial$. For each $\eta \in \prod_{i<\kappa} j_i$ let

$$w_\eta = \{\alpha \in u : i < \kappa \Rightarrow f_\alpha(i) \in v_{i,\eta(i)}\}.$$

Clearly $u = \cup\{w_\eta : \eta \in \prod_{i<\kappa} j_i\}$ as for any $\alpha \in u$ for each $i < \kappa$ we can define $\varepsilon_i(\alpha) < j_i$ such that $f_\alpha(i) \in v_{i,\varepsilon_i(\alpha)}$ and let $\eta_\alpha = \langle \varepsilon_i(\alpha) : i < \kappa \rangle$, clearly $\eta_\alpha \in \prod_{i<\kappa} j_i$ and so $\alpha \in w_{\eta_\alpha}$. By the assumption (f) as $i < \kappa \Rightarrow j_i < \partial$, clearly

$|\prod_{i<\kappa} j_i| < \partial$ and hence it is enough to prove that $\eta \in \prod_{i<\kappa} j_i \Rightarrow w_\eta \in \mathcal{P}$. As $u \in M \wedge |u| \leq \theta \Rightarrow \mathcal{P}(u) \subseteq M$ it is enough to prove, for $\eta \in \prod_{i<\kappa} j_i$, that

⊛ w_η is included in some $w \in M \cap [\lambda]^{\leq\theta}$.

Proof of ⊛: As $i < \kappa \Rightarrow |\mathcal{P}_i| = \lambda_i$ and $T_J(\bar{\lambda}) = \lambda$ by 0.6 there is $\mathcal{G} \subseteq \prod_{i<\kappa} \mathcal{P}_i$ satisfying $|\mathcal{G}| = \lambda$ and $(\forall g \in \prod_{i<\kappa} \mathcal{P}_i)(\exists g' \in \mathcal{G})(\{i : g(i) = g'(i)\} \in J^+)$. As $\langle \mathcal{P}_i : i < \kappa\rangle \in M$ without loss of generality $\mathcal{G} \in M$ and as $\lambda+1 \subseteq M$ we have $\mathcal{G} \subseteq M$. Apply the choice of $\mathcal{G}$ to $\langle v_{i,\eta(i)} : i < \kappa\rangle \in \prod_{i<\kappa} \mathcal{P}_i$; so for some $g \in \mathcal{G} \subseteq M$ the set $B =: \{i < \kappa : v_{i,\eta(i)} = g(i)\}$ belongs to J^+. Clearly $B \in M$ (as $B \subseteq \kappa, \mathcal{P}(\kappa) \in M$ and $|\mathcal{P}(\kappa)| \leq 2^\kappa \leq \theta^\kappa \leq \lambda \subseteq M$) and hence $\langle v_{i,\eta(i)} : i \in B\rangle \in M$ hence $w = \{\alpha < \lambda$: for every $i \in B$ we have $f_\alpha(i) \in v_{i,\eta(i)}\}$ belongs to M. Now $|w| \leq \prod_{i\in B} |v_{i,\eta(i)}| \leq \theta^\kappa = \theta$ because $\alpha < \beta < \lambda \Rightarrow f_\alpha \neq_J f_\beta \Rightarrow f_\alpha \restriction B \neq f_\beta \restriction B$. Lastly $w_\eta \subseteq w$ as $\alpha \in w_\eta \,\&\, i < \kappa \Rightarrow f_\alpha(i) \in v_{i,\eta(i)}$, so we are done. □

Remark 1.6. We could have used instead the w above the set $w' = \{\alpha < \lambda : \{i : f_\alpha(i) \in v_{i,\eta(i)}\} \in J^+\}$.

To make this section free of quoting the pcf theorem we use the following definition.

Definition/Observation 1.7. (1) For a set $\mathfrak{a}$ of regular cardinals and $\sigma = \mathrm{cf}(\sigma) \leq \mathrm{cf}(\lambda)$ let[1]

$$J^\sigma_{<\lambda}[\mathfrak{a}] = \{\mathfrak{b} \subseteq \mathfrak{a} : \text{there is a set } \mathcal{F} \subseteq \Pi\mathfrak{b} \text{ of cardinality } < \lambda \tag{4}$$

$$\text{such that for every } g \in \prod \mathfrak{b} \text{ we can find } j < \sigma \text{ and} \tag{5}$$

$$f_i \in \mathcal{F} \text{ for } i < j \text{ satisfying } \theta \in \mathfrak{b} \Rightarrow (\exists i < j)(g(\theta) < f_i(\theta))\}. \tag{6}$$

(2) Clearly $J^\sigma_{<\lambda}[\mathfrak{a}]$ is a σ-complete ideal on $\mathfrak{a}$ but possibly $\mathfrak{a} \in J^\sigma_{<\lambda}[\mathfrak{a}]$.

Remark 1.8. In fact, if $\mathrm{Min}(\mathfrak{a}) > |\mathfrak{a}|$, $J^\sigma_{<\lambda}[\mathfrak{a}] = \{\mathfrak{b} \subseteq \mathfrak{a}$: $\mathrm{pcf}_{\sigma\text{-complete}}(\mathfrak{b}) \subseteq \lambda\} = \{\mathfrak{b} \subseteq \mathfrak{a} : \mathfrak{b}$ is the union of $< \sigma$ members of $J_{<\lambda}[\mathfrak{a}]\}$ can be proved, but this is irrelevant here.

For completeness we recall and prove Claims 1.9–1.12, used in the proof of 1.13, the revised GCH.

Claim 1.9. $\lambda = \lambda^{[\sigma,<\theta]}$ <u>when</u>

(a) $\lambda \geq 2^{<\theta} \geq \sigma = \mathrm{cf}(\sigma) > \aleph_0$ *and* $\mathrm{cf}(\theta) \notin [\sigma, \theta)$,
(b) *for every set* $\mathfrak{a} \subseteq \mathrm{Reg} \cap \lambda^+ \backslash \theta$ *of cardinality* $< \theta$ *we have* $\mathfrak{a} \in J^\sigma_{<\lambda^+}[\mathfrak{a}]$.

Proof. Let χ be large enough; choose $M \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ of cardinality λ where $<^*_\chi$ is any well ordering of $\mathcal{H}(\chi)$ such that $\lambda + 1 \subseteq M$ and let $\mathcal{P} = M \cap [\lambda]^{<\theta}$; we shall prove that $\mathcal{P}$ exemplifies $\lambda = \lambda^{[\sigma,<\theta]}$.

Clearly $\mathcal{P} \subseteq [\lambda]^{<\theta}$ has cardinality λ so let $u \in [\lambda]^{<\theta}$ and as $2^{<\theta} \leq \lambda$ it is enough to show that u is included in a union of $< \sigma$ members of $\mathcal{P}$, thus finishing.

Let f be a one-to-one function from $\kappa =: |u|$ onto u so $\kappa < \theta$. By induction on n we shall choose $f_n, \bar{v}_n$ such that

⊛ (a) f_n is a function from κ to $\lambda + 1$,
(b) $\bar{v}_n = \langle v_{n,\varepsilon} : \varepsilon < \varepsilon_n\rangle$ is a partition of κ which satisfies $\varepsilon_n < \sigma$ and $\kappa = \cup\{v_{n,\varepsilon} : \varepsilon < \varepsilon_n\}$,
(c) $f_0(i) = \lambda$ for every $i < \kappa$,
(d) $f_{n+1}(i) \leq f_n(i)$ for $i < \kappa$,
(e) $f(i) \leq f_n(i)$ and if $f(i) < f_n(i)$ then $f_{n+1}(i) < f_n(i)$,
(f) $f_n \restriction v_{n,\varepsilon} \in M$ for each $\varepsilon < \varepsilon_n$.

[1] See the use in 3.1. In the notation of [17] this means that: $\mathfrak{b} \in J^\sigma_{<\lambda}[\mathfrak{a}] \leftrightarrow \mathrm{pcf}_{\sigma\text{-comp}}(\mathfrak{b}) \subseteq \lambda$.

This is sufficient: $\{\mathrm{Rang}(f_n \restriction v_{n,\varepsilon}) : n < \omega, \varepsilon < \varepsilon_n\}$ is a family of $< \sigma$ sets (as $\sigma = \mathrm{cf}(\sigma) > \aleph_0$ and $n < \omega \Rightarrow \sigma > \varepsilon_n$) each belonging to $\mathcal{P}$ (as $f_n \restriction v_{n,\varepsilon} \in M$) and their union includes u because for every $i < \kappa$, $f_n(i) = f(i)$ for every n large enough (by clauses (d) + (e) of $\circledast$).

So, all we need to do is to show, by induction, that we can choose the elements of $\circledast$. For $n = 0$, f_n is constantly λ. So assume $n = m + 1$ and f_m is given; let,

$$u_{n,0} = \{i < \kappa : f_m(i) = f(i)\}$$

$u_{n,1} = \{i < \kappa : f_m(i) > f(i)$ and is a successor ordinal or just has cofinality $< \theta\}$,

$$u_{n,2} = \kappa \backslash u_{n,0} \backslash u_{n,1}.$$

As $2^\kappa \le 2^{<\theta} \le \lambda$, clearly the partition $\langle u_{n,0}, u_{n,1}, u_{n,2}\rangle$ of κ belongs to M, so it is enough to choose $f_{n+1} \restriction u_{n,\ell}$ separately for $\ell = 0, 1, 2$.

Case 1: $\ell = 0$.

Let $f_n \restriction u_{n,0} = f_m \restriction u_{n,0}$.

Case 2: $\ell = 1$.

Let $\bar{C} = \langle C_\alpha : \alpha \le \lambda\rangle \in M$ be such that $C_0 = \emptyset$, $C_{\alpha+1} = \{\alpha\}$, C_δ is a club of δ of order type $\mathrm{cf}(\delta)$ for limit ordinal $\delta \le \lambda$. Let $f_n \restriction u_{n,1}$ be defined by $f_n(i) = \mathrm{Min}(C_{f_m(i)} \backslash f(i))$. For each $\varepsilon < \varepsilon_m$ the function $f_n \restriction (u_{n,1} \cap v_{m,\varepsilon})$ belongs to M and hence $\langle C_{f_m(i)} : i \in u_{n,1} \cap v_{m,\varepsilon}\rangle$ belongs to M, and $f_n \restriction (u_{n,1} \cap v_{m,\varepsilon}) \in \prod_{i \in u_{n,1} \cap v_{m,\varepsilon}} C_{f_m(i)}$; hence it is enough to prove that $\prod_{i \in u_{n,1} \cap v_{m,\varepsilon}} C_{f_m(i)}$ is $\subseteq M$. But as $u_{n,1}, v_{m,\varepsilon}, \bar{C}$ and $f_m \restriction v_{m,\varepsilon}$ belong to M, clearly $\prod_{i \in u_{n,1} \cap v_{m,\varepsilon}} C_{f_m(i)}$ belongs to M; hence it suffices to prove that it has cardinality $\le \lambda$.

Subcase 2A: $\mathrm{cf}(\theta) > \kappa$.

Note that $\sup\{|C_{f_m(i)}| : i \in u_{m,1} \cap v_{m,\varepsilon}\} < \theta$, so as $|u_{n,1} \cap v_{m,\varepsilon}| \le \kappa < \theta$ and $2^{<\theta} \le \lambda$ clearly $|\prod_{i \in u_{m,1} \cap v_{n,\varepsilon}} C_{f_m(i)}| \le \lambda$, so we are done.

Subcase 2B: $\mathrm{cf}(\theta) \le \kappa$; hence $\mathrm{cf}(\theta) < \sigma$.

Let $\theta = \Sigma\{\theta_\zeta : \zeta < \mathrm{cf}(\theta)\}$, $\theta_\zeta \in [\kappa, \theta)$ increasing with ζ and let $u_{n,1,\zeta} = \{i \in u_{n,1} : |C_{f_m(i)}| < \theta_\zeta\}$. So for each $\zeta < \mathrm{cf}(\theta)$ we have $(\theta_\zeta)^\kappa \le 2^{<\theta} \le \lambda$ and $f_n \restriction (u_{n,1,\zeta} \cap v_{m,\varepsilon}) \in M$. So we have a partition to $\mathrm{cf}(\theta) < \sigma$ cases.

Case 3: $\ell = 2$.

It is enough to define $f_n \restriction (v_{m,\varepsilon} \cap u_{n,2})$ for each $\varepsilon < \varepsilon_m$. Let $\lambda_{n,i} = \mathrm{cf}(f_m(i))$, so that $\langle \lambda_{n,i} : i \in v_{m,\varepsilon} \cap u_{n,2}\rangle \in M$ and hence there is a sequence $\langle h_{n,i} : i \in u_{n,2} \cap v_{m,\varepsilon}\rangle \in M$ where $h_{n,i}$ is an increasing continuous function from $\lambda_{n,i}$ onto some club of $f_m(i)$.

Let $\mathfrak{a} = \{\lambda_{n,i} : i \in u_{n,2} \cap v_{m,\varepsilon}\}$. Applying assumption (b) and Definition 1.7(1) it is easy to finish.

In detail, as $\mathfrak{a} \in J^\sigma_{<\lambda^+}[\mathfrak{a}]$ there is a set $\mathcal{F} \subseteq \prod_i \{\lambda_{n,i} : i \in u_{n,2} \cap v_{m,\varepsilon}\}$ of cardinality $\le \lambda$ witnessing it; without loss of generality $\mathcal{F} \in M$ and hence $\mathcal{F} \subseteq M$. Let $g \in \prod\{\lambda_{n,i} : i \in u_{n,2} \cap v_{m,\varepsilon}\}$ be such that $i \in u_{n,2} \cap v_{m,\varepsilon} \Rightarrow h_{n,i}(g(\lambda_{n,i})) \ge f(i)$ (e.g. $g(\lambda_{n,i})$ is the minimal ordinal such that this occurs).

By the choice of the family $\mathcal{F}$ there are $\zeta_{n,\varepsilon}(*) < \sigma$ and $f'_{m,\varepsilon,\zeta} \in \mathcal{F}$ for $\zeta < \zeta_{n,\varepsilon}(*)$ such that $(\forall i \in u_{n,2} \cap v_{m,\varepsilon})(\exists \zeta < \zeta_{n,\varepsilon}(*))(g(\lambda_{n,i}) < f'_{m,\varepsilon,\zeta}(\lambda_{n,i}))$.

Let $v_{m,\varepsilon,\zeta} = \{i \in v_{m,\varepsilon} : \zeta < \zeta_{n,\varepsilon}(*)$ is minimal such that $g(\lambda_{n,i}) < f'_{m,\varepsilon,\zeta}(\lambda_{n,i})\}$. Now we define $f_n \restriction (u_{n,2} \cap v_{m,\varepsilon})$ by choosing $f_n \restriction (u_{n,2} \cap v_{m,\varepsilon,\zeta})$ by $(f_n \restriction (u_{n,2} \cap v_{m,\varepsilon,\zeta}))(i) = h_{m,i}(f'_{m,\varepsilon,\zeta}(\lambda_{n,i}))$. $\square$

Claim 1.10. *There is $\bar{\lambda} = \langle \lambda_i : i < \kappa\rangle$ and a σ-complete ideal J on κ such that $T_J(\bar{\lambda}) \ge \lambda$ and $i < \kappa \Rightarrow 2^\kappa < \lambda_i < \lambda$ when*

$\circledast$ (a) $2^\kappa < \lambda$, $\aleph_0 < \sigma = \mathrm{cf}(\sigma) \le \kappa$,

(b) $\mathfrak{a} \subseteq \mathrm{Reg} \cap \lambda \backslash (2^\kappa)^+$ *has cardinality* $\le \kappa$ *and* $\mathfrak{a} \notin J^\sigma_{<\lambda}[\mathfrak{a}]$.

Proof. Let $\bar{\lambda} = \langle \lambda_i : i < \kappa\rangle$ list $\mathfrak{a}$ and let $J = J^\sigma_{<\lambda}[\mathfrak{a}]$, and by induction on $\alpha < \lambda$ we shall choose a function $f_\alpha \in \prod \mathfrak{a}$ such that $\beta < \alpha \Rightarrow f_\beta <_J f_\alpha$. Arriving at α for every $\mathfrak{b} \subseteq \mathfrak{a}$ let $\mathcal{F}^\alpha_\mathfrak{b} = \{f_\beta \restriction \mathfrak{b} : \beta < \alpha\}$; so by the definition of $J^\sigma_{<\lambda}[\mathfrak{a}]$, for every $\mathfrak{b} \in J^+ := \mathcal{P}(\mathfrak{a}) \backslash J$, there is $g^\alpha_\mathfrak{b} \in \prod \mathfrak{b}$ witnessing it because the set $\mathcal{F}^\alpha_\mathfrak{b}$ does not witness $\mathfrak{b} \in J^\sigma_{<\lambda}[\mathfrak{a}]$. Let $f_\alpha \in \Pi\mathfrak{a}$ be defined by $f_\alpha(\theta) = \sup\{g^\alpha_\mathfrak{b}(\theta) : \mathfrak{b} \in J^+$ and $\theta \in \mathfrak{b}\}$. Now $f_\alpha \in \Pi\mathfrak{a}$ as $\theta \in \mathfrak{a} \Rightarrow f_\alpha(\theta) < \theta$ which holds as $|J^+| \le 2^{|\mathfrak{a}|} \le 2^\kappa < \theta$. Also if $\beta < \alpha$ and we let $\mathfrak{b}^\alpha_\beta =: \{\theta \in \mathfrak{a} : f_\beta(\theta) \ge f_\alpha(\theta)\}$, then $\mathfrak{b}^\alpha_\beta \in J^+$ implies easy contradiction to the choice of $g^\alpha_{\mathfrak{b}^\alpha_\beta}$ (and f_α). So we can carry on the induction and so $\langle f_\alpha : \alpha < \lambda\rangle$, $f_\alpha \in \prod \bar\lambda$ where $f'_\alpha(i) = f_\alpha(\lambda_i)$ exemplify $T_J(\bar\lambda) \ge \lambda$ as required. $\square$

Remark 1.11. This is the case $\mathrm{Min}(\mathfrak{a}) > 2^{|\mathfrak{a}|}$ from [11, XIII].

Claim 1.12. *If $\circledast$ below holds, <u>then</u> we can get equality in 1.10, i.e., there is $\bar\lambda' = \langle \lambda'_i : i < \kappa\rangle$ such that*

(α) $2^\kappa < \lambda'_i \le \lambda_i$,
(β) *if* $f \in \prod_{i<\kappa} \lambda'_i$ *then* $T_J(f) < \lambda$,
(γ) $T_J(\bar\lambda') = \lambda$,

<u>where</u>

$\circledast$ (a) $2^\kappa < \lambda, \aleph_0 < \sigma = \mathrm{cf}(\sigma) \le \kappa$,
(b) $2^\kappa < \lambda_i < \lambda$,
(c) *J is a σ-complete ideal on κ,*
(d) $T_J(\bar\lambda) \ge \lambda$.

Proof. Clearly $\{i : \lambda_i \le (2^\kappa)^{+n}\} \in J$ for $n < \omega$ (as $((2^\kappa)^{+n})^\kappa = (2^\kappa)^{+n}$ by 0.6(2)); so by 0.6(1) without loss of generality $i < \kappa \Rightarrow \lambda_i > (2^\kappa)^{+2}$.

As $(\prod_{i<\kappa}(\lambda_i + 1), <_J)$ is well founded (i.e., has no (strictly) decreasing infinite sequence of members) and there is $f \in \prod_{i<\kappa}(\lambda_i + 1)$ satisfying $T_J(f) \ge \lambda$ (i.e. $\bar\lambda$ itself), clearly there is $f \in \prod_{i<\kappa}(\lambda_i + 1)$ for which $T_J(f) \ge \lambda$ satisfying $g \in \prod_{i<\kappa}(\lambda_i + 1)$, $g <_J f$ implies $T_J(g) < \lambda$. Now as above $\{i < \kappa : f(i) \le (2^\kappa)^{+2}\} \in J$, so without loss of generality $i < \kappa \Rightarrow f(i) > (2^\kappa)^{+2}$. Let $\lambda'_i = |f(i)|$; hence $\bar\lambda'$ satisfies demands $(\alpha) + (\beta)$ of the desired conclusion, and $T_J(\bar\lambda') = T_J(f) \ge \lambda$. So assume toward contradiction that it fails clause (γ), so by the last sentence we have $T_J(\bar\lambda') > \lambda$ and we shall derive a contradiction, thus finishing. So there is $\{f_\alpha : \alpha < \lambda^+\} \subseteq \prod_{i<\kappa} \lambda'_i$ such that $\alpha \ne \beta \Rightarrow f_\alpha \ne_J f_\beta$, and let $u_\alpha =: \{\beta : f_\beta <_J f_\alpha\}$. Note that for $\alpha < \beta < \lambda, (\beta \in u_\alpha \Rightarrow f_\alpha <_J f_\beta) \equiv (\beta \in u_\alpha \Rightarrow f_\alpha \le_J f_\beta)$, as $f_\alpha \ne_J f_\beta$. If for some $\alpha < \lambda, |u_\alpha| \ge \lambda$, then $\{f_\beta : \beta \in u_\alpha\}$ exemplifies that $T_J(f_\alpha) \ge \lambda$ and clearly $f_\alpha <_J \bar\lambda' \le f$, a contradiction to the choice of f. So $\alpha < \lambda^+ \Rightarrow |u_\alpha| < \lambda$. Hence by the Hajnal free subset theorem [5] there is $S \subseteq \lambda^+$ of cardinality λ^+ such that $(\forall \alpha \ne \beta \in S)(\beta \notin u_\alpha)$. So $\forall \alpha \ne \beta$ from $S \neg(f_\alpha \le_J f_\beta)$, contradicting 0.10(1). $\square$

The Revised GCH Theorem 1.13. *If θ is strong limit singular <u>then</u> for every $\lambda \ge \theta$ for some $\partial < \theta$ we have $\lambda = \lambda^{[\partial,\theta]}$.*

Remark 1.14. (1) Hence for every $\lambda \ge \theta$ for some $n < \omega$ and $\kappa_\ell < \theta(\ell < n)$, $\aleph_0 = \kappa_0 < \kappa_1 < \cdots < \kappa_n = \theta$ for each $\ell < n$, $2^{\kappa_\ell} \ge \kappa_{\ell+1}$ or $\lambda = \lambda^{[\kappa'_\ell, <\kappa_{\ell+1}]}$ where $\kappa'_\ell = (2^{\kappa_\ell})^+$.
(2) If $\sigma \in (\mathrm{cf}(\theta), \theta)$ and $\lambda \ge \theta$ then $\lambda^{[\sigma,\theta]} = \lambda^{[\sigma,<\theta]} = \Sigma\{\lambda^{[\sigma,\theta']} : \theta' \in [\sigma,\theta)\}$.
(3) Note that 1.13 with $\lambda = \lambda^{[\partial,<\theta]}$ + 1.14(1) holds also for regular θ strong limit uncountable by the Fodor lemma.

