Journal of Mathematical Logic, Vol. 10, Nos. 1 & 2 (2010) 45–81
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DOI: 10.1142/S0219061310000936

A DICHOTOMY FOR THE NUMBER OF ULTRAPOWERS

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Received 2 December 2009
Revised 6 July 2010

We prove a strong dichotomy for the number of ultrapowers of a given model of cardinality $\leq 2^{\aleph_0}$ associated with nonprincipal ultrafilters on $\mathbb{N}$. They are either all isomorphic, or else there are $2^{2^{\aleph_0}}$ many nonisomorphic ultrapowers. We prove the analogous result for metric structures, including C*-algebras and II_1 factors, as well as their relative commutants and include several applications. We also show that the C*-algebra $\mathcal{B}(H)$ always has nonisomorphic relative commutants in its ultrapowers associated with nonprincipal ultrafilters on $\mathbb{N}$.

Keywords: Dichotomy; ultrapower.

Mathematics Subject Classification 2010: Primary 03C20; Secondary 46M07

1. Introduction

In the following all ultrafilters are nonprincipal ultrafilters on $\mathbb{N}$. In particular, "all ultrapowers of A" always stands for "all ultrapowers associated with nonprincipal ultrafilters on $\mathbb{N}$".

The question of counting the number of nonisomorphic models of a given theory in a given cardinality was one of the main driving forces behind the development of

Model Theory (see Morley's Theorem and [20]). On the other hand, the question of counting the number of nonisomorphic ultrapowers of a given model has received more attention from functional analysts than from logicians.

Consider a countable structure A in a countable signature. By a classical result of Keisler, every ultrapower $\prod_{\mathcal{U}} A$ is countably saturated (recall that $\mathcal{U}$ is assumed to be a nonprincipal ultrafilter on $\mathbb{N}$). This implies that the ultrapowers of A are not easy to distinguish. Moreover, if the Continuum Hypothesis holds then they are all saturated and therefore isomorphic (this fact will not be used in the present paper; see [5]).

Therefore the question of counting nonisomorphic ultrapowers of a given countable structure is nontrivial only when the Continuum Hypothesis fails, and in the remaining part of this introduction we assume that it does fail. If we moreover assume that the theory of A is unstable (or equivalently, that it has the *order property* — see the beginning of Sec. 3) then A has nonisomorphic ultrapowers ([20, Theorem VI.3] and independently [6]). The converse, that if the theory of A is stable then all of its ultrapowers are isomorphic, was proved only recently ([10]) although main components of the proof were present in [20] and the result was essentially known to the second author.

The question of the isomorphism of ultrapowers was first asked by operator algebraists. This is not so surprising in the light of the fact that the ultrapower construction is an indispensable tool in Functional Analysis and in particular in Operator Algebras. The ultrapower construction for Banach spaces, C*-algebras, or II_1 factors is again an honest metric structure of the same type. These constructions coincide with the ultrapower construction for metric structures as defined in [2] (see also [10]). The Dow–Shelah result can be used to prove that C*-algebras and II_1 factors have nonisomorphic ultrapowers ([14] and [9], respectively), and with some extra effort this conclusion can be extended to the relative commutants of separable C*-algebras and II_1 factors in their utrapowers ([8] and [9, Theorem 5.1], respectively).

However, the methods used in [8, 9, 14] provide only as many nonisomorphic ultrapowers as there are uncountable cardinals $\leq \mathfrak{c} = 2^{\aleph_0}$ (with our assumption, two). In [15, Sec. 3] it was proved (still assuming only that CH fails) that $(\mathbb{N}, <)$ has $2^{\mathfrak{c}}$ nonisomorphic ultrapowers. As pointed out in [7], this proof could easily be modified to obtain the same conclusion for any infinite linear (sometimes called *total*) order in place of $(\mathbb{N}, <)$ but the proof does not cover even the case of an arbitrary partially ordered set with an infinite chain.

Theorem 1.1. *Assume the Continuum Hypothesis, CH, fails. If A is a model of cardinality $\leq \mathfrak{c}$ such that the theory of A is unstable, then there are $2^{\mathfrak{c}}$ isomorphism types of models of the form $\prod_{\mathcal{U}} A$, where $\mathcal{U}$ ranges over nonprincipal ultrafilters on $\mathbb{N}$.*

In Theorem 5.1, we prove a generalization of Theorem 1.1 for ultraproducts.

Corollary 1.2. *For a model A of cardinality $\leq \mathfrak{c}$ with a countable signature either all of its ultrapowers are isomorphic or there are $2^{\mathfrak{c}}$ isomorphism types of its ultrapowers.*

Proof. We may assume A is infinite. If the theory of A is stable, then $\prod_{\mathcal{U}} A$ is saturated and of cardinality $\mathfrak{c}$ and therefore all such ultrapowers are isomorphic ([10, Theorem 5.6]). If the Continuum Hypothesis holds, then all the ultrapowers are isomorphic by Keisler's result. In the remaining case when the Continuum Hypothesis fails and the theory of A is unstable use Theorem 1.1. $\square$

We also prove the analogue of Theorem 1.1 for metric structures (see [2] or [10]). The ultrapowers of metric structures are defined in Sec. 6. Recall that the *character density* of a metric space is the minimal cardinality of its dense subspace.

Theorem 1.3. *Assume CH fails. If A is a metric structure of character density $\leq \mathfrak{c}$ such that the theory of A is unstable, then there are $2^{\mathfrak{c}}$ isometry types of models of the form $\prod_{\mathcal{U}} A$, where $\mathcal{U}$ ranges over nonprincipal ultrafilters on $\mathbb{N}$.*

The proof is a modification of the proof of Theorem 1.1 and it will be outlined in Sec. 6. Although Theorem 1.3 implies Theorem 1.1, we chose to present the proof of Theorem 1.1 separately because it is the main case and because some of the main ideas are more transparent in the discrete case.

Corollary 1.4. *For a metric structure A of character density $\leq \mathfrak{c}$ with a countable signature either all of its ultrapowers are isomorphic or there are $2^{\mathfrak{c}}$ isomorphism types of its ultrapowers.*

Proof. We may assume A is infinite. If the theory of A is stable, then $\prod_{\mathcal{U}} A$ is saturated and of character density $\mathfrak{c}$ and therefore all such ultrapowers are isomorphic ([10, Theorem 5.6]). If the Continuum Hypothesis holds, then all ultrapowers are isomorphic by the analogue of Keisler's theorem for metric structures ([2]). In the remaining case, when the Continuum Hypothesis fails and the theory of A is unstable use Theorem 1.3. $\square$

Important instances of the ultraproduct construction for metric spaces include C*-algebras, II_1 factors (see, e.g. [10, Secs. 2.3.1 and 2.3.2]) and metric groups (see [18]).

Organization of the paper. The proof of Theorem 1.1 uses ideas from [20, Sec. VI.3], [15, Sec. 3] and [21, Sec. 3] and it will be presented in Secs. 2–5. Theorem 1.3 is proved in Sec. 6, and some applications will be given in Sec. 8. In Sec. 7 we prove local versions of Theorem 1.1 and Theorem 1.3, and in

Proposition 8.5 we use the latter to prove that $\mathcal{B}(H)$ always has nonisomorphic relative commutants in its ultrapowers associated with nonprincipal ultrafilters on $\mathbb{N}$. Secs. 2 and 3 are essentially a revision of [20, Sec. 3], and Sec. 4 has a small, albeit nonempty intersection with [15, Sec. 3] (and therefore with the latter half of [20, Sec. VI.3]).

Notation and terminology. If A denotes a model, then its universe is also denoted by A and the cardinality of its universe (or any other set A) is denoted by $|A|$. Hence what we denote by A is denoted by A or by $|A|$ in [20] and [21], and what we denote by $|A|$ is denoted by $||A||$ in [20] and [21] if A is a model. We also do not distinguish the notation for a formula $\phi(x)$ and its evaluation $\phi[a]$ in a model. It will always be clear from the context.

Letters I and J, possibly with subscripts or superscripts, will always denote *linear* (i.e. *total*) orders. The reverse of a linear order I will be denoted by I^*. The *cofinality* of a linear order I, $\mathrm{cf}(I)$, is the mininal cardinality of a cofinal subset of I. By $I + J$ we denote the order with domain $I \sqcup J$ in which copies of I and J are taken with the original ordering and $i < j$ for all $i \in I$ and all $j \in J$. If J and I_j, for $j \in J$, are linear orders then $\sum_{j\in J} I_j$ denotes the order with the underlying set $\bigcup_{j\in J}\{j\} \times I_j$ ordered lexicographically.

Following the notation common in Model Theory, an ultrapower of A associated with an ultrafilter $\mathcal{U}$ will be denoted by $\prod_{\mathcal{U}} A$, even in the case when A is an operator algebra, where the notation $A^{\mathcal{U}}$ for the ultrapower is standard. We refrain from using the symbol ω in order to avoid confusion.

By $\forall^\infty m$ we denote the quantifier "for all large enough $m \in \mathbb{N}$". More generally, if D is a filter on $\mathbb{N}$ then by $(\forall^D n)$ we denote the quantifier as a shortcut for "the set of all n such that… belongs to D".

An n-tuple of elements of A is always denoted by $\bar{a}$.

For $k \geq 1$ by $[X]^k$ we denote the set of all k-element subsets of X.

A cardinal κ will be identified with the least ordinal of cardinality κ, as well as the linear order $(\kappa, <)$. A cardinal κ is *regular* if $\kappa = \mathrm{cf}(\kappa)$ and *singular* otherwise. An increasing family of ordinals or cardinals λ_ξ, for $\xi < \gamma$, is *continuous* if $\lambda_\eta = \sup_{\xi<\eta} \lambda_\xi$ whenever η is a limit ordinal. Analogously, an increasing family A_ξ, for $\xi < \gamma$, of sets is *continuous* if $A_\eta = \bigcup_{\xi<\eta} A_\xi$ for every limit ordinal η.

2. Invariants of Linear Orders

The material of the present and the following sections is loosely based on [21, Sec. 3].

2.1. *The invariant* $\mathrm{inv}^m(J)$

In the following we consider the invariant $\mathrm{inv}^\alpha_\kappa(I)$ as defined in [21, Definition 3.4], or rather its special case when $\alpha = m \in \mathbb{N}$ and $\kappa = \aleph_1$. All the arguments presented here can straightforwardly be extended to the more general context of an arbitrary ordinal α and regular cardinal κ.

In certain cases we define the invariant to be `undefined`. The phrase "an invariant is defined" will be used as an abbreviation for "an invariant is not equal to `undefined`".

For a linear order $(I, \leq)$ define $\mathrm{inv}^m(I)$, for $m \in \mathbb{N}$, by recursion as follows. If $\mathrm{inv}^m(I)$ is `undefined` for some m, then $\mathrm{inv}^{m+1}(I)$ is also `undefined`. If $\mathrm{cf}(I) \leq \aleph_0$ then let $\mathrm{inv}^0(I)$ be `undefined`. Otherwise let

$$\mathrm{inv}^0(I) = \mathrm{cf}(I).$$

In order to define $\mathrm{inv}^m(I)$ for $m \geq 1$ write $\kappa = \mathrm{inv}^0(I)$. Although the definition when $m = 1$ is a special case of the general case, we single it out as a warmup. Fix a continuous sequence I_ξ, for $\xi < \kappa$, of proper initial segments of I such that $I = \bigcup_{\xi<\kappa} I_\xi$. Then let $\lambda_\xi = \mathrm{cf}((I \backslash I_\xi)^*)$, where J^* denotes the reverse order on J. Thus λ_ξ, for $\xi < \kappa$, is the sequence of coinitialities of end-segments of I corresponding to the sequence I_ξ, for $\xi < \kappa$.

Let $\mathcal{D}(\kappa, \aleph_1)$ be the filter on κ dual to the ideal generated by the nonstationary ideal and the set $\{\xi < \kappa\colon\ \mathrm{cf}(\xi) \leq \aleph_0\}$. Define $f\colon \kappa \to \mathrm{Card}$ by

$$f(\xi) = \begin{cases} \lambda_\xi, & \text{if } \lambda_\xi \geq \aleph_1, \\ 0, & \text{if }, \lambda_\xi \leq \aleph_0. \end{cases}$$

If the set $\{\xi\colon f(\xi) \neq 0\}$ belongs to $\mathcal{D}(\kappa, \aleph_1)$ then let $\mathrm{inv}^1(I)$ be the equivalence class of f modulo $\mathcal{D}(\kappa, \aleph_1)$, or in symbols

$$\mathrm{inv}^1(I) = f/\mathcal{D}(\kappa, \aleph_1).$$

Otherwise, $\mathrm{inv}^1(I)$ is `undefined`.

Assume $m \geq 1$ and $\mathrm{inv}^m(J)$ is defined for all linear orders J (allowing the very definition of $\mathrm{inv}^m(J)$ to be "`undefined`"). Assume I and I_ξ, for $\xi < \kappa = \mathrm{cf}(I)$, are as in the case $m = 1$. Define a function g_m with domain κ via

$$g_m(\eta) = \mathrm{inv}^m((I \backslash I_\eta)^*).$$

If $\{\eta\colon g_m(\eta) \text{ is defined}\}$ belongs to $\mathcal{D}(\kappa, \aleph_1)$ then let $\mathrm{inv}^{m+1}(I)$ be the equivalence class of g_m modulo $\mathcal{D}(\kappa, \aleph_1)$. Otherwise $\mathrm{inv}^{m+1}(I)$ is `undefined`.

This defines $\mathrm{inv}^m(I)$ for all I. For a (defined) invariant $\mathbf{d}$ we shall write $\mathrm{cf}(\mathbf{d})$ for $\mathrm{cf}(I)$, where I is any linear order with $\mathrm{inv}^m(I) = \mathbf{d}$. We also write

$$|\mathbf{d}| = \min\{|I|\colon \mathbf{d} = \mathrm{inv}^m(I) \text{ for some } m\}.$$

Our invariant $\mathrm{inv}^m(I)$ essentially corresponds to $\mathrm{inv}^m_{\aleph_1}(I)$ as defined in [21, Definition 3.4]. Although inv^η can be recursively defined for every ordinal η, we do not have applications for this general notion. As a matter of fact, only inv^m for $m \leq 3$ will be used in the present paper.

Example 2.1. Assume throughout this example that κ is a cardinal with $\mathrm{cf}(\kappa) \geq \aleph_1$.

(1) Then $\mathrm{inv}^0(\kappa) = \mathrm{cf}(\kappa)$ and $\mathrm{inv}^1(\kappa)$ is `undefined`.
(2) If λ is a cardinal with $\mathrm{cf}(\lambda) \geq \aleph_1$ then $\mathrm{inv}^0(\kappa \times \lambda^*) = \mathrm{cf}(\kappa)$ and $\mathrm{inv}^1(\kappa \times \lambda^*)$ is the equivalence class of the function on $\mathrm{cf}(\kappa)$ everywhere equal to $\mathrm{cf}(\lambda)$, modulo $\mathcal{D}(\mathrm{cf}(\kappa), \aleph_1)$.
(3) If $\mathrm{inv}^m(I_\xi)$ is defined for all $\xi < \kappa$ and κ is regular then with $I = \sum_{\xi<\kappa} I^*_\xi$ we have that $\mathrm{inv}^{m+1}(I)$ is the equivalence class of the function $g(\xi) = \mathrm{inv}^m(I_\xi)$ modulo $\mathcal{D}(\kappa, \aleph_1)$.

Example 2.1(3) above will be used to define linear orders with prescribed invariants.

Lemma 2.2. (1) *For every regular* $\lambda \geq \aleph_2$ *there are* 2^λ *linear orders of cardinality* λ *with pairwise distinct, defined, invariants* $\mathrm{inv}^1(I)$.
(2) *If* λ *is singular then for every regular uncountable* θ *such that*

$$\max(\aleph_2, \mathrm{cf}(\lambda)) \leq \theta < \lambda$$

there are 2^λ *linear orders of cardinality* λ *and cofinality* θ *with pairwise distinct, defined, invariants* $\mathrm{inv}^2(I)$.

Proof. These are cases (1–3) of [21, Lemma 3.8], with $\kappa = \aleph_1$ but we reproduce the proof for the convenience of the reader.

(1) If $\lambda \geq \aleph_2$ is regular, then the set $\{\xi < \lambda \colon \mathrm{cf}(\xi) \geq \aleph_1\}$ can be partitioned into λ disjoint stationary sets (see [20, Appendix, Theorem 1.3(2)] or [16, Corollary 6.12]). Denote these sets by S_η, for $\eta < \lambda$. For $Z \subseteq \lambda$ define a linear order L_Z as follows. For $\alpha < \lambda$ let

$$\kappa(\alpha) = \begin{cases} \aleph_1, & \text{if } \alpha \in \bigcup\limits_{\eta \in Z} S_\eta, \\ \aleph_2, & \text{if } \alpha \in \bigcup\limits_{\eta \notin Z} S_\eta, \\ 1, & \text{if } \mathrm{cf}(\alpha) \leq \aleph_0. \end{cases}$$

Let $L_Z = \sum_{\alpha<\lambda} \kappa(\alpha)^*$. More formally, let the domain of L_Z be the set $\{(\alpha, \beta) \colon \alpha < \lambda, \beta < \kappa(\alpha)\}$ ordered by $(\alpha_1, \beta_1) \prec_L (\alpha_2, \beta_2)$ if $\alpha_1 < \alpha_2$ or $\alpha_1 = \alpha_2$ and $\beta_1 > \beta_2$. Then $\mathrm{inv}^1(L_Z)$ is clearly defined. A standard argument using the stationarity of S_ξ for any $\xi \in Z \Delta Y$ shows that $\mathrm{inv}^1(L_Z) \neq \mathrm{inv}^1(L_Y)$ if $Z \neq Y$.
(2) Now assume λ is singular. Pick an increasing sequence of regular cardinals λ_i, for $i < \mathrm{cf}(\lambda)$, such that $\sum_{i<\mathrm{cf}(\lambda)} \lambda_i = \lambda$. Using (1) for each i fix linear orders I_{ij}, for $j < 2^{\lambda_i}$, of cardinality λ_i such that $\mathrm{inv}^1(I_{ij})$ are all defined and distinct. Since $|\prod_{i<\mathrm{cf}(\lambda)} 2^{\lambda_i}| = 2^\lambda$ it will suffice to associate a linear order

J_g to every $g \in \prod_{i<\mathrm{cf}(\lambda)} 2^{\lambda_i}$ such that $\mathrm{inv}^2(J_g)$ is defined for every g and $\mathrm{inv}^2(J_g) \neq \mathrm{inv}^2(J_h)$ whenever $g \neq h$.

