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Reflexive abelian groups and measurable cardinals and full MAD families

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Dedicated to George Grätzer and E. Tamas Schmidt

Abstract. Answering problem (DG) of [1], [2], we show that there is a reflexive group of cardinality equal to the first measurable cardinal.

1. Introduction

Definition 1.1. Let G be an abelian group.

(a) The dual of G is the abelian group $\mathrm{Hom}(G, \mathbb{Z})$, which we denote by G^*;
(b) the double dual of G is the abelian group $\mathrm{Hom}(G^*, \mathbb{Z})$, which we denote by G^{**}.

There is the canonical homomorphism from G into G^{**}, that is, $a \in G$ is mapped to $F_a \in \mathrm{Hom}(G^*, \mathbb{Z})$ defined by $F_a(f) = f(a)$. The best case, from our point of view, is when the canonical homomorphism is an isomorphism. There is a nice name for that phenomenon:

Definition 1.2. Let G be an abelian group. We say that G is *reflexive* if G is canonically isomorphic to G^{**}.

Basic results about reflexive groups appear in Eklof and Mekler (see [1] and [2] for a revised edition). They present a fundamental theorem of Łoś, generalized by Eda. Łoś' Theorem says that λ is smaller than the first ω-measurable cardinal (called the first measurable cardinal by set theorists) if and only if the dual of the direct product of λ copies of $\mathbb{Z}$ is the direct sum of λ copies of $\mathbb{Z}$. The converse is always true. It says that for all λ, the dual of the direct sum of λ copies of $\mathbb{Z}$ is the direct product of λ copies of $\mathbb{Z}$. For λ at least the first ω-measurable cardinal, Łoś' Theorem just says the abelian group $\mathbb{Z}^\lambda$ is not reflexive; Eda's Theorem describes $\mathrm{Hom}(\mathbb{Z}^\lambda, \mathbb{Z})$ in this case. A direct consequence of Łoś' Theorem is the existence of many reflexive groups, but still there is a cardinality limitation. Let us describe the problem. We use the terminology of Eklof and Mekler.

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Definition 1.3. Let μ be an infinite cardinal.

(a) μ is *measurable* if there exists a non-principal μ-complete ultrafilter on μ and μ is uncountable

(b) μ is *ω-measurable* if there exists a non-principal $\aleph_1$-complete ultrafilter on μ.

We would like to clarify one important point. Let μ be the first ω-measurable cardinal, and let D be a non-principal $\aleph_1$-complete ultrafilter on μ. It is well known that D is also μ-complete. So the first ω-measurable cardinal is, in fact, the first measurable cardinal. It is easy to extend any non-principal $\aleph_1$-complete ultrafilter on μ to an $\aleph_1$-complete ultrafilter on λ. So λ is ω-measurable for every $\lambda \geq \mu$.

Let us summarize:

Observation 1.4. *Let $\mu = \mu_{first}$ be the first measurable cardinal.*

(a) *For every $\theta < \mu$, θ is not ω-measurable.*

(b) *For every $\lambda \geq \mu$, λ is ω-measurable.*

This terminology enables us to formulate the result that we need. Recall that $\mathbb{Z}^\theta$ is $\prod_{i<\theta} \mathbb{Z}$ and $\mathbb{Z}^{(\theta)}$ is $\bigoplus_{i<\theta} \mathbb{Z}$. Łoś' Theorem deals with the existence of $\aleph_1$-complete ultrafilters. We will refer to the following corollary, also known as Łoś' Theorem:

Corollary 1.5. *Let $\mu = \mu_{first}$ be the first measurable cardinal.*

(a) *For any $\theta < \mu$, $\mathbb{Z}^{(\theta)}$ is reflexive (its dual being $\mathbb{Z}^\theta$).*

(b) *For every $\lambda \geq \mu$, $\mathbb{Z}^{(\lambda)}$ is not reflexive.*

The proof of 1.5(a) is based on the fact that every $\aleph_1$-complete ultrafilter is principal. So it does not work above μ_{first}. Naturally, we can ask - does there exist a reflexive group of large cardinality, i.e., of cardinality $\geq \mu_{\text{first}}$? This is problem (DG) of Eklof–Mekler [1], [2]. We can further conjecture:

Conjecture 1.6. There are reflexive abelian groups of arbitrarily large cardinalities.

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2. A reflexive group above the first measurable cardinal

We answer question (DG) of Eklof–Mekler [1]. There are reflexive groups of cardinality not smaller than the first measurable. Do we have it for arbitrarily large λ, i.e., Conjecture 1.6?

This is very likely, in fact it follows (in ZFC) from $2^{\aleph_0} > \aleph_\omega$ if some pcf conjecture holds. See the next section.

Theorem 2.1. *If $\mu = \mu_{first}$, the first measurable cardinal, then there is a reflexive $G \subseteq {}^\mu\mathbb{Z}$ of cardinality μ.*

Proof. By Claim 2.8(1) below (recall that μ is measurable, so it is strong limit with cofinality greater than $\aleph_0$) there are $\mathcal{A}_1, \mathcal{A}_2 \subseteq [\mu]^{\aleph_0}$ such that $\mathcal{A}_1 \subseteq \mathcal{A}_2^\perp$ and $\kappa^+(\mathcal{A}_1^\perp)+\kappa^+(\mathcal{A}_2^\perp) \leq \mu$, see Definition 2.3 below. By claim 2.7 below there is a G as required. $\square$

Convention 2.2. $\lambda \geq \aleph_0$ *is fixed in this section (we need to fix* λ *so that* $\mathcal{A}^\perp$ *is well defined).*

Definition 2.3.

(1) For $\mathcal{A} \subseteq \mathcal{P}(\lambda)$, let
  (a) $\mathrm{id}(\mathcal{A}) = \mathrm{id}_{\mathcal{A}}$ be the ideal of subsets of λ generated by $\mathcal{A} \cup [\lambda]^{<\aleph_0}$;
  (b) $\mathcal{A}^\perp = \{u \subseteq \lambda : u \cap v$ is finite for every $v \in \mathcal{A}\}$;
  (c) $c\ell(\mathcal{A}) = \{u \subseteq \lambda :$ every infinite $v \subseteq u$ contains some member of $\mathrm{sb}(\mathcal{A})\}$ (see below);
  (d) $\mathrm{sb}(\mathcal{A}) = \{u \subseteq \lambda : u$ is infinite and is included in some member of $\mathcal{A}\}$.

(2) For $\mathcal{A} \subseteq \mathcal{P}(\lambda)$ let
  (a) $G_{\mathcal{A}}$ be the subgroup of $\mathbb{Z}^\lambda$ consisting of $\{f \in \mathbb{Z}^\lambda : \mathrm{supp}(f) \in \mathrm{id}(\mathcal{A})\}$ where $\mathrm{supp}(f) = \{\alpha < \lambda : f(\alpha) \neq 0_{\mathbb{Z}}\}$;
  (b) $\mathbf{j}_{\mathcal{A}}$ is the function from $G^*_{\mathcal{A}} := \mathrm{Hom}(G_{\mathcal{A}}, \mathbb{Z})$ into $\mathbb{Z}^\lambda$ defined by:

$$(\mathbf{j}_{\mathcal{A}}(g))(\alpha) = g(\mathrm{ch}_{\{\alpha\}})$$

  where for $u \subseteq \lambda$, $\mathrm{ch}_u = \mathrm{ch}_{\lambda,u}$ is the function with domain λ mapping α to 1 if $\alpha \in u$ and to 0 if $\alpha \in \lambda \setminus u$.

Definition 2.4.

(a) $\kappa^+(\mathcal{A}) = \bigcup\{|u|^+ : u \in \mathcal{A}\}$;
(b) $\kappa(\mathcal{A}) = \bigcup\{|u| : u \in \mathcal{A}\}$.

Claim 2.5. *Let* $\mathcal{A} \subseteq \mathcal{P}(\lambda)$.

(1) (a) $\mathcal{A} \subseteq c\ell(\mathcal{A}) = c\ell(c\ell(\mathcal{A}))$;
  (b) $\mathcal{A}^\perp = \{u \subseteq \lambda : [u]^{\aleph_0} \cap \mathrm{sb}(\mathcal{A}) = \emptyset,$ *e.g.* u *is finite*$\}$;
  (c) $\mathcal{A} \subseteq \mathrm{sb}(\mathcal{A}) \cup [\lambda]^{<\aleph_0}$;
  (d) $\mathcal{A} \subseteq \mathrm{id}(\mathcal{A})$.