Proof. We prove this by induction on $\lambda \ge \theta$.

Let $\sigma =: (\mathrm{cf}(\theta))^+ < \theta$.

<u>Case 0</u>: $\lambda = \theta$.

Let $\mathcal{P}$ be the family of bounded subsets of θ, so $|\mathcal{P}| = \theta$ and every $u \in [\theta]^{\le\theta}$ is the union of $\le \mathrm{cf}(\theta)$ members of $\mathcal{P}$; hence (by Definition 1.1(1), (4)) we have $\lambda^{[\sigma,\theta]} = \lambda$.

<u>Case 1</u>: $\lambda > \theta$ and for every $\mathfrak{a} \subseteq \mathrm{Reg} \cap \lambda \backslash \theta$ of cardinality $< \theta$ we have $\mathfrak{a} \in J^\sigma_{<\lambda}[\mathfrak{a}]$.

By 1.9, we have $\lambda^{[\sigma,<\theta]} = \lambda$ (recalling $\mathrm{cf}(\theta) < \sigma$ and 1.3).

<u>Case 2</u>: Neither Case 0 nor Case 1.

Trivially for every $\kappa \in [\sigma,\theta)$, clause (a) of $\circledast$ of 1.10 holds. As this is not Case 1, the assumption (b) of $\circledast$ of Claim 1.10 holds for some κ for which $\sigma \le \kappa < \theta$, and hence the conclusion of 1.10 holds for some $\bar\lambda = \langle \lambda_i : i < \kappa\rangle$ and J;

we have $2^\kappa < \lambda_i < \lambda$ and $T_J(\bar\lambda) \geq \lambda$ where J is a σ-complete ideal on κ. So the assumption, i.e., $\circledast$ of Claim 1.12, holds, and hence also its conclusion, which means that for some $\bar\lambda'$ we have

$\circledast$ (i) J is a σ-complete ideal on κ,
(ii) $\bar\lambda' = \langle \lambda'_i : i < \kappa\rangle$,
(iii) $2^\kappa < \lambda'_i < \lambda$ (as $\lambda'_i \leq \lambda_i$),
(iv) $T_J(\bar\lambda') = \lambda$,
(v) $T_J(f) < \lambda$ if $f \in \prod_{i<\kappa} \lambda'_i$.

We can find an increasing sequence $\langle \theta_\varepsilon : \varepsilon < \mathrm{cf}(\theta)\rangle$ of regular cardinals from the interval (σ, θ) with limit θ. As we can replace this sequence by $\langle (\theta_\varepsilon)^\kappa : \varepsilon \in C\rangle$ for any unbounded $C \subseteq \mathrm{cf}(\theta)$, without loss of generality $\varepsilon < \mathrm{cf}(\theta) \Rightarrow \theta_\varepsilon^\kappa = \theta_\varepsilon$. By the induction hypothesis, for each $i < \kappa$ there is $\varepsilon(i) < \mathrm{cf}(\theta)$ such that $\lambda'_i = (\lambda'_i)^{[\theta_{\varepsilon(i)}, <\theta]} \geq \theta$ or $\lambda'_i \leq \theta_{\varepsilon(i)}$. For $\zeta < \mathrm{cf}(\theta)$ define $A_\zeta = \{i < \kappa : \lambda'_i \geq \theta$ and $\varepsilon(i) = \zeta\}$ and $A_{\mathrm{cf}(\theta)+\zeta} = \{i < \kappa : \lambda'_i < \theta$ and $\varepsilon(i) = \zeta\}$. So $\langle A_\varepsilon : \varepsilon < \mathrm{cf}(\theta) + \mathrm{cf}(\theta)\rangle$ is a partition of κ into $< \sigma$ sets and hence by 0.6(4) we know that

$$T^0_J(\bar\lambda') = \mathrm{Min}\{T^0_{J\restriction A_\varepsilon}(\bar\lambda' \restriction A_\varepsilon) : \varepsilon < \mathrm{cf}(\theta) + \mathrm{cf}(\theta) \text{ and } A_\varepsilon \in J^+\}.$$

Hence by 0.6(2) for some $\zeta < \mathrm{cf}(\theta) + \mathrm{cf}(\theta)$ we have $T_J(\bar\lambda') = T_{J\restriction A_\zeta}(\bar\lambda' \restriction A_\zeta)$ and $A_\zeta \in J^+$, so by renaming without loss of generality $A_\zeta = \kappa$. If $\zeta \geq \mathrm{cf}(\theta)$ as $\kappa < \theta, \theta$ strong limit we get $T_J(\bar\lambda') \leq \prod_{i<\kappa} \lambda'_i < (\theta_\zeta)^\kappa < \theta$, a contradiction, so $\zeta < \mathrm{cf}(\theta)$.

Now for each $\xi \in (\zeta, \mathrm{cf}(\theta))$ we would like to apply Claim 1.4 with $J, \bar\lambda', \sigma, \kappa, \theta^+_\zeta, \theta_\xi$ here standing for $J, \bar\lambda, \sigma, \kappa, \partial, \theta$ there. (But note that θ of 1.4 and θ of 1.13 are not the same.) Do the assumptions (a)–(g) of $\circledast$ of 1.4 hold?

Clause (a) there means $\aleph_0 < \sigma = \mathrm{cf}(\sigma) \leq \kappa < \theta^+_\zeta \leq \theta_\xi$ which holds as $\sigma = (\mathrm{cf}(\theta))^+, \theta^\kappa_\zeta = \theta_\zeta$ and $\zeta < \xi < \mathrm{cf}(\theta)$.

Clause (b) means J is a σ-complete ideal on κ which holds by clause (i) of $\circledast$ above.

Clause (c) there means $\bar\lambda' = \langle \lambda'_i : i < \kappa\rangle$ which holds by clause (ii) of $\circledast$ above.

Clause (d) there says $T_J(\bar\lambda') = \lambda$ which holds by clause (iv) of $\circledast$ above.

Clause (e) there means $(\lambda'_i)^{[\theta^+_\zeta, \theta_\xi]} = \lambda'_i$ which holds as $\varepsilon(i) = \zeta$, so by its choice $(\lambda'_i)^{[\theta_\zeta, <\theta]} = \lambda'_i$ but $\theta_\zeta < \theta^+_\zeta \leq \theta_\xi < \theta$ and hence, by the monotonicity in the definition, this gives $(\lambda'_i)^{[\theta^+_\zeta, \theta_\xi]} = \lambda'_i$ as required.

Clause (f) means "if $\partial_i < \theta^+_\zeta$ for $i < \kappa$ then $\prod_{i<\kappa} \partial_i < \theta^+_\zeta$" which holds as $\theta^\kappa_\zeta = \theta_\zeta$.

Clause (g) means $\theta^\kappa_\xi = \theta_\xi$.

So we get the conclusion of 1.4 which is $\lambda^{[\theta^+_\zeta, \theta_\xi]} = \lambda$. As this holds for every $\xi \in (\zeta, \mathrm{cf}(\theta))$ and $\langle \theta_\varepsilon : \varepsilon < \mathrm{cf}(\theta)\rangle$ is increasing with limit θ, by 1.3(3) we get $\lambda^{[\theta^+_\zeta, \theta]} = \lambda$. As $\theta^+_\zeta < \theta$, choosing $\partial =: \theta^+_\zeta$ we have finished. $\square$

Concluding Remark 1.15. We can in 1.4 assume less. Instead of $\theta = \theta^\kappa$, it is enough (which follows from [18, Section 3]; see 0.5) to assume:

$\circledast$ for every $\lambda' < \lambda$ we can find $\mathcal{F} \subseteq \prod_{i<\kappa} \lambda_i$ of cardinality λ' such that $f \neq g \in \mathcal{F} \Rightarrow f \neq_J g$.

This is seemingly a gain, but in the induction the case $(\forall \mathfrak{a} \subseteq \mathrm{Reg} \cap \lambda^+ \backslash \theta)(|\mathfrak{a}| \leq \kappa \Rightarrow \mathfrak{a} \in J^{\aleph_1}_{<\lambda^+}[\mathfrak{a}])$ is problematic.

2. The finitely many exceptions

What here is needed in later sections? Only 2.10 is essential. Definition 2.14 + Observation 2.15 tells us what the set of exceptional cardinals $\mathfrak{d}_{0,\mu}(\lambda)$ for λ is; and 2.3 proves it is finite. We do not succeed in proving e.g. $\lambda \geq \aleph_\omega \wedge \aleph_0 < \aleph_n \notin \mathfrak{d}_{0,\mu}(\lambda) \Rightarrow \lambda^{<\aleph_n>} = \lambda$; but we shall in Section 3 prove a consequence. Now all this is used in Section 3 only if we like to say explicitly what the finite set of possible exceptions is, i.e., in 3.3, but it is not used in 3.1 itself, which still uses Claim 2.10.

The rest clarifies the situation in various ways. In Definition 2.4 we define "$\bar\lambda$ is a D-representation of λ" and when such a representation is exact/true and in Definition 2.5 we give a name to the content of 2.3: i.e., we say that

$\mathbf{r} = \langle(\lambda_\eta, \mathcal{D}_\eta, D_\eta, \kappa_\eta) : \eta \in \mathcal{T}\rangle$ is a representation. In 2.6 we spell out basic properties of representations; in 2.8 we ask about possible improvements, which the rest supplies.

In 2.10, 2.11 we guarantee that every λ_η is regular if λ is. In 2.12 we deal with "$T_{D+A}(\bar\lambda) = T_D(\bar\lambda)$ for every $A \in D^+$" and in 2.13 we deal with how close we can get to "D_η is a co-bounded filter on κ_η". In 2.17, 2.18 we further investigate the possible representations of λ (needed for 3.3).

In 2.1 we prove a relative of 1.4 assuming only $i < \kappa \Rightarrow \lambda_i^{<\partial,\mu,\theta>} = \lambda_i$, replacing $2^\theta \le \lambda$ by $2^\kappa \le \lambda$ and getting $\lambda^{<\partial,\mu,\theta>} = \lambda$. But so far it has no conclusion parallel to 1.13. Note that Claim 2.1 is not needed for reading the rest of the paper.

In full:

Claim 2.1. *Assume*

(a) $\aleph_0 < \sigma = \mathrm{cf}(\sigma) \le \kappa < \partial \le \mu \le \theta$,
(b) *J is a σ-complete ideal on κ,*
(c) $\bar\lambda = \langle\lambda_i : i < \kappa\rangle$,
(d) *$T_J(\bar\lambda) = \lambda$ and moreover this is exemplified by a μ^+-free family,*
(e) *$\lambda_i^{<\partial,\mu,\theta>} = \lambda_i$ for $i < \kappa$,*
(f) *if $\partial_i < \partial$ for $i < \kappa$ then $\prod_{i<\kappa} \partial_i < \partial$,*
(g) *$\theta = \theta^\kappa$ and $2^\kappa \le \lambda$.*

Then $\lambda^{<\partial,\mu,\theta>} = \lambda$.

Remark 2.2. (1) Recall that $\mathcal{F} \subseteq {}^\kappa\mathrm{Ord}$ is (μ^*, J)-free when for every $\mathcal{F}' \subseteq \mathcal{F}, |\mathcal{F}'| < \mu^*$ we can find $\bar A = \langle A_f : f \in \mathcal{F}'\rangle$ such that $A_f \in J$ and $f_1 \ne f_2 \in \mathcal{F}' \wedge i \in \kappa\backslash(A_{f_1} \cup A_{f_2}) \Rightarrow f_1(i) \ne f_2(i)$ (we can use $f_1(i) < f_2(i)$).

(2) The addition to the assumption in clause (d) of 2.1 compared to clause (d) of 1.4 is mild.

Proof. Let $\bar f = \langle f_\alpha : \alpha < \lambda\rangle$ be μ^+-free, $f_\alpha \in \prod_{i<\kappa} \lambda_i$ pairwise J-different (i.e., $\alpha \ne \beta \Rightarrow \{i : f_\alpha(i) = f_\beta(i)\} \in J$ exists by clause (d) of the assumption).

For each $i < \kappa$ let $\mathcal{P}_i \subseteq [\lambda_i]^{\le\mu}$ be of cardinality λ_i and witness $\lambda_i^{<\partial,\mu,\theta>} = \lambda_i$; that is: every $u \in [\lambda_i]^{\le\theta}$ is included in the union of $< \partial$ members of

$$\mathrm{set}_{\theta,\mu}(\mathcal{P}_i) =: \{v : v \in [\lambda_i]^{\le\theta} \text{ and every } w \in [v]^{\le\mu} \text{ is included in some member of } \mathcal{P}_i\};$$

such a family exists by assumption (e). Let $M \prec (\mathcal{H}(\chi), \in)$ be of cardinality λ such that $\lambda + 1 \subseteq M$ and $\langle\lambda_i : i < \kappa\rangle, \langle\mathcal{P}_i : i < \kappa\rangle, J, \mathcal{P}(\kappa)$ belong to M.

Let $\mathcal{P} = M \cap [\lambda]^{\le\mu}$. We shall show that $\mathcal{P}$ exemplifies the desired conclusion. Now $\mathcal{P}$ is a family of $\le \|M\| = \lambda$ of subsets of λ each of cardinality $\le \mu$; hence it is enough to show

$(*)$ if $u \in [\lambda]^{\le\theta}$ then u is included in the union of $< \partial$ sets $v \in \mathrm{set}_{\theta,\mu}(\mathcal{P})$.

Proof of $(*)$: Let $u_i = \{f_\alpha(i) : \alpha \in u\}$; so $u_i \in [\lambda_i]^{\le\theta}$, and hence we can find $\langle v_{i,j} : j < j_i\rangle$ such that $v_{i,j} \in \mathrm{set}_{\theta,\mu}(\mathcal{P}_i)$ and $u_i = \cup\{v_{i,j} : j < j_i\}$ and $0 < j_i < \partial$. For each $\eta \in \prod_{i<\kappa} j_i$ let

$$w_\eta = \{\alpha \in u : i < \kappa \Rightarrow f_\alpha(i) \in v_{i,\eta(i)}\}.$$

Clearly $u = \cup\{w_\eta : \eta \in \prod_{i<\kappa} j_i\}$ as for any $\alpha \in u$ for each $i < \kappa$ we can choose $\varepsilon_i(\alpha) < j_i$ such that $f_\alpha(i) \in v_{i,\varepsilon_i(\alpha)}$ and let $\eta_\alpha = \langle\varepsilon_i(\alpha) : i < \kappa\rangle$ clearly $\eta_\alpha \in \prod_{i<\kappa} j_i$ and $\alpha \in w_{\eta_\alpha}$. By the assumption (f), as $i < \kappa \Rightarrow j_i < \partial$, clearly $|\prod_{i<\kappa} j_i| < \partial$; hence it is enough to prove that $\eta \in \prod_{i<\kappa} j_i \Rightarrow w_\eta \in \mathrm{set}_{\theta,\mu}(\mathcal{P})$. So it is enough to prove for $\eta \in \prod_{i<\kappa} j_i$ and $w \in [w_\eta]^{\le\mu}$ that

$\circledast$ w is included in some $w' \in M \cap [\lambda]^{\le\mu}$.

Proof of ⊛: As $i < \kappa \Rightarrow |\mathcal{P}_i| = \lambda_i$ and $T_J(\bar{\lambda}) = \lambda$ there is $\mathcal{G} \subseteq \prod_{i<\kappa} \mathcal{P}_i$ satisfying $|\mathcal{G}| = \lambda$ and $(\forall g \in \prod_{i<\kappa} \mathcal{P}_i)(\exists g' \in \mathcal{G})(\{i : g(i) = g'(i)\} \in J^+)$. As $\langle \mathcal{P}_i : i < \kappa \rangle \in M$, without loss of generality $\mathcal{G} \in M$ and as $\lambda + 1 \subseteq M$ we have $\mathcal{G} \subseteq M$. For each $i < \kappa$ we have $A_i = \{f_\alpha(i) : \alpha \in w\}$ is a subset of some $A'_i \in \mathcal{P}_i$. Apply the choice of $\mathcal{G}$ to $\langle A'_i : i < \kappa \rangle \in \prod_{i<\kappa} \mathcal{P}_i$; so for some $g \in \mathcal{G} \subseteq M$ the set $B =: \{i : A'_i = g(i)\}$ belongs to J^+. Clearly $w' = \{\alpha < \lambda$: for some $Y \in J^+$ for every $i \in Y$ we have $f_\alpha(i) \in g(i)\}$ belongs to M. Now $|w'| \leq \mu^\kappa$; as $\alpha < \beta < \lambda \Rightarrow f_\alpha \neq_J f_\beta$ but $\bar{f}$ is μ^+-free, we moreover have $|w'| \leq \mu$. Lastly, by the last two sentences $w' \in M \cap [\lambda]^{\leq\mu} = \mathcal{P}$; also $w \subseteq w'$ because $B \in J^+$ and $\alpha \in w \,\&\, i \in B \Rightarrow f_\alpha(i) \in A_i \subseteq A'_i = g(i)$, so we are done. □

Claim 2.3. *If $\theta > \sigma = \mathrm{cf}(\sigma) > \aleph_0$, $\mathrm{cf}(\theta) \in [\sigma, \theta)$ and $\lambda > \theta_* = 2^{<\theta}$ then there is $\langle (\lambda_\eta, \mathcal{D}_\eta, D_\eta, \kappa_\eta) : \eta \in \mathcal{T} \rangle$ such that*

(a) *$\mathcal{T}$ is a subtree of ${}^{\omega>}\theta$ (i.e. $<> \in \mathcal{T} \subseteq {}^{\omega>}\theta$, $\mathcal{T}$ is closed under initial segments) with no ω-branch; let $\max_{\mathcal{T}}$ be the set of maximal nodes of $\mathcal{T}$,*
(b) *λ_η is a cardinal $\in (2^{<\theta}, \lambda]$ and $\nu \lhd \eta \Rightarrow \lambda_\nu > \lambda_\eta$ and $\lambda_{<>} = \lambda$,*
(c) *κ_η is a regular cardinal $\in [\sigma, \theta)$ if $\eta \in \mathcal{T} \backslash \max_{\mathcal{T}}$ and κ_η is zero or undefined if $\eta \in \max_{\mathcal{T}}$ and $\eta^\frown\langle\alpha\rangle \in \mathcal{T} \Leftrightarrow \alpha < \kappa_\eta$,*
(d) *if $\eta \in \max_{\mathcal{T}}$ then*
 $(*)_{\lambda_\eta}$ for no $\kappa < \theta$ and σ-complete filter $\mathcal{D}$ on κ and cardinals $\lambda_i \in (2^{<\theta}, \lambda_\eta)$ for $i < \kappa$ do we have $T_{\mathcal{D}}(\langle \lambda_i : i < \kappa \rangle) \geq \lambda_\eta$,
(e) *$\mathcal{D}_\eta$ is a σ-complete filter on κ_η when $\eta \in \mathcal{T} \backslash \max_{\mathcal{T}}$,*
(f) *$T_{\mathcal{D}_\eta}(\langle \lambda_{\eta^\frown<\alpha>} : \alpha < \kappa_\eta \rangle) = \lambda_\eta$ if $\eta \in \mathcal{T} \backslash \max_{\mathcal{T}}$,*
(g) *if $f \in \prod_{\alpha<\kappa_\eta} \lambda_{\eta^\frown<\alpha>}$ then $T_{\mathcal{D}_\eta}(f) < \lambda_\eta$,*
(h) *D_η is the σ-complete filter on $\max_{\mathcal{T},\eta} = \{\nu \in \max_{\mathcal{T}} : \eta \trianglelefteq \nu\}$ such that*
 (α) if $\eta \in \max_{\mathcal{T}}$, $D_\eta = \{\{\eta\}\}$,
 (β) if $\eta \in \mathcal{T} \backslash \max_{\mathcal{T}}$ then

$$D_\eta = \{A \subseteq \max\nolimits_{\mathcal{T},\eta} : \textit{the following set belongs to } \mathcal{D}_\eta \tag{7}$$

$$\{\alpha < \kappa_\eta : A \cap \max\nolimits_{\mathcal{T},\eta^\frown<\alpha>} \in D_{\eta^\frown<\alpha>}\}\}, \tag{8}$$

(i) *if $\mathrm{cf}(\lambda) > \theta_*$ then $\eta \in \mathcal{T} \Rightarrow \mathrm{cf}(\lambda_\eta) > \theta_*$,*
(j) *we can replace "$\lambda > \theta_*$" above by $\lambda_\eta \geq \partial$ for any cardinal ∂ such that $\mathrm{cf}(\partial) \geq \theta \wedge (\forall \gamma < \partial)(\forall \alpha < \partial)|\alpha|^{|\gamma|} < \partial$.*

Proof. We leave clause (j) to the reader.