Since $\theta \geq \max(\aleph_2, \mathrm{cf}(\lambda))$, by [20, Appendix, Theorem 1.3(2)] or [16, Corollary 6.12] we may partition the set $\{\xi < \theta\colon \mathrm{cf}(\xi) \geq \aleph_1\}$ into $\mathrm{cf}(\lambda)$ stationary sets S_ξ, for $\xi < \mathrm{cf}(\lambda)$. Then (letting $\mathbf{S}(\xi) = \eta$ if $\xi \in S_\eta$):

$$J_g = \sum_{\xi<\theta} I^*_{\mathbf{S}(\xi),g(\mathbf{S}(\xi))}$$

has $\mathrm{inv}^0(J_g) = \theta$ and $\mathrm{inv}^2(J_g) = \langle \mathrm{inv}^1(I_{\xi,g(\xi)})\colon \xi < \theta\rangle/\mathcal{D}(\theta, \aleph_1)$. If ξ is such that $h(\xi) \neq g(\xi)$ then the representing sequences of $\mathrm{inv}^2(J_g)$ and $\mathrm{inv}^2(J_h)$ disagree on the stationary set S_ξ. Therefore $g \mapsto \mathrm{inv}^2(J_g)$ is an injection, as required. □

2.2. *A modified invariant* $\mathbf{inv}^{m,\lambda}(J)$

Fix a cardinal λ. For a linear order J of cardinality λ and $m \in \mathbb{N}$ we define an invariant that is a modification of $\mathrm{inv}^m(J)$, considering three cases.

2.2.1. *Assume λ is regular*

Then let $\mathrm{inv}^{m,\lambda}(J) = \mathrm{inv}^m(J)$ if $\mathrm{cf}(J) = \lambda$ and `undefined` otherwise.

2.2.2. *Assume λ is singular and* $\mathrm{cf}(\lambda) > \aleph_1$

Fix an increasing continuous sequence of cardinals λ_ξ, for $\xi < \mathrm{cf}(\lambda)$, such that $\lambda = \sup_{\xi<\mathrm{cf}(\lambda)} \lambda_\xi$.

Then let $\mathrm{inv}^{0,\lambda}(J) = \mathrm{inv}^0(J)$ if $\mathrm{cf}(J) = \mathrm{cf}(\lambda)$ and `undefined` otherwise. If $m \geq 1$ and $\mathrm{inv}^{0,\lambda}(J)$ is defined, then let $\mathrm{inv}^{m,\lambda}(J) = \mathrm{inv}^m(J)$ if $\mathrm{inv}^m(J) = \langle \mathbf{d}_\xi : \xi < \mathrm{cf}(\lambda)\rangle$ is such that

$$\{\xi < \mathrm{cf}(\lambda)\colon\ \mathrm{cf}(\mathbf{d}_\xi) > \lambda_\xi\} \in \mathcal{D}(\mathrm{cf}(\lambda), \aleph_1).$$

2.2.3. *Assume λ is singular and* $\aleph_1 \geq \mathrm{cf}(\lambda)$

This case will require extra work. Like above, fix an increasing continuous sequence of cardinals λ_ξ, for $\xi < \mathrm{cf}(\lambda)$, such that $\lambda = \sup_{\xi<\mathrm{cf}(\lambda)} \lambda_\xi$. By RegCard we denote the class of all regular cardinals.

Lemma 2.3. *If* $\mathrm{cf}(\lambda) \leq \aleph_1$ *then there is* $h = h_\lambda\colon \aleph_2 \to \lambda \cap \mathrm{RegCard}$ *such that* $h^{-1}([\mu, \lambda))$ *is* $\mathcal{D}(\aleph_2, \aleph_1)$*-positive for every* $\mu < \lambda$.

Proof. Partition $\aleph_2$ into $\mathcal{D}(\aleph_2, \aleph_1)$-positive sets S_ξ, $\xi < \mathrm{cf}(\lambda)$. Fix an increasing sequence of regular cardinals $\lambda_\eta, \eta < \mathrm{cf}(\lambda)$, cofinal in λ and let $h(\xi) = \lambda_\eta$ if $\xi \in S_\eta$. □

With $h = h_\lambda$ as in Lemma 2.3 let $\mathcal{D}_h(\aleph_2)$ be the filter generated by $\mathcal{D}(\aleph_2, \aleph_1)$ and the sets $h^{-1}([\mu, \lambda))$ for $\mu < \lambda$. In the following the function h_λ will be fixed for each λ such that $\mathrm{cf}(\lambda) \leq \aleph_1$. We shall therefore suppress writing h everywhere except in $\mathcal{D}_{h_\lambda}(\aleph_2)$, usually dropping the subscript λ which will be clear from the context.

Define $\mathrm{inv}^{m,\lambda}(J)$ (really $\mathrm{inv}^{m,\lambda,h}(J)$) as follows.

Let $\mathrm{inv}^{0,\lambda}(J) = \mathrm{inv}^0(J)$ if $\mathrm{cf}(J) = \aleph_2$ and `undefined` otherwise.

Assume $m \geq 1$ and

$$\mathrm{inv}^m(J) = \langle \mathbf{d}_\xi \colon \xi < \aleph_2 \rangle / \mathcal{D}(\aleph_2, \aleph_1).$$

If $\{\xi \colon \mathrm{cf}(\mathbf{d}_\xi) > h(\xi)\} \in \mathcal{D}_h(\aleph_2)$ then let

$$\mathrm{inv}^{m,\lambda}(J) = \langle \mathbf{d}_\xi \colon \xi < \aleph_2 \rangle / \mathcal{D}_h(\aleph_2)$$

and `undefined` otherwise.

Since $\mathcal{D}_h(\aleph_2)$ extends $\mathcal{D}(\aleph_2, \aleph_1)$, this invariant is well-defined.

Definition 2.4. Given a cardinal $\lambda \geq \aleph_2$ and $m \in \mathbb{N}$, an *m, λ-invariant* is any invariant $\mathrm{inv}^{m,\lambda}(J)$ for a linear order J of cardinality λ that is not equal to `undefined`.

Two representing sequences $\langle \mathbf{d}_\xi \colon \xi < \kappa \rangle$ and $\langle \mathbf{e}_\xi \colon \xi < \kappa \rangle$ of invariants of the same cofinality κ are *disjoint* if $\mathbf{d}_\xi \neq \mathbf{e}_\xi$ for all ξ. Note that this is not a property of the invariants since it depends on the choice of the representing sequences.

Lemma 2.5. *For every cardinal $\lambda \geq \aleph_2$ there exist $m \in \mathbb{N}$ and 2^λ disjoint representing sequences of m, λ-invariants of linear orders of cardinality λ.*

Proof. Assume first λ is regular. By Lemma 2.2, there are 2^λ linear orders of cardinality λ and with cofinality equal to λ, listed as I_ξ for $\xi < 2^\lambda$, with distinct (and defined) invariants $\mathrm{inv}^1(I_\xi)$. Let $J_\xi = \lambda \times J_\xi^*$. Then $|I_\xi| = \lambda$, $\mathrm{inv}^{2,\lambda}(J_\xi)$ is defined since $\mathrm{cf}(I_\xi) = \lambda$ for all ξ and it has constant representing sequence. Therefore all these representing sequences are disjoint.

Now assume λ is singular. By Lemma 2.2 for every sufficiently large regular $\theta < \lambda$, there are 2^λ linear orders, $J_{\theta,\xi}$, for $\xi < 2^\lambda$, of cardinality λ, cofinality θ, and with distinct and defined invariants $\mathrm{inv}^2(J_{\theta,\xi})$.

(a) Assume furthermore that $\mathrm{cf}(\lambda) \geq \aleph_2$. Fix an increasing continuous sequence λ_η, for $\eta < \mathrm{cf}(\lambda)$ with the supremum equal to λ, as in Sec. 2.2.2. Now fix an increasing sequence θ_η, for $\eta < \mathrm{cf}(\lambda)$, of regular cardinals with the supremum equal to λ and such that $\theta_\eta > \lambda_\eta$ for all η. For $\xi < 2^\lambda$ let

$$I_\xi = \sum_{\eta < \mathrm{cf}(\lambda)} (I_{\theta_\eta,\xi})^*$$

(see Example 2.1(3)). Then each linear order I_ξ, for $\xi < 2^\lambda$, has cardinality λ, $\mathrm{inv}^{3,\lambda}(I_\xi)$ is defined for all ξ, and the obvious representing sequences for $\mathrm{inv}^{3,\lambda}(I_\xi)$ are disjoint.

(b) Now assume $\mathrm{cf}(\lambda) \leq \aleph_1$ and consider $h = h_\lambda \colon \aleph_2 \to \lambda \cap \mathrm{RegCard}$ as in Lemma 2.3. For $\xi < 2^\lambda$ let $I_\xi = \sum_{\eta<\aleph_2} I_{h(\eta),\xi}{}^*$. Then each linear order I_ξ, for $\xi < 2^\lambda$, has cardinality λ, $\mathrm{inv}^{3,\lambda}(I_\xi)$ is defined, and the obvious representing sequences for $\mathrm{inv}^{3,\lambda}(I_\xi)$ are disjoint. □

3. Representing Invariants in Models of Theories with the Order Property

3.1. *The order property*

In the present section, A is a model of countable signature whose theory has the *order property*, as witnessed by formula $\phi(\bar{x}, \bar{y})$. Thus there is $n \geq 1$ such that ϕ is a $2n$-ary formula and in A^n there exist arbitrarily long finite $\prec_\phi$ chains, where $\prec_\phi$ is a binary relation on A^n defined by letting $\bar{a} \prec_\phi \bar{b}$ if

$$A \models \phi(\bar{a}, \bar{b}) \wedge \neg\phi(\bar{b}, \bar{a}).$$

It should be emphasized that $\prec_\phi$ is not required to be transitive.

The existence of such formula ϕ is equivalent to the theory of A being unstable ([20, Theorem 2.13]). This fact is the only bit of stability theory needed in the present paper.

We shall write $A \models \bar{a} \preceq_\phi \bar{b}$ to signify that $A \models \bar{a} \prec_\phi \bar{b}$ or $A \models \bar{a} = \bar{b}$. We shall frequently write $\bar{a} \prec_\phi \bar{b}$ and $\bar{a} \preceq_\phi \bar{b}$ instead of $A \models \bar{a} \prec_\phi \bar{b}$ and $A \models \bar{a} \preceq_\phi \bar{b}$ since at any given instance we will deal with a fixed A and its elementary substructures.

A *ϕ-chain* is a subset of A^n linearly ordered by $\preceq_\phi$. For $\bar{b}$ and $\bar{c}$ in A^n we write

$$[\bar{b}, \bar{c}]_\phi = \{\bar{d} \colon \bar{b} \preceq_\phi \bar{d} \wedge \bar{d} \preceq_\phi \bar{c}\}$$

and similarly

$$(-\infty, \bar{c}] = \{\bar{d} \colon \bar{d} \preceq_\phi \bar{c}\}, \quad \text{and}$$

$$[\bar{c}, \infty) = \{\bar{d} \colon \bar{c} \preceq_\phi \bar{d}\}.$$

If $\mathcal{C}$ is a ϕ-chain in A then we shall freely use phrases such as "large enough $\bar{c} \in \mathcal{C}$" with their obvious meaning. By $\mathrm{cf}(\mathcal{C})$ we denote the cofinality of $(\mathcal{C}, \preceq_\phi)$. We shall sometimes consider ϕ-chains with the reverse ordering, $\preceq_{\neg\phi}$. Whenever deemed necessary this will be made explicit by writing $(\mathcal{C}, \preceq_{\neg\phi})$ as in e.g. $\mathrm{cf}(\mathcal{C}, \preceq_{\neg\phi})$. Since $\preceq_\phi$ need not be transitive, one has to use this notation with some care.

3.2. *Combinatorics of the invariants*

The following is a special case of the definition of "weakly (κ, Δ)-skeleton like" where κ is an arbitrary cardinal and Δ is set of formulas as given in [21, Definition 3.1]. Readers familiar with [21] may want to know that we fix $\kappa = \aleph_1$ and $\Delta = \{\phi, \psi\}$ where $\psi(\bar{x}, \bar{y})$ stands for $\phi(\bar{y}, \bar{x})$.

Definition 3.1. A ϕ-chain $\mathcal{C}$ is *weakly* $(\aleph_1, \phi)$*-skeleton like* inside A if for every $\bar{a} \in A^n$ there is a countable $\mathcal{C}_{\bar{a}} \subseteq \mathcal{C}$ such that for all $\bar{b} \preceq_\phi \bar{c}$ in $\mathcal{C}$ with $[\bar{b}, \bar{c}]_\phi$ disjoint from $\mathcal{C}_{\bar{a}}$ we have

$$A \models \phi(\bar{b}, \bar{a}) \leftrightarrow \phi(\bar{c}, \bar{a})$$

and

$$A \models \phi(\bar{a}, \bar{b}) \leftrightarrow \phi(\bar{a}, \bar{c}).$$

Remark 3.2. One can weaken the definition of weakly $(\aleph_1, \phi)$-skeleton like by requiring only that (with $\bar{a}$, $\mathcal{C}_{\bar{a}}$, $\bar{b}$ and $\bar{c}$ as in Definition 3.1)

$$\bar{a} \preceq_\phi \bar{b} \quad \text{if and only if} \quad \bar{a} \preceq_\phi \bar{c}$$

and

$$\bar{b} \preceq_\phi \bar{a} \quad \text{if and only if} \quad \bar{c} \preceq_\phi \bar{a}.$$

All the statements about the notion of being weakly $(\aleph_1, \phi)$-skeleton like, except Lemma 3.7, remain true for the modified notion. As a matter of fact, it is transparent that even their proofs remain unchanged.

Remark 3.3. For $\bar{a} \in A^k$ and $\bar{b} \in A^n$ define

$$\mathrm{tp}(\bar{a}/\bar{b}) = \{\psi(\bar{x}, \bar{b})\colon \psi \text{ is a } k+n\text{-ary formula and } A \models \psi(\bar{a}, \bar{b})\}.$$

One may now consider a stronger indiscernibility requirement on a ϕ-chain $\mathcal{C}$ than being weakly $(\aleph_1, \phi)$-skeleton like, defined as follows.

(*) For every $k \in \mathbb{N}$ and $\bar{a} \in A^k$ there is a countable $\mathcal{C}_{\bar{a}} \subseteq \mathcal{C}$ such that for all $\bar{b} \preceq_\phi \bar{c}$ in $\mathcal{C}$ with $[\bar{b}, \bar{c}]_\phi \cap \mathcal{C}_{\bar{a}} = \emptyset$ we have that

$$\mathrm{tp}(\bar{a}/\bar{b}) = \mathrm{tp}(\bar{a}/\bar{c}).$$

The proofs of Theorems 1.1 and 1.3 can be easily modified to provide an ultrafilter $\mathcal{U}$ such that for a given linear order I the ultrapower $\prod_{\mathcal{U}} A$ includes a ϕ-chain $\mathcal{C}$ isomorphic to I and satisfying (*). See Remarks 4.5 and 6.9.

The nontrivial part of the following is a special case of [21, Claim 3.15] that will be needed in Sec. 3.3.

Lemma 3.4. *Assume $\mathcal{C}$ is a ϕ-chain that is weakly $(\aleph_1, \phi)$-skeleton like in A. Then $\mathcal{C}^*$ is weakly $(\aleph_1, \phi)$-skeleton like inside A, and every interval of $\mathcal{C}$ is weakly $(\aleph_1, \phi)$-skeleton like inside A. If $\mathcal{E} \subseteq \mathcal{C}$ is well-ordered* (*or conversely well-ordered*) *by $\preceq_\phi$ then $\mathcal{E}$ is weakly $(\aleph_1, \phi)$-skeleton like in A.*

Proof. Only the last sentence requires a proof. For $\bar{b} \in A^n$ define $\mathcal{E}_{\bar{b}} \subseteq \mathcal{E}$ as follows.

$$\mathcal{E}_{\bar{b}} = \{\min(\mathcal{E} \cap [\bar{c}, \infty)_\phi)\colon \bar{c} \in \mathcal{C}_{\bar{b}}\}.$$

Each $\mathcal{E}_{\bar{b}}$ is countable since every $\bar{c} \in \mathcal{C}_{\bar{b}}$ produces at most one element of $\mathcal{E}_{\bar{b}}$. For $\bar{a} \preceq_\phi \bar{c}$ in $\mathcal{E}$ such that $[\bar{a}, \bar{c}]_\phi \cap \mathcal{E}_{\bar{b}} = \emptyset$ we have that $[\bar{a}, \bar{c}]_\phi \cap \mathcal{C}_{\bar{b}} = \emptyset$ and therefore $\mathrm{tp}_\phi(\bar{a}/\bar{b}) = \mathrm{tp}_\phi(\bar{c}/\bar{b})$. $\square$

If $\mathcal{C}$ and $\mathcal{E}$ are $\preceq_\phi$-chains in A then we say $\mathcal{C}$ and $\mathcal{E}$ are *mutually cofinal* if for every $\bar{a} \in \mathcal{C}$ we have $\bar{a} \prec_\phi \bar{b}$ for all large enough $\bar{b} \in \mathcal{E}$ and for every $\bar{b} \in \mathcal{E}$ we have $\bar{b} \prec_\phi \bar{a}$ for all large enough $\bar{a} \in \mathcal{C}$.