(2) (a) $\mathcal{A}^\perp = \mathrm{id}(\mathcal{A}^\perp)$;
  (b) $(\mathcal{A}^\perp)^\perp = c\ell(\mathcal{A})$, *note that both include* $[\lambda]^{<\aleph_0}$;
  (c) $\mathcal{A}^\perp = c\ell(\mathcal{A}^\perp)$, *note that both include* $[\lambda]^{<\aleph_0}$;
  (d) $\mathcal{A}^\perp = (c\ell(\mathcal{A}))^\perp$.

(3) *If* $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{P}(\lambda)$, *then* $\mathcal{B}^\perp \subseteq \mathcal{A}^\perp$ *and* $\kappa^+(\mathcal{A}) \leq \kappa^+(\mathcal{B})$ *and* $\kappa^+(\mathcal{A}) + \aleph_0 = \kappa^+(\mathrm{id}(\mathcal{A}))$.

Proof. (1) and (3) are obvious.

(2) Clause (a): Clearly $\mathcal{A}^\perp \subseteq \mathrm{id}(\mathcal{A}^\perp)$ by the definition of id. For the other inclusion, as $\mathcal{A}^\perp$ includes all finite subsets of λ, assume $u \in \mathrm{id}(\mathcal{A}^\perp)$ is infinite; hence, for some $n < \omega$ and infinite $u_0, \dots, u_{n-1} \in \mathcal{A}^\perp$ we have: $u \setminus \bigcup\{u_\ell : \ell < n\}$ is finite. Hence,

$$\begin{aligned} v \in \mathcal{A} \quad &\Rightarrow \quad (\forall \ell < n)(v \cap u_\ell \text{ is finite }) \quad \Rightarrow \\ (v \cap \cup\{u_\ell : \ell < n\} &\text{ is finite }) \quad \Rightarrow \quad (v \cap u \text{ is finite }); \end{aligned}$$

hence, $u \in \mathcal{A}^\perp$.

Clause (b): Assume $u \in c\ell(\mathcal{A})$ and $v \in \mathcal{A}^\perp$. If $u \cap v$ is infinite, then by "$u \in c\ell(\mathcal{A})$" we know that $u \cap v$ includes some member of $\mathrm{sb}(\mathcal{A})$, but by "$v \in \mathcal{A}^\perp$" we know that $u \cap v$ includes no member of $\mathrm{sb}(\mathcal{A})$, a contradiction. So $u \cap v$ is finite.

Fixing $u \in c\ell(\mathcal{A})$ and varying $v \in \mathcal{A}^\perp$, this tells us that $u \in ((\mathcal{A})^\perp)^\perp$. Recalling that $[\lambda]^{<\aleph_0} \subseteq (\mathcal{A}^\perp)^\perp$ is obvious, we have shown $c\ell(\mathcal{A}) \subseteq (\mathcal{A}^\perp)^\perp$.

Next, if $u \subseteq \lambda$ and $u \notin c\ell(\mathcal{A})$, so that u is infinite, then there is an infinite $v \subseteq u$ such that $[v]^{\aleph_0} \cap \mathrm{sb}(\mathcal{A}) = \emptyset$. Hence, v is in $\mathcal{A}^\perp$, so u includes an infinite member of $\mathcal{A}^\perp$; hence, u is not in $(\mathcal{A}^\perp)^\perp$. This shows $u \notin c\ell(\mathcal{A}) \Rightarrow u \notin (\mathcal{A}^\perp)^\perp$. So we get the desired equality.

Clause (c): Similar to the proof of clause (b).

Clause (d): Similar to the proof of clause (b). □

Claim 2.6. *Let $\mathcal{A} \subseteq \mathcal{P}(\lambda)$.*

(1) *If $\kappa^+(\mathcal{A}) \le \mu_{\mathrm{first}} :=$ first measurable cardinal, then $\mathbf{j}_\mathcal{A}$ is an embedding of $G^*_\mathcal{A}$ into $\mathbb{Z}^\lambda$ with its image being $G_\mathcal{B}$ where $\mathcal{B} = \mathcal{A}^\perp$.*

(2) *$G_\mathcal{A}$ is reflexive iff $\mathrm{id}(\mathcal{A}) = c\ell(\mathcal{A})$ and $\kappa^+(\mathcal{A}) + \kappa^+(\mathcal{A}^\perp) \le \mu_{\mathrm{first}}$.*

(3) $|G_\mathcal{A}| = \Sigma\{2^{|u|} : u \in \mathrm{id}(\mathcal{A})\} \in [\lambda, 2^\lambda]$.

(4) *If $\kappa^+(\mathcal{A}) \le \mu$, then $\lambda \le |G_\mathcal{A}| = \lambda^{<\mu}$.*

Proof. (1) Clearly $\mathbf{j}_\mathcal{A}$ from Definition 2.3(2)(b) is a function from $G^*_\mathcal{A}$ into $\mathbb{Z}^\lambda$ and it is linear. If $g \in G^*_\mathcal{A}$ and $u \in \mathcal{A}$, then by Łoś' Theorem (as $|u| <$ the first measurable) necessarily $\{\alpha \in u : g(\mathrm{ch}_{\{\alpha\}}) \ne 0\}$ is finite. So $\mathrm{supp}(\mathbf{j}_\mathcal{A}(g)) \in \mathcal{A}^\perp$. Together, $\mathbf{j}_\mathcal{A}$ is a homomorphism from $G^*_\mathcal{A}$ into $G_\mathcal{B}$. Also if $\mathbf{j}_\mathcal{A}(g_1) = \mathbf{j}_\mathcal{A}(g_2)$ but $g_1 \ne g_2$, then for some $f \in G_\mathcal{A}$ we have $g_1(f) \ne g_2(f)$, and we can apply Łoś' Theorem for ${}^{\mathrm{supp}(f)}\mathbb{Z}$ to get a contradiction; hence $g_1 = g_2$, so we have deduced "$\mathbf{j}_\mathcal{A}$ is one-to-one". It is also easy to see that it is onto $G_\mathcal{B}$, so we are done.

(2) First assume $\mathrm{id}(\mathcal{A}) = c\ell(\mathcal{A})$. By Claim 2.5(2)(c), we have $\mathcal{A}^\perp = c\ell(\mathcal{A}^\perp)$. Applying part (1) to $\mathcal{A}$ and to $\mathcal{A}^\perp$, clearly if $\kappa^+(\mathcal{A}) + \kappa^+(\mathcal{A}^\perp) \le \mu_{\mathrm{first}}$, we get $G_\mathcal{A}$ is canonically isomorphic to $(G^*_\mathcal{A})^*$. Now $\mathbf{j}_\mathcal{A}$ is an isomorphism from $G_\mathcal{A}$ onto $G_\mathcal{B} = G_{\mathcal{A}^\perp}$, and $\mathbf{j}_\mathcal{B}$ is an isomorphism from $G_\mathcal{B}$ onto $G_{\mathcal{B}^\perp}$, but by 2.5(2)(c) by our assumption, $\mathcal{B}^\perp = (\mathcal{A}^\perp)^\perp = c\ell(\mathcal{A}) = \mathrm{id}(\mathcal{A})$. So we have proved the "if" implications.

If $\kappa^+(\mathcal{A}) > \mu_{\rm first}$, then there is $u \in \mathcal{A}$ of cardinality $\geq \mu_{\rm first}$; hence, by Łoś' Theorem we get $G_{\mathcal{A}}$ is not canonically isomorphic to $G^{**}_{\mathcal{A}}$.

Lastly if $\mathrm{id}(\mathcal{A}) \neq c\ell(\mathcal{A})$, necessarily there is $u \in c\ell(\mathcal{A}) \setminus \mathrm{id}(\mathcal{A})$; let $f = \mathrm{ch}_u$, so $u \in (\mathcal{A}^\perp)^\perp$ and f defines a member of $(G_{\mathcal{A}^\perp})^*$ not "coming from $G_{\mathcal{A}}$".