Case 1: Ignoring clause (i).

We prove this by induction for $\lambda > 2^{<\theta}$. If λ satisfies the requirement $(*)_\lambda$ from clause (d) let $\mathcal{T} = \{<>\}$; $\lambda_\eta = \lambda$ and $\kappa_{<>}$, $D_{<>}$ are trivial. If λ fails that demand use claims 1.10 + 1.12 to find $\mathcal{D}, \kappa, \bar{\lambda}$ such that

$(*)$ $\kappa \in [\sigma, \theta)$, $\mathcal{D}$ is a σ-complete filter on κ, $\bar{\lambda} = \langle \lambda_\alpha : \alpha < \kappa \rangle$ and $\lambda_\alpha \in (2^{<\theta}, \lambda)$, a cardinal $T_{\mathcal{D}}(\bar{\lambda}) = \lambda$, but $f \in \prod_{\alpha<\kappa} \lambda_\alpha \Rightarrow T_{\mathcal{D}}(f) < \lambda$.

Now for each $\alpha < \kappa$ we can use the induction hypothesis to find $\langle (\lambda^\alpha_\eta, \kappa^\alpha_\eta, \mathcal{D}^\alpha_\eta, D^\alpha_\eta) : \eta \in \mathcal{T}_\alpha \rangle$ as required in the claim for λ_α. Now we let:

⊛ (a) $\mathcal{T} = \{<>\} \cup \{\langle\alpha\rangle^\frown\eta : \eta \in \mathcal{T}_\alpha\}$,
(b) $\lambda_{<>} = \lambda$, $\kappa_{<>} = \kappa$,
(c) $\lambda_{\langle\alpha\rangle^\frown\eta} = \lambda^\alpha_\eta$ and $\kappa_{<\alpha>^\frown\eta} = \kappa^\alpha_\eta$ for $\alpha < \kappa$, $\eta \in \mathcal{T}_\alpha$,
(d) $\mathcal{D}_{<>} = \mathcal{D}$,
(e) $\mathcal{D}_{\langle\alpha\rangle^\frown\eta} = \mathcal{D}^\alpha_\eta$ for $\alpha < \kappa$, $\eta \in \mathcal{T}_\alpha$,
(f) $D_{<>} = \{A : A \subseteq \max_{\mathcal{T},<>}$ and $\{\alpha < \kappa : \{\eta : \langle\alpha\rangle^\frown\eta \in A \cap \max_{\mathcal{T}_\alpha,<>}\} \in \mathcal{D}^\alpha_{<>}\}$ belongs to $\mathcal{D}\}$,
(g) $D_{\langle\alpha\rangle^\frown\eta} = \{\{\langle\alpha\rangle^\frown\nu : \nu \in B\} : B \in D^\alpha_\eta\}$.

Easily, they are as required.

Case 2: Proving the claim with (i), so dealing with λ satisfying $\mathrm{cf}(\lambda) > \theta_*$.

If λ satisfies the requirement in clause (d) we finish as above. Otherwise, we can find $\kappa \in [\sigma, \theta)$, $\mathcal{D}, \bar{\lambda}$ such that

$(*)$ (i) $\mathcal{D}$ is a σ-complete filter $\mathcal{D}$ on κ, $\bar\lambda = \langle \lambda_\alpha : \alpha < \kappa\rangle$ and $\lambda_\alpha \in (2^{<\theta}, \lambda)$,
(ii) $\lambda \le T_{\mathcal{D}}(\langle \lambda_\alpha : \alpha < \kappa\rangle)$.
By 1.12 without loss of generality
(iii) $\lambda = T_{\mathcal{D}}(\langle \lambda_\alpha : \alpha < \kappa\rangle)$ and $f \in \prod_{\alpha<\kappa} \lambda_\alpha \Rightarrow T_{\mathcal{D}}(f) < \lambda$.

Let $B := \{\alpha : \mathrm{cf}(\lambda_\alpha) > \theta_*\}$. If $B \in \mathcal{D}^+$ and $T_{\mathcal{D}\restriction B}(f \restriction B) < \lambda$ for every $f \in \Pi\bar\lambda$ (hence $T_{\mathcal{D}\restriction B}(\bar\lambda \restriction B) = \lambda$), then we can use $\bar\lambda \restriction B, \mathcal{D} \restriction B$ (and renaming); hence we are done. So assume that this fails, i.e.,

$\boxtimes$ $B \notin \mathcal{D}^+$ or $B \in \mathcal{D}^+$, $T_{\mathcal{D}\restriction B}(f \restriction B) \ge \lambda$ for some $f \in \Pi\bar\lambda$.

In both cases $\bar\lambda \restriction (\kappa\backslash B), \mathcal{D} \restriction (\kappa\backslash B)$ are as required in $(*)$ (in the second case we use 0.6(5)), so by renaming, without loss of generality $B = \emptyset$. For each $\alpha < \kappa$ let $\langle \lambda_{\alpha,\varepsilon} : \varepsilon < \mathrm{cf}(\lambda_\alpha)\rangle$ be increasing continuous with limit λ_α, and let $\bar f = \langle f_\zeta : \zeta < \lambda\rangle$ witness $T_{\mathcal{D}}(\bar\lambda) \ge \lambda$. For each $\zeta < \lambda$ for some $h_\zeta \in \prod_{\alpha<\kappa} \mathrm{cf}(\lambda_\alpha)$ we have $f_\zeta < \langle \lambda_{\alpha,h_\zeta(\alpha)} : \alpha < \kappa\rangle$.

What is the number of possible h_ζ? At most $\prod_{\alpha<\kappa} \mathrm{cf}(\lambda_\alpha) \le (\theta_*)^\kappa$ but $\theta_* = 2^{<\theta}$, $\sigma \le \kappa < \theta$ and $\mathrm{cf}(\theta) = \theta \vee \mathrm{cf}(\theta) < \sigma$.

If $\mathrm{cf}(\theta) = \theta$ then $(\theta_*)^\kappa = (2^{<\theta})^\kappa = 2^{<\theta}$ and so $|\{h_\zeta : \zeta < \lambda\}| \le \theta_* < \mathrm{cf}(\lambda)$. If $\mathrm{cf}(\theta) \ne \theta$ then $\mathrm{cf}(\theta) < \sigma$; hence for each $\zeta < \lambda$ for some $\gamma_\zeta < \theta_*$ the set $A_\zeta = \{\alpha < \kappa : h_\zeta(\alpha) < \gamma_\zeta\}$ belongs to $\mathcal{D}^+$, and $(\forall f \in \prod_{\alpha<\kappa} \lambda_\alpha)(T_{\mathcal{D}\restriction A_\zeta}(f \restriction A_\zeta)) < \lambda$. As $\kappa < \theta$ and $|\{A_\zeta : \zeta < \lambda\}| \le 2^\kappa \le 2^{<\theta} = \theta_*$, clearly for some pair (A, γ) the set $\{\zeta < \lambda : (A_\zeta, \gamma_\zeta) = (A, \gamma)\}$ has cardinality λ, so renaming, without loss of generality $\zeta < \kappa \Rightarrow A_\zeta = \kappa$ and so again $|\{h_\zeta : \zeta < \lambda\}| \le \theta_* < \mathrm{cf}(\lambda)$.

So for some h, $|\{\zeta : h_\zeta = h\}| = \lambda$, a contradiction to clause (iii) of $(*)$ above.

We finish as in case (1). $\square$

Definition 2.4. (1) We say that $\bar\lambda = \langle \lambda_i : i < \kappa\rangle$ is a D-representation of λ <u>when</u>:
(a) D is a filter on κ,
(b) $T_D(\bar\lambda) = \lambda$,
(c) if $f \in \prod_{i<\kappa} \lambda_i$ then $T_D(f) < \lambda$.

(2) We say that $\bar\lambda$ is an exact D-representation of λ <u>when</u>:
(a) D is a filter on κ,
(b) $T_{D+A}(\bar\lambda) = \lambda$ for $A \in D^+$,
(c) if $f \in \prod_{i<\kappa} \lambda_i$ and $A \in D^+$ then $T_{D+A}(f) < \lambda$.

(3) We say that the D-representation is true <u>when</u>:
(d) $\mathrm{cf}(\lambda) = \mathrm{tcf}(\Pi\bar\lambda, <_D)$.

(4) We can replace the filter by the dual ideal.

Definition 2.5. (1) We say $\langle(\lambda_\eta, \mathcal{D}_\eta, D_\eta, \kappa_\eta) : \eta \in \mathcal{T}\rangle$ is a $(\partial, \theta, \sigma)$-representation if the conditions in Claim 2.3 hold; see clause (j) there. If $\partial = \theta$ we may omit it. Writing just σ means $\theta = |\mathcal{T}|^+$.
(2) We say it is an exact/true representation when each $\langle \lambda_{\eta^\frown<\alpha>} : \alpha < \kappa_\eta\rangle$ is an exact/true $\mathcal{D}_\eta$-representation of λ_η.

Claim 2.6. (1) *Assume*

$\circledast$ (a) $\bar\lambda^* = \langle \lambda_i : i < \kappa\rangle$ *is a* D_**-representation of* λ,
(b) $\bar\lambda^i = \langle \lambda_{i,j} : j < \kappa_i\rangle$ *is a* D_i*-representation of* λ_i,
(c) D *is* $\Sigma_{D_*}\langle D_i : i < \kappa\rangle$, *i.e., the filter on* $u = \{(i, j) : i < \kappa, j < \kappa_i\}$ *defined by* $D = \{A \subseteq u : \{i : \{j < \kappa_i : (i, j) \in A\} \in D_i\} \in D_*\}$,
(d) $\mathrm{cf}(\lambda)$, $\mathrm{cf}(\lambda_i)$ *are* $> |u|$ *and* $\lambda, \lambda_i, \lambda_{i,j}$ *are* $> 2^{|u|}$.

<u>Then</u> $\bar\lambda = \langle \lambda_{i,j} : (i, j) \in u\rangle$ *is a* D*-representation of* λ.
(2) *Like for exact representations, i.e., if in* $\circledast(a), (b)$ *we further assume that the representations are exact <u>then</u> also* $\bar\lambda$ *is an exact* D*-representation of* λ.
(3) *Like for true representations: if* $\lambda_i = \mathrm{tcf}(\prod_{j<\kappa_i} \lambda_{i,j}, <_{D_i})$, $\lambda = \mathrm{tcf}(\prod_{i<\kappa} \lambda_i, <_{D_*})$ *<u>then</u>* $\lambda = \mathrm{tcf}(\prod_{(i,j)} \lambda_{i,j}, <_D)$. *Similarly for* min-cf, *etc.*

(4) *Assume that D is an $\aleph_1$-complete filter on κ, $\bar\lambda = \langle \lambda_i : i < \kappa\rangle$ and $T_D(\bar\lambda) > \lambda > 2^\kappa$ and $i < \kappa \Rightarrow \lambda_i > 2^\kappa$. Then we can find $\bar\lambda'$ such that $i < \kappa \Rightarrow 2^\kappa \le \lambda'_i < \lambda_i$ and $\bar\lambda'$ is a D-representation of λ. If we demand only $T_D(\bar\lambda) \ge \lambda$ then we know only $\lambda'_i \le \lambda_i$.*

Proof. (1)

$(*)_1$ $\lambda = T_D(\langle \lambda_{i,j} : (i,j) \in u\rangle)$.

[Why? Let $\mathcal{G}^i = \{g^i_\alpha : \alpha < \lambda_i\}$ witness that $T^1_{D_i}(\bar\lambda^i) = \lambda_i$ and let $\mathcal{G}^* = \{g^*_\alpha : \alpha < \lambda\}$ witness that $T^1_{D_*}(\bar\lambda^*) = \lambda$. We now define $\mathcal{G} = \{g_\alpha : \alpha < \lambda\}$ where $g_\alpha \in \prod_{(i,j)\in u} \lambda_{i,j}$ is defined by $g_\alpha((i,j)) = g^i_{g^*_\alpha(i)}(j)$ and we can easily check that $\alpha < \beta < \lambda \Rightarrow g_\alpha \ne g_\beta \bmod D$, so $\mathcal{G}$ witnesses that $T^1_D(\bar\lambda) \ge \lambda$ and so by clause (d), $T_D(\bar\lambda) \ge \lambda$. Now if $g \in \prod_{(i,j)\in u} \lambda_{i,j}$ then for each i the function (i.e. sequence) $\langle g((i,j)) : j < \kappa_i\rangle$ belongs to $\prod_{j<\kappa_i} \lambda_{i,j}$, so for some $\gamma_i < \lambda_i$ we have $\{j : g((i,j)) = g^i_{\gamma_i}(j)\} \in D^+_i$. Similarly for some $\beta < \lambda$ we have $\{i < \kappa : \gamma_i = g^*_\beta(i)\} \in D^+_*$. Easily, $\{(i,j) \in u : g_\beta(i,j) = g(i,j)\} \in D^+$, so $\mathcal{G}$ witness that $T_D(\bar\lambda) = \lambda$ is as required.]

$(*)_2$ If $g \in \Pi\{\lambda_{i,j} : (i,j) \in u\}$ then $T_D(g) < \lambda$.

[Why? Without loss of generality $g((i,j)) > 0$ for every $(i,j) \in u$. For each $i < \kappa$, let $g_i \in \prod_{j<\kappa_i} \lambda_{i,j}$ be defined by $g_i(j) = g((i,j))$. So $g_i \in \prod_{j<\kappa_i} \lambda_{i,j}$ and hence $\mu_i =: T_{D_i}(g_i) < \lambda_i$; hence there is a sequence $\langle h^i_\alpha : \alpha < \mu_i\rangle$ such that $h^i_\alpha \in \prod_{j<\kappa_i} g_i(j)$ and $(\forall h \in \prod_{j<\kappa_i} g_i(j))(\exists \alpha < \mu_i)(\neg(h \ne_{D_i} h^i_\alpha))$. Clearly $\bar\mu = \langle \mu_i : i < \kappa\rangle \in \prod_{i<\kappa} \lambda_i$ and hence $\mu_* =: T_{D_*}(\bar\mu) < \lambda$; taking $\langle g^{**}_\alpha : \alpha < \mu_*\rangle$ exemplifies this. We now define $f^{**}_\alpha \in \prod_{(i,j)\in u} g((i,j))$ by $f^{**}_\alpha((i,j)) = h^i_{g^{**}_\alpha(i)}(j)$ and it suffices to show that $T_D(g) \le \mu_*(< \lambda)$ is exemplified by $\{f^{**}_\alpha : \alpha < \nu_*\}$ which is proved as in $(*)_1$, the second half of the proof.]

So we are done.

(2) Similarly.

(3) By [17, I].

(4) Easy (and proved above). $\square$

Remark 2.7. So if D is defined from $D_*, \langle D_i : i < \kappa\rangle$, as in 2.6, and $\bar\lambda = \langle \lambda_{i,j} : (i,j) \in u\rangle$, $\lambda_i = T_{D_i}(\langle \lambda_{i,j} : j < \kappa_i\rangle)$, $\lambda = T_{D_*}(\langle \lambda_i : i < \kappa\rangle)$, then $\lambda = T_D(\bar\lambda)$.

Question 2.8. *We may wonder whether, for Claim 2.3:*

(1) *If λ is regular can we add: Each λ_η is regular. Can we moreover get the representation to be true?*

(2) *Can we add the case of nice filters and get exact representations? (On nice filters/ideal, see [17, V], [15].)*

See below 2.11, 2.12(2)*, but first*

Observation 2.9. (1) *Assume that*

(a) *J_1, J_2 are ideals on κ with intersection J.*

(b) *$f \in {}^\kappa(Ord\backslash\omega)$.*

Then $T_J(f) = \mathrm{Min}\{T_{J_1}(f), T_{J_2}(f)\}$.

(2) *If* (a) *above holds and $\bar\lambda$ is a J-representation of λ, then for some $\ell \in \{1,2\}$, $\bar\lambda$ is a J_ℓ-representation of λ.*

(3) *Assume $\lambda = T_{J_1}(\bar\lambda)$ and J_1 a σ-complete ideal on $\kappa, \sigma > \aleph_0$ and $J_2 = \{A \subseteq \kappa : A \in J_1$ or $A \in J_1^+$ and $T_{J_1+(\kappa\backslash A)}(\bar\lambda) > \lambda\}$. Then J_2 is a σ-complete ideal on κ (extending J_1 and, consequently, $\kappa \notin J_2$).*

Proof. Easy; e.g.

(1) By using pairing functions. $\square$

Claim 2.10. *If $\bar{\lambda} = \langle \lambda_i : i < \kappa \rangle$ is a J-representation of λ, $\lambda \geq \mathrm{cf}(\lambda) > 2^\kappa$ and $\lambda > \mathrm{cf}(\lambda) \Rightarrow \mathrm{cf}(\lambda) > 2^{2^\kappa}$ and J is an $\aleph_1$-complete ideal on κ then for some $\aleph_1$-complete ideal $J' \supseteq J$, the sequence $\bar{\lambda}$ is a J'-representation of λ and $\prod_{i<\kappa} \lambda_i / J'$ has true cofinality $\mathrm{cf}(\lambda)$ (hence $\{i : \lambda_i \text{ singular}\} \in J'$ when λ is regular). We can replace $\aleph_1$ by $\sigma = \mathrm{cf}(\sigma)$.*

Proof. First assume λ is regular. By the pcf theorem there is $u^* \subseteq \kappa$ such that $\lambda \notin \mathrm{pcf}\{\mathrm{cf}(\lambda_i) : i \in \kappa \backslash u^*\}$ and $\mathrm{cf}(\lambda) \geq \mathrm{cf}(\prod_{i \in u^*} \lambda_i)$. First, assume that $\bar{\lambda}$ is a $(J + u^*)$-representation of λ, so $\lambda = T_{J+u^*}(\bar{\lambda})$, but this implies that for some $u \in (J + u^*)^+$ we have that $\prod_{i \in u} \lambda_i / ((J + u^*) \restriction u)$ has true cofinality $\mathrm{cf}(\lambda)$ by [20, 1.1], actually a variant of [20, 1.1](2); see the e-version.
[Why? Apply [20, 1.1](2) with $J + u^*$, $\langle \lambda_i : i < \kappa \rangle$, 2^κ here standing for J, $\langle f(i) : i < \kappa \rangle$, μ in the assumption there. This is acceptable, as clearly the assumption there holds, so by the conclusion of [20, 1.1] there are $u \in (J + u^*)^+$ and $\bar{\lambda}' = \langle \lambda'_i : i < \kappa \rangle$ satisfying $2^\kappa < \lambda'_i = \mathrm{cf}(\lambda'_i) \leq \lambda_i$ such that $\lambda = \mathrm{tcf}(\prod_{i \in u} \lambda'_i, \leq_{J+u^*})$. By the choice of u^*, $\{i \in u : \lambda'_i = \lambda_i\} \in J + u^*$, a contradiction to "$\bar{\lambda}$ is a $(J + u)$-presentation".]

So "$\bar{\lambda}$ is a $(J + u^*)$-representation of λ" is impossible. Hence by 2.9(3) we have $\bar{\lambda} \restriction u^*$ is a $(J \restriction u^*)$-representation of λ, so without loss of generality $u^* = \kappa$, so $\lambda \geq \max \mathrm{pcf}\{\lambda_i : i < \kappa\}$. Let $J_1 = \{u \subseteq \kappa : u \in J$ or $u \notin J$ and $\mathcal{P}(u) \cap J_2 \subseteq J\}$ where $J_2 = \{u \subseteq \kappa : u \in J$ or for some $v \in J$ we have $\lambda > \max \mathrm{pcf}\{\lambda_i : i \in u \backslash v\}\}$. Clearly J_1, J_2 are ideals on κ extending J and by the definition we have $J_1 \cap J_2 = J$. So by 2.9 for some $\ell \in \{1, 2\}$, $\bar{\lambda}$ is a J_ℓ-representation of λ.

<u>Case 1</u>: $\ell = 1$.