Lemma 3.5. *Assume $\mathcal{C}$ and $\mathcal{E}$ are mutually cofinal ϕ-chains in A. Then* $\mathrm{cf}(\mathcal{C}) = \mathrm{cf}(\mathcal{E})$.

Of course this is standard but since $\prec_\phi$ is not assumed to be a partial ordering on A we shall prove it. Also note that if the condition "for every $\bar{a} \in \mathcal{C}$ we have $\bar{a} \prec_\phi \bar{b}$ for all large enough $\bar{b} \in \mathcal{E}$" is replaced by "for every $\bar{a} \in \mathcal{C}$ we have $\bar{a} \prec_\phi \bar{b}$ for some $\bar{b} \in \mathcal{E}$" and the condition "for every $\bar{b} \in \mathcal{E}$ we have $\bar{b} \prec_\phi \bar{a}$ for all large enough $\bar{a} \in \mathcal{C}$" is replaced by is replaced by "for every $\bar{b} \in \mathcal{E}$ we have $\bar{b} \prec_\phi \bar{a}$ for some $\bar{a} \in \mathcal{C}$" then we cannot conclude $\mathrm{cf}(\mathcal{C}) = \mathrm{cf}(\mathcal{E})$ in general.

Proof. Assume $\kappa = \mathrm{cf}(\mathcal{C}) < \mathrm{cf}(\mathcal{E}) = \lambda$ and fix a cofinal $X \subseteq \mathcal{C}$ of cardinality κ. For each $\bar{a} \in X$ pick $f(\bar{a}) \in \mathcal{E}$ such that $\bar{a} \prec_\phi \bar{b}$ for all $\bar{b}$ such that $f(\bar{a}) \preceq_\phi \bar{b}$. The set $\{f(\bar{a})\colon a \in X\}$ is not cofinal in $\mathcal{E}$ and we can pick $\bar{b} \in \mathcal{E}$ such that $f(\bar{a}) \preceq_\phi \bar{b}$ for all $\bar{a} \in X$. Now let $\bar{a} \in \mathcal{C}$ be such that for all $\bar{c} \in \mathcal{C}$ such that $\bar{a} \prec_\phi \bar{c}$ we have $\bar{b} \prec_\phi \bar{c}$. But there is $\bar{c} \in X$ such that $\bar{a} \prec_\phi \bar{c}$, and this is a contradiction. □

The following is [21, Lemma 3.7] in the case $\kappa = \aleph_1$. We reproduce the proof for the convenience of the reader.

Lemma 3.6. *Assume $\mathcal{C}_0, \mathcal{C}_1$ are increasing, weakly $(\aleph_1, \phi)$-skeleton like, ϕ-chains in A. Also assume these two chains are mutually cofinal and m is such that both* $\mathrm{inv}^m(\mathcal{C}_0)$ *and* $\mathrm{inv}^m(\mathcal{C}_1)$ *are defined. Then* $\mathrm{inv}^m(\mathcal{C}_0) = \mathrm{inv}^m(\mathcal{C}_1)$.

Proof. The proof is by induction on m. If $m = 0$ then this is Lemma 3.5. Now assume the assertion has been proved for m and all pairs $\mathcal{C}_0$ and $\mathcal{C}_1$. Fix $\mathcal{C}_0, \mathcal{C}_1$ satisfying the assumptions for $m+1$ in place of m and let $\kappa = \mathrm{cf}(\mathcal{C}_0) = \mathrm{cf}(\mathcal{C}_1)$. Since $\mathrm{inv}^m(\mathcal{C}_0)$ is defined, $\kappa \geq \aleph_1$. Since $\mathrm{inv}^{m+1}(\mathcal{C}_0)$ is defined, $\mathcal{D}(\kappa, \aleph_1)$ is a proper filter and $\kappa \geq \aleph_2$.

For an elementary sumbodel N of $(A, \mathcal{C}_0, \mathcal{C}_1)$ consider

$$\mathcal{C}_N^0 = \{\bar{b} \in \mathcal{C}_0 \colon A \models \bar{c} \preceq_\phi \bar{b} \text{ for all } \bar{c} \in N^n \cap \mathcal{C}_0\}, \quad \text{and}$$

$$\mathcal{C}_N^1 = \{\bar{b} \in \mathcal{C}_1 \colon A \models \bar{c} \preceq_\phi \bar{b} \text{ for all } c \in N^n \cap \mathcal{C}_1\}.$$

By our assumption that $\mathrm{inv}^{m+1}(\mathcal{C}_0)$ and $\mathrm{inv}^{m+1}(\mathcal{C}_1)$ are defined we have that for any regular $\mu < \kappa$ the set of $N \prec (A, \mathcal{C}_0, \mathcal{C}_1)$ of cardinality μ such that $\mathrm{cf}(N^n \cap \mathcal{C}_0) \geq \aleph_1$ implies $\mathrm{inv}^m(\mathcal{C}_N^0, \preceq_{\neg\phi})$ is defined includes a club. In particular, for club many N of size μ such that $\mathrm{cf}(N^n \cap \mathcal{C}_0) \geq \aleph_1$ we have $\mathrm{cf}(\mathcal{C}_N^0, \preceq_{\neg\phi}) \geq \aleph_1$. Similarly, for club many N of size μ such that $\mathrm{cf}(N^n \cap \mathcal{C}_1) \geq \aleph_1$ we have that $\mathrm{inv}^m(\mathcal{C}_N^1)$ is defined and $\mathrm{cf}(\mathcal{C}_N^1, \preceq_{\neg\phi}) \geq \aleph_1$.

Now pick $N \prec A$ such that $\mathrm{cf}(N^n \cap \mathcal{C}_0)$, $\mathrm{cf}(N^n \cap \mathcal{C}_1)$, $\mathrm{cf}(\mathcal{C}_N^0, \preceq_{\neg\phi})$ and $\mathrm{cf}(\mathcal{C}_N^1, \preceq_{\neg\phi})$ are all uncountable and $\mathrm{inv}^m(\mathcal{C}_N^0, \preceq_{\neg\phi})$ and $\mathrm{inv}^m(\mathcal{C}_N^1, \preceq_{\neg\phi})$ are defined. We shall prove that in this case $(\mathcal{C}_N^0, \preceq_{\neg\phi})$ and $(\mathcal{C}_N^1, \preceq_{\neg\phi})$ are mutually cofinal.

By the elementarity $N^n \cap \mathcal{C}_0$ and $N^n \cap \mathcal{C}_1$ satisfy the assumptions of Lemma 3.5, and in particular $\mathrm{cf}(N^n \cap \mathcal{C}_0) = \mathrm{cf}(N^n \cap \mathcal{C}_1)$. Pick $\bar{a} \in \mathcal{C}_N^0$. Since $N^n \cap \mathcal{C}_1$ and $N^n \cap \mathcal{C}_0$ are mutually cofinal, by elementarity for all $\bar{c} \in N^n \cap \mathcal{C}_1$ we have that $\bar{c} \preceq_\phi \bar{a}$.

Let $\mathcal{E}_{\bar{a}} \subseteq \mathcal{C}_1$ be a countable set such that for all $\bar{b}$ and $\bar{c}$ in $\mathcal{C}_1$ satisfying $\bar{b} \preceq_\phi \bar{c}$ and $[\bar{b}, \bar{c}]_\phi \cap \mathcal{E}_{\bar{a}} = \emptyset$ we have that $A \models \phi(\bar{b}, \bar{a}) \leftrightarrow \phi(\bar{c}, \bar{a})$ and $A \models \phi(\bar{a}, \bar{b}) \leftrightarrow \phi(\bar{a}, \bar{c})$. Since $\mathcal{E}_{\bar{a}}$ is countable, by our assumptions on the cofinalities of $N^n \cap \mathcal{C}_1$ and $(\mathcal{C}_N^1, \preceq_{\neg\phi})$ for $\preceq_\phi$ end-segment many $\bar{c} \in N^n \cap \mathcal{C}_1$ and for $\preceq_{\neg\phi}$-end-segment many $\bar{d} \in \mathcal{C}_N^1$ we have

$$A \models \bar{c} \preceq_\phi \bar{a} \leftrightarrow \bar{d} \preceq_\phi \bar{a}.$$

Therefore for $\preceq_{\neg\phi}$-end-segment many $\bar{d} \in \mathcal{C}_N^1$ we have $\bar{d} \preceq_\phi \bar{a}$, i.e. $\bar{a} \preceq_{\neg\phi} \bar{d}$.

An analogous proof shows that for every $\bar{e} \in \mathcal{C}^1$ and $\preceq_{\neg\phi}$-cofinally many $\bar{d} \in \mathcal{C}^0$ we have $\bar{e} \preceq_{\neg\phi} \bar{d}$. We have therefore proved that the ϕ-chains $(\mathcal{C}_N^0, \preceq_{\neg\phi})$ and $(\mathcal{C}_N^1, \preceq_{\neg\phi})$ are mutually cofinal. They are both obviously weakly $(\aleph_1, \phi)$-skeleton like, and by the inductive hypothesis in this case we have $\mathrm{inv}^m(\mathcal{C}_N^0, \preceq_{\neg\phi}) = \mathrm{inv}^m(\mathcal{C}_N^1, \preceq_{\neg\phi})$ if both of these invariants are defined.

By the inductive hypothesis we have $\mathrm{inv}^{m+1}(\mathcal{C}_0) = \mathrm{inv}^{m+1}(\mathcal{C}_1)$. □

3.3. *Defining an invariant over a submodel*

Assume Z is an elementary submodel of A. By $\mathrm{tp}_\phi(\bar{a}/Z)$ we denote the ϕ-type of $\bar{a} \in A^n$ in the signature $\{\phi\}$ over Z, or in symbols

$$\mathrm{tp}_\phi(\bar{a}/Z) = \{\phi(\bar{x}, \bar{b})\colon \bar{b} \in Z, A \models \phi(\bar{a}, \bar{b})\} \cup \{\phi(\bar{b}, \bar{x})\colon \bar{b} \in Z, A \models \phi(\bar{b}, \bar{a})\}.$$

If $B \subseteq A$ (in particular, if B is an elementary submodel of A) we shall write $\mathrm{tp}_\phi(\bar{a}/B)$ for $\mathrm{tp}_\phi(\bar{a}/B^n)$. Write $\mathrm{tp}_\phi(\bar{a}/\bar{e})$ for $\mathrm{tp}_\phi(\bar{a}/\{\bar{e}\})$.

Lemma 3.7. *A ϕ-chain $\mathcal{C}$ in A is weakly $(\aleph_1, \phi)$-skeleton like in A if and only if for every $\bar{a} \in A^n$ there exists a countable $\mathcal{C}_{\bar{a}} \subseteq \mathcal{C}$ with the property that for $\bar{c}$ and $\bar{d}$ in $\mathcal{C}$ the condition*

$$\mathcal{C}_{\bar{a}} \cap (-\infty, \bar{c}]_\phi = \mathcal{C}_{\bar{a}} \cap (-\infty, \bar{d}]_\phi$$

implies $\mathrm{tp}_\phi(\bar{a}/\bar{c}) = \mathrm{tp}_\phi(\bar{a}/\bar{d})$.

Proof. Immediate from Definition 3.1. □

Definition 3.8. Assume B is an elementary submodel of A, $m \in \mathbb{N}$, and $\mathbf{d}$ is an m-invariant. We say that $\bar{c} \in A^n \backslash B^n$ *defines* an (A, B, ϕ, m)-invariant $\mathbf{d}$ if there are

(1) (nonempty) linear orders J and I, and
(2) $\bar{a}_j \in B^n$ for $j \in J$ and $\bar{a}_i \in A^n \backslash B^n$ for $i \in I$, such that

(3) $\langle \bar{a}_i : i \in J + I^* \rangle$ is a ϕ-chain in A that is weakly $(\aleph_1, \phi)$-skeleton like in A,
(4) $\mathrm{tp}_\phi(\bar{a}_i/B) = \mathrm{tp}_\phi(\bar{c}/B)$ for all $i \in I$,
(5) $\mathbf{d} = \mathrm{inv}^m(I)$, and
(6) if $J', I', \bar{a}'_i$ for $i \in J' \cup I'$ and $\mathbf{d}'$ satisfy conditions (1)–(5) then $\mathbf{d}' = \mathbf{d}$.

Let $\mathrm{INV}^m(A, B, \phi)$ denote the set of all m-invariants $\mathbf{d}$ such that some $\bar{c}$ defines an (A, B, ϕ, m)-invariant $\mathbf{d}$.

The point of Definition 3.8 and the conclusion of the Lemma 3.9 is that, once A, ϕ, and m are fixed, the invariant $\mathbf{d}$ depends only on the submodel B and the element $\bar{c}$ outside of this submodel, and not on the ϕ-chain $\mathcal{C}$. Conditions (1)–(5) of Definition 3.8 imply (6) of Definition 3.8. This is a consequence of Lemma 3.10 and the fact that cofinalities occurring in invariants that are "defined" in the sense of Sec. 2.1 or Sec. 2.2 are uncountable.

The following notation will be useful. Assume $\mathcal{C}$ is a ϕ-chain that is weakly $(\aleph_1, \phi)$-skeleton like in A and B is an elementary submodel of A. For $\bar{c} \in \mathcal{C} \backslash B^n$ let

$$\mathcal{C}[B, \bar{c}] = \{\bar{a} \in \mathcal{C} : (\forall \bar{b} \in B^n \cap \mathcal{C}) \bar{c} \preceq_\phi \bar{b} \leftrightarrow \bar{a} \preceq_\phi \bar{b}\}.$$

We shall always consider $\mathcal{C}[B, \bar{c}]$ with respect to the reverse order, $\preceq_{\neg\phi}$.

Lemma 3.9. *Assume $\mathcal{C} = \langle a_i : i \in I \rangle$ is a ϕ-chain that is weakly $(\aleph_1, \phi)$-skeleton like in A. Assume B is an elementary submodel of A and $\bar{c} \in \mathcal{C} \backslash B^n$ are such that*

(1) *$\mathcal{C}_{\bar{b}} \cap \mathcal{C}[B, \bar{c}] \cap (-\infty, \bar{c}]_\phi = \emptyset$ for all $\bar{b} \in B^n$, and*
(2) *$\mathbf{d} = \mathrm{inv}^m(\mathcal{C}[B, \bar{c}], \preceq_{\neg\phi})$ is well-defined.*
(3) *$(\mathcal{C} \cap B \cap [-\infty, \bar{c}]) \geq \aleph_1$*

Then $\bar{c}$ defines the (A, B, ϕ, m)-invariant $\mathbf{d}$.

Proof. Let J_0 be a well-ordered $\preceq_\phi$-cofinal subset of

$$\{i \in I : \bar{a}_i \in B^n \text{ and } \bar{a}_i \preceq_\phi \bar{c}\}$$

of minimal order type. By Lemma 3.4, the ϕ-chain $\langle a_i : i \in J_0 \rangle$ is weakly $(\aleph_1, \phi)$-skeleton like in A. Let $I_0 = \{i \in I : \bar{a}_i \in \mathcal{C}[B, \bar{c}] \text{ and } \bar{a}_i \leq \bar{c}\}$. We need to check that I_0, J_0 and $\langle \bar{a}_i : i \in J_0 + I_0^* \rangle$ satisfy (1)–(6) of Definition 3.8.

Clauses (1)–(2) are immediate. As an interval of a weakly $(\aleph_1, \phi)$-skeleton like order, $\langle a_i : i \in I_0 \rangle$ is weakly $(\aleph_1, \phi)$-skeleton like. Therefore clause (3) follows. In order to prove (4) pick $\bar{b} \in B^n$ and $\bar{d} \in \mathcal{C}[B, \bar{c}] \cap (-\infty, \bar{c}]_\phi$. Then $[\bar{d}, \bar{c}]_\phi \cap \mathcal{C}_{\bar{b}} = \emptyset$, hence $\mathrm{tp}_\phi(\bar{c}/\bar{b}) = \mathrm{tp}_\phi(\bar{d}/\bar{b})$. Since $\bar{b} \in B^n$ was arbitrary, we have $\mathrm{tp}_\phi(\bar{c}/B) = \mathrm{tp}_\phi(\bar{d}/B)$ and we have proved (4). Clause (5) is automatic, and (6) follows by Lemma 3.10 below. $\square$

Lemma 3.10. *Assume I_0, I_1, J_0, J_1 are linear orders and $\langle \bar{a}_i : i \in J_0 + I_0^* \rangle$ and $\langle \bar{b}_i : i \in J_1 + I_1^* \rangle$ are weakly $(\aleph_1, \phi)$-skeleton like ϕ-chains in A such that*

(1) *$\bar{a}_i \in B^n$ if and only if $i \in J_0$ and $\bar{b}_i \in B^n$ if and only if $i \in J_1$,*
(2) *$\mathrm{tp}_\phi(\bar{a}_i/B) = \mathrm{tp}_\phi(\bar{b}_j/B)$ for all $i \in I_0$ and all $j \in I_1$,*

(3) *each of* $\mathrm{cf}(I_0), \mathrm{cf}(I_1), \mathrm{cf}(J_0),$ *and* $\mathrm{cf}(J_1)$ *is uncountable.*

If $\mathrm{inv}^m(I_0)$ *and* $\mathrm{inv}^m(I_1)$ *are both defined then* $\mathrm{inv}^m(I_0) = \mathrm{inv}^m(I_1)$.