(3) and (4) are easy. □

Claim 2.7. *A sufficient condition for the existence of a reflexive group G of cardinality $\geq \lambda$, in fact $\subseteq \mathbb{Z}^\lambda$ but $\supseteq \mathbb{Z}^{(\lambda)}$ such that $|G| + |G^*| \leq \lambda^{<\mu_{\rm first}}$, is $\circledast_{\lambda,\mu_{\rm first}}$, where we define (for cardinals $\lambda \geq \mu$):*

$\circledast_{\lambda,\mu}$: there are $\mathcal{A}_1, \mathcal{A}_2 \subseteq [\lambda]^{\aleph_0}$ such that

(a) $\mathcal{A}_1 \subseteq \mathcal{A}_2^\perp$, i.e., $u_1 \in \mathcal{A}_1 \wedge u_2 \in \mathcal{A}_2 \quad\Rightarrow\quad u_1 \cap u_2$ is finite,

(b) $\kappa^+(\mathcal{A}_1^\perp) + \kappa^+(\mathcal{A}_2^\perp) \leq \mu$.

Proof. Let $\mathcal{A} = c\ell(\mathcal{A}_1)$ and $\mathcal{B} = c\ell(\mathcal{A}^\perp)$. By Claim 2.5(2)(c), we have $\mathcal{A}^\perp = \mathcal{B}$, and by Claim 2.5(2)(b), we have $\mathcal{B}^\perp = \mathcal{A}$, and lastly by Claim 2.5(1)(a), we have $\mathcal{A}_1 \subseteq \mathcal{A}$ and by $\circledast_{\lambda,\mu}(a)$, we have $\mathcal{A}_2 \subseteq \mathcal{A}_1^\perp$. But $\mathcal{A}_1^\perp = (c\ell(\mathcal{A}))^\perp = \mathcal{A}^\perp = c\ell(\mathcal{A}^\perp) = \mathcal{B}$ by the definitions of $\mathcal{A}$ and by Claim 2.5(2)(d) and together we have $\mathcal{A}_2 \subseteq \mathcal{B}$.

Now $\mathcal{A}_1 \subseteq \mathcal{A}$; hence, $\mathcal{A}^\perp \subseteq \mathcal{A}_1^\perp$, so $\kappa^+(\mathcal{A}^\perp) \leq \kappa^+(\mathcal{A}_1^\perp) \leq \mu_{\rm first}$. Also $\mathcal{A}_2 \subseteq \mathcal{B}$; hence, $\mathcal{B}^\perp \subseteq \mathcal{A}_2^\perp$ and hence $\kappa^+(\mathcal{B}^\perp) \leq \kappa^+(\mathcal{A}_2^\perp) \leq \mu_{\rm first}$. But $\mathcal{A}^\perp = \mathcal{B}$ and $\mathcal{B}^\perp = \mathcal{A}$, so we have shown $\kappa^+(\mathcal{A}), \kappa^+(\mathcal{B}) \leq \mu_{\rm first}$. So by Claim 2.6(1) and (2), $G_{\mathcal{A}}, G_{\mathcal{B}}$ are reflexive and by Claim 2.6(4) the cardinality inequalities hold. □

Claim 2.8.

(1) *If $\lambda > \aleph_0$ has uncountable cofinality, then there are $\mathcal{A}_1, \mathcal{A}_2 \subseteq [\lambda]^{\aleph_0}$ such that $\mathcal{A}_1 \subseteq \mathcal{A}_2^\perp$ and $\kappa^+(\mathcal{A}_1^\perp) + \kappa^+(\mathcal{A}_2^\perp) \leq \lambda$.*

(2) *Assume $\lambda > \mu > \aleph_0$, cf. $(\lambda) > \aleph_0$ and $S_1 \subseteq S^\lambda_{\aleph_0} = \{\delta < \lambda : \mathrm{cf}(\delta) = \aleph_0\}$, $S_2 = S^\lambda_{\aleph_0} \setminus S_1$ are such that for every ordinal $\delta < \lambda$ of cofinality $\geq \mu$, the set $\delta \cap S_\ell$ is stationary in δ for $\ell = 1, 2$. Then for some $\mathcal{A}_1, \mathcal{A}_2 \subseteq [\lambda]^{\aleph_0}$, we have*

$$\kappa^+(\mathcal{A}_\ell^\perp) \leq \mu \text{ for } \ell = 1,2 \quad \text{and} \quad \mathcal{A}_1 \subseteq \mathcal{A}_2^\perp.$$

(3) *If $\mathbf{V} = \mathbf{L}$ (or, e.g. just $\neg\exists 0^\#$), then for every $\lambda > \aleph_0 = \mu$, the assumption of (2) holds.*

Proof. (1) For each $\delta \in S^\lambda_{\aleph_0} = \{\delta < \lambda : \delta \text{ limit of cofinality } \aleph_0\}$, let

$$\mathcal{P}_\delta = \{u \subseteq \delta : \mathrm{otp}(u) = \omega \text{ and } \sup(u) = \delta\}.$$

Let $S_1, S_2 \subseteq S^\lambda_{\aleph_0}$ be stationary disjoint subsets of λ and $\mathcal{A}_\ell = \bigcup\{\mathcal{P}_\delta : \delta \in S_\ell\}$ for $\ell = 1, 2$. Now check.

(2) The same proof as the proof of part (1).

(3) Well known. □

Remark 2.9. Also it is well known that we can force an example as in 2.8(2) for $\lambda = \mu_{\rm first}$, $\mu = \aleph_1$.

Without loss of generality, let $\mathbf{V} \models \mathrm{GCH}$ and $\theta = \mathrm{cf}(\theta) \leq \mu_{\mathrm{first}}$ for $\theta > \aleph_0$. Let $\langle (\mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha) : \alpha \in \mathrm{Ord}\rangle$ be a full support iteration, where $\underset{\sim}{\mathbb{Q}}_\alpha$ is defined as follows: $\underset{\sim}{\mathbb{Q}}_\alpha = \{f : \text{for some } \gamma < \aleph_\alpha,\ f \in {}^\gamma\{1,2\}\}$, provided that for no increasing continuous sequence $\langle \alpha_\varepsilon : \varepsilon < \theta\rangle$ of ordinals $< \gamma$ and $\ell \in \{1,2\}$ do we have $\varepsilon < \theta \Rightarrow f(\alpha_\varepsilon) = \ell$ when $\aleph_\alpha$ is regular and uncountable; otherwise $\underset{\sim}{\mathbb{Q}}_\alpha = \{\emptyset\}$ is trivial.

Claim 2.10. *Assume $\mathbf{V} = \mathbf{L}$ or much less: for every singular μ above $2^{\aleph_0}$ with countable cofinality, we have $\mu^{\aleph_0} = \mu^+$ and $\square_\mu$. Then for every λ there is a pair $(\mathcal{A}_1, \mathcal{A}_2)$ as in $\circledast_{\lambda,\aleph_1}$ from 2.7.*

Proof. See Goldstern–Judah–Shelah [3]. □

Remark 2.11.

(1) The assumption of Claim 2.10 holds in models with many measurable cardinals.

(2) Note that if $\mu_1 \leq \mu_2$, then clearly $\circledast_{\lambda,\mu_1} \Rightarrow \circledast_{\lambda,\mu_2}$.

3. Arbitrarily large reflexive groups

In this section we shall show that it is "hard" to fail the assumptions needed in the previous section in order to prove that there are reflexive groups of arbitrarily large cardinality. A typical result is Conclusion 3.1. Its proof uses parameters $\mathbf{x}$ (see Definition 3.7). It is close to an application in [7] to the Cantor discontinuum partition problem, but as the needed Lemma 3.8 is only close to [7], we give a complete proof in the appendix (the next section).

A characteristic conclusion is

Conclusion 3.1. There is a reflexive subgroup G of ${}^\lambda\mathbb{Z}$ if $(*)_\mu$ below holds; moreover, G, G^* have cardinality $\in [\lambda, \lambda^\mu]$ in this case:

$(*)_\mu$: κ is strong limit singular $< \mu_{\mathrm{first}}$ of cofinality $\aleph_0$ with $\kappa < \kappa^* < 2^\kappa$, and for no $\chi \geq 2^\kappa$ is there a subfamily $\mathcal{A} \subseteq [\chi]^{\kappa^*}$ of cardinality $> \chi$ such that the intersection of any two members is of cardinality $< \kappa$.

Remark 3.2. Alternatively, assume $\kappa = \aleph_0 < \kappa^* < 2^{\aleph_0}$, $\mathfrak{a} = 2^{\aleph_0}$.

Definition 3.3.

(1) We say that the triple (κ, κ^*, μ) is *admissible* when $\mu = \mu^\kappa$ (here usually $\mu = 2^\kappa$), $\kappa \leq \kappa^* < \mu$, and the triple is λ-admissible for every $\lambda \geq \mu$, see below.