So $\lambda = T_{J_1}(\bar{\lambda})$ and hence by [20, 1.1](1) for some $v \in (J_1)^+$ we have that $\prod_{i \in v} \lambda_i / (J_1 \restriction v)$ has true cofinality λ. So if $u \in J_2 \backslash J$, then for some $u' \subseteq u$, $u' \in J$ and $\lambda > \max \mathrm{pcf}(\{\lambda_i : i \in u \backslash u'\})$, but by the definition of J_1 we have $J_1 \restriction (u \backslash u') = J \restriction (u \backslash u')$ and hence $(v \cap (u \backslash u')) \bigcup (v \cap u') = v \cap u \in J$. But this means $v \cap u \in J$ for every $u \in J_2 \backslash J$ and hence $v \in J_1$, a contradiction.

<u>Case 2</u>: $\ell = 2$.

By the pcf theorem, $\prod_{i<\kappa} \lambda_i / J_2$ has true cofinality λ.

So we have finished the proof for the case λ is regular; hence we are left with the case $\lambda > \mathrm{cf}(\lambda) > 2^{2^\kappa}$. Let $\langle \lambda_\epsilon : \epsilon < \mathrm{cf}(\lambda) \rangle$ be an increasing sequence of regular cardinals $> 2^\kappa$ with limit λ. For every $\epsilon < \mathrm{cf}(\lambda)$ there is $\bar{\lambda}^\epsilon = \langle \lambda^\varepsilon_i : i < \kappa \rangle \in \Pi_{i<\kappa} \lambda_i$ such that $T_J(\bar{\lambda}^\varepsilon) = \lambda_\varepsilon$ and $f <_J \bar{\lambda}^\varepsilon \Rightarrow T_J(f) < \lambda_\varepsilon$. Hence there is an $\aleph_1$-complete ideal J_ϵ on κ extending J such that $T_{J_\epsilon}(\bar{\lambda}^\epsilon) = \lambda_\epsilon$ but $f \in \Pi_{i<\kappa}(\bar{\lambda}^\epsilon) \Rightarrow T_{J_\epsilon}(f) < \lambda_\epsilon$ and $\mathrm{tcf}(\Pi_{i<\kappa} \lambda^\epsilon_i) = \lambda_\varepsilon$.

As we are assuming $\mathrm{cf}(\lambda) > 2^{2^\kappa}$, clearly for some ideal J_* on κ the set $\{\epsilon < \mathrm{cf}(\lambda) : J_\epsilon = J_*\}$ is unbounded in $\mathrm{cf}(\lambda)$.

Without loss of generality $J_\epsilon = J_*$ for every $\epsilon < \mathrm{cf}(\lambda)$. Clearly $\varepsilon < \zeta \Rightarrow \{i : \lambda^\varepsilon_i = \lambda^\zeta_i\} \in J_*$, so by 0.10(1) it follows that without loss of generality $\langle \bar{\lambda}^\epsilon : \epsilon < \mathrm{cf}(\lambda) \rangle$ is a $\leq_{J_*}$-increasing sequence and hence by 0.10(2) it has a lub f modulo J; without loss of generality f is $\leq \bar{\lambda}$, and without loss of generality it is a sequence of cardinals — call it $\bar{\lambda}' = \langle \lambda'_i : i < \kappa \rangle$.

Clearly $\mathrm{cf}(\prod_{i<\kappa} \lambda'_i / J_*) = \mathrm{cf}(\lambda)$ and $T_J(\bar{\lambda}') = \lambda = T_{J_*}(\bar{\lambda}')$.

Let $A = \{i < \kappa : \lambda'_i = \lambda_i\}$. Now if $A \in (J_*)^+$ and $I = J_* + (\kappa \setminus A)$ satisfies $f \in \prod_{i<\kappa} \lambda'_i \Rightarrow T_I(f) < \lambda$, i.e., I is as required, we are done. Otherwise, by monotonicity $T_I(\bar{\lambda}) > \lambda$ and there is $f_1 \in \Pi_{i<\kappa} \lambda_i$ satisfying $T_I(f_1) \geq \lambda$.

Note that if $\kappa \setminus A \in J^+_*$ then $T_{I+A}(\bar{\lambda}') \geq \lambda$; hence letting $f_2 = (f_1 \restriction A) \cup (\bar{\lambda}' \restriction (\kappa \setminus A))$ we have $f_2 \in \prod_{i<\kappa} \lambda_i$ but $T_{J_*}(f_2) \geq \lambda$; but by the choice of $f = \bar{\lambda}'$, for some $\varepsilon < \mathrm{cf}(\lambda)$ we have $\bar{\lambda}' \leq \bar{\lambda}^\varepsilon \mod J$. But we have $T_{J_*}(\bar{\lambda}^\varepsilon) = \lambda'_\varepsilon$, $T_{J_*}(\bar{\lambda}') = \lambda > \lambda_\varepsilon$; contradiction. $\square$

Conclusion 2.11. In 2.3 we can add:

(j) if λ is regular then every λ_η is regular and for $\eta \in \mathcal{T} \backslash \max_{\mathcal{T}}$ we have $\lambda_\eta = \mathrm{tcf}(\prod_{\alpha<\kappa_\eta} \lambda_{\eta^\frown\langle\alpha\rangle} / \mathcal{D}_\eta)$.

Now 2.8(2) (and also 2.8(1)) are answered by:

Claim 2.12. *Assume*[2] *that the pair* $(\mathbf{K}[S], \mathbf{V})$ *fails the covering lemma for every* $S \subseteq \beth_2(\kappa)$ *(or less). Then in* 2.3 *we can add:*
(1) *If* $\mathfrak{a} \subseteq \mathrm{Reg} \cap \lambda \backslash (2^{<\theta})^+$ *and* $|\mathfrak{a}| < \theta$ *and* $\mathfrak{a} \in J^\sigma_{<\lambda}[\mathfrak{a}]$, $\lambda > 2^{<\theta}$ *then for some* $\kappa = \mathrm{cf}(\kappa) \in [\sigma, \theta)$ *and* κ*-complete ideal* J *on* κ *and* $\bar{\lambda} = \langle \lambda_i : i < \kappa \rangle$ *we have:*
(a) $\mathrm{cf}(\lambda) > 2^{<\theta} \Rightarrow \mathrm{cf}(\lambda_i) > 2^{<\theta}$,
(b) $\langle \lambda_i : i < \kappa \rangle$ *is an exact true* J*-representation of* λ,
(c) *if* λ *is regular then every* λ_i *is regular.*
(2) *For any normal filter* D *on* κ *we can further demand in part* (1) *that for some function* $\iota : \kappa \to \kappa$ *the pair* (J, ι) *is nice and* $A \in D \Rightarrow \iota^{-1}(\kappa \backslash A) \in J$.
(2A) *If* $\sigma \geq \partial = \mathrm{cf}(\partial) > \aleph_0$ *and* D *is a normal filter on* ∂ *we can add in part* (1) *that the pair* (J, ι) *is nice and* $A \in D \Rightarrow \iota^{-1}(\kappa \backslash A) \in J$. *Similarly for normal filters on* $[\sigma]^{<\partial}$.
(3) *So in* 2.3*, we can strengthen clauses* (f), (g) *to*
(f)$^+$ *if* $A \in \mathcal{D}^+_\eta$, $\eta \in \mathcal{T} \backslash \max_{\mathcal{T}}$ *then* $T_{\mathcal{D}_\eta + A}(\langle \lambda_{\eta^\frown<\alpha>} : \alpha < \kappa_\eta \rangle) = \lambda_\eta$ *(and hence the parallel result for* D_η*),*
(g)$^+$ *if* $\eta \in \mathcal{T} \backslash \max_{\mathcal{T}}$, $A \in \mathcal{D}^+_\eta$ *and* $f \in \prod_{\alpha < \kappa_\eta} \lambda_{\eta^\frown<\alpha>}$ *then* $T_{\mathcal{D}_\eta + A}(f) < \lambda_\eta$ *(and hence the parallel result for* D_η*), this being an exact representation; see Definition* 2.4(2),
(h)$^+$ *for each* $\eta \in \mathcal{T} \backslash \max_{\mathcal{T}}$ *for some function* $\iota_\eta : \kappa_\eta \to \kappa_\eta$ *the pair* $(\mathcal{D}_\eta, \iota_\eta)$ *is nice,*
(j) *if* λ *is regular then every* λ_η *is regular.*

Proof. By [15, Section 3], very close to [16].

(1) There are D a κ-complete filter on κ and $\lambda_i < \lambda$ such that $T_D(\langle \lambda_i : i < \kappa \rangle) \geq \lambda$ (by the pcf theorem). By the results quoted above without loss of generality D is a normal filter on $\kappa \times \kappa$ for the function ι defined by $\iota(\alpha, \beta) = \alpha$. Now we can choose $(D, \bar{\lambda})$ such that D is a nice filter on $\kappa \times \kappa$, $T_D(\bar{\lambda}) \geq \lambda$ and $\mathrm{rk}^3_D(\bar{\lambda})$ is minimal. As $D_1 \subseteq D_2 \Rightarrow T_{D_1}(\bar{\lambda}) \leq T_{D_2}(\bar{\lambda})$ without loss of generality $\mathrm{rk}^3_D(\bar{\lambda}) = \mathrm{rk}^2_D(\bar{\lambda})$ and so $A \in D^+ \Rightarrow \mathrm{rk}^3_{D+A}(\bar{\lambda}) = \mathrm{rk}^2_{D+A}(\bar{\lambda}) = \mathrm{rk}^3_D(\bar{\lambda})$ and $T_{D+A}(\bar{\lambda}) \geq T_D(\bar{\lambda})$. If $T_{D+A}(\bar{\lambda}) > \lambda$ then for some $f \in \prod \bar{\lambda}$, $T_{D+A}(f) \geq \lambda$, let $\bar{\lambda}' = \langle f(i) : i < \kappa \rangle$, so $\bar{\lambda}' <_D \bar{\lambda}$; hence $\mathrm{rk}^3_{D+A}(\bar{\lambda}') < \mathrm{rk}^3_{D+A}(\bar{\lambda})$ and we get a contradiction).

(2), (2A), (3) Left to the reader. □

Claim 2.13. *We can add in* 2.3

(k) *for each* $\eta \in \mathcal{T} \backslash \max_{\mathcal{T}}$*, for every unbounded* $A \subseteq \kappa_\eta$ *the set* $\cup\{[\omega\alpha, \omega\alpha + \omega) : \alpha < \kappa\}$ *belongs to* $\mathcal{D}^+_\eta$.

Proof. By [17, VII, Section 1]. □

Definition 2.14. Assume $\aleph_1 \leq \mathrm{cf}(\sigma) = \sigma < \theta < \lambda$.

(1) Let $\mathfrak{d}_0(\lambda) = \mathfrak{d}^0_{\sigma,\theta}(\lambda) = \{\kappa : \kappa \in \mathrm{Reg} \cap \theta \backslash \sigma$ such that we cannot find $\langle (\lambda_\eta, \mathcal{D}_\eta, D_\eta, \kappa_\eta) : \eta \in \mathcal{T} \rangle$ as in 2.3 with $\mathcal{D}_\eta$ being κ_η-complete for $\eta \in \mathcal{T}$ satisfying $\kappa \notin \{\kappa_\eta : \eta \in \mathcal{T}\}\}$ (and so finite!; see below). If $\sigma = \aleph_1$ we may omit it. If $\sigma = \aleph_1, \theta = \lambda$ we may omit both.

(2) Let $\mathfrak{d}_1(\lambda) = \mathfrak{d}^1_{\sigma,\theta}(\lambda) = \{\kappa : \kappa = \mathrm{cf}(\kappa) < \lambda$ and for arbitrarily large $\alpha < \lambda$ we have $\kappa \in \mathfrak{d}_0(|\alpha|)\}$; note that if $\mathrm{cf}(\lambda) > \aleph_0$ we can deduce the finiteness of $\mathfrak{d}_1(\lambda)$ from the finiteness of $\mathfrak{d}_0(\lambda)$.

(3) Let $\mathfrak{d}'_\ell(\lambda) = \mathfrak{d}'_{\ell,\sigma,\theta}(\lambda) = \mathfrak{d}_\ell(\lambda) \cup \{\aleph_0\}$ for $\ell = 0, 1$; similarly $\mathfrak{d}'_{\ell,\theta}(\lambda)$.
If we omit σ we mean $\sigma = \aleph_1$.

Observation 2.15. (1) *If* $\aleph_1 \leq \sigma = \mathrm{cf}(\sigma) < \theta < \lambda$ *then* $\mathfrak{d}^0_{\sigma,\theta}(\lambda)$ *is finite.*
(2) *If* $\mathrm{cf}(\lambda) > \aleph_0$ *then* $\mathfrak{d}^1_{\sigma,\theta}(\lambda)$ *is finite; we use* 2.17(1), 2.18(4).

Proof. (1) Let $\langle (\lambda_\eta, \mathcal{D}_\eta, D_\eta, \kappa_\eta) : \eta \in \mathcal{T} \rangle$ be as in 2.3. If $\mathfrak{d}^0_{\sigma,\theta}(\lambda)$ is infinite we can find pairwise distinct $\kappa^n \in \mathfrak{d}^0_{\sigma,\theta}(\lambda)$ for $n < \omega$. For every $\eta \in \max_{\mathcal{T}}$ there is a finite $w_\eta \subseteq \omega$ such that $\{\kappa_{\eta \restriction \ell} : \ell < \ell g(\eta)\} \cap \{\kappa^n : n < \omega\} \subseteq \{\kappa^n : n \in w_\eta\}$; in fact, $|w_\eta| \leq \ell g(\eta)$.

By an easy partition theorem on trees we can finish. (That is, we use $\mathrm{dp}_{\mathcal{T}} : \mathcal{T} \to \mathrm{Ord}$ which is defined by $\mathrm{dp}(\eta) = \cup\{\mathrm{dp}_{\mathcal{T}}(\eta^\frown\langle\alpha\rangle) : \eta^\frown\langle\alpha\rangle \in \mathcal{T}\}$; it is well defined as $\mathcal{T}$ has no ω-branch (as $\eta \triangleleft \nu \Rightarrow \lambda_\eta > \lambda_\nu$). Now by induction on the ordinal α we can observe that if $\rho \in \mathcal{T}$ and $\mathrm{dp}_{\mathcal{T}}(\rho) \leq \alpha$ then there is $\mathcal{T}' = \mathcal{T}'_\rho \subseteq \mathcal{T}$ and $w \subseteq \omega$

[2] Without this assumption much more follows; see [17, V].

finite such that $\rho \in \mathcal{T}', \mathcal{T}'$ closed under initial segments and $\rho \trianglelefteq \nu \in \mathcal{T}' \Rightarrow \{\alpha < \kappa_\nu : \nu\frown\langle\alpha\rangle \in \mathcal{T}'\} \in \mathcal{D}^+_\nu$ and $\max_{\mathcal{T}'} \subseteq \max_{\mathcal{T}}$ and $\nu \in \max_{\mathcal{T}'} \Rightarrow w_\nu = w$. For $\rho \in \max_{\mathcal{T}}$ this is trivial; otherwise use that $\mathcal{D}$ is $\aleph_1$-complete. For $\rho =<>$ we get $\mathcal{T}' = \mathcal{T}'_{<>}$; let $D'_\eta = D_\eta \restriction \{\nu : \eta \trianglelefteq \nu \in \max_{\mathcal{T}'}\}, \mathcal{D}'_\eta = \mathcal{D}_\eta \restriction \{\alpha : \eta\frown\langle\alpha\rangle \in \mathcal{T}'\}$ for $\eta \in \mathcal{T}'$, so that for every $n \in \omega\backslash w$, $\langle\lambda_\eta, \mathcal{D}'_\eta, D'_\eta, \kappa_\eta : \eta \in \mathcal{T}'\rangle$ exemplifies $\kappa^n \notin \mathfrak{d}^0_{\sigma,\theta}(\lambda)$ (on stronger partition theorems see [6]).
(2) Similar. $\square$

Remark 2.16. Note that if $\langle(\lambda_\eta, \mathcal{D}_\eta, D_\eta, \kappa_\eta) : \eta \in \mathcal{T}\rangle$ represent λ strictly (see Definition 2.17(1)), the regular cardinal κ does not belong to $\{\kappa_\eta : \eta \in \mathcal{T}\}$ and $\langle\mathcal{U}_i : i < \kappa\rangle$ is an increasing sequence of subsets of $\max_{\mathcal{T}}$, then $\cup\{\mathcal{U}_i : i < \kappa\} \in D^+_{<>} \Rightarrow (\exists i < \kappa)(\mathcal{U}_1 \in D^+_{<>})$. We can make this central.

Definition 2.17. Let $\mathbf{r} = \langle(\lambda_\eta, \mathcal{D}_\eta, D_\eta, \kappa_\eta) : \eta \in \mathcal{T}\rangle$ be a σ-representation of λ.

(1) We say $\mathbf{r}$ is strict if $\mathcal{D}_\eta$ is κ_η-complete for each $\eta \in \mathcal{T}$ (for $\eta \in \max_{\mathcal{T}}$ this is uninteresting).
(2) We say that $\bar{\mathfrak{d}} = \langle\mathfrak{d}_\eta : \eta \in \mathcal{T}\rangle$ is a strong/weak witness for $\mathbf{r}$ when:
  (a) each $\mathfrak{d}_\eta$ is a set of regular cardinals,
  (b) if $\theta \in \mathrm{Reg}\backslash\mathfrak{d}_\eta$ and $\eta \in \mathcal{T}\backslash \max_{\mathcal{T}}$ <u>then</u>
    <u>stronger version</u>: $\{\alpha : \alpha < \kappa_\eta \text{ and } \theta \in \mathfrak{d}_{\eta\frown<\alpha>} \vee \theta = \lambda_{\eta\hat{}<\alpha>}\} = \emptyset \bmod \mathcal{D}_\eta$,
    <u>weak version</u>: $A = \{\alpha : \alpha < \kappa_\eta \text{ and } \theta \notin \mathfrak{d}_\eta \wedge \theta \neq \lambda_{\eta\frown<\alpha>}\}$ belongs to $\mathcal{D}^+_\eta$ and $T_{\mathcal{D}^*_\eta+A}(\langle\lambda_{\eta\frown<\alpha>} : \alpha < \kappa_\eta\rangle) = \lambda_\eta$.
(3) We say above that $\bar{\mathfrak{d}}$ is finitary when each $\mathfrak{d}_\eta$ is finite.
(4) We say that $\mathbf{r}$ has a $\mathfrak{d}$-witness if it has a finitary weak witness $\bar{\mathfrak{d}}$ with $\mathfrak{d}_{<>} = \bar{\mathfrak{d}}$.

Observation 2.18. *Assume $\theta > \sigma = \mathrm{cf}(\sigma) > \aleph_0$ and $\mathrm{cf}(\theta) \notin [\sigma, \theta)$ and $\lambda \geq \partial$, $\mathrm{cf}(\partial) \geq \theta$ and $(\forall\alpha < \theta)(\forall\beta < \partial)(|\beta|^{|\alpha|} < \partial)$.*

(1) *If $\mathbf{r}$ is a $(\partial, \theta, \sigma)$-representation of λ <u>then</u> for some $\mathbf{s}$:*
  (a) *$\mathbf{s}$ is a (∂, σ)-representation,*
  (b) *$\mathcal{T}^{\mathbf{s}} = \mathcal{T}^{\mathbf{r}}$,*
  (c) *$\mathcal{D}^{\mathbf{s}}_\eta \supseteq \mathcal{D}^{\mathbf{r}}_\eta$ for $\eta \in \mathcal{T}^{\mathbf{r}}$ (moreover $\mathcal{D}^{\mathbf{s}}_\eta = D^{\mathbf{r}}_\eta + A_\eta$ for some $A_\eta \in D^+_\eta$),*
  (d) *$\mathbf{s}$ has a weak witness $\bar{\mathfrak{d}}$.*
(2) *If we waive the moreover in clause* (c) *<u>then</u> we can add*
  (e) *$\mathbf{s}$ is true.*
(3) *There is a sequence $\langle\bar{\mathfrak{d}}_n : n < n_*\rangle$ when $n_* < \omega$ such that*
  (a) *$\mathfrak{d}_n \subseteq Reg\ \cap \theta\backslash\sigma$ is finite,*
  (b) *λ has a $(\partial, \theta, \sigma)$-representation $\mathbf{x}_n$ with $\mathfrak{d}_n$-witness for each n (and moreover is true),*
  (c) *if $\kappa \in \mathrm{Reg} \cap \theta\backslash(\sigma \cup \mathfrak{d}'_{\sigma,\theta}(\lambda))$ then for some $n, \kappa \notin \mathfrak{d}_n$.*
(4) *λ has a strict $(\partial, \theta, \sigma)$-representation.*

Proof. (1) We choose to proceed by induction on γ: for $\eta \in \mathcal{T}$ with $\mathrm{dp}_{\mathcal{T}}(\eta) = \gamma$ choose $(A_\eta, \mathfrak{d}_\eta)$ such that

(*) (a) $\mathfrak{d}_\eta$ is a finite subset of $\mathrm{Reg} \cap \theta\backslash\sigma$,
  (b) if $\eta \in \max_{\mathcal{T}}$ then $\mathfrak{d}_\eta = A_\eta = \emptyset$ (or is not defined),
  (c) if $\eta \in \mathcal{T}\backslash \max_{\mathcal{T}}$ then
    [(α) $A_\eta \in \mathcal{D}^+_\eta$,
    (β) $\kappa_\eta \in \mathfrak{d}_\eta$,
    (γ) if $\kappa \in \mathrm{Reg} \cap \theta\backslash(\sigma \cup \mathfrak{d}_\eta)$ then $\lambda_\eta = T_{\mathcal{D}_\eta+A_\eta}(\langle\lambda_{\eta\frown<\alpha>} : \alpha < \kappa_\eta\rangle)$ and $\lambda_\eta < T_{\mathcal{D}_\eta+A_\eta+\{\alpha<\kappa_\eta:\kappa\in\mathfrak{d}_{\eta\hat{}<\alpha>}\}}(\langle\lambda_{\eta\frown<\alpha>} : \alpha < \kappa_\eta\rangle)$.