Proof. Pick $i(0) \in I_0$. Since $\mathrm{tp}_\phi(\bar{a}_{i(0)}/B) = \mathrm{tp}_\phi(\bar{b}_j/B)$ for some (any) $j \in I_1$, we have that $\bar{b}_i \preceq_\phi \bar{a}_{i(0)}$ for all $i \in J_1$. Since $\mathrm{cf}(J_1)$ and $\mathrm{cf}(I_1)$ are both uncountable and since $\langle b_i : i \in J_1 + I_1^* \rangle$ is weakly $(\aleph_1, \phi)$-skeleton like, we conclude that for large enough $i \in I_1$ we have $\bar{a}_{i(0)} \preceq_{\neg\phi} \bar{b}_i$.

The analogous argument shows that for every $i(1) \in I_1$ and all large enough $i \in I_0$ we have $\bar{a}_{i(1)} \preceq_{\neg\phi} \bar{b}_i$. Then $\langle \bar{a}_i : i \in I_0 \rangle$ and $\langle \bar{b}_i : i \in I_1 \rangle$ are, when ordered by $\preceq_{\neg\phi}$, mutually cofinal.

By Lemma 3.6 we have that $\mathrm{inv}^m(I_0) = \mathrm{inv}^m(I_1)$ if both of these invariants are defined, and the claim follows. □

3.4. *Representing invariants*

In addition to A, ϕ and m fixed in Sec. 3.1 we distinguish $\lambda = |A|$. A *representation* of A is a continuous chain of elementary submodels A_ξ, for $\xi < \mathrm{cf}(\lambda)$, of A such that $|A_\xi| < |A|$ for all ξ and $\bigcup_{\xi < \mathrm{cf}(\lambda)} A_\xi = A$.

Define a set $\mathrm{INV}^{m,\lambda}(A, \phi)$ of m, λ-invariants (see Sec. 2.2) by cases as follows. Whenever $\mathbf{d}$ is an m-invariant, or an m, λ-invariant, for $m \geq 1$ we write $\langle \mathbf{d}_\xi : \xi < \mathrm{cf}(\mathbf{d}) \rangle$ for its representation. Although this representation is not unique, it is unique modulo the appropriate filter $\mathcal{D}(\mathrm{cf}(\lambda), \aleph_1)$ or $\mathcal{D}_{h_\lambda}(\aleph_2)$.

3.4.1. *Assume* λ *is regular*

Then $\mathbf{d}$ is an *m, λ-invariant* of A, ϕ if $\mathbf{d}$ is an m, λ-invariant and for every representation A_ξ, $\xi < \lambda$ of A we have

$$\{\xi : \mathbf{d}_\xi \in \mathrm{INV}^{m-1}(A, A_\xi, \phi)\} \in \mathcal{D}(\lambda, \aleph_1).$$

3.4.2. *Assume* λ *is singular and* $\mathrm{cf}(\lambda) > \aleph_1$

Then $\mathbf{d}$ is an *m, λ-invariant* of A, ϕ if $\mathbf{d}$ is an m, λ-invariant and for every representation A_ξ, $\xi < \mathrm{cf}(\lambda)$ of A we have

$$\{\xi : \mathbf{d}_\xi \in \mathrm{INV}^{m-1}(A, A_\xi, \phi)\} \in \mathcal{D}(\mathrm{cf}(\lambda), \aleph_1).$$

3.4.3. *Assume* λ *is singular and* $\aleph_1 \geq \mathrm{cf}(\lambda)$

Fix $h : \aleph_2 \to \lambda \cap \mathrm{Reg}$ as in Lemma 2.3. Then $\mathbf{d}$ is an *m, λ-invariant* of A, ϕ if $\mathbf{d}$ is an m, λ-invariant and for every representation $A = \bigcup_{\xi < \mathrm{cf}(\lambda)} A_\xi$ there is $\xi < \mathrm{cf}(\lambda)$ such that

$$\{i < \aleph_2 : \mathbf{d}_i \in \mathrm{INV}^{m-1}(A, A_\xi, \phi) \text{ and } h(i) > |A_\xi|\} \in \mathcal{D}_h(\aleph_2).$$

Lemma 3.11. *Assume A, ϕ, m and $\lambda = |A|$ are as above. Also assume $\mathcal{C} = \langle \bar{a}_j : j \in J \rangle$ is a ϕ-chain in A that is weakly $(\aleph_1, \phi)$-skeleton like in A. If $\mathrm{inv}^{m,\lambda}(J)$ is defined then $\mathrm{inv}^{m,\lambda}(J) \in \mathrm{INV}^{m,\lambda}(A)$.*

Proof. This is really three lemmas wrapped up in one. We prove each of the three cases, depending on the cofinality of λ (Secs. 3.4.1–3.4.3) separately.

3.4.4. *Assume λ is regular*

Fix a representation A_ξ, $\xi < \lambda$, of A. Let $\mathbf{C} \subseteq \lambda$ be the club consisting of all ξ such that for every $\bar{a} \in A^n_\xi$ we have $\mathcal{C}_{\bar{a}} \subseteq A^n_\xi$. By the assumption $\mathrm{cf}(J) = \lambda$ we may clearly assume $m \geq 1$. Let

$$\mathbf{d} = \langle \mathbf{d}_\xi : \xi < \lambda \rangle / \mathcal{D}(\lambda, \aleph_1).$$

Fix $\xi \in \mathbf{C}$ such that $\mathrm{cf}(\xi) = \mathrm{cf}(\mathcal{C} \cap A^n_\xi) \geq \aleph_1$ and $\mathbf{d}_\xi$ is defined. Since $\mathrm{cf}(J) = \lambda$ by Sec. 2.2.1 the set of such ξ belongs to $\mathcal{D}(\lambda, \aleph_1)$. It will therefore suffice to show that for every such ξ some $\bar{c}$ defines the $(A, A_\xi, \phi, m-1)$-invariant $\mathbf{d}_\xi$.

Pick $\bar{c} \in \mathcal{C}$ such that $(-\infty, \bar{c}]_\phi \cap A^n_\xi \supseteq \mathcal{C} \cap A^n_\xi$. Let I^ξ be the order with the underlying set $\{i \in J : \bar{a}_i \in \mathcal{C}[A_\xi, \bar{c}]\}$, so that $\mathrm{inv}^{m-1}(I^\xi) = \mathbf{d}_\xi$. Then

$$\mathrm{cf}(\mathcal{C} \cap A^n_\xi) = \mathrm{cf}(\xi) \geq \aleph_1$$

and

$$\mathrm{cf}(\mathcal{C}[A_\xi, \bar{c}], \preceq_{\neg\phi}) = \mathrm{cf}(\mathbf{d}_\xi) \geq \aleph_1.$$

Since $\bar{a} \in A^n_\xi$ implies $\mathcal{C}_{\bar{a}} \subseteq A^n_\xi$, Lemma 3.9 implies that $\bar{c}$ defines the $(A, A_\xi, \phi, m-1)$-invariant $\mathbf{d}_\xi$.

3.4.5. *Assume λ is singular and $\aleph_1 < \mathrm{cf}(\lambda)$*

Fix a representation A_ξ, $\xi < \mathrm{cf}(\lambda)$, of A. By the assumption $\mathrm{cf}(J) = \mathrm{cf}(\lambda)$ and we may clearly assume $m \geq 1$.

Let $\mathbf{d} = \langle \mathbf{d}_\xi : \xi < \mathrm{cf}(\lambda) \rangle / \mathcal{D}(\mathrm{cf}(\lambda), \aleph_1)$. By Sec. 2.2.2 we have $J = \sum_{\xi < \mathrm{cf}(\lambda)} J^*_\xi$ with $\mathrm{inv}^{m-1}(J_\xi) = \mathbf{d}_\xi$ for $\mathcal{D}(\mathrm{cf}(\lambda), \aleph_1)$-many ξ. We identify J^*_ξ with the corresponding subset of J.

Recall that $A = \bigcup_{\xi < \mathrm{cf}(\lambda)} A_\xi$, where this is an increasing sequence of elementary submodels each of cardinality $< \lambda$. Let L_ξ be the maximal initial segment of J such that $\{\bar{a}_i : i \in L_\xi\} \subseteq A_\xi$. Let $\mathbf{C}_0$ be the club in $\mathrm{cf}(\lambda)$ consisting of all ξ such that $|A_\xi| = \lambda_\xi$, with λ_ξ as fixed in Sec. 2.2.2, and that (L_ξ, J^*_ξ) forms a *gap* in J. That is, for every $i \in L_\xi$ and every $j \in J^*_\xi$ we have $i < j$ and there is no $l \in J$ such that $i < l < j$ for all $i \in L_\xi$ and all $j \in J^*_\xi$.

By Sec. 2.2.2 the set $\mathbf{C}_1$ of ξ such that $\mathrm{cf}(\mathbf{d}_\xi) = \mathrm{cf}(J_\xi) > |A_\xi|$ and $\mathrm{inv}^{m-1}(J_\xi) = \mathbf{d}_\xi$ is in $\mathcal{D}(\mathrm{cf}(\lambda), \aleph_1)$. It will therefore suffice to show that for every $\xi \in \mathbf{C}_0 \cap \mathbf{C}_1$ some $\bar{c}$ defines the $(A, A_\xi, \phi, m-1)$-invariant $\mathbf{d}_\xi$.

Since $\mathrm{cf}(J_\xi) > |A_\xi|$, for such ξ we can pick $j(0) \in J_\xi$ such that

$$\{\bar{a}_i : i \in J_\xi, i > j(0)\} \cap \left(A^n_\xi \cup \bigcup\{\mathcal{C}_{\bar{a}} : \bar{a} \in A^n_\xi\}\right) = \emptyset.$$

Let $\bar{c} = \bar{a}_{j(0)}$. Then

$$\mathrm{cf}(A^n_\xi \cap \mathcal{C} \cap (-\infty, \bar{c}]_\phi, \preceq_\phi) = \mathrm{cf}(\xi) \geq \aleph_1$$

and

$$\mathrm{cf}(\mathcal{C}[A_\xi, \bar{c}]) = \mathrm{cf}(\mathbf{d}_\xi) \geq \aleph_1.$$

By Lemma 3.9 we have that $\bar{c}$ defines the $(A, A_{\xi(0)}, \phi, m-1)$-invariant $\mathbf{d}_\eta$.

3.4.6. *Assume* λ *is singular and* $\mathrm{cf}(\lambda) \leq \aleph_1$

Fix a representation $A_\xi, \xi < \mathrm{cf}(\lambda)$, of A. Since $\mathrm{cf}(J) = \aleph_2$ we may assume $m \geq 1$. Let $\mathbf{d} = \langle \mathbf{d}_\xi : \xi < \aleph_2\rangle / \mathcal{D}_{h_\lambda}(\aleph_2)$ and write $J = \sum_{\zeta < \aleph_2} J^*_\zeta$ so that $\mathrm{inv}^{m-1}(J_\zeta) = \mathrm{inv}^{m-1}(\mathbf{d}_\zeta)$ for $\mathcal{D}_{h_\lambda}(\aleph_2)$-many ζ.

Let $I_\eta = \bigcup_{\xi < \eta} J^*_\xi$. Pick $L \subseteq J$ such that $L \cap J_\xi$ is nonempty for all ξ and $|L| \leq \aleph_2$. Then for every $\xi < \aleph_1$ we have that $L \cap I_\eta$ is cofinal in I_η for every limit η. Since λ is uncountable and regular we have $\lambda > \aleph_2$ and we can fix $\xi(0) < \mathrm{cf}(\lambda)$ such that $A_{\xi(0)} \supseteq \{\bar{a}_i : i \in L\}$.

The set of $\eta < \aleph_2$ such that $h(\eta) > \xi(0)$ and $\mathrm{cf}(\mathbf{d}_\eta) > |A_{\xi(0)}|$ belongs to $\mathcal{D}_h(\aleph_2)$, and it will suffice to show that for such η some $\bar{c}$ defines the $(A, A_{\xi(0)}, \phi, m-1)$-invariant $\mathbf{d}_\eta$. Since $\mathrm{cf}(\mathbf{d}_\eta) = \mathrm{cf}(J_\eta) > |A_{\xi(0)}|$, we can pick $j(0) \in J_\eta$ such that

$$\{\bar{a}_i : i \in J_\eta, i > j(0)\} \cap \left(A^n_{\xi(0)} \cup \bigcup\{\mathcal{C}_{\bar{a}} : \bar{a} \in A^n_{\xi(0)}\}\right) = \emptyset.$$

Let $\bar{c} = \bar{a}_{j(0)}$. Then

$$\mathrm{cf}(A^n_{\xi(0)} \cap \mathcal{C} \cap (-\infty, \bar{c}]_\phi, \preceq_\phi) = \mathrm{cf}(\eta) \geq \aleph_1$$

and

$$\mathrm{cf}(\mathcal{C}[A_{\xi(0)}, \bar{c}], \preceq_{\neg\phi}) = \mathrm{cf}(\mathbf{d}_\eta) \geq \aleph_1.$$

By Lemma 3.9 we have that $\bar{c}$ defines the $(A, A_{\xi(0)}, \phi, m-1)$-invariant $\mathbf{d}_\eta$.

This exhausts the cases and concludes the proof of the lemma. □

3.5. *Counting the number of invariants of a model*

We would like to prove the inequality $|\mathrm{INV}^{m,\lambda}(A, \phi)| \leq |A|$ for every model A of cardinality $\geq \aleph_2$. Instead we prove a sufficiently strong approximation to this inequaity. As a courtesy to the reader we start by isolating the following triviality.

Lemma 3.12. *For every cardinal* λ *and every* $\mathcal{X} \subseteq \mathcal{P}(\lambda)$ *of cardinality* $> \lambda$ *there is* $\xi < \lambda$ *such that* $|\{x \in \mathcal{X} : \xi \in x\}| > \lambda$.

Proof. We may assume $|\mathcal{X}| = \lambda^+$ and enumerate $\mathcal{X}$ as $\{x_\eta \colon \eta < \lambda^+\}$. If the conclusion of lemma fails then $f(\xi) = \sup\{\eta < \lambda^+ \colon \xi \in x_\eta\}$ defines a cofinal function from λ to λ^+. $\square$

See the paragraph before Lemma 2.5 for the definition of disjoint representing sequences.

Lemma 3.13. *For A, ϕ, m as usual and $\lambda = |A|$ every set of disjoint representing sequences of invariants in $\mathrm{INV}^{m,\lambda}(A, \phi)$ has size at most λ.*

Proof. Let us prove the case when λ is regular. We may assume $m \geq 1$ since the case $m = 0$ is trivial. Assume the contrary and let $\mathbf{d}(\eta)$, for $\eta < \lambda^+$, be disjoint representing sequences of elements of $\mathrm{INV}^{m,\lambda}(A, \phi)$. Let $\mathbf{d}(\eta) = \langle \mathbf{d}(\eta)_\xi \colon \xi < \lambda \rangle / \mathcal{D}(\lambda, \aleph_1)$. Fix a representation A_ξ, for $\xi < \lambda$, of A.

For each $\eta < \lambda^+$ fix $S_\eta \in \mathcal{D}(\lambda, \aleph_1)$ such that for every $\xi \in S_\eta$ some $\bar{c}_\xi$ defines an (A, A_ξ, ϕ, m)-invariant $\mathbf{d}(\eta)_\xi$. By Lemma 3.12 there is $\xi < \lambda$ such that λ^+ distinct (A, A_ξ, ϕ, m)-invariants are defined by elements of A^n. Since $|A| = \lambda$, this is impossible.

The proofs of the two cases when λ is singular are almost identical to the above proof and are therefore omitted. $\square$

The following application, proved in [21], is not concerned with ultraproducts.