(2) The triple (κ, κ^*, μ) is λ-*admissible* when there is θ witnessing it, which means:

  (a) $\mu = \mu^\kappa, \kappa \leq \kappa^* < \mu \leq \lambda$;

  (b) $\kappa^* \leq \theta < \mu$;

  (c) there is no family of more than λ members of $[\lambda]^{\geq\theta}$ such that the intersection of any two has cardinality (strictly) less than κ^*.

(3) The triple (κ, κ^*, μ) is *weakly λ-admissible* when:

   (a) as above, i.e., $\mu = \mu^\kappa, \kappa \le \kappa^* < \mu \le \lambda$;

   (b) there is no family of more than λ members of $[\lambda]^\mu$ with any two of intersection of cardinality (strictly) less than κ^*.

Remark 3.4.

(1) We may allow (κ, κ^*) to be ordinals.
(2) In the proof of [7, 3.8], "θ witnesses (κ, κ^*, μ) is λ-admissible" was written $\otimes^\theta_\lambda$.

For the next claim, recall that $\mathrm{pp}_J(\theta) = \sup(\cup\{\mathrm{pcf}_J(\bar\theta) : \bar\theta = \langle \theta_\varepsilon : \varepsilon \in S\rangle\})$, where $\theta_\varepsilon = \mathrm{cf}(\theta_\varepsilon) \in (|S|, \theta)$, $\theta = \lim_J \langle \theta_\varepsilon : \varepsilon \in S\rangle$, and $S =$ Dom(J).

Claim 3.5. *The triple (κ, κ^*, μ) is admissible when at least one of the following occurs:*

$(*)_1$: (a) $\mu = 2^{\aleph_0} \ge \aleph_\delta > \kappa^* \ge \kappa = \aleph_0, \delta$ *a limit ordinal;*
(b) *for every* $\lambda > \mu = 2^{\aleph_0}$ *we have:*

$$\delta > \sup\{\alpha < \delta : \textit{for some } \theta \in (\mu, \lambda), \mathrm{cf}(\theta) = \aleph_\alpha \textit{ and } \mathrm{pp}_J(\theta) > \lambda \textit{ for some } \aleph_\alpha\textit{-complete ideal } J \textit{ on } \aleph_\alpha\}.$$

$(*)_2$: $\kappa > \mathrm{cf}(\kappa) = \aleph_0$ *is strong limit,* δ *a limit ordinal and we have:*
(a) $\mu = \mu^\kappa \ge \kappa^{+\delta} > \kappa^* \ge \kappa$*;*
(b) *for every* $\lambda > \mu$ *we have:*

$$\delta > \sup\{\alpha < \delta : \textit{ for some } \theta \in (\mu, \lambda), \mathrm{cf}(\theta) = (\kappa^*)^{+\alpha} \textit{ and } \mathrm{pp}_J(\theta) > \lambda \textit{ for some } (\kappa^*)^{+\alpha}\textit{-complete ideal on } (\kappa^*)^{+\alpha}\}.$$

Remark 3.6. In Claim 3.5, clause (b) of $(*)_2$ we can ask less because in clause (c) of Definitrion 3.3(2) the intersection has cardinality $< \kappa^*$ not just $< \theta$. The proof of this should be clear.

Definition 3.7.

(1) The quintuple $\mathbf{x} = (X, c\ell, \kappa, \kappa^*, \mu)$ is a *parameter* when:
   (a) $c\ell : \mathcal{P}(X) \to \mathcal{P}(X)$;
   (b) $\kappa \le \kappa^* \le \mu = \mu^\kappa$.
(2) The quintuple $\mathbf{x}$ is an *admissible parameter* when in addition:
   (c) the triple (κ, κ^*, μ) is an admissible triple (see Definition 3.3(1)).
(3) We define

$$\mathcal{P}^*_{\mathbf{x}} := \{A \subseteq X : |A| = \mu \text{ and for every } B \subseteq A \text{ satisfying } |B| = \kappa^* \text{ there is } B' \subseteq B, |B'| = \kappa \text{ such that } c\ell(B') \subseteq A, \text{ and } |c\ell(B')| = \mu\}, \text{ and}$$

$$\mathcal{Q}^*_{\mathbf{x}} := \{B : B \subseteq X, |B| = \kappa \text{ and } |c\ell(B)| = \mu\}.$$

For $A \in \mathcal{P}^*_{\mathbf{x}}$ we define $\mathcal{Q}^*_{\mathbf{x},A} = \{B \in \mathcal{Q}^*_{\mathbf{x}} : c\ell(B) \subseteq A\}$.

(4) We say $\mathbf{x}$ is a *strongly solvable parameter* when:
  (a) as in part (1);
  (b) as in part (1);
  (c) if $\bar{h} = \langle h^1_B, h^2_B : B \in \mathcal{Q}^*_{\mathbf{x}}\rangle$ and for all $B \in \mathcal{Q}^*_{\mathbf{x}}$ we have $h^\ell_B : c\ell(B) \to \mu$ for $\ell = 1, 2$ and $(\forall \alpha < \mu)(\exists^\mu \beta \in c\ell(B))(h^2_B(\beta) = \alpha)$, then there is a function $h : X \to \mu$ such that:
    $\odot$: if $A \in \mathcal{P}^*_{\mathbf{x}}$, so $|A| = \mu$, then for some $B \in \mathcal{Q}^*_{\mathbf{x},A}$ for every $\beta < \mu$, the set $\{x \in c\ell(B) : h^2_B(x) = \beta, h(x) = h^1_B(x)\}$ has cardinality μ.

(5) We say $\mathbf{x}$ is *solvable* if above in (4)(c) we restrict to the case $h^2_B = h^1_B$.

Lemma 3.8. *If $\mathbf{x} = (X, c\ell, \kappa, \kappa^*, \mu)$ is an admissible parameter, then $\mathbf{x}$ is strongly solvable.*

Proof. The proof is similar to [7, 3.8(2)], see a full proof in the next section. □

We need the following for stating the main result:

Definition 3.9.

(1) We say $\mathcal{A} \subseteq \mathcal{P}(\lambda)$ is (σ, κ^*, μ)-*full* in λ when $\mathcal{A} \subseteq [\lambda]^\sigma$ and for every $A \in [\lambda]^{\kappa^*}$ we have: $|A \cap B| \geq \sigma$ for at least μ members B of $\mathcal{A}$, or $\sigma = \kappa^*$ and $\{B \in \mathcal{A} : |B \cap A| \geq \sigma\}$ has cardinality $< \kappa^*$.

(2) We say $\mathcal{A} \subseteq [\lambda]^\sigma$ is (σ, θ)-MAD or θ-MAD in λ when $|\mathcal{A}| \geq \sigma$ and $B_1 \neq B_2 \in \mathcal{A} \Rightarrow |B_1 \cap B_2| < \theta$ and $B \in [\lambda]^\sigma \Rightarrow (\exists A \in \mathcal{A})(|A \cap B| \geq \theta)$.

(2A) If $\theta = \sigma$, we may omit θ, writing "MAD". We may omit "in λ" and we may replace "in λ" by "in A_*".

(3) For $\theta \leq \sigma \leq \chi$ let $\mathfrak{a}_{\chi,\sigma,\theta} = \mathrm{Min}\{|\mathcal{A}| : \mathcal{A} \subseteq [\chi]^\sigma$ is θ-MAD$\}$ and let $\mathfrak{a}_{\chi,\sigma} = \mathfrak{a}_{\chi,\sigma,\sigma}$.

Claim 3.10.