If we succeed in that we define $\mathbf{s}$ as $\langle(\lambda_\eta, \mathcal{D}_\eta + A_\eta, D'_\eta, \kappa_\eta) : \eta \in \mathcal{T}^{\mathbf{r}}\rangle$ with D'_η computed from the rest and $\bar{\mathfrak{d}} = \langle\mathfrak{d}_\eta : \eta \in \mathcal{T}^{\mathbf{r}}\rangle$, clearly they are as required.

So let us carry out the definition. If $\eta \in \max_{\mathcal{T}}$ this is trivial. Otherwise $\langle\mathfrak{d}_{\eta\frown<\alpha>} : \alpha < \kappa_\eta\rangle$ is well defined and we let $A^n_\eta = \{\alpha < \kappa_\eta : |d_{\eta\frown<\alpha>}| = n\}$, so $\langle A^n_\eta : n < \omega\rangle$ is a partition of κ_η, but D_η is σ-complete, $\sigma > \aleph_0$ and hence by 2.9 for some $n = n(\eta)$ we have $\lambda_\eta = T_{D_\eta+A^n_\eta}(\langle\lambda_{\eta\frown<\alpha>} : \alpha < \kappa_\eta\rangle)$. Now we can choose A_η from $\{A : A \subseteq A^n_\eta, A \in D^+_\eta$ and $\lambda_\eta = T_{D_\eta+A}(\langle\lambda_{\eta\frown<\alpha>} : \alpha < \kappa_\eta\rangle)\}$ such that $\cap\{\mathfrak{d}_{\eta\hat{}<\alpha>} : \alpha \in A_\eta\}$ has minimal size.

Lastly, let $\mathfrak{d}_\eta = \cap\{\mathfrak{d}_{\eta\frown<\alpha>} : \alpha \in A_\eta\}$; it is easy to check that it is as required.
(2) Use each time Claim 2.10 in the end.
(3) We try to choose $\mathfrak{d}_n$ by induction on $n < \omega$ such that

⊛ (a) $\mathfrak{d}_n \subseteq \text{Reg} \cap \theta \backslash \sigma$ is finite,
(b) λ has a $(\partial, \theta, \sigma)$-representation with a $\mathfrak{d}_n$-witness,
(c) if $n > 0$ then $\cap\{\mathfrak{d}_m : m < n\} \nsubseteq \mathfrak{d}_n$,
(d) under (a) + (b) + (c), the set $\cap\{\mathfrak{d}_m : m \leq n\}$ has minimal size.

By part (1) and 2.3 we can choose $\mathfrak{d}_0$ and clearly for some $n^* \leq |\mathfrak{d}_0| + 1$, $\mathfrak{d}_n$ is defined iff $n < n^*$; so we are done.
(4) We repeat the proof of 2.3; however using 1.10 we need to ask there somewhat more: for some $\kappa_1 \in [\sigma, \kappa]$, the ideal J is κ_1-complete and $\kappa \backslash \kappa_1 \in J$ (so we can use $\langle \lambda_i : i < \kappa_1 \rangle$. As in the proof of 1.10, we use [17] without loss of generality $\kappa_1 = |\mathfrak{a}|$ is minimal. Now if $\mathfrak{a}$ is not the union of any $< \kappa_1$ member of $\{\mathfrak{b}_\theta[\mathfrak{a}] : \theta \in \lambda \cap \text{pcf}(\mathfrak{a})\}$, let $\langle \lambda_i : i < \kappa_1 \rangle$ list $\mathfrak{a}$ and let J be the κ_1-complete ideal on κ_1 generated by $\{\{i < \kappa_1 : \lambda_i \in \mathfrak{b}_\theta[\mathfrak{a}]\} : \theta \in \lambda \cap \text{pcf}(\mathfrak{a})\}$. If $\mathfrak{a}$ is $\cup\{\mathfrak{b}_{\theta_\varepsilon}[\mathfrak{a}] : \varepsilon < \varepsilon^*\}$ where $\varepsilon^* < \kappa_1$ and $\theta_\varepsilon \in \text{pcf}(\mathfrak{a}) \cap \lambda$ for $\varepsilon < \varepsilon^*$ then, by [17, I,Section 1], we can replace $\mathfrak{a}$ by $\{\theta_\varepsilon : \varepsilon < \varepsilon^*\}$. □

3. The main results ($\mathbf{Pr}_\ell$, $\mathbf{Ps}_\ell$)

In this section we prove the main theorem:

Theorem 3.1. *Assume that $\mu > \aleph_0$ is strong limit and $\lambda \geq \text{cf}(\lambda) > \mu$. Then for some $\kappa < \mu$ and finite $\mathfrak{d} \subseteq \text{Reg} \cap \mu$ there is $\bar{\mathcal{P}}$ such that*

$(*)_{\lambda,\bar{\mathcal{P}}}$ $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \lambda \rangle, \mathcal{P}_\alpha \subseteq [\alpha]^{<\mu}$ *and* $|\mathcal{P}_\alpha| < \lambda$, $\mathcal{P}_\alpha$ *is increasing,*
$(*)^{\mathfrak{d},\kappa}_{\lambda,\bar{\mathcal{P}},h}$ *for every set $A \subseteq \lambda$ of cardinality $< \mu$ there is $\mathbf{c} : [A]^2 \to \kappa$ such that:*
if $B \subseteq A$ has no last element, $\mathbf{c} \restriction [B]^2$ is constant and $\delta = \sup(B)$ satisfies $\text{cf}\delta \notin \mathfrak{d}$, then $B \in \mathcal{P}_\delta$.

The theorem states that for all cardinals λ with cofinality greater than μ, there is a "good" sequence $\langle \mathcal{P}_\delta : \delta < \lambda \rangle$, which, in spite of each $\mathcal{P}_\delta$ being small, captures many small subsets of λ. "Many" here means that for every small set $A \subseteq \lambda$ there is a pair-coloring $\mathbf{c} : [A]^2 \to \kappa$ such that each monochromatic $B \subseteq A$ with no last element and with supremum δ belongs to $\mathcal{P}_\delta$ — provided that $\text{cf}\delta$ is not one of the finitely many exceptional cofinalities.

Thus, if $\theta^+ < \mu$ is not one of the exceptional cofinalities for λ, then, by the Erdős–Rado theorem, for every $A \subset \lambda$ with $(2^\theta)^+ \leq |A| < \mu$ there is some $B \in [A]^{\theta^+}$ with $\sup B = \delta$ which belongs to $\mathcal{P}_\delta$, and, moreover, each of the initial segments of B with no last element belongs to a suitable $\mathcal{P}_{\delta'}$ — provided that the cofinality of δ' is not one of the exceptional cofinalities.

Note that the theorem is closely related to the RGCH in the following way. By the RGCH, for some $\kappa < \mu$ there is a family $\mathcal{P} \subseteq [\lambda]^{<\mu}$ of cardinality λ and closed under taking subsets such that every subset of λ of cardinality $< \mu$ is the union of $\leq \kappa$ members of $\mathcal{P}$. So if we define, for $\delta < \lambda$ of cofinality $< \mu$, the family $\mathcal{P}_\delta$ as the family of $u \in \mathcal{P}$ which are unbounded subsets of δ, then we get $|\mathcal{P}_\delta| \leq \lambda$ and the sequence $\langle \mathcal{P}_\delta : \delta < \lambda \rangle$ has a property stronger than what we promise in the present theorem: if $A \subseteq \lambda$ has cardinality $< \mu$ then there is a *unary* function $\mathbf{c}$ from A to κ (obtained by partitioning A to κ cells from $\mathcal{P}$) such that if $B \subseteq A$ is $\mathbf{c}$-monochromatic and without a last element then $B \in \mathcal{P}_{\sup B}$ (with no exceptions on $\text{cf} \sup B$).

So what we gain in the present theorem in comparison with the RGCH is mainly the strict inequality $|\mathcal{P}_\delta| < \lambda$. In return we have to exclude finitely many "exceptional" cofinalities and settle for a weaker sense of "many subsets of A" — rather than all monochromatic sets with respect to some unary coloring, we take all monochromatic sets with respect to some *binary* coloring.

Remark 3.2. (1) The proof of 3.1 is simpler if λ is regular.
(2) The conclusion of 3.1 implies that for $\lambda > \mu$, for all but finitely many $\kappa = \text{cf}(\kappa) < \mu$, $\text{Pr}_1(\lambda, \text{cf}(\lambda), \kappa)$ holds (see Definition 3.9(b)).

Similarly

Claim 3.3. *In fact in 3.1 we can choose $\mathfrak{d} = \mathfrak{d}'_{0,\mu}(\lambda)$; see Definition 2.14(1).*

Proof of 3.1: Without loss of generality, $\text{cf}(\mu) = \aleph_0$ (this is no loss by the Fodor lemma; if μ is singular we may use $\mu > \text{cf}(\mu)$ or replace $\aleph_1$ by $(\text{cf}(\mu))^+$).

We choose $h : \text{cf}(\lambda) \to \lambda$ such that

$\boxdot_1$ (a) If λ is regular then h is the identity.
(b) If λ is singular then $\langle h(\alpha) : \alpha < \mathrm{cf}(\lambda)\rangle$ is an increasing continuous sequence of cardinals with limit λ.

We shall choose below $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \mathrm{cf}(\lambda)\rangle$ such that $\mathcal{P}_\alpha \subseteq [h(\alpha)]^{<\mu}$, $|\mathcal{P}_\alpha| < \lambda$ and $\mathcal{P}_\alpha$ is increasing with α. Now for each $\alpha < \mathrm{cf}(\lambda)$ we can find $\mathcal{P}^1_\alpha \subseteq [h(\alpha+1)]^{<\mu}$ of cardinality $< \lambda$ such that for some $\kappa_0(*) < \mu$

$\boxdot_2$ If $A \subseteq h(\alpha+1)$, $|A| < \mu$, then there is $\mathbf{c} : A \to \kappa_0(*)$ such that every $B \subseteq A$ for which $\mathbf{c} \restriction B$ is constant belongs to $\mathcal{P}^1_\alpha$.

We then, for $\gamma < \lambda$, let $\mathcal{P}'_\gamma = (\mathcal{P}_{\alpha(\gamma)} \cup \mathcal{P}^1_{\alpha(\gamma)}) \cap [\gamma]^{<\mu}$ where $\alpha(\gamma) = \mathrm{Min}\{\alpha < \mathrm{cf}(\lambda) : \gamma \le h(\alpha)\}$. Now

$\boxdot_3$ for $\langle \mathcal{P}'_\gamma : \gamma < \lambda\rangle$ to be as required it is enough that, for some $\kappa < \mu$ and $\bar{\mathcal{P}}_\alpha$ and finite $\mathfrak{d} \subseteq \mathrm{Reg} \cap \mu$, we have
$(**)_{\lambda,\bar{\mathcal{P}}}$ $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \mathrm{cf}(\lambda)\rangle$, $\mathcal{P}_\alpha \subseteq [h(\alpha)]^{<\mu}$, $|\mathcal{P}_\alpha| < \lambda$, $\mathcal{P}_\alpha$ increasing,
$(**)^{\mathfrak{d},\kappa}_{\lambda,\bar{\mathcal{P}},h}$ for every A satisfying $A \subseteq \mathrm{cf}(\lambda)$ or (more generally) $A \subseteq \lambda$& $(\forall \alpha \in A)[\mathrm{Min}(A \backslash (\alpha+1)) < \mathrm{Min}(\mathrm{Rang}(h) \backslash (\alpha+1))]$ and satisfying $|A| < \mu$ <u>there is</u> $\mathbf{c} : [A]^2 \to \kappa$ such that:
if $B \subseteq A$ has no last element, $\mathbf{c} \restriction [B]^2$ is constant and $\delta = \cup\{\mathrm{Min}\{(\alpha+1) : \gamma < h(\alpha)\} : \gamma \in B\}$ has cofinality $\in (\mathrm{Reg} \cap \mu \backslash \mathfrak{d})$ and so $B \subseteq h(\delta)$, then $B \in \mathcal{P}_\delta$.

So let us turn to proving $(**)_{\lambda,\bar{\mathcal{P}}}$, $(**)^{\mathfrak{d},\kappa}_{\lambda,\bar{\mathcal{P}},h}$.

We first prove the desired conclusions for cardinal λ such that

$\boxtimes_\lambda$ $\mathfrak{a} \subseteq \lambda \cap \mathrm{Reg} \backslash \mu$& $|\mathfrak{a}| < \mu \Rightarrow \mathfrak{a} \in J^{\aleph_1}_{<\lambda^+}[\mathfrak{a}]$.

Let $\bar{M} = \langle M_\alpha : \alpha < \mathrm{cf}(\lambda)\rangle$ be such that

$\circledast_1$ (a) $M_\alpha \prec (\mathcal{H}(\chi), \in)$ is increasing continuous,
(b) $\lambda \in M_\alpha$, $\|M_\alpha\| < \lambda$, $h(\alpha) \subseteq M_\alpha$,
(c) $\langle M_\alpha : \alpha \le \beta\rangle \in M_{\beta+1}$,
(d) (α) if λ is regular then $M_\alpha \cap \lambda \in \lambda$,
(β) if λ is singular then $\lambda_\alpha + 1 \subseteq M_{\alpha+1}$,
where $\lambda_\alpha = \mathrm{Min}\{\chi$: if $\mathfrak{a} \subseteq (h(\alpha+1)+1) \cap \mathrm{Reg} \backslash \mu$
and $|\mathfrak{a}| < \mu$
then $\mathfrak{a} \in J^{\aleph_1}_{<\chi}[\mathfrak{a}]$ and $\chi \ge \|M_\alpha\|\}$.

We let $\mathcal{P}_\alpha =: M_{\alpha+1} \cap [h(\alpha)]^{<\mu}$ and $\mathfrak{d} = \{\aleph_0\}$ and $\kappa = \aleph_0$, and will show that $\langle \mathcal{P}_\alpha : \alpha < \mathrm{cf}(\lambda)\rangle, \mathfrak{d}$ are as required. Now $(*)_{\lambda,\bar{\mathcal{P}}}$ of the claim holds trivially. To prove $(*)^{\mathfrak{d},\kappa}_{\lambda,\bar{\mathcal{P}},h}$ let $A \subseteq \lambda$, $\mathrm{otp}(A) < \mu$ be as there and let $\{\alpha_\varepsilon : \varepsilon < \varepsilon(*)\}$ list A in increasing order. Hence there is $\langle \beta_\varepsilon : \varepsilon < \varepsilon(*)\rangle$ increasing continuous such that $\beta_\varepsilon < \mathrm{cf}(\lambda), h(\beta_\varepsilon) \le \alpha_\varepsilon < h(\beta_{\varepsilon+1})$. By the assumption (and 1.9, i.e., [17, II,5.4]), if λ is regular then for each $\varepsilon < \varepsilon(*)$ there is a set $\mathcal{P}^\varepsilon \subseteq [h(\beta_\varepsilon)]^{<\mu}$ of cardinality $< \lambda$ such that every $a \in [h(\beta_\varepsilon)]^{<\mu}$ is equal to the union of $\le \kappa$ of them (by the choice of κ and $\boxtimes$); hence without loss of generality $\mathcal{P}^\varepsilon \in M_{\beta_\varepsilon+1}$ and hence $\mathcal{P}^\varepsilon \subseteq M_{\beta_\varepsilon+1} \cap [h(\beta_\varepsilon)]^{<\mu} = \mathcal{P}_{\beta_\varepsilon}$. If λ is singular, using clause $(d)(\beta)$ we get the same conclusion. So there is a sequence $\langle A_{\varepsilon,i} : i < \kappa\rangle$ such that $A_{\varepsilon,i} \in \mathcal{P}_{\beta_\varepsilon}$, $A \cap \alpha_\varepsilon = A \cap h(\beta_\varepsilon) = \cup\{A_{\varepsilon,i} : i < \kappa\}$. We defined $\mathbf{c} : [A]^2 \to \kappa$ as follows: for $\varepsilon < \zeta < \varepsilon(*)$, $\mathbf{c}(\{\alpha_\varepsilon, \alpha_\zeta\}) := \mathrm{Min}\{i : \alpha_\varepsilon \in A_{\zeta,i}\}$. So assume $B \subseteq A$ and $\mathbf{c} \restriction [B]^2$ is constantly $j < \kappa$ and $\delta = \sup(B)$ has cofinality $\theta \in \mathrm{Reg} \cap \mu \backslash \mathfrak{d}$. Clearly $\alpha_\varepsilon \in B \Rightarrow \alpha_\varepsilon \cap B \subseteq \{\alpha_\zeta : \zeta < \varepsilon$ and $\mathbf{c}\{\alpha_\zeta, \alpha_\varepsilon\} = j\} \subseteq A_{\varepsilon,j} \in \mathcal{P}_{\beta_\varepsilon}$. But $\mathcal{P}_\alpha = M_{\alpha+1} \cap [h(\alpha)]^{<\mu}$ is closed under subsets and hence $\alpha_\varepsilon \in B \Rightarrow \alpha_\varepsilon \cap B \in \mathcal{P}_{\beta_\varepsilon}$.

Now in $M_{\delta+1}$ we can define a tree $\mathcal{T}$; it has $\mathrm{otp}(B)$ levels;

$$\text{the } i - \text{ level is } \{a \in M_\delta : a \subseteq \delta \text{ and } \mathrm{otp}(a) = i\}$$

and the order is $\triangleleft$, as they are initial segments.

So by the assumptions (and [20, Section 2]), as $\aleph_1 \le \mathrm{cf}(\delta) < \mu$, the number of δ-branches of $\mathcal{T}$ is $< \lambda$, so as $\mathcal{T} \in M_{\delta+1}$, every δ-branch of $\mathcal{T}$ belongs to $M_{\delta+1}$, and hence $B \in M_{\delta+1}$, which implies that $B \in \mathcal{P}_\delta$, as required.

Now we prove the statement in general.

We prove this by induction on λ. For $\lambda = \mu^+$ this is trivial by the first part of the proof. So assume $\lambda > \mu^+$ and the conclusion fails, but the first part does not apply.

In particular, for some $\mathfrak{a} \subseteq \mathrm{Reg} \cap \lambda \backslash \mu$, $|\mathfrak{a}| < \mu$ and $\mathfrak{a} \notin J^{\aleph_1}_{<\lambda^+}[\mathfrak{a}]$. Hence recalling $\mathrm{cf}(\lambda) > \mu$, by 1.10 + 1.12 + 2.3 + 2.10 + proof of 2.18(4), for some $\kappa = \mathrm{cf}(\kappa) \in [\aleph_1, \mu)$ we have:

$(*)_1$ there is a sequence $\langle \lambda_i : i < \kappa \rangle$ and an κ-complete filter D on κ such that
(a) $T_D(\prod_{i<\kappa} \lambda_i) = \lambda$,
(b) $\lambda_i < \lambda$ and $\mathrm{cf}(\lambda_i) > \mu$ (see 2.3),
(c) if $\lambda'_i < \lambda_i$ for $i < \kappa$, <u>then</u> $T_D(\langle \lambda'_i : i < \kappa \rangle) < \lambda$,
(d) $\mathrm{tcf}(\prod_{i<\kappa} \lambda_i, <_D) = \mathrm{cf}(\lambda)$.

Clearly we can find $\langle h_i : i < \kappa \rangle$ such that

$(*)_2$ h_i is an increasing continuous function from $\mathrm{cf}(\lambda_i)$ to λ_i.

Let

$$D_1 = \Big\{ A : A \in D \text{ or } A \notin D,\ A \in D^+ \text{ and} \tag{9}$$

$$T_{D+(\kappa \setminus A)}(\bar\lambda') \geq \lambda \text{ for some } \bar\lambda' \in \prod_{i<\kappa} \lambda_i \Big\}. \tag{10}$$

Clearly (by 2.9),

$(*)_3$ D_1 is an $\aleph_1$-complete filter on κ extending D and we can replace D by $D + A$ whenever $A \in D_1^+$.

By the induction hypothesis applied to λ_i, as $\lambda_i > \mu$ there is a pair $(\kappa_i, \mathfrak{d}_i)$ as in the conclusion. Without loss of generality $\kappa_i^\kappa = \kappa_i$. So for some $m(*) < \omega$ and $\kappa(*) < \mu$ the set $\{i < \kappa : |\mathfrak{d}_i| = m(*), \kappa_i \leq \kappa(*)\} \in D_1^+$, so without loss of generality

$(*)_4$ $i < \kappa \Rightarrow |\mathfrak{d}_i| = m(*) \& \kappa_i = \kappa(*)$.