Proposition 3.14. *Assume $\lambda \geq \aleph_2$ and $\mathbb{K}$ is a class of models of cardinality λ. If there are n and a $2n$-ary formula ϕ such that for every linear order I of cardinality λ there exists a model $A \in \mathbb{K}$ such that I is isomorphic to a weakly $(\aleph_1, \phi)$-skeleton like ϕ-chain in A^n, then there are 2^λ nonisomorphic models in $\mathbb{K}$.*

Proof. Let I be a linear order and let A be a model such that I is isomorphic to a weakly $(\aleph_1, \phi)$-skeleton like ϕ-chain in A. By Lemma 3.11, $\mathrm{inv}^{m,\lambda}(I) \in \mathrm{INV}^{m,\lambda}(A)$ and by Lemma 3.13, $\mathrm{INV}^m(A)$ has cardinality at most λ for every $A \in \mathbb{K}$. By Lemma 2.5 there are 2^λ disjoint representing sequences of m, λ-invariants of linear orders of cardinality λ. By the pigeonhole principle there are 2^λ nonisomorphic elements of $\mathbb{K}$. $\square$

4. Construction of Ultrafilters

The main result of this section is Proposition 4.2 below. Its version in which $M_i = (\mathbb{N}, <)$ for all i was proved in [15, Lemma 4.7] and some of the ideas are taken from this proof. Recall that if D is a filter on λ then D^+ is the coideal of all sets positive with respect to D, or in symbols

$$D^+ = \{X \subseteq \lambda \colon X \cap Y \neq \emptyset \text{ for all } Y \in D\}.$$

If D is a filter on λ and $\mathcal{G} \subseteq \mathbb{N}^\lambda$ then we say $\mathcal{G}$ is *independent mod D* if for all $k \in \mathbb{N}$, all distinct $g_0, \dots, g_{k-1}$ in $\mathcal{G}$ and all $j_0, \dots, j_{k-1}$ in $\mathbb{N}$ the set

$$\{\xi < \lambda \colon g_0(\xi) = j_0, \dots g_{k-1}(\xi) = j_{k-1}\}$$

belongs to D^+. Note that it is not required that j_i be distinct.

Write $\mathrm{FI}(\mathcal{G})$ for the family of all finite partial functions h from $\mathcal{G}$ into $\mathbb{N}$. For $h \in \mathrm{FI}(\mathcal{G})$ write

$$A_h = \{\lambda \in \mathbb{N} \colon f(\lambda) = h(f) \text{ for all } f \in \mathrm{dom}(h)\}.$$

Let

$$\mathrm{FI}_s(\mathcal{G}) = \{A_h \colon h \in \mathrm{FI}(\mathcal{G})\}.$$

We shall write $X \subseteq^D Y$ for $X \setminus Y = \emptyset$ mod D and $X =^D Y$ for $X \Delta Y = \emptyset$ mod D. Forcing-savvy readers will recognize both where the following paragraph is coming from and that Lemma 4.1 simply states that the poset for adding λ Cohen reals densely embeds into $\mathcal{P}(\mathbb{N})/D$.

For h and h' in $\mathrm{FI}(\mathcal{G})$ say that h and h' are *incompatible* if $h \cap h'$ is not a function. Note that if $\mathcal{G}$ is independent mod D then $h \perp h'$ if and only if $A_h \cap A_{h'} =^D \emptyset$ and $A_h \subseteq^D A_{h'}$ whenever $h \supseteq h'$, for all h and h' in $\mathrm{FI}(\mathcal{G})$.

A standard Δ-system argument (see [20] or [16]) shows that every family of pairwise incompatible elements of $\mathrm{FI}(\mathcal{G})$ is countable. Lemma 4.1 below is a special case of [20, Claim VI.3.17(5)]. We include its proof for convenience of the reader.

Lemma 4.1. *Assume D is a filter on λ and $\mathcal{G} \subseteq \mathbb{N}^\lambda$ is a family of functions independent mod D. Furthermore, assume D is a maximal filter such that $\mathcal{G}$ is independent mod D. Then for every $X \subseteq \lambda$ there is a countable subset $\mathcal{A} \subseteq \mathrm{FI}(\mathcal{G})$ such that*

(1) *For every $h \in \mathcal{A}$ either $A_h \subseteq^D X$ or $A_h \cap X =^D \emptyset$.*
(2) *For every $h' \in \mathrm{FI}(\mathcal{G})$ there is $h \in \mathcal{A}$ such that $A_{h'} \cap A_h \neq^D \emptyset$.*

Proof. Assume for a moment that for every $h \in \mathrm{FI}(\mathcal{G})$ there is $h' \supseteq h$ such that

$$A_{h'} \subseteq^D X \quad \text{or} \quad A_{h'} \cap X =^D \emptyset. \tag{$*$}$$

Let $\mathcal{A}$ be a maximal family of incompatible elements of $\mathrm{FI}(\mathcal{G})$ such that $(*)$ holds. Then $\mathcal{A}$ is countable and it satisfies (1). By our assumption and the maximality of $\mathcal{A}$, it satisfies (2) as well.

Now assume there is h such that for every $h' \supseteq h$ in $\mathrm{FI}(\mathcal{G})$ we have both $A_{h'} \setminus X \neq^D \emptyset$ and $A_{h'} \cap X \neq^D \emptyset$. Let D' be the filter generated by D and the complement of $X \cap A_h$. Since $A_h \cap X \neq^D \emptyset$, for h, we have that D' is a proper extension of D. Since $A_{h'} \setminus X \neq^D \emptyset$ for every h' extending h, we have that $\mathcal{G}$ is independent modulo D'. This contradicts the assumed maximality of D. $\square$

Lemma 4.1 implies that for every $X \subseteq \mathbb{N}$ there is a countable $\mathcal{G}_0 \subseteq \mathcal{G}$ such that $\mathcal{A}$ satisfying the above conditions is included in $\mathrm{FI}_s(\mathcal{G}_0)$. In this situation we say

X is *supported* by $\mathcal{G}_0$. The question of uniqueness of a support $\mathcal{G}_0$ for given X is irrelevant for us and it will be ignored.

Proposition 4.2. *Assume $\phi(\bar{x}, \bar{y})$ is a formula and M_i, for $i \in \mathbb{N}$, are models of the same signature such that in M_i there is a $\preceq_\phi$-chain of length i. Then for every linear order I of cardinality $\leq \mathfrak{c}$ there exists an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that $\prod_{\mathcal{U}} M_n$ includes a weakly $(\aleph_1, \phi)$-skeleton like ϕ-chain $\mathcal{C}$ isomorphic to I.*

Proof. In order to simplify the notation and release the bound variable n we shall assume that ϕ is a binary formula and hence the elements of the ϕ-chain $\mathcal{C}$ will be elements of A instead of n-tuples of elements from A. Let $a_i(n)$, for $0 \leq i < n$, be a $\preceq_\phi$-chain in M_n. For convenience of notation, we may assume

$$a_i(n) = i$$

for all i and n, and we also write $a_i(n) = n - 1$ if $i \geq n$. Fix an independent family $\bar{\mathcal{G}}$ of size $\mathfrak{c}$ of functions $f \colon \mathbb{N} \to \mathbb{N}$ (see [20, Appendix, Theorem 1.5(1)]). Let $\mathcal{G} = \{\min(f, id - 1) : f \in \bar{\mathcal{G}}\}$. Then $\mathcal{G}$ is still independent. Fix a filter D on $\mathbb{N}$ such that $\mathcal{G}$ is independent with respect to D and D is a maximal (under the inclusion) filter with this property. Let $\mathrm{FI}(\mathcal{G})$, A_h for $h \in \mathrm{FI}(\mathcal{G})$ and $\mathrm{FI}_s(\mathcal{G})$ be as introduced before Lemma 4.1. The following is an immediate consequence of Lemma 4.1 (i.e. of [20, Claim VI.3.17(5)]). Fix an enumeration of $\mathcal{G}$ by elements of I and write $\mathcal{G} = \{f_i \colon i \in I\}$.

Claim 4.3. *For every $g \in \prod_{n\in\mathbb{N}} M_n$ there is a countable set $\mathbb{S}_g \subseteq I$ such that for all $l \in \mathbb{N}$ both sets*

$$X_{g,l} = \{n \colon M_n \models \phi(a_l(n), g(n))\},$$
$$Y_{g,l} = \{n \colon M_n \models \phi(g(n), a_l(n))\}$$

are supported by $\{f_i \colon i \in \mathbb{S}_g\}$.

Proof. Apply Lemma 4.1 to each $X_{g,l}$ and each $Y_{g,l}$ and take union of the supports. □

For $i < j$ in I write $[i, j]_I$ for the interval $\{k \in I \colon i \leq k \leq j\}$. For elements $a \preceq_\phi b$ in a model M write

$$[a, b]_\phi = \{c \in M \colon a \preceq_\phi c \text{ and } c \preceq_\phi b\}.$$

Since $\preceq_\phi$ is not necessarily transitive, this notation should be taken with a grain of salt. For $i < j$ in I write

$$B_{ij} = \{n \colon f_i(n) \prec_\phi f_j(n)\}.$$

Note that by our convention about $a_i(n)$ we have that $f_i(n) \prec_\phi f_j(n)$ is equivalent to $f_i(n) < f_j(n)$. For $g \in \prod_{n\in\mathbb{N}} M_n$ and $i < j$ in I such that $[i,j]_I \cap \mathbb{S}_g = \emptyset$ let

$$C_{gij} = \{n\colon M_n \models \phi(f_i(n), g(n)) \leftrightarrow \phi(f_j(n), g(n)) \\ \text{and } M_n \models \phi(g(n), f_i(n)) \leftrightarrow \phi(g(n), f_j(n))\}.$$

In other words, $C_{gij} = \{n\colon \operatorname{tp}_\phi(f_i(n)/g(n)) = \operatorname{tp}_\phi(f_j(n)/g(n))\}$, with tp_ϕ as computed in M_n.

Claim 4.4. *The family of all sets B_{ij} for $i < j$ in I and C_{gij} for $g \in \prod_{n\in\mathbb{N}} M_n$ and $i < j$ in I such that $[i,j]_I \cap \mathbb{S}_g = \emptyset$ has the finite intersection property.*

Proof. It will suffice to show that for $\bar{k} \in \mathbb{N}$, $i(0) < \ldots < i(\bar{k}-1)$ in I, and $g(0), \ldots, g(\bar{k}-1)$ in $\prod_{n\in\mathbb{N}} M_n$ the set

$$\bigcap_{l<m<\bar{k}} B_{i(l),i(m)} \cap \bigcap \{C_{g(k),i(l),i(m)}\colon k < \bar{k}, l < m < \bar{k}, \text{ and } [i(l), i(m)]_I \cap \mathbb{S}_{g(k)} = \emptyset\}$$

is nonempty. Let

$$\mathbb{S} = \bigcup_{k<\bar{k}} \mathbb{S}_{g(k)}.$$

Write $\mathcal{T} = \{i(k)\colon k < \bar{k}\}$, also $\mathcal{T}^{\mathcal{G}} = \{f_i\colon i \in \mathcal{T}\}$ and $\mathbb{S}^{\mathcal{G}} = \{f_i\colon i \in \mathbb{S}\}$.

Pick h_m, for $m \in \mathbb{N}$, in $\mathrm{FI}(\mathbb{S}^{\mathcal{G}} \backslash \mathcal{T}^{\mathcal{G}})$ so that

(1) $h_m \subseteq h_{m+1}$ for all m and
(2) For all $h \in \mathrm{FI}(\mathcal{T}^{\mathcal{G}})$, with $\operatorname{dom}(h) \supseteq \mathcal{T}^{\mathcal{G}}$ all $l \in \mathbb{N}$ and all $k < \bar{k}$, for all but finitely many m we have either

 (i$_X$) $(\forall^D n \in A_{h_m \cup h}) M_n \models \phi(a_l(n), g(k)(n))$, or
 (ii$_X$) $(\forall^D n \in A_{h_m \cup h}) M_n \models \neg\phi(a_l(n), g(k)(n))$

 and also either

 (i$_Y$) $(\forall^D n \in A_{h_m \cup h}) M_n \models \phi(g(k)(n), a_l(n))$, or
 (ii$_Y$) $(\forall^D n \in A_{h_m \cup h}) M_n \models \neg\phi(g(k)(n), a_l(n))$.

The construction of h_m proceeds recursively as follows. Enumerate all triples (h, k, l) in $\mathrm{FI}(\mathcal{T}^{\mathcal{G}}) \times \bar{k} \times \mathbb{N}$ by elements of $\mathbb{N}$. Let $h_0 = \emptyset$. If h_m has been chosen and (h, k, l) is the mth triple then use the fact that $X_{g(k),l}$ and $Y_{g(k),l}$ are supported by $\mathbb{S}$ (Claim 4.3) to find $h_{m+1} \in \mathrm{FI}(\mathbb{S}^{\mathcal{G}} \backslash \mathcal{T}^{\mathcal{G}})$ such that $A_{h_{m+1} \cup h}$ satisfies one of (i$_X$) and (ii$_X$) and one of (i$_Y$) or (ii$_Y$). Then the sequence of h_m constructed as above clearly satisfies the requirements.

In order to complete the proof it suffices to show that there exist $h \in \mathrm{FI}(\mathcal{T}^{\mathcal{G}})$ and n such that

$$A_{h_n \cup h} \subseteq^D \bigcap_{l<m<\bar{k}} B_{i(l),i(m)} \cap \bigcap \{C_{g(k),i(l),i(m)} \colon k < \bar{k}, l < m < \bar{k}, \text{ and } [i(l), i(m)]_I \cap \mathbb{S}_{g(k)} = \emptyset\}. \quad (1)$$

In order to have $A_{h_n \cup h} \subseteq^D B_{i(l),i(m)}$ it is necessary and sufficient to have $h(f_{i(l)}) < h(f_{i(m)})$. We shall therefore consider only h that are *increasing* in this sense. An increasing function in $\mathrm{FI}(\mathcal{T}^{\mathcal{G}})$ is uniquely determined by its range. For $t \in [\mathbb{N}]^{\bar{k}}$ let h_t denote the increasing function in $\mathrm{FI}(\mathcal{T}^{\mathcal{G}})$ whose range is equal to t.

Assume for a moment that for every $t \in [\mathbb{N}]^{\bar{k}}$ there are k, l, m such that for all n we have $A_{h_n \cup h_t} \not\subseteq^D C_{g(k^*),i(l^*),i(m^*)}$ and therefore by the choice of the sequence $\{h_n\}$ that

$$A_{h_n \cup h_t} \cap C_{g(k),i(l),i(m)} =^D \emptyset.$$

For $t \in [\mathbb{N}]^{\bar{k}}$ let $\psi(t)$ be the lexicographically minimal triple (k, l, m) such that this holds for a large enough n. By Ramsey's theorem, there are an infinite $Z \subseteq \mathbb{N}$ and a triple (k^*, l^*, m^*) such that for every $t \in [Z]^{\bar{k}}$ we have $A_{h_n \cup h_t} \cap C_{g(k^*),i(l^*),i(m^*)} =^D \emptyset$.

Let $N = |[i(l^*), i(m^*)]_I \cap \mathcal{T}|$ and find $t \in [Z]^{\bar{k}}$ such that the set

$$[h_t(f_{i(l^*)}), h_t(f_{i(m^*)})] \cap Z$$

has at least $4N - 3$ elements. Let $h' = h_t \restriction (\mathcal{T}^{\mathcal{G}} \cap \mathbb{S}^{\mathcal{G}}_{g(k^*)})$. Then for each $p \in \mathbb{N}$ there is a large enough $m = m(p)$ such that either (i_X) or (ii_X) holds, and either (i_Y) or (ii_Y) holds. We say that such m *decides the* k^**-type of* p.

Pick m large enough to decide the k^*-type of each $p \in [h'(f_{i(l^*)}), h'(f_{i(m^*)})] \cap Z$. Since there are only four different k^*-types, by the pigeonhole principle there are N elements of $[h'(f_{i(l^*)}), h'(f_{i(m^*)})] \cap Z$ with the same k^*-type. There is therefore $t^* \in [Z]^{\bar{k}}$ such that all N elements of $t^* \cap [h'(f_{i(l^*)}), h'(f_{i(m^*)})]$ have the same k^*-type. This means that $A_{h_m \cup h_{t^*}} \subseteq^D C_{g(k^*),i(l^*),i(m^*)}$, contradicting $\psi(t^*) = (k^*, l^*, m^*)$.

Therefore there exists $t \in [\mathbb{N}]^{\bar{k}}$ such that for every $k < \bar{k}$ and all $l < m < \bar{k}$ such that $[i(l), i(m)]_I \cap \mathbb{S}_{g(k)} = \emptyset$ for some $n = n(k, l, m)$ we have

$$A_{h_n \cup h_t} \subseteq^D C_{g(k),i(l),i(m)}.$$

Then h_t and $n = \max_{k,l,m} n(k, l, m)$ satisfy (1) and this completes the proof. $\square$

By Claim 4.4 we can find an ultrafilter $\mathcal{U}$ such that the sets B_{ij} for $i < j$ in I and C_{gij} for $g \in \prod_{n \in \mathbb{N}} M_n$ and $i < j$ in I such that $[i, j]_I \cap \mathbb{S}_{\mathcal{G}} = \emptyset$ all belong to $\mathcal{U}$. Let $\mathbf{a}_i$ be the element of the ultrapower $\prod_{\mathcal{U}} M_n$ with the representing sequence f_i if $i \in I$ for $n \in \mathbb{N}$, if $i \in \mathbb{N}$. Since the relevant A_{ki} and B_{ij} belong to $\mathcal{U}$ we have that $\mathbf{a}_i, i \in I$, is a ϕ-chain in the ultraproduct.

In order to check it is weakly $(\aleph_1,\phi)$-skeleton like fix $\mathbf{g}\in\prod_{\mathcal U} M_n$ and a representing sequence $g\in\prod_n M_n$ of $\mathbf{g}$. Let $J_g=\{f_i\colon i\in\mathbb{S}_{\mathcal G}\}$. If $i<j$ are such that $[i,j]_I\cap J_g=\emptyset$, then $C_{gij}\in\mathcal U$, which implies that $\prod_{\mathcal U} M_n\models\phi(\mathbf{a}_i,\mathbf{g})\leftrightarrow\phi(\mathbf{a}_j,\mathbf{g})$ and $\prod_{\mathcal U} M_n\models\phi(\mathbf{g},\mathbf{a}_i)\leftrightarrow\phi(\mathbf{g},\mathbf{a}_j)$, as required.