(1) *Assume $\mathcal{A} \subseteq [\lambda]^\sigma$ is* MAD, *i.e., $|\mathcal{A}| \geq \sigma, A \neq B \in \mathcal{A} \Rightarrow |A \cap B| < \sigma$ and there is no $A \in [\lambda]^\sigma$ such that $B \in \mathcal{A} \Rightarrow |B \cap A| < \sigma$. Then the family $\mathcal{A}$ is (σ, κ^*, μ)-full (in λ) when*

$$\boxplus_{\sigma,\kappa^*,\mu} : \quad \sigma \leq \kappa^* < \mu \text{ and } \mathfrak{a}_{\kappa^*,\sigma} \geq \mu.$$

(2) *The statement $\boxplus_{\sigma,\kappa^*,\mu}$ holds when at least one of the following occurs:*
  $(*)_1$: *$\sigma = \aleph_0 \leq \kappa^* < \mu = 2^{\aleph_0}$ and $\mathfrak{a} = 2^{\aleph_0}$ (or just $\mathfrak{a}_{\kappa^*,\aleph_0} = 2^{\aleph_0}$);*
  $(*)_2$: *σ is regular and for some strong limit singular cardinal $\chi > \sigma$ of cofinality σ we have $\chi \leq \kappa^* < \mu = 2^\chi$.*

Proof. (1) Let $A \in [\lambda]^{\kappa^*}$, so if $\kappa^* > \kappa$, then by "$\mathcal{A}$ is MAD" necessarily $(\exists^{\geq \kappa^*} B \in \mathcal{A})(B \cap A$ has cardinality $\geq \sigma)$; hence, $(\exists^{\geq \kappa} B \in \mathcal{A})(B \cap A$ has cardinality $\geq \sigma)$. Now $\mathcal{A}' := \{u \cap A : u \in \mathcal{A}$ and $u \cap A$ has cardinality $\geq \sigma\}$ is a MAD family of subsets of A; hence, $|\mathcal{A}'| \geq \mathfrak{a}_{\kappa^*,\sigma} \geq \mu$, as required. Note that $\mathfrak{a}_{\kappa^*,\sigma} \geq \mathfrak{a}_{\sigma,\sigma}$.

(2) Case 1: $(*)_1$ holds. Obvious.

Case 2: We have $(*)_2$ so $\sigma, \chi, \kappa^*, \mu$ are as there. Verifying $\boxplus_{\sigma,\kappa^*,\mu}$, the first demand "$\sigma \le \kappa^* < \mu$" is obvious - just check $(*)_2$, but we have to prove $\mathfrak{a}_{\kappa^*,\sigma} \ge \mu$; see Definition 3.9(3). So assume $\mathcal{A} \subseteq [\kappa^*]^\sigma$ is σ-MAD in κ^*; we need to prove that $|\mathcal{A}| \ge \mu$.

Let $A \in [\kappa^*]^\chi$ and $\mathcal{A}' := \{u \cap A : u \in \mathcal{A}$; then $|u \cap A| = \sigma\}$ has cardinality $\ge \kappa^*$ and clearly $\mathcal{A}$ is a MAD subfamily of $[A]^\sigma$. But:

$\odot_1$: there is a MAD family $\mathcal{A}_0 \subseteq [A]^\sigma$ of cardinality $\chi^\sigma = 2^\chi$;

$\odot_2$: if $u \in \mathcal{A}'$ and even $u \in [A]^\sigma$, then $|\{v \in \mathcal{A}_0 : |v \cap u| \ge \sigma\}| \le 2^\kappa$.

Hence, necessarily $|\mathcal{A}'| = 2^\chi = \mu$, and so $\{B \in \mathcal{A} : A \cap B$ of cardinality $\sigma\}$ has cardinality μ, as required. □

We shall use the following definition for $\sigma = \aleph_0$ in the proof of the main result in this section:

Definition 3.11. Assume λ is an infinite cardinal, $\mathcal{A} \subseteq [\lambda]^\sigma$ a MAD family, $|\mathcal{A}| = \lambda^\sigma$, and $\bar{u}^* = \langle u^*_\alpha : \alpha < \lambda^\sigma \rangle$ enumerates $\mathcal{A}$ with no repetitions. For every $A \subseteq \lambda^\sigma$ we define $\mathrm{set}(A) = \mathrm{set}(A, \bar{u}^*)$ as

$$\bigcup \Big\{ u^*_\alpha : \text{ the set } u^*_\alpha \cap (\bigcup \{u^*_\beta : \beta \in A\}) \text{ is an infinite set} \Big\} \cup (\lambda \cap A).$$

Claim 3.12.

(1) *There is a reflexive group $G \subseteq {}^\lambda\mathbb{Z}$ of cardinality $\in [\lambda, \lambda^\mu]$ when:*
 (a) *(κ, κ^*, μ) is an admissible triple, $\mu < \mu_{\mathrm{first}}$;*
 (b) *at least one of the following holds:*
  (α) *$\mathfrak{a} = 2^{\aleph_0} = \mu$ and $\kappa = \aleph_0$;*
  (β) *κ is strong limit singular of cofinality $\aleph_0$ and $\mu = 2^\kappa$;*
  (γ) *there is a MAD family $\mathcal{A} \subseteq [\mu]^{\aleph_0}$ which is $(\aleph_0, \kappa^*, \mu)$-full, i.e., such that: if $A \in [\mu]^{\kappa^*}$, then*

$$|\{u \in \mathcal{A} : u \cap B \text{ is infinite }\}| = \mu.$$

(2) *Given μ, for every $\lambda \ge \mu$ there are $\mathcal{A}_1, \mathcal{A}_2$ as in $\circledast_{\lambda,\mu}$ of 2.7 provided that there are an admissible triple, (κ, κ^*, μ) and a $(\aleph_0, \kappa^*, \mu)$-full MAD family $\mathcal{A} \subseteq [\lambda]^{\aleph_0}$.*

Remark 3.13. (1) Concerning Claim 3.12(2), if $\kappa < \mu_{\mathrm{first}}$, then trivially Claim 2.7 applies.

(2) Actually [8, §3] deals essentially with equiconsistency results for such properties.

Proof of Claim 3.12. (1) First, there is a MAD family $\mathcal{A} \subseteq [\lambda]^{\aleph_0}$. It is (κ, κ^*, μ)-full. [Why? If assumption $(b)(\alpha)$, then by Claim 3.10 using $(*)_1$ in part (2) there; if $(b)(\beta)$, then by Claim 3.10 using $(*)_2$ of part (2) there, with κ here standing for χ there; of course also (b)(γ) implies this.] Second, the result follows from part (2) and Claim 2.7.

(2) Without loss of generality $\lambda > \mu$, as otherwise the conclusion is trivial. We use the Lemma 3.8.

To apply it we shall choose $X, c\ell$ and let $\mathbf{x} = (X, c\ell, \kappa, \kappa^*, \mu)$ to show that the demands there hold. Let $\mathcal{A} \subseteq [\lambda]^{\aleph_0}$ be a MAD family of cardinality $\lambda^{\aleph_0}$ which is (κ, κ^*, μ)-full and, without loss of generality, $\mathcal{A} \cap \lambda = \emptyset$, i.e., no $u \in \mathcal{A}$ is a countable ordinal; let $\bar{u}^* = \langle u^*_\alpha : \alpha < \lambda^{\aleph_0} \rangle$ list $\mathcal{A}$ with no repetitions.

Recall that by the claim's assumption, $\mu = 2^\kappa$ and let $X = \lambda \cup \mathcal{A}$, i.e., if α is an ordinal of cardinality $\aleph_0$, then $\alpha \notin \mathcal{A}$. We define a function $c\ell \colon \mathcal{P}(\lambda \cup \mathcal{A}) \to \mathcal{P}(\lambda \cup \mathcal{A})$ by:

$$c\ell(A) := A \cup \{B \in \mathcal{A} : B \cap \operatorname{set}(A, \bar{u}^*) \text{ is infinite}\},$$

where $\operatorname{set}(A, \bar{u}^*)$ is defined in Definition 3.11 with $\aleph_0$ here standing for σ there.

We shall prove

$\otimes$: the quintuple $\mathbf{x} = (X, c\ell, \kappa, \kappa^*, \mu)$ is an admissible parameter.

We should check the demands of Definition 3.7(1)(2).

Clause (a): that $c\ell : \mathcal{P}(X) \to \mathcal{P}(X)$ is trivial by our choices of $X, c\ell$

Clause (b): $\kappa \leq \kappa^* \leq \mu = \mu^\kappa$, which is trivial.

Clause (c): that (κ, κ^*, μ) is an admissible triple holds by our assumption (a) of Claim 3.12.

So we can apply Lemma 3.8; hence, $\mathbf{x}$ is strongly solvable, see Definition 3.7(3). To apply it, we should choose $\bar{h} = \langle h^1_u, h^2_u : u \in \mathcal{Q}^*_{\mathbf{x}} \rangle$.

Given $u \in \mathcal{Q}^*_{\mathbf{x}}$, hence $u \in [X]^\kappa$, we let $h^2_u : c\ell(u) \to \mu$ be such that

$$(\forall \alpha < \mu)(\exists^\mu \beta \in c\ell(u))[h^2_u(\beta) = \alpha].$$

Let $h^1_u(x)$ be $h^2_u(x)$.