By (d) of $(*)_1$ there is $\bar f$ such that

$(*)_5$ $\bar f = \langle f_\alpha : \alpha < \mathrm{cf}(\lambda) \rangle$ is $<_D$-increasing and cofinal in $\prod_{i<\kappa} \lambda_i$ and if $\delta < \mathrm{cf}(\lambda)$, $\mathrm{cf}(\delta) < \mu$ and $\bar f \restriction \delta$ has a $<_D$-eub, then f_δ is such a $<_D$-eub and we let $f'_\alpha \in \prod_{i<\kappa} \lambda_i$ be $f'_\alpha(i) = \mathrm{Min}(\mathrm{Rang}(h_i) \setminus f_\alpha(i))$ and $f''_\alpha \in \prod_{i<\kappa} \mathrm{cf}(\lambda_i)$ be defined by $f''_\alpha(i) = h_i^{-1}(f'_\alpha(i))$.

For each i let $\bar{\mathcal{P}}^i = \langle \mathcal{P}^i_\alpha : \alpha < \mathrm{cf}(\lambda_i) \rangle$ be such that $(**)_{\lambda_i, \bar{\mathcal{P}}^i} + (**)^{\mathfrak{d}_i, \kappa(*)}_{\lambda_i, \bar{\mathcal{P}}^i, h_i}$ holds. We now choose M_α for $\alpha < \mathrm{cf}(\lambda)$ such that

$\circledast_2$ (a) $M_\alpha \prec (\mathcal{H}(\chi), \in)$, $M_\alpha \cap \mathrm{cf}(\lambda) \in \mathrm{cf}(\lambda) + 1$
(b) $\|M_\alpha\| < \lambda$, M_α is increasing continuous, $\beta < \alpha \Rightarrow h(\beta) \subseteq M_{\alpha+1}$ and
$\beta < \alpha \Rightarrow \langle M_\beta : \beta \leq \alpha \rangle \in M_{\alpha+1}$,
(c) the following objects belong to M_α : $\langle \bar{\mathcal{P}}^i : i < \kappa \rangle$:
$\langle \lambda_i, h_i : i < \kappa \rangle$, $\bar f$, D and μ,
(d) if $A \in D_1^+$, and so $T_{D+A}(\langle |\mathcal{P}^i_{f_\alpha(i)}| : i < \kappa \rangle) < \lambda$, then $T_{D+A}(f_\alpha) + 1 \subseteq M_{\alpha+1}$
(remember $\mathrm{cf}(\lambda) > \mu > 2^\kappa$).

Let $\mathfrak{d}^* = \{\theta : \theta = \kappa$ or $\{i < \kappa : \theta \notin \mathfrak{d}_i\} = \emptyset \bmod D_1\}$; it should be clear that $|\mathfrak{d}^*| \leq m(*) + 1$.
Let $\mathcal{P}_\alpha = M_{\alpha+1} \cap [h(\alpha)]^{<\mu}$ and $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \mathrm{cf}(\lambda) \rangle$.

It is enough now to prove that $(*)^{\mathfrak{d}^*, \kappa(*)}_{\lambda, \bar{\mathcal{P}}, h}$ holds.

Let $A \subseteq \lambda$, $|A| < \mu$ be as in the assumption and we should find $\mathbf{c} : [A]^2 \to \kappa(*)$ as required. For $i < \kappa$ let $A_i = \{f_\alpha(i) : \alpha \in A\}$, so $A_i \in [\lambda_i]^{<\mu}$ and hence there is $\mathbf{c}_i : [A_i]^2 \to \kappa(*)$ as required. Recalling that $\kappa(*)^\kappa = \kappa(*)$, we can choose $\mathbf{c} : [A]^2 \to \kappa(*)$ such that

$\circledast_3$ if $\alpha_1 < \beta_1, \alpha_2 < \beta_2$ are from A and $\mathbf{c}\{\alpha_1, \beta_1\} = \mathbf{c}\{\alpha_2, \beta_2\}$ <u>then</u>
(i) if $i < \kappa$ then $f_{\alpha_1}(i) < f_{\beta_1}(i) \equiv f_{\alpha_2}(i) < f_{\beta_2}(i)$,
(ii) if $i < \kappa$ then $f_{\alpha_1}(i) > f_{\beta_1}(i) \equiv f_{\alpha_2}(i) > f_{\beta_2}(i)$,
(iii) if $i < \kappa$ and $f_{\alpha_1}(i) < f_{\beta_1}(i)$ then $\mathbf{c}_i\{f_{\alpha_1}(i), f_{\beta_1}(i)\} = \mathbf{c}_i\{f_{\alpha_2}(i), f_{\beta_2}(i)\}$.

Let $\theta \in \mathrm{Reg} \cap \mu \backslash \mathfrak{d}^*$ and let $\delta < \mathrm{cf}(\lambda)$ and $B \subseteq A \cap h(\delta)$ be such that $\mathbf{c} \restriction [B]^2$ is constantly j and $\theta = \mathrm{cf}(\delta)$ and $\delta = \sup(B)$. We can replace D by $D + \{i < \kappa : \theta \notin \mathfrak{d}_i\}$. So for some set $a \subseteq \kappa$ we have

$\circledast_4$ if $\alpha < \beta$ are from B then $a = \{i < \kappa : f_\alpha(i) < f_\beta(i)\}$.

Clearly $a \in D$ and $\langle f_\alpha(i) : \alpha \in B\rangle$ is increasing for each $i \in a$. Note that by $\circledast_4$ for each $i \in a$ the sequence $\langle f_\alpha(i) : \alpha \in B\rangle$ is increasing and let $B_i = \{f_\alpha(i) : \alpha \in B\}$, so $\delta_i =: \sup(B_i)$ has cofinality θ and $\mathbf{c}_i \restriction [B_i]^2$ is constant. Hence by the choice of $\bar{\mathcal{P}}^i$ clearly $B_i \in \mathcal{P}^i_{\delta_i}$. Also as $a \in D$ by $\circledast_4$ and D being κ-compact $\bar{f} \restriction \delta$ has a $\leq_D$-eub f', $f'(i) =: \cup\{f_\alpha(i) : i \in B\}$, and hence[3] $a' := \{i \in a : f_\delta(i) = \delta_i\}$ belongs to D. Now $|\mathcal{P}^i_{f_\delta(i)}| < \lambda_i$, and hence $T_D(\langle \mathcal{P}^i_{f_\delta(i)} : i < \kappa\rangle) < \lambda$, so there is $\mathcal{F} \subseteq \prod_{i<\kappa} \mathcal{P}^i_{f_\delta(i)}$, $|\mathcal{F}| < \lambda$ such that for every $g \in \prod_{i<\kappa} \mathcal{P}^i_{f_\delta(i)}$ there is $g' \in \mathcal{F}$ such that $\{i : g(i) = g'(i)\} \in D^+$. So $\bar{f}, \bar{\mathcal{P}} \in M_0 \subseteq M_{\delta+1}$ and hence $f_\delta \in M_{\delta+1}$; hence without loss of generality $\mathcal{F} \in M_{\delta+1}$. By the choice of $M_{\delta+1}$, i.e., clause (b) of $\circledast_2$, it follows that $\mathcal{F} \subseteq M_{\delta+1}$. We can define $g \in \prod_{i<\kappa} \mathcal{P}^i_{f_\delta(i)}$ by letting $i \in a' \Rightarrow g(i) = B_i$. So there is $g' \in \mathcal{F} \subseteq M_{\delta+1}$ such that $b = \{i : g(i) = g'(i)\} \in D^+$ and hence $b \cap a' \in D^+$. That is $b' =: \{i \in a' : g'(i) = B_i\} \in D^+$. Clearly $b' \in M_{\delta+1}$ (as $\mu \in M_{\delta+1}$ and hence $\mathcal{H}(\mu) \subseteq M_{\delta+1}$) and $g' \in M_{\delta+1}$; hence $g' \restriction b' \in M_{\delta+1}$, and hence also the set B^* belongs to $M_{\delta+1}$ where

$$B^* =: \{\gamma < \lambda : \{i \in b' : f_\gamma(i) \in g'(i) = g(i) = B_i\} \in D^+\}.$$

Now $|B^*| \leq \prod_{i<\kappa} B_i < \mu$ and $\alpha \in B \Rightarrow \alpha \in B^*$. But as $B^* \in M_{\delta+1}$ every subset of B^* belongs to $M_{\delta+1}$; hence $B \in M_{\delta+1}$ and so $B \in \mathcal{P}_\delta$, as required.

Proof of 3.3.

The proof is a variant of the proof of 3.1. In the case where $\boxtimes_\lambda$ holds, recall that $\aleph_0 \in \mathfrak{d}(= \mathfrak{d}'_{0,\mu}(\lambda))$, so what is proved there suffices.

In the general case, when $\neg\boxtimes_\lambda$, there is $\langle \lambda_i : i < \kappa\rangle$ as in $(*)_1$, but we would like to choose $\mathfrak{a}$ carefully. By 2.18 we can find $\bar{\mathfrak{d}}, \bar{\lambda}_n, \bar{d}_n$ for $n < n^*$ such that

$\boxtimes$ (a) $\bar{\mathfrak{d}} = \mathfrak{d}_n : n < n^*\rangle$ where $\mathfrak{d}_n \subseteq \mathrm{Reg} \cap \mu$ is finite,
(b) $\mathfrak{d}'_{0,\mu}(\lambda) = \cap\{\mathfrak{d}_n : n < n^*\}$,
(c) the $\bar{\lambda}_n = \langle \lambda^n_i : i < \kappa\rangle$ satisfy
 (α) $T_D(\prod_{i<\kappa} \lambda^n_i) = \lambda$,
 (β) $\lambda^n_i < \lambda$ and $\mathrm{cf}(\lambda^n_i) > \mu$,
 (γ) if $\lambda'_i < \lambda^n_i$ for $i < \kappa$ then $\lambda > T_D(\langle \lambda'_i : i < \kappa\rangle)$,
 (δ) $\mathrm{tcf}(\prod_{i<\kappa} \lambda^n_i, <_D) = \mathrm{cf}(\lambda)$,
(d) $\bar{d}_n = \langle d^n_i : i < \kappa\rangle$ satisfies
(e) if $\theta \in \mathrm{Reg} \cap \mu \backslash d_n$ then $\{i < \kappa : \theta \in d^n_i\} = \emptyset \bmod D$.

We then continue as there using $\bar{f}^n = \langle f^n_\alpha : \alpha < \lambda\rangle$ for $n < n^*$ as there (so $\mathbf{c}\{\alpha, \beta\}$ will be defined f^n_α, f^n_β for $n < n^*$).

Discussion 3.4. (1) Note that in a sense what was done in [10], i.e., $I[\lambda]$ large for $\lambda = \mu^+$, is done in 3.1 for any λ with $\mathrm{cf}(\lambda) > \mu$.
(2) We may consider replacing $\mathfrak{d}$ by $\{\aleph_0\}$ in 3.1. The base of the induction is clear ($\mathrm{pcf}_{\aleph_1}$-inaccessibility). So eventually we have f_δ for it as above $\langle f_\alpha : \alpha \in B\rangle$, the hard case is $\mathrm{cf}(\mathrm{otp}(B)) = \kappa$; we have the induced $h_* \in {}^\kappa\kappa$ such that $\alpha < \kappa \Rightarrow \{i : d < h_*(i)\} \in D$, but $(\forall^D i)[\mathrm{cf}(h_*(i)) = \aleph_0]$ (otherwise using niceness of the filter (which without loss of generality holds), etc., we are done).

Note that this problem appears even in the simplest version of our problem: "assume μ is the strong limit of cofinality $\aleph_1$ (or $\kappa \in [\aleph_1, \mu)$) and $2^\mu = \mu^+$; does it follow that $\diamondsuit_{S^{\mu^+}_{\mathrm{cf}(\mu)}}$ holds?" See [12], Cummings–Dzamonja–Shelah [1], Dzamonja–Shelah [3]; and [23], Section 1, for a positive answer for a somewhat weaker property.

But if $\kappa = \mathrm{cf}(\kappa) > \aleph_0$ and in 2.14 we use $D = D_\kappa + S^\kappa_{\aleph_1}$, for each $\alpha < \kappa$ we should consider $\iota(t)$; if D-positively we have $\iota(t) \leq h_*(t)$ we are done. But if $\iota(t) > h_*(t)$, D-positively, then on some $A \in D^+$, $h_* \restriction A$ is constant.

[3] Note that here we use $\theta \neq \kappa$ — in fact this is the only point that we use it at; if we could avoid it, then $\mathfrak{d}$ could be chosen as $\{\aleph_0\}$.

Conclusion 3.5. Assume $\mu < \lambda$, μ is strong limit $> \aleph_0$, λ is regular (or just $\mathrm{cf}(\lambda) > \mu$). Then for some $\kappa < \mu$ and finite $\mathfrak{d} \subseteq \mathrm{Reg} \cap \mu$ to which $\aleph_0$ belongs (in fact $(\mathfrak{d}^0_\mu(\lambda) \cup \{\aleph_0\})$ is acceptable), there is $\bar{\mathcal{F}}$ such that

$\circledast^{\mu,\mathfrak{d},\kappa}_{\lambda,\bar{\mathcal{F}}}$ (a) $\bar{\mathcal{F}} = \langle \mathcal{F}_\alpha : \alpha < \lambda \rangle$, $|\mathcal{F}_\alpha| < \lambda$ for $\alpha < \lambda$, $E = \lambda$ if $\lambda = \mathrm{cf}(\lambda)$, E is a club of λ if $\mathrm{cf}(\lambda) < \lambda$,
(b) $\mathcal{F}_\alpha \subseteq \{f : f$ a partial function from α to α, $|\mathrm{Dom}(f)| < \mu\}$, $\mathcal{F}_\alpha$ closed under restriction,
(c) for every $A \subseteq \lambda$, $|A| < \mu$ and $f : A \to \lambda$ for some $\mathbf{c} : [A]^2 \to \kappa$ we have

$\boxdot_1$ if $B \subseteq A$, $\delta = \sup(B) \in E$, $\mathbf{c} \restriction [B]^2$ is constant,
$[\alpha \in B \Rightarrow f(\alpha) < \delta]$ and $\mathrm{cf}(\delta) \notin \mathfrak{d}$ then $f \restriction B \in \mathcal{F}_\delta$
and $\alpha \in B \Rightarrow f \restriction (B \cap \alpha) \in \mathcal{F}_\alpha$.

Proof. We use the result of 3.1.

For clause (c) we use the pairing function pr on λ such that $\mathrm{pr}(\alpha, \beta) < \mathrm{Max}\{\omega, \alpha + |\alpha|, \beta + |\beta|\}$ to replace the function f in clause (c) by the set $\{\mathrm{pr}(\alpha, f(\alpha)) : \alpha \in A\}$ and first we restrict ourselves to δ in some club E of λ (the range of h in 3.1's notation) such that $\delta \in E \Rightarrow |\delta|$ divides δ (and hence δ is closed under pr); so if $B \subseteq \lambda$, $\sup(B) \in E$ we are done. The other cases are easier as without loss of generality if $\alpha < \delta \in E$, then $\alpha +$ $\mathrm{Min}\{\chi : \mu \geq |\alpha|$ and if $\mathfrak{a} \subseteq$ Reg $\cap\, \chi^+$, $|\mathfrak{a}| < \mu$, $\mathrm{pcf}_{\chi^+\text{-comp}}(\mathfrak{a}) \subseteq \mu^+\} < \delta$, and it is easy to finish as in the proof of 3.1. $\square$

Conclusion 3.6. Assume that μ is strong limit, $\lambda = \lambda^{<\mu}$ (equivalently $\lambda = \lambda^\mu$) and $c\ell : [\lambda]^{<\mu} \to [\lambda]^{<\mu}$ satisfies for notational simplicity $c\ell(B) = \cup\{c\ell(B \cap (\alpha + 1)) : \alpha \in B\}$ and $B_1 \trianglelefteq B_2 \Rightarrow B_1 \subseteq c\ell(B_1) \subseteq c\ell(B_2)$.

Then in 3.5 we can add to (a), (b) and (c) also

(d) $\mathbf{g}$ is a function from $\{f \restriction u : f \in {}^\lambda\lambda$ and $u \in [\lambda]^{<\mu}\}$ to λ,
(e) for every $f : \mathrm{cf}\,\lambda \to \lambda$ for some $g_f : [\lambda]^{<\mu} \to \lambda$ (in fact $g_f(u) = \mathbf{g}(f \restriction c\ell(u))$ we have
   $\boxtimes$ for every $A \subseteq \mathrm{cf}\,\lambda$ of cardinality $< \mu$ such that $\alpha \in A \Rightarrow g_f(A \cap \alpha) < \alpha$, for some function $\mathbf{c} : [A]^2 \to \kappa$ we have
   $\otimes$ if $B \subseteq A$, $\mathbf{c} \restriction [B]^2$ is constant and B has no last element,

$$\delta = \sup(B) \text{ has cofinality} \notin \mathfrak{d} \text{ } \underline{\text{then}}\ f \restriction c\ell(B)$$
$$\text{belong to } \mathcal{F}_\delta \text{ and } \alpha \in B \Rightarrow f \restriction c\ell(B \cap \alpha) \in \cup\{\mathcal{F}_\beta : \beta < \delta\},$$

(f) if λ is regular then there is a sequence $\bar{C} = \langle C_\delta : \delta \in S \rangle$ such that
   (α) $S \subseteq S^* = \{\delta < \lambda$: $\mathrm{cf}(\delta) \in [\aleph_1, \mu)\}$,
   (β) C_δ is a club of δ of order type $\mathrm{cf}(\delta)$
   and in clause (e) we can add:
   (γ) $f \restriction c\ell(C_\delta) \in \mathcal{F}_\delta$ and
   (δ) $\alpha \in C_\delta \Rightarrow f \restriction c\ell(C_\delta \cap \alpha) \in \bigcup\limits_{\beta<\delta} \mathcal{F}_\beta$ and
   (ε) $\alpha \in$ $\mathrm{nacc}(C_{\delta_1}) \cap$ $\mathrm{nacc}(C_{\delta_2}) \Rightarrow C_{\delta_1} \cap \alpha = C_{\delta_2} \cap \alpha$,
   (ζ) if $\alpha <$ $\mathrm{cf}(\lambda)$ is limit, $\mathrm{cf}(\alpha) \notin \mathfrak{d}$ then $\{C_\delta \cap \alpha : \alpha \in$ $\mathrm{acc}(C_\delta)\}$ has cardinality $< \lambda$,
   (η) if $B \subseteq \lambda$, $|B| < \mu$ then for some $\mathbf{c} : [B]^2 \to \kappa$ if $B' \subseteq B$ has no last member and $\mathbf{c} \restriction [B']^2$ is constant and cf $(\sup(B'))\notin \mathfrak{d}$ then $\sup(B') \in S$.

Proof. We repeat the proof of 3.1.

Choose $h : \mathrm{cf}(\lambda) \to \lambda$ and $\langle M_\alpha : \alpha <$ $\mathrm{cf}(\lambda)\rangle$ as in the proof of 3.1 but add the requirement that $c\ell \in M_0$ and still use $\mathcal{F}_\alpha = M_{\alpha+1} \cap \{f : f$ a partial function from α to α with domain of cardinality $< \mu\}$.

Choose $\mathbf{g}$ such that

$\boxtimes$ (a) $\mathbf{g}$ is a function from $\{f \restriction u : f \in {}^\lambda\lambda$ and $u \in [\lambda]^{<\mu}\}$ onto λ,
(b) $f_1 \subseteq f_2 \in$ $\mathrm{Dom}(\mathbf{g}) \Rightarrow \mathbf{g}(f_1) \leq \mathbf{g}(f_2)$ and
(c) for each $\alpha < \lambda$ for some $f \in$ $\mathrm{Dom}(\mathbf{g})$ we have $\mathbf{g}(f) = \alpha \&$
$$(\forall f')[\mathbf{g}(f') = \alpha \Rightarrow f' \subseteq f]$$
(d) if $f : B_2 \to \lambda$ and $B_1 \triangleleft B_2$ then $\mathbf{g}(B_1) < \mathbf{g}(B_2)$,
(e) $\mathbf{g}(f) = \alpha \Rightarrow$ $\mathrm{Dom}(f) \subseteq \alpha$.

Without loss of generality $\mathbf{g} \in M_0$, so clause (d) (of the conclusion of 3.6) holds trivially; let us prove clause (e). As $\mathbf{g}$ has already been chosen, we are given $A \subseteq \operatorname{cf}(\lambda)$ of cardinality $< \mu$ and $f : A \to \lambda$ such that $\alpha \in A \Rightarrow \mathbf{g}(f \restriction c\ell(A \cap \alpha)) < \alpha$.

Now $\alpha \mapsto \mathbf{g}(f \restriction c\ell(A \cap \alpha))$ is an increasing function from A to λ; let $A' = \{\mathbf{g}(f \restriction c\ell(A \cap \alpha)) : \alpha \in A\}$ and let $\mathbf{c}' : [A']^2 \to \kappa$ be as proved to exist in 3.1 and by $\mathbf{c} : [A]^2 \to \kappa$ be defined by $\mathbf{c}\{\alpha, \beta\} = \mathbf{c}'\{\mathbf{g}(f \restriction c\ell(A \cap \alpha)), \mathbf{g}(f \restriction c\ell(A \cap \beta))\}$.