Remark 4.5. As pointed out in Remark 3.3, the proof of Proposition 4.2 can be easily modified to obtain $\mathcal U$ such that $\prod_{\mathcal U} M_i$ includes a ϕ-chain $\mathcal C$ isomorphic to I that satisfies the indiscernibility property $(*)$ stronger than being weakly $(\aleph_1,\phi)$-skeleton like stated there. In order to achieve this, we only need to add a variant $D_{ijg\psi}$ of the set C_{ijg} to the filter basis from Claim 4.4 for every $k\in\mathbb N$, every $k+n$-ary formula $\psi(\bar x,\bar y)$ and every $g\in A^k$. Let

$$D_{ijg\psi}=\{n\colon M_n\models\psi(f_i(n),g(n))\leftrightarrow\psi(f_j(n),g(n))\}.$$

The obvious modification of the proof of Claim 4.4 shows that the augmented family of sets still has the finite intersection property. It is clear that any ultrafilter $\mathcal U$ extending this family is as required.

5. The Proof of Theorem 1.1

Fix a model A of cardinality $\le\mathfrak c$ whose theory is unstable. By [20, Theorem 2.13] the theory of A has the order property and we can fix ϕ in the signature of A such that A includes arbitrarily long finite ϕ-chains. Therefore Theorem 1.1 is a special case of the following with $A_i=A$ for all i.

Theorem 5.1. *Assume CH fails. Assume $\phi(\bar x,\bar y)$ is a formula and A_i, for $i\in\mathbb N$, are models of cardinality $\le\mathfrak c$ such that in A_i there is a $\preceq_\phi$-chain of length i. Then there are $2^{\mathfrak c}$ isomorphism types of models of the form $\prod_{\mathcal U} A_n$, where $\mathcal U$ ranges over nonprincipal ultrafilters on $\mathbb N$.*

Proof. Since $|A_i|\ge i$ for all i, the ultrapower $\prod_{\mathcal U} A$ has cardinality equal to $\mathfrak c$ whenever $\mathcal U$ is a nonprincipal ultrafilter on $\mathbb N$. By Lemma 2.5, there are $2^{\mathfrak c}$ linear orders I of cardinality $\mathfrak c$ with disjoint representing sequences corresponding to (defined) invariants $\mathrm{inv}^{m,\mathfrak c}(I)$ (with $m=2$ or $m=3$ depending on wheher $\mathfrak c$ is regular or not). Use Proposition 4.2 to construct an ultrafilter $\mathcal U(I)$ such that I is isomorphic to a weakly $(\aleph_1,\phi)$-skeleton like ϕ-chain $\mathcal C$ in $\prod_{\mathcal U(I)} A_i$. The conclusion follows by Proposition 3.14. $\square$

6. Ultrapowers of Metric Structures

6.1. *Metric structures*

In this section, we prove a strengthening of Theorem 1.3 which is the analogue of Theorem 5.1 for metric structures. First we include the definitions pertinent to understanding the statement of Theorem 1.3. Assume $(A,d,f_0,f_1,\dots,R_0,R_1,\dots)$ is a metric structure. Hence d is a complete metric on A such that the diameter of

A is equal to 1, each f_i is a function from some finite power of A into A, and each R_i is a function from a finite power of A into $[0, 1]$. All f_i and all R_i are required to be uniformly continuous with respect to d, with a fixed modulus of uniform continuity (see [2] or [10, Sec. 2]). In the interesting cases, such as (unit balls of) C*-algebras, tracial von Neumann algebras, and Banach spaces, this requirement follows from the uniform continuity of algebraic operations on bounded balls.

If $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$ then on $A^{\mathbb{N}}$ we define a quasimetric $d_{\mathcal{U}}$ by letting, for $\mathbf{a} = (a_i)_{i\in\mathbb{N}}$ and $\mathbf{b} = (b_i)_{i\in\mathbb{N}}$,

$$d_{\mathcal{U}}(\mathbf{a}, \mathbf{b}) = \lim_{i\to\mathcal{U}} d(a_i, b_i).$$

Identify pairs $\mathbf{a}$ and $\mathbf{b}$ such that $d_{\mathcal{U}}(\mathbf{a}, \mathbf{b}) = 0$. The uniform continuity implies that $f_n(\mathbf{a}) = (f_n(a_i))_i$ and $R_n(\mathbf{a}) = \lim_{i\to\mathcal{U}} R_n(a_i)$ are uniformly continuous functions with respect to the quotient metric. The quotient structure is denoted by $\prod_{\mathcal{U}}(A, d, \ldots)$ (or shortly $\prod_{\mathcal{U}} A$ if the signature is clear from the context) and called the *ultrapower of A associated with $\mathcal{U}$*. An ultraproduct of metric structures of the same signature is defined analogously.

The assumption that the metric d is finite is clearly necessary in order to have $d_{\mathcal{U}}$ be a metric. However, one can show that the standard ultrapower constructions of C*-algebras and of II_1 factors can essentially be considered as special cases of the above definition (see [10, Sec. 4] for details). These two constructions served as a motivation for our work (see Sec. 8).

More information on the logic of metric structures is given in [2], and [10] contains an exposition of its variant suitable for C*-algebras and II_1 factors.

Let $A = (A, d, \ldots)$ be a metric structure. Interpretations of formulas are functions uniformly continuous with respect to d, and the value of an n-ary formula ψ at an n-tuple $\bar{a}$ is denoted by

$$\psi(\bar{a})^A.$$

We assume that the theory of A is unstable, and therefore by [10, Theorem 5.5] it has the order property. Fix n and a $2n$-ary formula ϕ that witnesses the order property of the theory of A. Define the relation $\preceq_\phi$ on every model such that ϕ is a formula in its signature by letting $\bar{a} \preceq_\phi \bar{b}$ if and only if

$$\phi(\bar{a}, \bar{b}) = 0 \quad \text{and} \quad \phi(\bar{b}, \bar{a}) = 1.$$

A *ϕ-chain* in A is a subset of A^n linearly ordered by $\preceq_\phi$. Theorem 1.3 is a consequence of the following.

Theorem 6.1. *Assume CH fails. Assume $\phi(\bar{x}, \bar{y})$ is a formula and A_i, for $i \in \mathbb{N}$, are metric structures of cardinality $\leq \mathfrak{c}$ of the same signature such that in A_i there is a $\preceq_\phi$-chain of length i. Then there are $2^{\mathfrak{c}}$ isometry types of models of the form $\prod_{\mathcal{U}} A_n$, where $\mathcal{U}$ ranges over nonprincipal ultrafilters on $\mathbb{N}$.*

The proof proceeds along the same lines as the proof of Theorem 5.1 and we shall only outline the novel elements, section by section.

6.2. *Combinatorics of the invariants*

For $\bar{a} \in A^n$ and $\bar{b} \in A^n$ write

$$\mathrm{tp}_\phi(\bar{a}/\bar{b}) = \langle \phi(\bar{a},\bar{b})^A, \phi(\bar{b},\bar{a})^A \rangle.$$

For $\bar{a} \in A^n$ and $X \subseteq A^n$, let $\mathrm{tp}_\phi(\bar{a}/X)$ be the function from X into $[0,1]^2$ defined by

$$\mathrm{tp}_\phi(\bar{a}/X)(\bar{b}) = \mathrm{tp}_\phi(\bar{a}/\bar{b}).$$

The notation and terminology such as $[\bar{a},\bar{b}]_\phi$ have exactly the same interpretation as in Sec. 3.1.

Definition 6.2. A ϕ-chain $\mathcal{C}$ is *weakly* $(\aleph_1,\phi)$*-skeleton like in* A if for every $\bar{a} \in A^n$ there is a countable $\mathcal{C}_{\bar{a}} \subseteq \mathcal{C}$ such that for all $\bar{b}$ and $\bar{c}$ in $\mathcal{C}$ satisfying

$$[\bar{b},\bar{c}]_\phi \cap \mathcal{C}_{\bar{a}} = \emptyset$$

we have $\mathrm{tp}_\phi(\bar{a}/\bar{b}) = \mathrm{tp}_\phi(\bar{a}/\bar{c})$.

Note that $(\mathcal{C}, \preceq_\phi)$ is an honest (discrete) linear ordering. Because of this a number of the proofs in the discrete case work in the metric case unchanged. In particular, Lemmas 3.4–3.7 are true with the new definitions and the old proofs. Definition 3.8 and the definition of $\mathcal{C}[B,\bar{c}]$ are transferred to the metric case unmodified, using the new definition of tp_ϕ. As a matter of fact, the analogue of Remark 3.2 applies in the metric context. That is, even if weakly $(\aleph_1,\phi)$-skeleton like is defined by requiring only that (with $\bar{a}$, $\mathcal{C}_{\bar{a}}$, $\bar{b}$ and $\bar{c}$ as in Definition 6.2) we only have

$$\bar{a} \preceq_\phi \bar{b} \quad \text{if and only if } \bar{a} \preceq_\phi \bar{c}$$

and

$$\bar{b} \preceq_\phi \bar{a} \quad \text{if and only if } \bar{c} \preceq_\phi \bar{a},$$

then all of the above listed lemmas remain true, with the same proofs, in the metric context. However, Lemma 6.5 below requires the original, more restrictive, notion of weakly $(\aleph_1,\phi)$-skeleton like.

6.3. *Defining an invariant over a submodel*

Definition 3.8 is unchanged. The statement and the proof of Lemma 3.9 remain unchanged. However, in order to invoke it in the proof of the metric analogue of Lemma 3.10 we shall need Lemma 6.3 below. For a metric structure B its *character density*, the smallest cardinality of a dense subset, is denoted by $\chi(B)$. Note that $\chi(A) \geq |\mathcal{C}|$ for every ϕ-chain $\mathcal{C}$ in A, since each ϕ-chain is necessarily discrete.

Lemma 6.3. *Assume* $\mathcal{C} = \langle a_i : i \in I \rangle$ *is a* ϕ*-chain that is weakly* $(\aleph_1,\phi)$*-skeleton like in a metric structure* A. *Assume* B *is an elementary submodel of* A *and* $\bar{a} \in \mathcal{C} \backslash B^n$ *is such that*

$$\mathrm{cf}(\mathcal{C}[B,\bar{a}], \leq_{\neg\phi}) > \chi(B).$$

Then there is $\bar{c} \in \mathcal{C}[B, \bar{a}]$ *such that for all* $\bar{d} \in \mathcal{C}[B, \bar{a}] \cap (-\infty, \bar{c}]_\phi$ *we have* $\mathrm{tp}_\phi(\bar{d}/B) = \mathrm{tp}_\phi(\bar{c}/B)$.

Proof. Pick a dense $B_0 \subseteq B$ of cardinality $\chi(B)$. Let $\bar{c} \in \mathcal{C}[B, \bar{a}]$ be such that

$$\mathcal{C}[B, \bar{a}] \cap \bigcup\{\mathcal{C}_{\bar{b}} \colon \bar{b} \in B_0^n\} \cap (-\infty, \bar{c}]_\phi = \emptyset.$$

Then for every $\bar{d} \in \mathcal{C}[B, \bar{c}] \cap (-\infty, \bar{c}]_\phi$ and every $\bar{b} \in B_0^n$ we have that $[\bar{d}, \bar{c}]_\phi \cap \mathcal{C}_{\bar{b}} = \emptyset$, and therefore $\mathrm{tp}_\phi(\bar{c}/\bar{b}) = \mathrm{tp}_\phi(\bar{d}/\bar{b})$. Since the maps $\bar{x} \mapsto \mathrm{tp}_\phi(\bar{c}/\bar{x})$ and $\bar{x} \mapsto \mathrm{tp}_\phi(\bar{d}/\bar{x})$ are continuous, they agree on all of B^n and therefore $\mathrm{tp}_\phi(\bar{c}/B) = \mathrm{tp}_\phi(\bar{d}/B)$. □

6.4. *Representing invariants*

The definition of $\mathrm{INV}^{m,\lambda}(A, \phi)$ from Sec. 3.4 transfers to the metric context verbatim, and Lemma 3.11 and its proof are unchanged.

6.5. *Counting the number of invariants over a model*

Lemma 3.12 is unchanged but Lemma 3.13 needs to be modified, since the right analogue of cardinality of a model is its character density.

Lemma 6.4. *For* A, ϕ, m *as usual every set of disjoint representing sequences of invariants in* $\mathrm{INV}^{m,\chi(A)}(A, \phi)$ *has size at most* $\chi(A)$.

Proof. In this paper we shall only need the trivial case when $\chi(A) = |A| = \mathfrak{c}$, but the general case is needed in [12]. It will follow immediately from the proof of Lemma 3.13 with Lemma 6.5 below applied at the right moment. □

Lemma 6.5. *For* A, ϕ, m *as usual and an elementary submodel* B *of* A *there are at most* $\chi(A)$ *distinct* (A, B, ϕ, m)*-invariants.*

Proof. Let $\lambda = \chi(A)$. Let $h \colon \mathbb{R} \to [0, 1]$ be the continuous function such that $h(x) = 0$ for $x \leq 1/3$, $h(x) = 1$ for $x \geq 2/3$, and h linear on $[1/3, 2/3]$. Let $\psi = h \circ \phi$.

Note that every ϕ-chain is a ψ-chain. Also, $\phi(\bar{a}_1, \bar{b}_1) = \phi(\bar{a}_2, \bar{b}_2)$ implies $\psi(\bar{a}_1, \bar{b}_1) = \psi(\bar{a}_2, \bar{b}_2)$, and therefore every weakly $(\aleph_1, \phi)$-skeleton like ϕ-chain is weakly $(\aleph_1, \psi)$-skeleton like, with the same witnessing sets $\mathcal{C}_{\bar{a}}$. This implies the following, for every elementary submodel B of A and $m \in \mathbb{N}$.

(*) If $\bar{c} \in A^n$ defines the (A, B, ϕ, m)-invariant $\mathbf{d}$ then $\bar{c}$ defines the (A, B, ψ, m)-invariant $\mathbf{d}$.

Denote the sup metric on A^n by d^n. Since ϕ^A is a uniformly continuous function, there is $\delta > 0$ sufficiently small so that $d^n(\bar{c}_1, \bar{c}_2) < \delta$ implies $|\phi(\bar{a}, \bar{c}_1) - \phi(\bar{a}, \bar{c}_2)| < 1/3$ for all $\bar{a}$. Therefore we have the following.

(**) For every $\bar{a} \in A^n$ we have that $\bar{a} \preceq_\phi \bar{c}_1$ implies $\bar{a} \leq_\psi \bar{c}_2$, and $\bar{a} \preceq_\phi \bar{c}_2$ implies $\bar{a} \leq_\psi \bar{c}_1$.

Assume B is an elementary submodel of A and $\bar{c}_i$ defines the (A, B, ϕ, m)-invariant $\mathbf{d}_i$, for $i = 1, 2$. By (*) we have that $\bar{c}_i$ defines the (A, B, ψ, m)-invariant $\mathbf{d}_i$, for $i = 1, 2$. If $d^n(\bar{c}_1, \bar{c}_2) < \delta$ then (**) implies $\mathbf{d}_1 = \mathbf{d}_2$. □

Proposition 3.14 applies in the metric case literally.

6.6. *Construction of ultrafilters*

It is the construction of the ultrafilter in Sec. 4 that requires the most drastic modification. Although the statement of Proposition 4.2 transfers unchanged, the proof of its analogue, Proposition 6.6, requires new ideas.

Proposition 6.6. *Assume $\phi(\bar{x}, \bar{y})$ is a formula and M_i, for $i \in \mathbb{N}$, are metric structures of the same signature such that in M_i there is a $\preceq_\phi$-chain of length i. Assume I is a linear order of cardinality $\leq \mathfrak{c}$. Then there is an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that $\prod_\mathcal{U} M_n$ includes a ϕ-chain $\{\mathbf{a}_i : i \in I\}$ that is weakly $(\aleph_1, \phi)$-skeleton like.*

Proof. Like in the proof of Proposition 4.2, we assume ϕ is a binary formula in order to simplify the notation. Fix a ϕ-chain $a_i(n)$, for $0 \leq i < n$, in M_n. Like in Sec. 4 fix an independent family $\mathcal{G}$ of size $\mathfrak{c}$ and a filter D such that $\mathcal{G}$ is independent with respect to D and D is a maximal filter with this property. Define $\mathcal{G}$, $\mathrm{FI}(\mathcal{G})$ and $\mathrm{FI}_s(\mathcal{G})$ exactly as in Sec. 4. Since the diameter of each M_n is ≤ 1, each element of $\prod_n M_n$ is a representing sequence of an element of the ultrapower. Claim 4.3 is modified as follows.

Claim 6.7. *For every $g \in \prod_{n\in\mathbb{N}} M_n$ there is a countable set $\mathbb{S}_g \subseteq I$ such that for all $l \in \mathbb{N}$ and all $r \in \mathbb{Q} \cap [0, 1]$ all sets of the form*

$$X_{g,l,r} = \{n : \phi(a_l(n), g(n))^{M_n} < r\},$$
$$Y_{g,l,r} = \{n : \phi(g(n), a_l(n))^{M_n} < r\}$$

are supported by $\mathbb{S}_g$.