So by clause (c) of Definition 3.7(4), there is a function $h : X \to \mu$ satisfying $\odot$ from Definition 3.7(4). We define $\mathcal{A}_\ell := \{A \in \mathcal{A} : h(A) = \ell\} \subseteq \mathcal{A}$ for $\ell = 1, 2$ and it suffices to check that $(\mathcal{A}_1, \mathcal{A}_2)$ are as required in $\circledast_{\lambda,\mu}$ of Claim 2.7.

First, as $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{A}$ and $\mathcal{A}$ is $\subseteq [\lambda]^{\aleph_0}$, clearly $\mathcal{A}_1, \mathcal{A}_2 \subseteq [\lambda]^{\aleph_0}$.

Second, clause (a) there says "$u_1 \in \mathcal{A}_1 \wedge u_2 \in \mathcal{A}_2 \Rightarrow u_1 \cap u_2$ is finite" and this holds as $\mathcal{A}_1, \mathcal{A}_2$ are disjoint subsets of $\mathcal{A}$ which is a MAD subset of $[\lambda]^{\aleph_0}$.

Third and lastly, clause (b) from $\circledast_{\lambda,\mu}$ of Definition 2.7 says that $\kappa^+(\mathcal{A}^\perp_\ell) \leq \mu$. So, towards a contradiction, assume $A \subseteq \lambda, |A| = \mu$ and $A \in \mathcal{A}^\perp_\ell$, i.e., "$u \in \mathcal{A}_\ell \Rightarrow A \cap u$ is finite". Let $\mathcal{A}' := \{u^*_\alpha \in \mathcal{A} : A \cap u^*_\alpha \text{ is infinite}\}$.

Now if $A \in \mathcal{P}^*_{\mathbf{x}}$, then by the definition of "$A \in \mathcal{P}^*_{\mathbf{x}}$" in Definition 3.7(3), there is $B \subseteq A$ which belongs to $\mathcal{Q}^*_{\mathbf{x},A}$; hence, $|B| = \kappa$ and recall that $\kappa < \mu$. Clearly $c\ell(B) \backslash \lambda \subseteq \mathcal{A}' \subseteq \mathcal{A}$ and by the choice of h for some such B, there is $u^*_{\alpha_\ell} \in c\ell(B)$ satisfying $h(u^*_{\alpha_\ell}) = \ell$. Also as $u^*_{\alpha_\ell} \in c\ell(B) \cap \mathcal{A} \subseteq c\ell(A) \cap \mathcal{A}$; clearly $h(u^*_{\alpha_\ell}) = \ell$ and $u^*_{\alpha_\ell} \in \mathcal{A}_\ell$, a contradiction to "$A \in \mathcal{A}^\perp_\ell$". So we are left with proving $A \in \mathcal{P}^*_{\mathbf{x}}$. This follows from the choice of $\mathcal{A} \subseteq [\lambda]^{\aleph_0}$ as MAD (κ, κ^*, μ)-full. $\square$

Note that it is very hard to violate $(\forall\lambda)(\circledast_{\lambda,\mu_{\text{first}}})$.

Claim 3.14.

(1) *If χ is strong limit (uncountable), $\mathbb{P}$ is a (set) forcing and,*

$$\Vdash_{\mathbb{P}} \text{“}2^{\aleph_0} > \chi \textit{ and } \chi \textit{ is still a limit cardinal”},$$

where $\mathbb{P}$ has cardinality $\le \chi$ or at least satisfies the χ^+-c.c., then in $\mathbf{V}^{\mathbb{P}}$ the triple $(\aleph_0, \chi, 2^{\aleph_0})$ is admissible.

(1A) *If $\mathbb{P}$ is a (set) forcing, $\kappa < \chi$ are strong limit cardinals, $\Vdash_{\mathbb{P}}$ “χ is a limit cardinal and κ is a strong limit cardinal of cofinality $\aleph_0$, $\chi < \kappa^{\aleph_0}$”, and $\mathbb{P}$ satisfies the χ-c.c., then the tuple $(\kappa, \chi, \kappa^{\aleph_0})$ is admissible.*

(2) *If $\circledast_{\lambda,\mu}$ of Claim 2.7 holds, $\mu = \mu_{\text{first}}$ or just μ is regular and $\mathbb{P}$ is a forcing notion of cardinality $< \mu$, then we have $\circledast_{\lambda,\mu}$ in $\mathbf{V}^{\mathbb{P}}$ also.*

Proof. (1) Without loss of generality, there is δ, a limit ordinal such that $\Vdash_{\mathbb{P}}$ “$\mu = \aleph_\delta$”, and the first demand of Definition 3.3(1) and clause (a) of Definition 3.3(2) hold.

By [7], or see [9], in $\mathbf{V}$ we have $\odot_1$: for every $\lambda > \chi$ for some $\theta = \theta_\lambda < \mu$ we have $\text{cov}(\lambda, < \chi, < \chi, \theta_\lambda) = \lambda$. This continues to hold in $\mathbf{V}^{\mathbb{P}}$ if we use $\theta^1_\lambda = \theta_\lambda + (\text{cf}(\chi))^+$ when χ is singular and $\theta'_\lambda = \theta_\lambda$ when χ is regular. This is more than required in clauses (b), (c) of Definition 3.3.

(1A) and (2) are easy. □

Remark 3.15. (1) The holding of “θ witnesses (κ, κ^*, μ) is λ-admissible” is characterized in [6, §6].

(2) On earlier results concerning such problems and earlier history see Hajnal–Juhasz–Shelah [4].

4. Appendix: the proof of Lemma 3.8

Proof of Lemma 3.8. We are assuming $\mathbf{x} = (X, c\ell, \kappa, \kappa^*, \mu)$ is an admissible parameter and we shall prove that it is strongly solvable. In Definition 3.7(4), clauses (a), (b) hold trivially so it suffices to prove clause (c). So let $\bar{h} = \langle h^1_B, h^2_B : B \in \mathcal{Q}^*_{\mathbf{x}} \rangle$ as there be given.

We prove by induction on $\lambda \in [\mu, |X|]$ that:

$(*)_\lambda$ if Z, Y are disjoint subsets of X such that $|Y| \le \lambda$, <u>then</u> there are h, Y^+ such that

(a) $Y \subseteq Y^+ \subseteq X \backslash Z$

(b) $|Y^+| \le \lambda$

(c) h is a function from Y^+ to μ

(d) if $A \in \mathcal{P}^*_{\mathbf{x}}$, $\kappa^* \le \theta < \mu$, the cardinal θ is a witness to (κ, κ^*, μ) being λ-admissible, $|A \cap Y^+| \ge \theta, |A \cap Z| < \mu$ and $\beta < \mu$ <u>then</u> $|\{x : h^2_B(x) = \beta \text{ and } h(x) = h^1_B(x)\}| = \mu$ for some $B \in \mathcal{Q}^*_A$.

Case A: $\lambda = \mu$, so $|Y| \le \mu$.
As $|Y| \le \mu = \mu^\kappa$, there is a set Y^+ of cardinality $\le \mu$ such that $Y \subseteq Y^+ \subseteq X \backslash Z$ and

$\odot_1$: if $B \subseteq Y^+$ and $|B| \le \kappa$ and $|c\ell(B)| = \mu$, then $c\ell(B) \backslash Z \subseteq Y^+$.

Let $\mathcal{P} =$

$$\{B \subseteq Y^+ : |B| \le \kappa \text{ and} (h^2_B)^{-1}(\{\beta\}) \backslash Z \text{ has cardinality } \mu \text{ for every } \beta < \mu\}.$$

Clearly $|\mathcal{P}| \le |\{B : B \subseteq Y^+ \text{ and } |B| \le \kappa\}| \le |Y^+|^\kappa \le \mu^\kappa = \mu$, and for every $B \in \mathcal{P}$ and $\beta < \mu$ the set $(h^2_B)^{-1}(\{\beta\}) \backslash Z$ is included in Y^+ and has cardinality μ. So $\langle (h^2_B)^{-1}(\{\beta\}) \backslash Z : B \in \mathcal{P} \text{ and } \beta < \mu \rangle$ is a sequence of μ subsets of Y^+ each of cardinality μ. Hence there is a sequence $\langle C_{B,\beta} : B \in \mathcal{P}, \beta < \mu \rangle$ of pairwise disjoint sets such that $C_{B,\beta} \subseteq (h^2_B)^{-1}(\{\beta\})$ and $|C_{B,\beta}| = \mu$.