It is easy to check that $\mathbf{c}$ is as required. We turn to proving clause (f) of the claim. Now there is a function $F : {}^{\omega}\lambda \to \lambda$ such that for any $\bar{\alpha} \in {}^{\omega}\lambda$ for every large enough $n < \omega$ there are $m_0 < m_1 < m_2 < \ldots < \omega$ which are $> n$ and $\alpha_n = F(\alpha_{m_0}, \alpha_{m_1}, \ldots)$, by [4]. For any $u \in [\lambda]^{<\mu}$ we define $c\ell_*(u)$ as follows: let $u^{+\mathbf{g}} = u \cup \{\mathbf{g}(1_v) : v \subseteq u \cap \alpha$ for some $\alpha \in u\}$ and let $c\ell_*(u)$ be the minimal set v such that $u^{+\mathbf{g}} \subseteq v$ and $[\delta = \sup(v \cap \delta) < \sup(u^{+\mathbf{g}}) \& \operatorname{cf}(\delta) \leq |u| \Rightarrow \delta \in v]$ and $[\mathbf{g}(1_w) \in v \& |w| \leq |u| \Rightarrow w \subseteq v]$ and $\bar{\alpha} \in {}^{\omega}v \Rightarrow F(\bar{\alpha}) \in v$; so $|c\ell_*(u)| \leq (|u|^+ + 2)^{\aleph_0}$.

In the proof above we can replace $c\ell$ by $c\ell_* \circ c\ell$. Now if $\delta < \lambda$, $\aleph_0 < \operatorname{cf}(\delta) < \mu$ for some club C^*_δ of δ of order type $\operatorname{cf}(\delta)$ we have: if $C \subseteq C^*_\delta$ is a club of δ then $c\ell_* \circ c\ell(C) = c\ell_* \circ c\ell(C^*_\delta)$ (which exists by the choice of F). Alternatively, let $C'_\delta = \cap\{c\ell_*(C) : C$ a club of $\delta\}$; however, C'_δ seemingly has order type just $< (\operatorname{cf}(\delta)^{\aleph_0})^+$. Now if C^*_δ satisfies $(\forall \alpha \in C^*_\delta)(\mathbf{g}(1_{C^*_\delta \cap \alpha}) < \delta)$ then we can find C^{**}_δ, C_δ such that:

$\circledast_1$ $C^{**}_\delta \subseteq c\ell_* \circ c\ell(C^*_\delta)$ is a club of δ of order type $\operatorname{cf}(\delta)$ such that $\alpha \in \operatorname{nacc}(C^{**}_\delta) \Rightarrow \sup((C^{**}_\delta \cup \{0\}) \cap \alpha) < \mathbf{g}(1_{((C^{**}_\delta \cup \{0\}) \cap \alpha)}) < \alpha$,

$\circledast_2$ C_δ is $\{\mathbf{g}(1_{((C^{**}_\delta \cup \{0\}) \cap \alpha)}) : \alpha \in \operatorname{nacc}(C^{**}_\delta)\} \cup \operatorname{acc}(C^{**}_\delta)$.

Clearly

$\circledast_3$ $C_\delta \subseteq c\ell_*(B)$ whenever $B \subseteq \delta = \sup(B)$,

$\circledast_4$ if $\alpha \in \operatorname{nacc}(C_{\delta_1}) \cap \operatorname{nacc}(C_{\delta_2})$ then $C_{\delta_1} \cap \alpha = C_{\delta_2} \cap \alpha$.

We are done, as we have used $c\ell_* \circ c\ell$ and

$(*)$ if $\delta < \lambda$, $\aleph_0 < \operatorname{cf}(\delta) < \mu$ and B is an unbounded subset of δ then $C_\delta \subseteq c\ell_*(B)$. $\square$

Remark 3.7. (1) In 3.1, 3.5, 3.6 if λ is regular, then

$$A_{\bar{M}} = \{\delta : \delta < \lambda, \operatorname{cf}(\delta) < \delta \text{ and there is} \tag{11}$$

$$u \subseteq \delta = \sup(u), \operatorname{otp}(u) < \delta \text{ and } (\forall \alpha < \delta)(u \cap \alpha \in M_\alpha)\} \tag{12}$$

belongs to $I[\lambda]$ and the δ mentioned in $(*)^{\mathfrak{d},\kappa}_{\lambda,\bar{\mathcal{P}}}$ of 3.1,(c) of 3.5 necessarily belongs to $A_{\bar{M}}$. So $A_{\bar{M}}$, for ordinals of cofinality $\in \operatorname{Reg} \cap \mu \setminus \mathfrak{d}$, contains "almost all of them" in the appropriate sense.

(2) We can use them to upgrade if $\{\delta < \omega_2 : S_\kappa^{\beth^+_\delta} \in I(\beth^+_\delta)\}$ then $S_\kappa^{\beth^+_{\omega_1}} \in I[\beth^+_{\omega_1+1}]$ when $\kappa = \operatorname{cf}(\kappa) > \aleph_1$; see [20].

Main Conclusion 3.8. (1) If μ is strong limit and $\lambda = \lambda^{<\mu}$ <u>then</u> for all but finitely many regular $\kappa < \mu$ (actually $\kappa \notin \mathfrak{d}^0_\mu(\lambda) \cup \{\aleph_0\}$ is enough) we have $\operatorname{Ps}_1(\lambda, \kappa)$, see Definition 3.9 below.

(2) We also get $\operatorname{Ps}_1(\operatorname{cf}(\lambda), \lambda, \kappa)$ when $\kappa > \aleph_0$.

Proof. By 3.5, 3.6. $\square$

Definition 3.9. (1) $\operatorname{Ps}_1(\lambda, \kappa)$ means that $\operatorname{Ps}_2(\lambda, S)$ for some stationary $S \subseteq S^\lambda_\kappa$.

(2) $\operatorname{Ps}_2(\lambda, S)$ means that for some $\bar{C} = \langle C_\delta : \delta \in S\rangle$ we have $\operatorname{Ps}_3(\lambda, \bar{C})$.

(3) $\operatorname{Ps}_3(\lambda, \bar{C})$ means that for some $\bar{\mathcal{F}}$ we have $\operatorname{Ps}_4(\lambda, \bar{C}, \bar{\mathcal{F}})$.

(4) $\operatorname{Ps}_4(\lambda, \bar{C}, \bar{\mathcal{F}})$ means that for some S:

- (a) S is a stationary subset of λ,
- (b) $\bar{C}$ has the form $\langle C_\delta : \delta \in S\rangle$,
- (c) $\bar{\mathcal{F}}$ has the form $\bar{\mathcal{F}} = \langle \mathcal{F}_\alpha : \alpha \in S\rangle$,
- (d) C_δ is a club of δ of order type $\operatorname{cf}(\delta)$ and $\alpha \in \operatorname{nacc}(C_{\delta_1}) \cap \operatorname{nacc}(C_{\delta_2}) \Rightarrow \alpha \cap C_{\delta_1} = \alpha \cap C_{\delta_2}$,
- (e) $\mathcal{F}_\delta$ is a set of functions from C_δ to δ of cardinality $< \lambda$,
- (f) if $f : \lambda \to \lambda$ then for stationarily many $\delta \in S$ we have $f \restriction C_\delta \in \mathcal{F}_\delta$.

(5) $\operatorname{Ps}_4(\lambda, \mu, h, \bar{C}, \bar{\mathcal{F}})$ is defined similarly (and λ is regular) except that

(e)$_1$ h is an increasing continuous function from λ to μ with limit μ,
(e)$_2$ $\mathcal{F}_\delta$ is a set of functions from δ to $h(\delta)$ of cardinality $< \mu$,
(f) if $f : \lambda \to \mu$ then for stationarily many $\delta \in S$ we have $f \restriction C_\delta \in \mathcal{F}_\delta$.

(6) If in (5) we omit h we mean some h.
(7) $\mathrm{Ps}_1(\lambda, \mu, \kappa)$, $\mathrm{Ps}_2(\lambda, \mu, S)$, $\mathrm{Ps}_3(\lambda, \mu, \bar{C})$ are defined in parallel.

Definition 3.10. Pr_ℓ are defined similarly except not using $\bar{C}$ and $\mathcal{F}_\delta$ is a set of functions from some unbounded subset of δ into δ (or $h(\delta)$), that is:

(1) $\mathrm{Pr}_1(\lambda, \kappa)$ means that $\mathrm{Pr}_2(\lambda, S)$ for some stationary $S \subseteq S^\lambda_\kappa$.
(2) $\mathrm{Pr}_2(\lambda, S)$ means that for some $\bar{\mathcal{F}} = \langle \mathcal{F}_\alpha : \alpha \in S\rangle$ we have $\mathrm{Pr}_4(\lambda, \bar{\mathfrak{F}})$.
(3) $\mathrm{Pr}_4(\lambda, \bar{\mathcal{F}})$ means that for some S:
  (a) S is a stationary subset of λ,
  (b) $\bar{\mathcal{F}}$ has the form $\bar{\mathcal{F}} = \langle \mathcal{F}_\alpha : \alpha \in S\rangle$,
  (c) $\mathcal{F}_\delta$ is a set of cardinality $< \lambda$ of functions from some unbounded subset of δ to δ,
  (d) if $f : \lambda \to \lambda$ then for stationarily many $\delta \in S$ we have $f \restriction A \in \mathcal{F}_\delta$ for some $A \subseteq \delta = \sup(A)$.
(4) $\mathrm{Pr}_4(\lambda, \mu, h, \bar{\mathcal{F}})$ is defined similarly except that
  (c)$_1$ h is an increasing continuous function from λ to μ with limit μ,
  (c)$_2$ $\mathcal{F}_\delta$ is a set of cardinality $< \lambda$ of functions from some unbounded subset of δ to $h(\delta)$,
  (d) if $f : \lambda \to \mu$ then for stationarily many $\delta \in S$ we have $f \restriction A \in \mathcal{F}_\delta$ for some $A \subseteq \delta = \sup(A)$.
(5) If in (4) we omit h we mean some h.

Observation 3.11. *If* $Ps_4(\lambda, \bar{C}, \bar{\mathcal{F}})$, $\lambda_1 = \mathrm{cf}(\lambda) < \lambda$, $\bar{C} = \langle C_\delta : \delta \in S\rangle$, $(\forall \delta \in S)[\mathrm{cf}(\delta) > \aleph_0]$, $h : \lambda_1 \to \lambda$ *is increasing continuous with limit* λ, $S' = \{\delta < \lambda_1 : h(\delta) \in S\}$, $C'_\delta = \{\alpha < \delta : h(\alpha) \in C_\delta\}$, $\bar{C}' = \langle C'_\delta : \delta \in S'\}$, $\mathcal{F}'_\delta = \{h \circ f : f \in \mathcal{F}_\delta\}$ *<u>then</u>* $Ps_4(\lambda_1, \lambda, h, \bar{C}', \bar{\mathcal{F}}')$.

BB: We may phrase what we have for the ideal $I[\lambda]$.

Conclusion 3.12. (1) If $\lambda = \mathrm{cf}(\lambda) > \mu > \aleph_0$, μ strong limit singular <u>then</u> for some $A \in I[\lambda]$, $\kappa < \mu$ and finite $\mathfrak{d} \subseteq \mathrm{Reg} \cap \mu$ (in fact $\mathfrak{d} = \mathfrak{d}'_{0,\mu}(\lambda)$) we have:

(∗) for every $\kappa(2) = \kappa(2)^{\kappa(1)} < \mu$, $\kappa(1) > \kappa$ and increasing continuous sequence $\langle \alpha_\varepsilon : \varepsilon < \kappa(2)^+\rangle$ we have: there is a club C of $\kappa(2)^+$ such that $\{\alpha \in C : \mathrm{cf}(\alpha) \notin \mathfrak{d}$ and $\mathrm{cf}(\alpha) \le \kappa(1)^+\} \subseteq A$.

(2) For above $\lambda = \lambda^{<\lambda}$ we can add: $\kappa \in \mathrm{Reg} \cap \mu \backslash \mathfrak{d} \Rightarrow (D\ell)_{S^\lambda_\kappa}$ (and even $(D\ell)_S$ for any $S \subseteq S^\lambda_\kappa$ which is $\neq \emptyset$ modulo for a suitable filter similarly to in (3)).

On diamond from instances of GCH and its history, see [21]. Whereas $\lambda = \mu^+$ a successor of regular cardinals has strong partial squares [14, Section 4], for a successor of singular we have much less. If $\lambda = \mu^+$, $\mu^\theta = \mu$ for cofinalities $\le \theta$, we still have this.

Conclusion 3.13. Assume $\lambda = \mathrm{cf}(\lambda) > \mu > \aleph_0$, μ strong limit and $\mathfrak{d} = \mathfrak{d}'_{0,\mu}$ which is finite. If $\lambda = \chi^+ = 2^\chi$ and $\kappa \in \mathrm{Reg} \cap \mu \backslash \mathfrak{d}$ then $\diamondsuit_{S^\lambda_\kappa}$.

Proof. Follows easily from 3.8. □

Recall that the previous approach gives 3.14. In particular if $\lambda = 2^\mu$ is singular, see 3.15.

Claim 3.14. *Assume* $\mu > \kappa = \mathrm{cf}(\kappa)$ *is strong limit and* $\mathrm{cf}(\lambda) > \mu$ *and* $h : \mathrm{cf}(\lambda) \to \lambda$ *is increasing continuous with limit* λ. *<u>Then</u> for any regular* $\chi < \mu$ *large enough,* $(A)_{\lambda,\mu,\kappa} \Rightarrow (B)_{\lambda,\mu,\kappa,h}$, *and* $(B)^+_{\lambda,\mu,\kappa,h}$ *where*

$(A)_{\lambda,\mu,\kappa}$ *there is* $\bar{A}$ *such that*
  (a) $\bar{A} = \langle \mathcal{A}_\alpha : \alpha < \mathrm{cf}\,(\lambda)\rangle$,
  (b) $\mathcal{A}_\alpha \subseteq [h(\alpha)]^{<\mu}$ *has cardinality* $< \lambda$ *(we can add* $A \in \mathcal{A}_\alpha \Rightarrow A$ *closed subset of* $\sup(A)$*; it does not matter),*
  (c) *if* E *is a club of* $\mathrm{cf}(\lambda)$ *<u>then</u> for some increasing continuous* $\alpha_\varepsilon \in E$ *for* $\varepsilon < \chi$ *we have* $\{\varepsilon < \chi : \mathrm{cf}(\varepsilon) = \kappa$ *and* $\{\alpha_\zeta : \zeta < \varepsilon\} \in \mathcal{A}_{\alpha_\varepsilon}\}$ *is a stationary subset of* χ,

$(B)_{\lambda,\mu,\kappa,h}$ *there is* $\bar{\mathcal{F}}$ *such that*

(a) $\bar{\mathcal{F}} = \langle \mathcal{F}_\alpha : \alpha < cf(\lambda)\rangle$,

(b) $\mathcal{F}_\alpha \subseteq \{f : f$ *a partial function from* α *to* $h(\alpha)\}$ *has cardinality* $< \lambda$,

(c) *for every club* E *of* $\mathrm{cf}(\lambda)$ *and function* $f : \mathrm{cf}(\lambda) \to \lambda$ *there is an increasing continuous* $\alpha_\varepsilon \in E$ *for* $\varepsilon < \chi$ *for which the set* $\{\varepsilon < \chi : f{\restriction}\{\alpha_\zeta : \zeta < \varepsilon\} \in \mathcal{F}_{\alpha_\varepsilon}\}$ *is a stationary subset of* χ,

$(B)^+_{\lambda,\mu,\kappa,h}$ *there is* $\bar{\mathcal{F}}$ *such that*

(a),(b) *as above,*

(c) *if* $\alpha_\varepsilon < \mathrm{cf}(\lambda)$ *for* $\varepsilon < \chi_1$ *and* $\langle \alpha_\varepsilon : \varepsilon < \chi_1\rangle$ *is increasing continuous* $\chi_1 \in [\chi, \mu)$ *and* $f : \{\alpha_\varepsilon : \varepsilon < \chi_1\} \to \lambda$ *and* $f(\alpha_\varepsilon) < h(\alpha_{\varepsilon+1})$ *for* $\varepsilon < \chi_1$ *for simplicity, then we can find* $\bar{u} = \langle u_i : i < \chi\rangle$ *such that* $\chi_1 = \cup\{u_i : i < \chi\}$ *and for every* $\varepsilon < \chi_1$ *and* $i < \chi$, $f \restriction \{\alpha_\zeta : \zeta < \varepsilon$ *and* $\zeta \in u_i\}$ *belongs to* $\mathcal{F}_{\alpha_\varepsilon}$.

Conclusion 3.15. Assume $\mu > \aleph_0$ is strong limit, $\chi \geq \mu$ and $\lambda = 2^\chi$ is singular. Then for every $\kappa \in \mu \cap \mathrm{Reg}\backslash\{\aleph_0\}$ we have $\mathrm{Ps}_1(\mathrm{cf}(\lambda), \lambda, \kappa)$.

4. Middle diamonds and black boxes

We use Section 3 to improve the main results of [7]. The point is that there we use [21], while here we use Section 3 instead. Towards our aim we quote some results and definitions. See 4.4 and 4.3.

The Special Black Box Claim 4.0. *Assume*

(a) $\lambda = \mathrm{cf}(2^\mu)$, *$D$ is a μ^+-complete filter on λ extending the club filter,*

(b) $\kappa = \mathrm{cf}(\kappa) < \lambda$ *and* $S \subseteq S^\lambda_\kappa$,

(c) $\bar{C} = \langle C_\delta : \delta \in S\rangle$, *$C_\delta$ a club of δ of order type κ and* $\lambda = \mathrm{cf}(2^\mu) = 2^\mu \,\&\, \delta \in S \Rightarrow \lambda > |\{C_\delta \cap \alpha : \alpha \in \mathrm{nacc}(C_\delta)\}|$ *and* $S \in D$,

(d) $2^{<\chi} \leq 2^\mu$ *and* $\theta \leq \mu$,

(e) $\mathrm{Ps}_1(\lambda, 2^\mu, \bar{C})$ *(see Definition 3.9),*

(f) $\mathrm{Sep}(\mu, \theta)$ *(see Definition 4.1 below and 4.2 on sufficient conditions).*

Then λ has the (κ,θ)-BB exemplified by some $\langle \bar{C} \restriction S_i : i < \lambda\rangle$ and $\bar{C}$ has the $(D, 2^\mu, \theta)$-Md-property (see Definitions 4.3 and 4.4 below).

Proof. By the proof of [7, 1.10]. □

Definition 4.1. (1) $\mathrm{Sep}(\mu, \theta)$ means that for some $\bar{f}$ and Υ:

(a) $\bar{f} = \langle f_\varepsilon : \varepsilon < \mu\rangle$,

(b) f_ε is a function from ${}^\mu\theta$ to θ,

(c) for every $\varrho \in {}^\mu\theta$ the set $\{\nu \in {}^\mu\theta$: for every $\varepsilon < \mu$ we have $f_\varepsilon(\nu) \neq \varrho(\varepsilon)\}$ has cardinality $< \Upsilon$,

(d) $\Upsilon = \mathrm{cf}(\Upsilon) \leq 2^\mu$.

(2) $\mathrm{Sep}_\sigma(\mu, \theta)$ means that for some $\bar{f}$, R and Υ we have

(a) $\bar{f} = \langle f^i_\varepsilon : \varepsilon < \mu$ and $i < \sigma\rangle$,

(b) f^i_ε is a function from ${}^R\theta$ to ${}^\mu\theta$,

(c) $R \subseteq {}^\mu\theta$; $|R| = 2^\mu$ (if $R = {}^\mu\theta$ we may omit it),

(d) $\bar{\mathcal{I}} = \langle \mathcal{I}_i : i < \sigma\rangle, \mathcal{I}_i \subseteq \mathcal{P}(\mu)$ and if $A_j \in \mathcal{I}_j$ for $j < j^* < \sigma$ then $\mu \neq \cup\{A_j : j < j^*\}$ (e.g. $\mathcal{I}_i$ is a σ-complete ideal on μ),

(e) if $\eta \in {}^\mu\theta$ and $i < \sigma$ then $\Upsilon > |\mathrm{Sol}_\eta|$ where

$$\mathrm{Sol}_\eta = \{\rho \in R : \text{ the set } \{\varepsilon < \mu : \text{if } i < \theta \text{ then } (f^i_\alpha(\eta))(\varepsilon) \neq \eta(\varepsilon)\} \text{ belong to } \mathcal{I}_i\}.$$

We may wonder whether clause (f) of the assumption is reasonable; the following claim gives some sufficient conditions for clause (f) to hold.

Claim 4.2. *Clause* (f) *of* 4.0 *holds, i.e.,* $\mathrm{Sep}(\mu, \theta)$ *holds, if at least one of the following holds:*

(a) $\mu = \mu^\theta$,

(b) $\mathbf{U}_\theta(\mu) = \mu + 2^\theta \leq \mu$,

(c) $\mathbf{U}_J(\mu) = \mu$ *where for some* σ *we have* $J = [\sigma]^{<\theta}, \theta \le \sigma, 2^{<\sigma} < \mu$,
(d) μ *is a strong limit of cofinality* $> \theta$,
(e) $\mu \ge \beth_\omega(\theta)$.