Proof. Since there are only countably many relevant sets, this is an immediate consequence of Lemma 4.1. □

For $i < j$ in I the definitions of sets

$$B_{ij} = \{n : f_i(n) \preceq_\phi f_j(n)\}$$

is unchanged, but we need to modify the definition of C_{gij}. For $g \in \prod_{n\in\mathbb{N}} M_n$, $i < j$ in I such that $[i,j]_i \cap \mathbb{S}_g = \emptyset$ and $\varepsilon > 0$ let

$$C_{gij\varepsilon} = \{n\colon |\phi(f_i(n), g(n))^{M_n} - \phi(f_j(n), g(n))^{M_n}| < \varepsilon \\ \text{and } |\phi(g(n), f_i(n))^{M_n} - \phi(g(n), f_j(n))^{M_n}| < \varepsilon\}.$$

Claim 6.8. *The family of all sets B_{ij} for $i < j$ in I and $C_{gij\varepsilon}$ for $g \in \prod_{n\in\mathbb{N}} M_n$, $i < j$ in I such that $[i,j]_i \cap \mathbb{S}_g = \emptyset$ and $\varepsilon > 0$ has the finite intersection property.*

Proof. It will suffice to show that for $\bar{k} \in \mathbb{N}, i(0) < \cdots < i(\bar{k}-1)$ in I, and $g(0), \ldots, g(\bar{k}-1)$ in $\prod_{n\in\mathbb{N}} M_n$ and $\varepsilon > 0$ the set

$$\bigcap_{l<m<\bar{k}} B_{i(l),i(m)} \cap \bigcap\{C_{g(k),i(l),i(m),\varepsilon}\colon k < \bar{k}, l < m < \bar{k}, \quad \text{and} \\ [i(l), i(m)]_I \cap \mathbb{S}_{g(k)} = \emptyset\}$$

is nonempty. Pick $M \in \mathbb{N}$ such that $M > 2/\varepsilon$. Let

$$\mathbb{S} = \bigcup_{k<\bar{k}} \mathbb{S}_{g(k)}.$$

Write $\mathcal{T} = \{i(k)\colon k < \bar{k}\}$, also $\mathcal{T}^{\mathcal{G}} = \{f_i\colon i \in \mathcal{T}\}$ and $\mathbb{S}^{\mathcal{G}} = \{f_i\colon i \in \mathbb{S}\}$.

Pick h_m, for $m \in \mathbb{N}$, in $\mathrm{FI}(\mathbb{S}^{\mathcal{G}} \backslash \mathcal{T}^{\mathcal{G}})$ so that

(1) $h_m \subseteq h_{m+1}$ for all m and
(2) For all $h \in \mathrm{FI}(\mathcal{T}^{\mathcal{G}})$, all $l \in \mathbb{N}$, and all $k < \bar{k}$ there exist r and s in $\mathbb{N}$ such that $0 \le r \le M$, $0 \le s \le M$ and for all but finitely many m we have
   (i$_X$) $(\forall^D n \in A_{h_m \cup h})|\phi(a_l(n), g(k)(n))^{M_n} - r/M| < \varepsilon/2$ and
   (i$_Y$) $(\forall^D n \in A_{h_m \cup h})|\phi(g(k)(n), a_l(n))^{M_n} - s/M| < \varepsilon/2$.

The construction of h_m is essentially the same as in the proof of Claim 4.4, except that it uses Claim 6.7 in place of Claim 4.3.

In order to complete the proof we need to show that there exist $h \in \mathrm{FI}(\mathcal{T}^{\mathcal{G}})$ and n such that

$$A_{h_n \cup h} \subseteq^D \bigcap_{l<m<\bar{k}} B_{i(l),i(m)} \cap \\ \bigcap\{C_{g(k),i(l),i(m),\varepsilon}\colon k < \bar{k}, l < m < \bar{k}, \text{ and } [i(l), i(m)]_I \cap \mathbb{S}_{g(k)} = \emptyset\}. \quad (2)$$

In order to have $A_{h_n \cup h} \subseteq^D B_{i(l),i(m)}$ it is necessary and sufficient to have $h(i(l)) < h(i(m))$. We shall therefore consider only h that are *increasing* in this sense. An increasing function in $\mathrm{FI}(\mathcal{T}^{\mathcal{G}})$ is uniquely determined by its range. For $t \in [\mathbb{N}]^{\bar{k}}$ let h_t denote the increasing function in $\mathrm{FI}(\mathcal{T}^{\mathcal{G}})$ whose range is equal to t.

Assume for a moment that for every $t \in [\mathbb{N}]^{\bar{k}}$ there are k, l, m such that for all n we have $A_{h_n \cup h_t} \not\subseteq^D C_{g(k),i(l),i(m),\varepsilon}$ and therefore by the choice of the sequence $\{h_n\}$ that

$$A_{h_n \cup h_t} \cap C_{g(k),i(l),i(m),\varepsilon} =^D \emptyset.$$

For $t \in [\mathbb{N}]^{\bar{k}}$ let $\psi(t)$ be the lexicographically minimal triple (k, l, m) such that this holds for a large enough n. By Ramsey's theorem, there are an infinite $Z \subseteq \mathbb{N}$ and a triple (k^*, l^*, m^*) such that for every $t \in [\mathbb{N}]^{\bar{k}}$ we have $A_{h_n \cup h_t} \cap C_{g(k),i(l),i(m),\varepsilon} =^D \emptyset$.

Let $N = |[i(l^*), i(m^*)]_I \cap \mathcal{T}|$ and find $t \in [Z]^{\bar{k}}$ such that the set

$$[h_t(i(l^*)), h_t(i(m^*))] \cap Z$$

has at least $(M^2 + 2M)N + 1$ elements. Let $h' = h \restriction (\mathcal{T}^{\mathcal{G}} \cap \mathbb{S}^{\mathcal{G}}_{g(k^*)})$. Then for each $p \in \mathbb{N}$ there are a large enough $m = m(p)$ such that for some $r = r(p)$ and $s = s(p)$ we have

$$(\forall^D n \in A_{h_m \cup h}) |\phi(a_l(n), g(k)(n))^{M_n} - r/M| < \varepsilon/2$$

and

$$(\forall^D n \in A_{h_m \cup h}) |\phi(g(k)(n), a_l(n))^{M_n} - s/M| < \varepsilon/2.$$

We say that such m *decides the k^*-type of p*. Pick m large enough to decide the k^*-type of each $p \in [h'(i(l^*)), h'(i(m^*))] \cap Z$. Since there are only $(M+1)^2$ different k^*-types, by the pigeonhole principle there are N elements of $[h'(i(l^*)), h'(i(m^*))] \cap Z$ with the same k^*-type. There is therefore $t^* \in [Z]^{\bar{k}}$ such that h_{t^*} extends t' and all N elements of $t^* \cap [h'(i(l^*)), h'(i(m^*))]$ have the same k^*-type. This means that $h_n \cup h_{t^*} \subseteq^D C_{g(k^*),i(l^*),i(m^*),\varepsilon}$, contradicting $\psi(t^*) = (k^*, l^*, m^*)$.

Therefore there exists $t \in [\mathbb{N}]^{\bar{k}}$ such that for every $k < \bar{k}$ and all $l < m < \bar{k}$ such that $[i(l), i(m)]_I \cap \mathbb{S}_{g(k)} = \emptyset$ for some $n = n(k, l, m)$ we have

$$A_{h_n \cup h_t} \subseteq^D C_{g(k),i(l),i(m),\varepsilon}.$$

Then h_t and $n = \max_{k,l,m} n(k, l, m)$ satisfy (2). ☐

Let $\mathcal{U}$ be any ultrafilter that extends the family of sets from the statement of Claim 6.8. Since M_n are assumed to be bounded metric spaces, each f_i is a representing sequence of an element of the ultraproduct $\prod_{\mathcal{U}} M_n$. Denote this element by $\mathbf{a}_i$ and let $\mathcal{C}$ denote $\langle \mathbf{a}_i : i \in I \rangle$. Since $B_{i,j} \in \mathcal{U}$ for all $i < j$ in I, $\mathcal{C}$ is a ϕ-chain isomorphic to I. For $\mathbf{b} \in \prod_{\mathcal{U}} M_n$ fix its representing sequence g and let $\mathcal{C}_{\mathbf{b}} \subseteq \mathcal{C}$ be $\{\mathbf{a}_i : i \in \mathbb{S}_g\}$. Since $C_{g,i,j,\varepsilon} \in \mathcal{U}$ whenever $[i, j] \cap \mathbb{S}_g = \emptyset$ and $\varepsilon > 0$, we conclude that $\mathcal{C}$ is a weakly $(\aleph_1, \phi)$-skeleton like ϕ-chain as in the proof in Sec. 4.

6.7. *The proof of Theorem 6.1*

Compiling the above facts into the proof of Theorem 6.1 proceeds exactly like in Sec. 5.

Remark 6.9. Remark 4.5 applies to Proposition 6.6 in place of Proposition 4.2 verbatim.

7. Types with the Order Property

In this section we prove local versions of Theorem 1.1 and Theorem 1.3 in which the ϕ-chain is contained in the set of n-tuples realizing a prescribed type $\mathbf{t}$ (the definition of a type in the logic of metric structures is given below). We will make use of this in case when $\mathbf{t}$ is the set of all n-tuples all of whose entries realize a given 1-type, and the set of these realizations is a substructure. In order to conclude that a ϕ-chain is still a ϕ-chain when evaluated in this substructure, we will consider a formula ϕ that is quantifier-free. Throughout this section we assume A is a model, $\phi(\bar{x}, \bar{y})$ is a $2n$-ary formula in the same signature and $\mathbf{t}$ is an n-ary type over A.

Although the motivation for this section comes from the metric case, we shall first provide the definitions and results in the classical case of discrete models. An n-ary type $\mathbf{t}$ over A has the *order property* if there exists a $2n$-ary formula ϕ such that for every finite $\mathbf{t}_0 \subseteq \mathbf{t}$ and for every $m \in \mathbb{N}$ there exists a ϕ-chain of length m in A all of whose elements realize $\mathbf{t}_0$.

Proposition 7.1. *Assume A is countable and type $\mathbf{t}$ over A has the order property, as witnessed by ϕ. Assume I is a linear order of cardinality $\leq \mathfrak{c}$. Then there is an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that $\prod_{\mathcal{U}} A$ includes a weakly $(\aleph_1, \phi)$-skeleton like ϕ-chain isomorphic to I consisting of n-tuples realizing $\mathbf{t}$.*

Proof. Since $\mathbf{t}$ is countable we may write it as a union of finite subtypes, $\mathbf{t} = \bigcup_{i \in \mathbb{N}} \mathbf{t}_i$. Let $a_i(k)$, for $0 \leq i < k$, be a $\preceq_\phi$ chain in A of elements realizing $\mathbf{t}_k$. Let $\mathcal{G}$ be an independent family of functions of cardinality $\mathfrak{c}$. Unlike the proof of Proposition 4.2, we cannot identify $\mathcal{G}$ with functions in $\prod_k \{a_i(k) \colon i < k\}$, since we cannot assume $a_i(k) = a_i(l)$ for all $i < \min(k, l)$. Therefore to each $g \in \mathcal{G}$ we associate a function $\hat{g}$ such that

$$\hat{g}(k) = a_{g(k)}(k)$$

if $g(k) < k$ and $\hat{g}(k) = a_{k-1}(k)$, otherwise. Then by the Fundamental Theorem of Ultraproducts $\hat{g}$ is a representing sequence of an element that realizes $\mathbf{t}$. The rest of the proof is identical to the proof of Proposition 4.2. $\square$

In order to state the metric version of Proposition 7.1 we import some notation from [8, 9]. Given $0 \leq \varepsilon < 1/2$ define relation $\preceq_{\phi,\varepsilon}$ on A^n via

$$\bar{a}_1 \preceq_{\phi,\varepsilon} \bar{a}_2 \quad \text{if } \phi(\bar{a}_1, \bar{a}_2) \leq \varepsilon \quad \text{and} \quad \phi(\bar{a}_2, \bar{a}_1) \geq 1 - \varepsilon.$$

Note that $\preceq_{\phi,0}$ coincides with $\preceq_\phi$. A *ϕ, ε-chain* is defined in a natural way.

We shall now define a type in the logic of metric structures, following [2] and [10, Sec. 4.3]. A *condition* over a model A is an expression of the form $\phi(\bar{x}, \bar{a}) \leq r$ where ϕ is a formula, $\bar{a}$ is a tuple of elements of A and $r \in \mathbb{R}$. A type $\mathbf{t}$ over A is a set of conditions over A. A condition $\phi(\bar{x}, \bar{a}) \leq r$ is *ε-satisfied* in A by $\bar{b}$ if $\phi(\bar{b}, \bar{a})^A \leq r + \varepsilon$. Clearly a condition is satisfied by $\bar{b}$ in A if and only if it is ε-satisfied by $\bar{b}$ for all $\varepsilon > 0$. A type $\mathbf{t}$ is *ε-satisfied* by $\bar{b}$ if all conditions in $\mathbf{t}$ are ε-satisfied by $\bar{b}$.

An n-ary type $\mathbf{t}$ over a metric structure A has the *order property* if there exists a $2n$-ary formula ϕ such that for every finite $\mathbf{t}_0 \subseteq \mathbf{t}$ and for every $m \in \mathbb{N}$ there exists a $\phi, 1/m$-chain of length m in A consisting of n-tuples each of which $1/m$-satisfies $\mathbf{t}_0$.

Proposition 7.2. *Assume A is separable metric structure and type $\mathbf{t}$ over A has the order property, as witnessed by ϕ. Assume I is a linear order of cardinality $\leq \mathfrak{c}$. Then there is an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that $\prod_{\mathcal{U}} A$ includes a weakly $(\aleph_1, \phi)$-skeleton like ϕ-chain isomorphic to I and consisting of n-tuples realizing $\mathbf{t}$.*

Proof. For elements $\mathbf{a}$ and $\mathbf{b}$ of $\prod_{\mathcal{U}} A$ and their representing sequences $(a_i)_{i\in\mathbb{N}}$ and $(b_i)_{i\in\mathbb{N}}$ we have $\mathbf{a} \preceq_\phi \mathbf{b}$ in $\prod_{\mathcal{U}} A$ if and only if $\{i\colon a_i \preceq_{\phi,\varepsilon} b_i\} \in \mathcal{U}$ for every $\varepsilon > 0$. Modulo this observation and replacing $\mathbf{t}$ with its restriction to a countable dense subset of A, the proof is identical to the proof of Proposition 7.1. $\square$

In order to prove versions of Propositions 7.1 and 7.2 for uncountable (respectively, nonseparable) structures we shall need the following well-known lemma.

Lemma 7.3. *Assume D is a meager filter on $\mathbb{N}$ extending the Frechét filter. Then there is a family $\mathcal{G}_D$ of cardinality $\mathfrak{c}$ of functions in $\mathbb{N}^\mathbb{N}$ that is independent mod D.*

Proof. Let $\mathcal{G}$ be a family of cardinality $\mathfrak{c}$ that is independent mod the Fréchet filter ([20, Appendix, Theorem 1.5(1)]). Since D is meager there is a surjection $h\colon \mathbb{N} \to \mathbb{N}$ such that the h-preimage of every finite set is finite and the h-preimage of every infinite set is D-positive (see, e.g. [1]). Then $\mathcal{G}_D = \{h \circ f\colon f \in \mathcal{G}\}$ is independent mod D because the h-preimage of every infinite set is D-positive. $\square$

Again A, ϕ and $\mathbf{t}$ are as above and $A^{<\mathbb{N}}$ denotes the set of all finite sequences of elements of A. Note that A is not assumed to be countable.

Proposition 7.4. *Let A be a model and let $\mathbf{t}$ be a type over A. Assume there is a function $\mathbf{h} \in \prod_{k\in\mathbb{N}} A^{k\cdot n}$ such that the sets*

$$X[\mathbf{t}_0, k] = \{i\colon \mathbf{h}(i) \text{ is a } \phi\text{-chain of } n\text{-tuples satisfying } \mathbf{t}_0\}$$

for $\mathbf{t}_0 \subseteq \mathbf{t}$ finite and $k \in \mathbb{N}$ generate a meager filter extending the Frechét filter.

Assume I is a linear order of cardinality $\leq \mathfrak{c}$. Then there is an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that $\prod_{\mathcal{U}} A$ includes a ϕ-chain $\{\mathbf{a}_i\colon i \in I\}$ that is weakly $(\aleph_1, \phi)$-skeleton like and consists of elements realizing $\mathbf{t}$.

Proof. Let D_0 denote the filter generated by all $X[\mathbf{t}_0, k]$ for $\mathbf{t}_0 \subseteq \mathbf{t}$ finite and $k \in \mathbb{N}$. By Lemma 7.3 there is a family $\mathcal{G}_0$ of cardinality $\mathfrak{c}$ that is independent mod D. For each $k \in \mathbb{N}$ enumerate the ϕ-chain $\mathbf{h}(k)$ as $a_i(k)$, $i < k$. Like in the proof of Proposition 7.1 for $g \in \mathcal{G}_0$ define $\hat{g} \in A^\mathbb{N}$ by $\hat{g}(k) = a_{g(k)}(k)$ if $g(k) < k$ and $a_{k-1}(k)$ otherwise.

The construction described in the proof of Proposition 4.2 results in $\mathcal{U}$ such that all elements of the resulting ϕ-chain $\mathbf{a}_i$, for $i \in I$, realize $\mathbf{t}$. $\square$

The proof of the following metric version is identical to the proof of Proposition 7.4. Note that A is not assumed to be separable.