Define a function h from Y^+ to μ such that $h \restriction C_{B,\beta} \subseteq h^1_B$ for $B \in \mathcal{P}, \beta < \mu$ and $h \restriction \big(Y^+ \backslash \bigcup \{C_{B,\beta} : B \in \mathcal{P} \text{ and } \beta < \mu\}\big)$ is constantly zero.

Clearly clauses (a), (b), (c) of $(*)_\lambda$ hold. For clause (d), assume $A \in \mathcal{P}^*_{\mathbf{x}}$, and $|A \cap Z| < \mu, \theta \in [\kappa^*, \mu)$ witness that the tuple (κ, κ^*, μ) is μ-admissible, and $|A \cap Y^+| \ge \theta$. Then by Definition 3.7(3), there is a set $B \in \mathcal{Q}^*_{\mathbf{x},A}$, so $B \subseteq A, |B| \le \kappa$ and $|c\ell(B)| = \mu$. Clearly $B \in \mathcal{P}$ and so clause (d) holds by the choice of h. So the function h is as required.

Case B: $\lambda > \mu$.
Let $\chi = \big(2^\lambda\big)^+$ and choose $\langle N_i : i \le \lambda \rangle$ an increasing continuous sequence of elementary submodels of $(\mathcal{H}(\chi), \in, <^*_\chi)$ such that $X, c\ell, Y, Z, \lambda, \kappa, \kappa^*, \mu$ belong to the set N_0, $\mu + 1$ is included in N_0, and the sequence $\langle N_i : i \le j \rangle$ belongs to N_{j+1} (when $j < \lambda$) and $\|N_i\| = \mu + |i|$.

Choose $\theta \in [\kappa^*, \mu)$ which witnesses that the triple (κ, κ^*, μ) is λ-admissible. We define by induction on $i < \lambda$, a set Y^+_i and a function h_i as follows:
$\circledast$: (Y^+_i, h_i) is the $<^*_\chi$–first pair (Y^*, h^*) such that:

(a) $Y^* \subseteq X \setminus (Z \cup \bigcup_{j<i} Y^+_j)$;

(b) $Y \cap N_i \backslash \bigcup_{j<i} Y^+_j \backslash Z \subseteq X \cap N_i \backslash \bigcup_{j<i} Y^+_j \backslash Z \subseteq Y^*$;

(c) $|Y^*| \le \mu + |i|$;

(d) $h^* : Y^* \to \mu$;

(e) $h^* \restriction ((h^2_B)^{-1}(\{\beta\}) \cap Y^*)$ coincides with h^1_B on a set of cardinality μ for some $B \in \mathcal{Q}^*_{\mathbf{x},A}$, and every $\beta < \mu$, when for some θ':

(α) $A \in \mathcal{P}^*_{\mathbf{x}}$,

(β) $\kappa^* \le \theta' < \mu$, moreover θ' is a witness for the triple (κ, κ^*, μ) being $(\mu + |i|)$-admissible,

(γ) $|A \cap Y^*| \ge \theta'$,

(δ) $|A \cap (Z \cup \bigcup_{j<i} Y^+_j)| < \mu$.

Note: (Y_i^+, h_i) exists by the induction hypothesis applied to the cardinal $\lambda' := \mu + |i|$ and the sets $Z' := Z \cup \bigcup_{j<i} Y_j^+$ and $Y' := X \cap N_i \backslash \bigcup_{j<i} Y_j^+$ so we can carry out the induction. Also it is easy to prove by induction on i that

$\oplus$: (a) $\langle (Y_j^+, h_j) : j \leq i \rangle \in N_{i+1}$

(b) $Y_j^+ \subseteq N_{j+1}$

[Why? First we show $\langle (Y_j^+, h_j) : j < i \rangle \in N_{i+1}$ as the induction can be carried inside N_{i+1}. Now $Y_i^+, h_i \in N_{i+1}$ as we always have chosen "the $<^*_\chi$-first", so clause (a) above holds. As for $Y_i^+ \subseteq N_{i+1}$, i.e., clause (b), note that $|Y_j^+| = \mu + |i|$ and $(\mu + 1) \subseteq N_{j+1}, j + 1 \subseteq N_{j+1}$ by the choice of $\langle N_i : i < \lambda \rangle$.]

Let $Y^+ = \bigcup_{i<\lambda} Y_i^+$ and $h = \bigcup_{i<\lambda} h_i$. Clearly $Y \subseteq N_\lambda = \bigcup_{i<\lambda} N_i$ as $Y \in N_0, i < \lambda \Rightarrow i \subseteq N_i \subseteq N$ and $|Y| = \lambda$, so $\lambda \in N_0$ and $\lambda \subseteq N_\lambda$; hence, by requirement (b) of $\circledast$, $Y \subseteq Y^+$ (and even $X \cap N_\lambda \backslash Z \subseteq Y^+$). By requirements (c) (and (a)) of $\circledast$, $|Y^+| \leq \lambda$; by requirement (a) of $\circledast$, $Y^+ \subseteq X \backslash Z$, and by rquirement (b) of $\circledast$, even $Y^+ = X \cap N_\lambda \backslash Z$.

By requirements (a) and (d) of $\circledast$, h is a function from Y^+ to μ. So in $(*)_\lambda$ for Y, Z, demands (a), (b), (c) on Y^+, h are satisfied; so it suffices to prove demand (d) there. Suppose $A \in \mathcal{P}^*_{\mathbf{x}}, \kappa^* \leq \theta < \mu$ and moreover, θ witnesses that the triple (κ, κ^*, μ) is λ-admissible, $|A \cap Y^+| \geq \theta$, $|A \cap Z| < \mu$, and $\beta < \mu$; we should prove "for every $\beta < \mu$, $h \restriction ((h^2_B)^{-1}(\{\beta\}) \cap Y^+)$ coincides with h^1_B on a set of cardinality μ for some $B \in \mathcal{Q}^*_{\mathbf{x},A}$". Then $|A \cap N_\lambda| \geq \theta$. Choose a pair (δ^*, θ^*) such that:

$\otimes$: (i) $\delta^* \leq \lambda$,

(ii) θ^* witnesses that (κ, κ^*, μ) is $(\mu + |\delta^*|)$-admissible, hence $\kappa^* \leq \theta^* < \mu$,

(iii) $|A \cap N_{\delta^*}| \geq \mu$ or $\delta^* = \lambda$,

(iv) under (i), (ii), and (iii), δ^* is minimal.

This pair is well defined as (λ, θ) satisfies requirements (i), (ii), and (iii).

Subcase B1: δ^* is zero.
So $|Y_0^+ \cap A| \geq \theta^* \geq \kappa^*$ hence by the choice of h_0, i.e. clause (e) of $\circledast$, recalling $A \in \mathcal{P}^*_{\mathbf{x}}$ we are done.

Subcase B2: $\delta^* = i + 1$.
So $\delta^* < \lambda$; clearly the pair (i, θ^*), standing for (δ^*, θ^*), satisfies clauses (i) and (ii) of $\otimes$, so it cannot satisfy clause (iii) there, as then (δ^*, θ^*) fails clause (iv). This means that $|A \cap N_i| < \mu$, but $\bigcup \{Y_j^+ : j < i\} \subseteq N_i$; hence, $|A \cap \bigcup_{j<i} Y_j^+| < \mu$, but also $|A \cap Z| < \mu$, and hence $|A \cap (Z \cup \bigcup_{j<i} Y_j^+)| < \mu$. Clearly θ^* witnesses that (κ, κ^*, μ) is $(\mu + |i|)$ admissible holds (as $\mu + |i| = \mu + |i+1| = \mu + |\delta^*|$), so if $|A \cap Y_i^+| \geq \theta^*$, we are done by the choice of h_i, i.e., by clause (e) of $\circledast$; if not, then $|A \cap (Z \cup \bigcup_{j<i+1} Y_j^+)| < \mu$ and so necessarily $A \cap Y_{i+1}^+ \supseteq A \cap N_{i+1} \setminus \cup \{Y_j^+ : j < i+1\} = A \cap N_{\delta^*} \setminus \cup \{Y_j^+ : j < i+1\}$

has cardinality $\geq \theta^*$ (and "θ^* witnesses that (κ, κ^*, μ) is $\mu + |i+1| = |Y^+_{i+1}|$–admissible" holds) so we are done by the choice of h_{i+1}.