Proof. This is [7, 1.11]. □

Definition 4.3. (1) We say that $\bar{C}$ exemplifies $\mathrm{Md}^+(\lambda, \kappa, \theta, \Upsilon, D)$ when

(a) $\lambda > \kappa$ are regular cardinals, Υ an ordinal (or a function with domain λ or ${}^{\omega>}\lambda$ in this case a function f from X to Υ means that f is a function with domain X and $f(x) \in \Upsilon(x)$, so ${}^C\Upsilon = \{f : f$ is a function with $\mathrm{Dom}(f) = C$ and $\alpha \in C \Rightarrow f(\alpha) \in \Upsilon(\alpha)\}$),
(b) $\bar{C} = \langle C_\delta : \delta \in S\rangle$, S a stationary subset of λ such that $\delta \in S \Rightarrow \mathrm{cf}(\delta) = \kappa$,
(c)$^+$ C_δ is a club of δ disjoint from S and $\alpha \in \mathrm{nacc}(C_{\delta_1}) \cap \mathrm{nacc}(C_{\delta_2}) \Rightarrow C_{\delta_1} \cap \alpha = C_{\delta_2} \cap \alpha$ so we may define $C_\alpha = C_\delta \cap \alpha$ when $\alpha \in \mathrm{nacc}(C_\delta)$,
(d) if $\mathbf{F}$ is a function from $\bigcup_{\delta \in S}\{f : f$ is a function from ${}^{\omega>}(C_\delta)$ to $\Upsilon\}$ to θ <u>then</u> for some $\mathbf{c} \in {}^S\theta$ for every $f \in {}^\lambda\Upsilon$ the set $\{\delta \in S : \mathbf{F}(f \restriction C_\delta)) = \mathbf{c}(\delta)\} \in D^+$.

(2) We write Md instead Md^+ if we weaken (c)$^+$ to
(c) C_δ is an unbounded subset of δ.

(3) We say $\bar{C}$ has the (D, Υ, θ)-Md property <u>when</u> clauses (a), (b), (c), (d) above hold; we say λ has this property if some $\bar{C} = \langle C_\delta : \delta \in S\rangle$ has it, $S \subseteq S^\lambda_\theta$ stationary.

The following is a variant of the silly black box (trying to reconcile the definitions of [13, III], [8, IV] with [7]).

Definition 4.4. (1) We say that λ has the (κ, θ)-SBB^+ (= Special Black Box) property <u>when</u> there are $\bar{C}^i = \langle C_\delta : \delta \in S_i\rangle$ for $i < \lambda$ such that

$\boxdot^{\lambda,\kappa}_{\bar{C}}$ (a) S_i are pairwise disjoint stationary subsets of λ,
(b) $\delta \in S_i \Rightarrow \mathrm{cf}(\delta) = \kappa$,
(c) C_δ is a club of δ of order type κ and every $\alpha \in \mathrm{nacc}(C_\delta)$ is a successor ordinal,
(d) if $\alpha \in \mathrm{nacc}(C_{\delta_1}) \cap \mathrm{nacc}(C_{\delta_2})$ then $C_{\delta_1} \cap \alpha = C_{\delta_2} \cap \alpha$,
(e) $\bar{C}^i$ has the θ-BB property which means that there is $\bar{f} = \langle f_\delta : \delta \in S_i\rangle$ such that $f_\delta : {}^{\omega>}(C_\delta) \to \theta$ and for every $f \in {}^{\omega>}\lambda \to \theta$ for stationarily many $\delta \in S_i$ we have $f_\delta = f \restriction C_\delta$.

(2) We write SBB instead of SBB^+ if we omit clause (d); we write $\mathrm{SBB}^\pm$ if we replace "C_δ a club of δ" by "$C_\delta \subseteq \delta = \sup(C_\delta)$" and SBB^- if we make both changes.

Remark 4.5. (1) How strong is the demand that S can be divided into λ sets S_i with the property? It is hard not to have it.
(2) In 4.6 to have more than one exception is a heavy demand on $\mathcal{H}(\mu)$.
(3) We can improve 4.6 including the case $\mathrm{cf}(\mu_*) = \aleph_0$, even $\mu_* = \beth_{\alpha+\omega}$. Then probably in part (2) we have to distinguish λ a successor of regular (easy), successor of singular (harder), rest (hardest).

The Main Theorem 4.6. (1) *If* μ_* *is strong limit* $> \aleph_0, \mu \ge \mu_* > \theta, \lambda = \mathrm{cf}(2^\mu)$ *and* $\Upsilon = 2^\mu$ <u>*then*</u> *for all but finitely many* $\kappa \in \mathrm{Reg} \cap \mu_*$ *(even every* $\kappa \in \mathrm{Reg} \cap \mu_* \backslash \mathfrak{d}'_{0,\mu_*}(2^\mu)$*), there is* $\bar{C} = \langle C_\delta : \delta \in S\rangle$ *exemplifies* $Md^+(\lambda, \kappa, \theta, \Upsilon)$*; hence* (κ, θ)*-*SBB^+.
(2) *Assume* μ_* *is strong limit singular of uncountable cofinality and* $\lambda = \mathrm{cf}(\lambda) > \mu_*$ *is not strongly inaccessible.* <u>*Then*</u> *for all but finitely many* $\kappa \in \mathrm{Reg} \cap \mu_*$ *for every* $\theta < \mu_*$, λ *has* (κ, θ)*-SBB; hence* (κ, θ)*-*SBB^+ *(moreover only one of the exceptions depends on* λ*).*

Proof. (1) Let $\mathfrak{d} = \mathfrak{d}'_{0,\mu_*}(\lambda)$. So by Section 3 we have $\kappa \in \mathrm{Reg} \cap \mu_* \backslash \mathfrak{d} \Rightarrow \mathrm{Ps}_1(\lambda, 2^\mu, \bar{C})$ for some $\bar{C}$ satisfying clause (c) of 4.0, and moreover clauses (c) and (d) of 4.4(1). So we apply 4.0.
(2) Let $\langle \mu_i : i < \mathrm{cf}(\mu_*)\rangle$ be increasing continuous with limit μ_*; each μ_i is strong limit singular. For each $i < \mathrm{cf}(\mu_*)$ let $\mathfrak{d}_i = \mathfrak{d}'_{0,\mu_i}(\mathrm{cf}(2^{\mu_i}))$, so it is finite and let $\mathfrak{d} = \{\kappa : \kappa = \mathrm{cf}(\kappa) < \mu_*$ and $\kappa \in \mathfrak{d}_i$ for every $i < \mathrm{cf}(\mu_*)$ large enough$\}$.

Case 1: $(\forall\alpha < \lambda)[|\alpha|^{<\mu_*} < \lambda]$.

So we can find $\mu < \lambda \leq 2^\lambda$; let $\mu_1 = ((\mu)^{<\mu_*})^{<\mu_*}$; this cardinal is $< \lambda$ and $\mu_1 = (\mu_1)^{\mu_*}$.
Now use [7, Section 2].

Case 2: $(\exists\alpha < \lambda)[|\alpha|^{<\mu_*} \geq \lambda]$.

As λ is regular for some $\kappa < \lambda$, $\mu < \lambda$ we have $\mu^\kappa \geq \lambda$. Let $\mu = \mathrm{Min}\{\mu : \mu^\kappa \geq \lambda$ for some $\kappa < \mu_*\}$.
NOTE: Here getting λ pairwise disjoint S_i should be done. Again we use [7, Section 2]. $\square$

Remark 4.7. $\aleph_0 \in \mathfrak{d}$ as we need $F : {}^\omega\lambda \to \lambda$ as in Section 3!!

Definition 4.8. We say that $\bar{C}$ exemplifies $\mathrm{SBB}_0(\lambda, \kappa, \theta)$ when

(a) $\lambda > \kappa$ are regular,
(b) $\bar{C} = \langle C_\delta : \delta \in S\rangle$, S a stationary subset of λ such that $\delta \in S \Rightarrow \mathrm{cf}(\delta) = \kappa$,
(c) C_δ is an unbounded subset of δ disjoint from S such that $\alpha \in C_{\delta_1} \cap C_{\delta_2} \Rightarrow C_{\delta_1} \cap \alpha = C_{\delta_2} \cap \alpha$,
(d) assume $\tau_0 \subseteq \tau_1 \subseteq \tau_2$ are vocabularies of cardinality $\leq \theta$, $\tau_1\backslash\tau_0$ has only predicates, $\tau_2\backslash\tau_1$ has only function symbols (allowed to be partial), $\mathfrak{B}$ is a τ_0-model with universe λ (but not individual constants), then we can find $\langle \mathcal{M}_\delta : \delta \in S\rangle$ such that
  (α) every $M \in \mathcal{M}_\delta$ is a τ_2-model of cardinality θ expanding $\mathfrak{B} \restriction |\mathcal{M}_\delta|$,
  (β) if $M \in \mathcal{M}_\delta$, $F \in \tau_2\backslash\tau_1$ then F^M has domain $\subseteq C_\delta$ (i.e., ${}^{\mathrm{arity}(F)}(D_\delta)$),
  (γ) every $M \in \mathcal{M}_\delta$ has a universe which includes C_δ and is included in δ and the universe of M is the $\mathfrak{B}$-closure of $C_\delta \cup \{F(\bar{\alpha}) : F \in \tau_2\backslash\tau_1$ and $\bar{\alpha} \in {}^{\mathrm{arity}(F)}(C_\delta)\}$,
  (δ) if $M', M'' \in \mathcal{M}_\delta$ then $(M', \gamma)_{\gamma\in C_\delta}$, $(M'', \gamma)_{\gamma\in C_\delta}$ are isomorphic,
  (ε) if $\mathfrak{B}^+$ is a τ_2-expansion of $\mathfrak{B}$ then for stationarily many $\delta \in S$ for some $M \in \mathcal{M}_\delta$ we have:
    (i) $F \in \tau_2\backslash\tau_1 \Rightarrow F^{\mathfrak{B}^+} \restriction C_\delta = F^M \restriction C_\delta\ (= F^M)$,
    (ii) $M \restriction \tau_1 \subseteq \mathfrak{B}^+ \restriction \tau_1$.

Observation 4.9. (1) *In 4.8 if the order $<$ on λ is a relation of $\mathfrak{B}$ (which is no loss) then the isomorphism is unique as it is necessarily the unique order preserving function from $|M'|$ onto $|M''|$.*
(2) *In 4.8, if the function F_i where $\alpha < \beta \in C_\delta, \alpha \in C_\delta$, $\mathrm{otp}(C_\delta \cap \alpha) = i \Rightarrow F_i(\beta) = \alpha$, then for any $M \in \cup\{\mathcal{M}_\delta : \delta \in S\}$ and δ, $M \cap C_\delta$ is an initial segment of C_δ.*

Definition 4.10. We say that $\bar{C}$ exemplifies $\mathrm{BB}_1(\lambda, \kappa, \theta)$ when (a), (b), (d), (e) from 4.8 hold + (ε) below. $\mathrm{BB}_2(\lambda, \kappa, \theta)$ holds when we add (ζ) to clause (d) where

(ϵ) the isomorphism type of $(M, \gamma)_{\gamma\in C_\delta}$ for $M \in \mathcal{M}_\delta$ depends on τ_0, τ_1, τ_2 but not on $\mathfrak{B}$,
(ζ) if $M', M'' \in \mathcal{M}_\delta$ and Π is an isomorphism from M' onto M'' and $\delta', \delta'' \in S$, $C_{\delta'} \subseteq M'$, $C_{\delta''} \subseteq M''$ and Π maps $C_{\delta'}$ onto $C_{\delta''}$, then for any $N' \in \mathcal{M}_{\delta'}$, $N'' \in \mathcal{M}_{\delta''}$ we have $(N', \gamma)_{\gamma\in C_{\delta'}} \cong (N'', \gamma)_{\gamma\in C_{\delta''}}$.

Claim 4.11. *If $\mu > \aleph_0$ is strong limit and $\lambda = \mathrm{cf}(2^\mu)$ or $\lambda > 2^{2^\mu}$ is not strongly inaccessible then for all but finitely many $\kappa \in \mathrm{Reg} \cap \theta$ ($\kappa \in \mathrm{Reg} \cap \mu\backslash\mathfrak{d}'_0(2^\mu)$) for every $\theta < \mu$, $\mathrm{BB}_1(\lambda, \kappa, \theta)$ holds.*

Proof. Use also 4.13 below. $\square$

Observation 4.12. (1) *If $\bar{C}$ exemplifies $BB_\ell(\lambda, \kappa, \theta)$ then for some pairwise disjoint $\langle S_\varepsilon : \varepsilon < \lambda\rangle$ we have that each $\bar{C} \restriction S_\varepsilon$ exemplifies $BB_\ell(\lambda, \kappa, \theta)$.*
(2) *If $\lambda = \lambda^\theta$ we can allow in $\tau_1\backslash\tau_0$ individual constants.*
We delay their proof as we first use them.

Now we turn to proving 4.11, 4.12.

Claim 4.13. (1) *If $\bar{C}$ exemplifies $\mathrm{SBB}(\lambda, \kappa, 2^\theta, \lambda)$ then $\bar{C}$ exemplifies $\mathrm{BB}_1(\lambda, \kappa, \theta)$. [Rethink: if we use $C * \chi$, $\chi = \beth_\kappa$ enough to have many guesses.]*
(2) *$\bar{C}$ exemplifies $\mathrm{BB}_1(\lambda, \kappa, \theta)$ when there are $\lambda_1, \bar{C}^1$:*
  (a) *$\bar{C}$ exemplifies $\mathrm{SBB}(\lambda, \kappa, 2^\theta, \lambda)$ (hence $\bar{C}^1 = \langle C^1_\delta : \delta \in S_1\rangle$ exemplifies $\mathrm{BB}_1(\lambda, \kappa, \theta)$ but apparently we need more),*
  (b) *$\bar{h} = \langle h_\delta : \delta \in S_1\rangle$ where h_δ is an increasing function from C_δ onto some $\gamma = \gamma(\delta) \in S_1$,*

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(c) *for every club C of λ there is an increasing continuous function g from λ_1 into C such that $\alpha \in S_1 \Rightarrow g(\alpha) \in S$ & $\gamma_{g(\alpha)} = \alpha$.*

(3) *If $\bar{C}$ exemplifies* $\mathrm{MD}(\lambda, \kappa, 2^\theta)$ *then $\bar{C}$ exemplifies* $\mathrm{BB}_2(\lambda, \kappa, \theta)$.

Proof. (1) $\bar{C}$ has the $(D, 2^\mu, \theta)$-Md-property (which is like the desired conclusion except that we write $F_\delta(\nu \restriction C_\delta)$ instead of $F(\nu \restriction C_\delta, \bar{C} \restriction C_\delta)$. But let $\beta = \alpha/\theta$ mean that $\theta\beta \leq \alpha < \theta\beta + 1$. But define $F'_\delta(\nu) = F_\delta(\langle \nu(\alpha)/\theta : \alpha \in C_\delta \rangle, \langle \nu(\alpha) - \theta(\nu(\alpha)/\theta) : \alpha \in C_\delta \rangle)$. So for $\langle F'_\delta : \delta \in S \rangle$ we have $\bar{C}$ as required in the original requirement; the same $\bar{C}$ is as required for our $\bar{F}$.

(2), (3) Left to the reader. □

Conclusion 4.14. If $\lambda = \mathrm{cf}(\lambda) > \beth_{\omega+3}$ is not strongly inaccessible, <u>then</u> for every regular $\kappa < \beth_\omega$ except possibly finitely many we have:

⊛ for some topological space X and $\bar{C} = \langle C_\delta : \delta \in S \rangle$ we have
  (a) X is Hausdorff having λ points with a clopen basis set,
  (b) every $Y \subseteq X$ of cardinality $< \kappa$ is closed,
  (c) every point has a neighborhood of cardinality $\leq \kappa$,
  (d) there is $f : X \to \kappa$ such that:
  if $X = \bigcup_{\alpha<\beta} X_\alpha$, $\beta < \kappa$ <u>then</u> some non-isolated point x has a neighborhood included in $X_{f(x)}$ (so $f(x) < \beta$).

Remark 4.15. It is natural to add in Definition 2.14 (but is not useful here): For regular λ let $\mathfrak{d}_2(\lambda) = \mathfrak{d}^2_{\sigma,\theta,\theta_*}(\lambda)$ be defined as in part (1) of 2.14 omitting clauses (d), (f) and (g) of 2.3 adding (j) of 2.11 and: if $\eta \in \max_{\mathcal{T}}$, $\mathfrak{a} \subseteq \mathrm{Reg} \cap \lambda_\eta \setminus \theta_*$ and $|\mathfrak{a}| < \theta$ then $\lambda_\eta \notin \mathrm{pdf}_{\sigma\text{-com}}(\mathfrak{a})$ (it too is finite).

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References

[1] James Cummings, Mirna Dzamonja, Saharon Shelah, A consistency result on weak reflection, Fundamenta Mathematicae 148 (1995) 91–100. math.Lo/9504221.
[2] Mirna Dzamonja, Saharon Shelah, On squares, outside guessing of clubs and $I_{<f}[\lambda]$, Fundamenta Mathematicae 148 (1995) 165–198. math.LO/9510216.
[3] Mirna Dzamonja, Saharon Shelah, Weak reflection at the successor of a singular cardinal, Journal of the London Mathematical Society 67 (2003) 1–15. math.LO/0003118.
[4] Paul Erdős, Andras Hajnal, Unsolved problems in set theory, in: Axiomatic Set Theory, in: Proc. of Symp. in Pure Math., vol. XIII, Part I, AMS, Providence, RI, 1971, pp. 17–48.
[5] Andras Hajnal, Peter Hamburget, Set theory, in: London Math. Soc. Student Texts, vol. 48, Cambridge, 1999.
[6] Matatyahu Rubin, Saharon Shelah, Combinatorial problems on trees: partitions, δ-systems and large free subtrees, Annals of Pure and Applied Logic 33 (1987) 43–81.
[7] Saharon Shelah, Middle diamond. Archive for Mathematical Logic (in press). math.LO/0212249.
[8] Saharon Shelah, Non-structure Theory, Oxford University Press (in press).
[9] Saharon Shelah, Pcf without choice, Archive for Mathematical Logic (preprint 835) (submitted for publication).
[10] Saharon Shelah, On successors of singular cardinals, in: Logic Colloquium'78 (Mons, 1978), in: Stud. Logic Foundations Math, vol. 97, North-Holland, Amsterdam, New York, 1979, pp. 357–380.
[11] Saharon Shelah, Proper Forcing, in: Lecture Notes in Mathematics, vol. 940, Springer-Verlag, Berlin, New York, 1982, pp. xxix+496.
[12] Saharon Shelah, Diamonds, uniformization, The Journal of Symbolic Logic 49 (1984) 1022–1033.
[13] Saharon Shelah, Universal classes, in: John T. Baldwin (Ed.), Classification theory (Chicago, IL, 1985), (Proceedings of the USA–Israel Conference on Classification Theory, Chicago, December, 1985), in: Lecture Notes in Mathematics, vol. 1292, Springer, Berlin, 1987, pp. 264–418.
[14] Saharon Shelah, Reflecting stationary sets and successors of singular cardinals, Archive for Mathematical Logic 31 (1991) 25–53.
[15] Saharon Shelah, Advances in cardinal arithmetic, in: Norbert W. Sauer et al. (Eds.), Finite and Infinite Combinatorics in Sets and Logic, Kluwer Academic Publishers, 1993, pp. 355–383.
[16] Saharon Shelah, Bounding $\mathrm{pp}(\mu)$ when $\mathrm{cf}(\mu) > \mu > \aleph_0$ using ranks and normal ideals, in: Cardinal Arithmetic, in: Oxford Logic Guides, vol. 29, Oxford University Press, 1994.

[17] Saharon Shelah, Cardinal Arithmetic, in: Oxford Logic Guides, vol. 29, Oxford University Press, 1994.
[18] Saharon Shelah, The pcf-theorem revisited, in: Ron Graham, Jaroslav Nešetřil (Eds.), The Mathematics of Paul Erdős, II, in: Algorithms and Combinatorics, vol. 14, Springer, 1997, pp. 420–459. math.LO/9502233.
[19] Saharon Shelah, Set theory without choice: not everything on cofinality is possible, Archive for Mathematical Logic 36 (1997) 81–125. (A special volume dedicated to Prof. Azriel Levy) math.LO/9512227.
[20] Saharon Shelah, Applications of pcf theory, Journal of Symbolic Logic 65 (2000) 1624–1674. math.LO/9804155.
[21] Saharon Shelah, The generalized continuum hypothesis revisited, Israel Journal of Mathematics 116 (2000) 285–321. math.LO/9809200.
[22] Saharon Shelah, Pcf and infinite free subsets in an algebra, Archive for Mathematical Logic 41 (2002) 321–359. math.LO/9807177.
[23] Saharon Shelah, Successor of singulars: combinatorics and not collapsing cardinals $\leq \kappa$ in $(< \kappa)$-support iterations, Israel Journal of Mathematics 134 (2003) 127–155. math.LO/9808140.
[24] Saharon Shelah, Pcf arithmetic with little choice (preprint F728).
[25] Saharon Shelah, κ-free silly λ-black n-boxes (preprint F691).