Proposition 7.5. *Let A be a metric structure and let $\mathbf{t}$ be a type over A. Assume there is a function $\mathbf{h} \in \prod_{k\in\mathbb{N}} A^{k\cdot n}$ such that the sets*

$$X[\mathbf{t}_0, k] = \{i\colon\ \mathbf{h}(i) \textit{ is a } \phi, 1/k\textit{-chain consisting of } n\textit{-tuples } 1/k\textit{-satisfying } \mathbf{t}_0\}$$

for $\mathbf{t}_0 \subseteq \mathbf{t}$ finite and $k \in \mathbb{N}$ generate a meager filter extending the Frechét filter.

Assume I is a linear order of cardinality $\leq \mathfrak{c}$. Then there is an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that $\prod_{\mathcal{U}} A$ includes a ϕ-chain $\{\mathbf{a}_i : i \in I\}$ that is weakly $(\aleph_1, \phi)$-skeleton like and consists of elements realizing $\mathbf{t}$.

8. Applications

Recall that $\mathrm{Alt}(n)$ is the alternating group on $\{0, \ldots, n-1\}$. The following is the main result of [7] (see also [24]).

Theorem 8.1 (Ellis–Hachtman–Schneider–Thomas). *If CH fails then there are $2^{\mathfrak{c}}$ ultrafilters on $\mathbb{N}$ such that the ultraproducts $\prod_{\mathcal{U}} \mathrm{Alt}(n)$ are pairwise nonisomorphic.*

Proof. Let $\phi(x_1, x_2, y_1, y_2)$ be the formula asserting that $x_1y_2 = y_2x_1$ and $x_2y_1 \neq y_1x_2$. It is then easy to see that for all natural numbers $k \geq 2n+4$ the group $\mathrm{Alt}(k)$ includes a ϕ-chain of length n. Therefore the conclusion follows by Theorem 5.1. $\square$

8.1. *Applications to operator algebras*

Theorems 1.3 and 6.1 were stated and proved for the case of bounded metric structures. However, the original motivation for the present paper came from a question about the of ultrapowers of C*-algebras and II_1 factors stated in early versions of [9, 10]. An excellent reference for operator algebras is [4].

In the following propositions and accompanying discussion we deal with the ultrapower constructions for C*-algebras and II_1-factors, as well as the associated relative commutants. Although Theorem 1.3 was proved for bounded metric structures, it applies to the context of C*-algebras and II_1 factors. Essentially, one applies the result to the unit ball of the given algebra. All the pertinent definitions can be found in [9] or [10].

The classes of C*-algebras and of II_1 factors are axiomatizable in the logic of metric structures. Both proofs can be found in [10, Sec. 3], and the (much more difficult) II_1 factor case was first proved in [3], using a rather different axiomatization from the one given in [10]. Extending results of [8, 14], in [9, Lemma 5.4] it was also proved that the class of infinite dimensional C*-algebras has the order property, as witnessed by the formula

$$\phi(x, y) = \|xy - x\|.$$

Assume $a_i, i \in \mathbb{N}$, is a sequence of positive operators of norm one such that $a_i - a_j$ is positive and of norm one whenever $j < i$. Then this sequence forms a $\preceq_\phi$-chain. Such a sequence exists in every infinite-dimensional C*-algebra (see the proof of [9, Lemma 5.2]). Note that it is important to have this $\preceq_\phi$-chain inside the unit ball of the algebra. In [9, Lemma 5.4] it was also proved that the relative commutant type (see below for the definition) of every infinite-dimensional C*-algebra has the order property, and that this is witnessed by the same ϕ as above.

In [9, Lemma 3.2(3)], it was proved that the class of II_1 factors has the order property, as witnessed by the formula

$$\psi(x_1, y_1, x_2, y_2) = \|x_1 y_2 - y_2 x_1\|_2.$$

Unlike the case of C*-algebras, the relative commutant type of some II_1 factors does not have the order property. For a II_1 factor N, the relative commutant type having the order property is equivalent to having a nonabelian relative commutant in some (equivalently, all) of its ultrapowers associated with nonprincipal ultrafilters on $\mathbb{N}$ ([9, Theorem 4.8]). Such II_1 factors are called *McDuff factors*. We emphasize that, similarly to the case of C*-algebras, an arbitrarily long finite ψ-chain can be found inside the unit ball of a II_1 factor. This is necessary in order to have the proof work. Note that without this requirement even $\mathbb{C}$ includes an infinite ψ-chain, although $\mathbb{C}$ clearly does not have the order property.

Recall that two C*-algebras are (algebraically) isomorphic if and only if they are isometric, and that the same applies to II_1 factors. The following is a quantitative improvement to the results of [14], [8] (for C*-algebras) and [9] (for II_1 factors).

Proposition 8.2. *Assume A is a separable infinite-dimensional C*-algebra or a separably acting II_1-factor. If the Continuum Hypothesis fails, then A has $2^{\mathfrak{c}}$ nonisomorphic ultrapowers associated with ultrafilters on $\mathbb{N}$.*

In Proposition 8.2 it suffices to assume that the character density of A is $\leq \mathfrak{c}$. This does not apply to Proposition 8.4 below where the separability assumption is necessary (cf. the last paragraph of [10, Sec. 4] or [13]).

Proof. Since by the above discussion both classes are axiomatizable with unstable theories, Theorem 1.3 implies that in all of these cases there are $2^{\mathfrak{c}}$ ultrapowers with nonisomorphic unit balls. Therefore the result follows. □

In the light of Proposition 8.2, it is interesting to note that the theory of abelian tracial von Neumann algebras is stable (this is a consequence of [9, Theorem 4.7] and the characterization of stability from [10, Theorem 5.6]). More precisely, a tracial von Neumann algebra M has the property that it has nonisomorphic ultrapowers (and therefore by Theorem 1.3 it has $2^{\mathfrak{c}}$ nonisomorphic ultrapowers) if and only if it is not of type I. This is a consequence of [9, Theorem 4.7].

The following is a quantitative improvement of [9, Proposition 3.3], confirming a conjecture of Sorin Popa in the case when the Continuum Hypothesis fails. The intended ultrapower is the tracial ultrapower, and the analogous result for norm ultrapower is also true.

Proposition 8.3. *Assume the Continuum Hypothesis fails. Then there are* $2^{\mathfrak{c}}$ *ultrafilters on* $\mathbb{N}$ *such that the* II_1 *factors* $\prod_{\mathcal{U}} M_n(\mathbb{C})$ *are all nonisomorphic.*

Proof. This is a direct application of Theorem 1.3, using $\preceq_\phi$-chains obtained in [9, Lemma 3.2]. □

Assume M is a C*-algebra or a II_1 factor and $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$. Identify M with its diagonal copy inside $\prod_{\mathcal{U}} M$. The *relative commutant* of M inside its ultrapower is defined as

$$M' \cap \prod_{\mathcal{U}} M = \{a \in \prod_{\mathcal{U}} M \colon (\forall\, a \in M) ab = ba\}.$$

Thus the relative commutant is the set of all elements of $\prod_{\mathcal{U}} M$ realizing the *relative commutant type* of M, consisting of all conditions of the form $\|xb - bx\| = 0$, for $b \in M$. (Here $\|\cdot\|$ stands for $\|\cdot\|_2$ in case when M is a II_1 factor.) The relative commutant is a C*-algebra (II_1 factor, respectively) and it is fair to say that most applications of ultrapowers in operator algebras are applications of relative commutants. A relative commutant is said to be *trivial* if it is equal to the center of M. From a model-theoretic point of view, a relative commutant is a submodel consisting of all realizations of a definable type over M.

The original motivation for the work in [8–10] came from the question whether all relative commutants of a given operator algebra in its ultrapowers associated with ultrafilters on $\mathbb{N}$ are isomorphic. This was asked by Kirchberg in the case of C*-algebras and McDuff in the case of II_1-factors. Here is a quantitative improvement to the answer to these questions given in the above references.

Proposition 8.4. *Assume* A *is a separable infinite-dimensional C*-algebra or a separably acting McDuff* II_1*-factor. If the Continuum Hypothesis fails, then* A *has* $2^{\mathfrak{c}}$ *nonisomorphic relative commutants in ultrapowers associated with ultrafilters on* $\mathbb{N}$*.*

Proof. In [9, Theorem 4.8], it was proved that the relative commutant type of a McDuff factor has the order property, witnessed by ψ given in the introduction to Sec. 8.1. In [9, Lemma 5.4], it was proved that the relative commutant type of any infinite-dimensional C*-algebra has the order property, witnessed by ϕ given in the introduction to Sec. 8.1. Hence applying Proposition 7.2 concludes the proof. □

By $\mathcal{B}(H)$ we shall denote the C*-algebra of all bounded linear operators on an infinite-dimensional, separable, complex Hilbert space H. In [13] it was proved that for certain ultrafilters on $\mathbb{N}$ the relative commutant of $\mathcal{B}(H)$ in $\prod_{\mathcal{U}} \mathcal{B}(H)$ is nontrivial. These ultrafilters exist in ZFC. It was also proved in [13] that the relative commutant of $\mathcal{B}(H)$ in an ultrapower associated to a selective ultrafilter is trivial. Therefore CH implies that not all relative commutants of $\mathcal{B}(H)$ in its ultrapowers associated with ultrafilters on $\mathbb{N}$ are isomorphic. This fact motivated Juris Steprāns and the first author to ask whether this statement can be proved in ZFC. Since $\mathcal{B}(H)$ is not a separable C*-algebra, the following is not a consequence of Proposition 8.4.

Proposition 8.5. *Assume that the Continuum Hypothesis fails. Then $\mathcal{B}(H)$ has $2^{\mathfrak{c}}$ nonisomorphic relative commutants associated with its ultrapowers.*

Proof. We shall apply Proposition 7.5. The following construction borrows some ideas from the proof of [13, Theorems 3.3 and 4.1]. Let $\mathbb{F}^{<\mathbb{N}}$ be the countable set of all finite sequences of nonincreasing functions $h\colon \mathbb{N} \to \mathbb{Q} \cap [0,1]$ that are eventually zero and such that $h(0) = 1$. We shall construct a filter D on $\mathbb{F}^{<\mathbb{N}}$. For f and g in $\mathbb{R}^{\mathbb{N}}$ write $\|f - g\|_\infty = \sup_i |f(i) - g(i)|$. For $f\colon \mathbb{N} \nearrow \mathbb{N}$ and $m \in \mathbb{N}$ let $X_{f,m}$ be the set of all k-tuples $\langle h_0, h_1, \ldots, h_{k-1}\rangle$ in $\mathbb{F}$ such that

(1) $k \geq m$,
(2) $\max_{i<k} \|h_i - h_i \circ f\|_\infty \leq 1/m$,
(3) $h_i(j) \leq h_{i+1}(j)$ for all $i < k-1$ and all j,
(4) for all $i < k-2$ there is $j \in \mathbb{N}$ such that $h_i(j) = 0$ and $h_{i+1}(j) = 1$.

We claim that $X_{f,m}$ is always infinite. This is essentially a consequence of the proof of [13, Lemma 3.4] but we shall sketch a proof. Fix a sequence $n(j)$, for $j \in \mathbb{N}$, such that $n(l+1) \geq f(n_l)$ for all l. For $Z \subseteq \mathbb{N}$ by χ_Z we denote the characteristic function of Z. For $i < k$ set

$$h_i = \chi_{[0,mi)} + \sum_{l=im}^{(i+1)m-1} \frac{(i+1)m - l}{m} \chi_{[n(l),n(l+1))}.$$

A straightforward computation shows that $\langle h_0, h_1, \ldots, h_{k-1}\rangle \in X_{f,m}$. Since $X_{f,m} \cap X_{g,n} \supseteq X_{\max(f,g),\,\max(m,n)}$, the collection of all $X_{f,m}$, for $f\colon \mathbb{N} \nearrow \mathbb{N}$ and $\varepsilon > 0$, has the finite intersection property. Since the filter generated by these sets is analytic, proper, and includes all cofinite sets, it is meager (see e.g. [1]). Fix a basis e_j, for $j \in \mathbb{N}$, of H. For $h\colon \mathbb{N} \to [0,1]$ define a positive operator a_h in $\mathcal{B}(H)$ via

$$a_h = \sum_{j \in \mathbb{N}} h(j) e_j.$$

In other words, a_h is the operator with the eigenvalues $h(j)$ corresponding to the eigenvectors e_j. Fix an enumeration $\mathbb{F}^{<\mathbb{N}} = \{s_i \colon i \in \mathbb{N}\}$. Let $\mathbf{h}$ be a function from

$\mathbb{N}$ into the finite sequences of positive operators in the unit ball of $\mathcal{B}(H)$ defined by $\mathbf{h}(i) = \langle a_h : h \in s_i \rangle$. With

$$\phi(x, y) = \|xy - y\|$$

conditions (3) and (4) above imply that each $\mathbf{h}(i)$ is a ϕ-chain.

Let $\mathbf{t}$ be the relative commutant type of $\mathcal{B}(H)$, i.e. the set of all conditions of the form $\|ax - xa\| < \varepsilon$ for a in the unit ball of $\mathcal{B}(H)$ and $\varepsilon > 0$. Let $\mathbf{t}_0$ be a finite subset of $\mathbf{t}$, let $\varepsilon > 0$, and let $a_0, \dots, a_{k-1}$ list all elements of $\mathcal{B}(H)$ occurring in $\mathbf{t}_0$. Let $\delta = \varepsilon/6$. [13, Lemma 4.6] implies that there are g_0 and g_1 such that for each $i < k$ we can write $a_i = a_i^0 + a_i^1 + c_i$ so that

(1) a_i^0 commutes with a_h for every h that is constant on every interval of the form $[g_0(m), g_0(m+1))$,
(2) a_i^1 commutes with a_h for every h that is constant on every interval of the form $[g_1(m), g_1(m+1))$, and
(3) $\|c_i\| < \delta$.

Then for $i < k$, $j \in X_{g_0,\delta} \cap X_{g_1,\delta}$, and h an entry of $\mathbf{h}(j)$ we have

$$[a_i, a_h] = [a_i^0, a_h] + [a_i^1, a_h] + [c_i, a_h]$$

and since $\|a_i^0\|$, $\|a_i^1\|$ and $\|a_h\|$ are all ≤ 1 we conclude that $\|[a_i, a_h]\| < 6\delta$.

Therefore a_h realizes $\mathbf{t}_0$, and Proposition 7.5 implies that for every linear order I of cardinality $\mathfrak{c}$ there is an ultrafilter $\mathcal{U}$ such that $\prod_{\mathcal{U}} \mathcal{B}(H)$ contains a ϕ-chain $\mathcal{C}$ isomorphic to I which is $(\aleph_1, \phi)$-skeleton like and included in the relative commutant of $\mathcal{B}(H)$. Since ϕ is quantifier-free, $\mathcal{C}$ remains a ϕ-chain in the relative commutant $\mathcal{B}(H)' \cap \prod_{\mathcal{U}} \mathcal{B}(H)$. Since $\mathcal{C}$ is $(\aleph_1, \phi)$-skeleton like in $\prod_{\mathcal{U}} \mathcal{B}(H)$, it is $(\aleph_1, \phi)$-skeleton like in the substructure. Using Lemmas 2.5, 3.11, 3.13 and a counting argument as in the proof of Theorem 5.1 we conclude the proof. □

8.2. *Concluding remarks*

Before Theorem 1.1 was proved the following test question was asked in a preliminary version of [10]: Assume A and B are countable models with unstable theories. Also assume $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters on $\mathbb{N}$ such that $\prod_{\mathcal{U}} A \not\cong \prod_{\mathcal{V}} A$. Can we conclude that $\prod_{\mathcal{U}} B \not\cong \prod_{\mathcal{V}} B$? A positive answer would, together with [15, Sec. 3], imply Theorem 1.1. However, the answer to this question is consistently negative. Using the method of [22] one can show that in the model obtained there are countable graphs G and H and ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$ such that $\prod_{\mathcal{U}} G, \prod_{\mathcal{V}} G$ and $\prod_{\mathcal{V}} H$ are saturated but $\prod_{\mathcal{U}} H$ is not. This model has an even more remarkable property: Every automorphism of $\prod_{\mathcal{U}} H$ lifts to an automorphism of $H^{\mathbb{N}}$. An interesting and related application of [22] was recently given in [17].

Methods of the present paper was adapted to the class of all *approximately matricial* (shortly AM) C*-algebras in [12]. A C*-algebra is AM if and only if it is an inductive limit of finite-dimensional matrix algebras ([11]). In [12] it was proved that in every uncountable character density λ there are 2^λ nonisomorphic AM algebras. Unlike the classes of C*-algebras and II_1 factors, the class of AM algebras is not elementary. This is because AM algebras are not closed under taking ultrapowers (by the proof of [10, Proposition 6.1]).

Results related to our Sec. 6 were proved in [23], where it was shown that an unstable theory in logic of metric structures has maximal possible number of models in every uncountable cardinality. In the general case, treated in [23], there is a distinction between "isomorphic" and "isometric". For C*-algebras and II_1 factors treated here the two notions are equivalent.

Acknowledgments

The first author was partially supported by NSERC and he would like to thank Takeshi Katsura for several useful remarks.

The second author would like to thank the Israel Science Foundation for partial support of this research (Grant no. 710/07). Part of this work was done when the authors visited the Mittag-Leffler Institute. No. 954 on Shelah's list of publications.

We would like to thank Gabor Sagi and the anonymous referee for a number of useful comments on the first version of this paper. Our special thanks go to Heike Mildenberger and the Freiburg logic group for carefully reading our paper and sending an extensive list of last minute corrections.

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