Subcase B3: δ^* is a limit ordinal below λ.
For some $i < \delta^*$, $|A \cap N_i| \geq \theta^*$. [Why? As $\theta^* < \mu \leq |A \cap N_{\delta^*}|$]. Now in N_{i+1} there is a maximal family $\mathcal{Q} \subseteq [X \cap N_i]^{\theta^*}$ satisfying $[B_1 \neq B_2 \in \mathcal{Q} \Rightarrow |B_1 \cap B_2| < \kappa^*]$; hence, by clause (ii) of $\otimes$ and clause (c) of Definition 3.3(2), we have $|\mathcal{Q}| \leq \mu + |\delta^*|$. Choosing the $<^*_\chi$-first such $\mathcal{Q}$, clearly $\mathcal{Q} \in N_{i+1}$, so recalling $\mathcal{Q} \in N_{i+1} \subseteq N_{\delta^*}$, we have $\mathcal{Q} \subseteq N_{\delta^*}$. By the choice of $\mathcal{Q}$, necessarily there is $B \in \mathcal{Q}$ such that $|B \cap A| \geq \kappa^*$ (if $A \notin \mathcal{Q}$, by the maximality of $\mathcal{Q}$, and if $A \in \mathcal{Q}$, one can choose $B = A$). But as $B \in \mathcal{Q}$ clearly $B \in N_{\delta^*}$ and $|B| = \theta^* < \mu = \mu^\kappa$; hence, $[B' \in [B \cap A]^\kappa \Rightarrow B \cap A \in N_{\delta^*}]$. As $A \in \mathcal{P}^*_{\mathbf{x}}$ and $|B \cap A| \geq \kappa^*$, there is $B' \in [B \cap A]^\kappa$ satisfying $c\ell(B') \subseteq A$, $|c\ell(B')| = \mu$. Clearly $c\ell(B') \in N_{\delta^*}$; hence, $c\ell(B') \in N_j$ for some $j \in (i, \delta^*)$, and hence $c\ell(B') \subseteq X \cap N_j$. So $|A \cap N_j| \geq \mu$. By assumption, for some $\theta' \in [\kappa^*, \mu)$, the triple (κ, κ^*, μ) is $(\mu + |j|)$-admissible, see Definition 3.3, so the pair (j, θ') contradicts the choice of (δ^*, θ^*).

Subcase B4: $\delta^* = \lambda$.
As $\lambda \in N_0$, there is a maximal family $\mathcal{Q} \subseteq [\lambda]^{\theta^*}$ satisfying

$$[B_1 \neq B_2 \in \mathcal{Q} \quad \Rightarrow \quad |B_1 \cap B_2| < \kappa^*]$$

which belongs to N_0. By the assumption $\otimes(ii)$ on θ^* and clause (c) of Definition 3.3(2), we know that $|\mathcal{Q}| \leq \lambda$. But $\lambda + 1 \subseteq N_\lambda$; hence, $\mathcal{Q} \subseteq N_\lambda$, and hence $(\forall B \in \mathcal{Q})(\exists i < \lambda)(B \in N_i)$. We define by induction on $j \leq \lambda$, a one-to-one function g_j from $N_j \cap X \backslash Z$ onto an initial segment of λ increasing continuous with j; g_j is the $<^*_\chi$-first such function; clearly $g_j \in N_{j+1}$. Let $\mathcal{Q}' = \{g^{-1}_\lambda(B) : B \in \mathcal{Q}\}$, (i.e. $\{\{g^{-1}_\lambda(x) : x \in B\} : B \in \mathcal{Q}\}$). Clearly for any $B \in \mathcal{Q}$, there is $i < \lambda$ such that $B \in N_i \cap \mathcal{Q}$. Let $\mathbf{i}(B)$ be the first such i, so $B \subseteq \mathrm{Dom}(g^{-1}_{\mathbf{i}(B)})$, and so $g^{-1}_{\mathbf{i}(B)}(B) \in N_{i+1}$ and g_λ is necessarily a one-to-one function from $N_\lambda \cap X \backslash Z$ onto λ. Recall that $A \cap Y^+ = A \cap (X \cap N_\lambda) \backslash Z$ has cardinality $\geq \theta^*$. Hence, $|B \cap A| \geq \kappa^*$ for some $B \in \mathcal{Q}'$, so as in subcase B3, for some $B' \in N_\lambda$, we have $B' \subseteq B \cap A$, $|B'| = \kappa$, $c\ell(B') \subseteq A$, and $|c\ell(B')| = \mu$. Clearly $B \in N_{\mathbf{i}(B)+1}$; hence, $[B]^{\leq\kappa} \in N_{\mathbf{i}(B)+1}$. But its cardinality is $\leq \mu$; hence, $[B]^{\leq\kappa} \subseteq N_{\mathbf{i}(B)+1}$, so $B' \in N_{\mathbf{i}(B)+1}$, and so $c\ell(B') \subseteq N_{\mathbf{i}(B)+1}$. But $|A \cap Z| < \mu$, so by the last two sentences, $|A \cap Y^+_{\mathbf{i}(B)+1}| = \mu$ and by assumption $\otimes$(ii), some θ is a witness to (κ, κ^*, μ) being $(\mu + |i|)$-admissible (stipulating $i = \mathbf{i}(B) + 1$), contradicting the choice of (δ^*, θ^*) (i.e., minimality of δ^*).

This completes the proof of Lemma 3.8. □

Discussion 4.1. (1) If we would like to include the case $\mu = 2^{\aleph_0} = \aleph_2$, $\kappa = \aleph_0, \kappa^* = \aleph_1$, we should consider a revised framework. We have a family $\mathfrak{I}$ of ideals on cardinals θ from $[\kappa^*, \mu)$ which are κ-based (i.e. if $A \in I^+$, $I \in \mathfrak{I}$ (similar to [4]), then $(\exists B \in [A]^\kappa)(B \in I^+)$) and in Definition 3.7(3); hence in the proof of 3.8 replace $\mathcal{P}^*_{\mathbf{x}}$ by

$$\mathcal{P}^* = \mathcal{P}^*_{\mathfrak{I}} := \big\{A \subseteq X : \ |A| = \text{ and for every pairwise distinct } x_\alpha \in A, \text{ for } \alpha < \theta \text{ the set } \{u \subseteq \theta : |c\ell\{x_\alpha : \alpha \in u\}| < \mu\} \text{ is included in some } I \in \mathfrak{I}\big\}.$$

and in Definition 3.3(1), (2) we replace the triple (κ, κ^*, μ) by the quadruple $(\kappa, \kappa^*, \mu, \mathfrak{I})$ and clause (c) of Definition 3.3(2) by

$(c)'_\lambda$: $\lambda \geq \mu$, and $|\mathcal{F}| \leq \lambda$ whenever
$\mathcal{F} \subseteq \{(\theta, I, f) : I \in \mathfrak{I},\ \theta = \mathrm{Dom}(I),\ f : \theta \longrightarrow \lambda \text{ is one to one}\}$,
and if $(\theta_\ell, I_\ell, f_\ell) \in \mathcal{F}$ are distinct for $\ell = 1, 2$ then
$\{\alpha < \theta_2 : f_2(\alpha) \in \mathrm{Rang}(f_1)\} \in I_2$.

Note that the present $\mathcal{P}^*$ fits for repeating the proof of Lemma 3.8.

(2) Discussion of the Consistency of NO:

There are some restrictions on such theorems. Suppose

$(*)$: GCH and there is a stationary $S \subseteq \{\delta < \aleph_{\omega+1} : \mathrm{cf}(\delta) = \aleph_1\}$ and $\langle A_\delta : \delta \in S\rangle$ such that:

- $A_\delta \subseteq \delta = \sup A_\delta$,
- $\mathrm{otp}(A_\delta) = \omega_1$, and
- $\delta_1 \neq \delta_2 \quad \Rightarrow \quad |A_{\delta_1} \cap A_{\delta_2}| < \aleph_0$.

(This statement is consistent by [4, 4.6, p.384] which continues [5]; see more in [8].)

Now on $\aleph_{\omega+1}$ we define a closure operation:

$$\alpha \in c\ell(u) \quad \Leftrightarrow \quad (\exists \delta \in S)[\alpha \in A_\delta \text{ and } |u \cap A_\delta| \geq \aleph_0].$$

This certainly satisfies the demands in Definition 3.7 with $\kappa = \kappa^* = \aleph_0, \mu = \aleph_1$ except the pcf assumptions, i.e., clause (c) of Definition 3.3(2). However, this is not a case of our theorem.

(3) We may consider in the proof of Lemma 3.8 strengthening clause (e) of $\circledast$ by weakening clause $(e)(\delta)$ of $\circledast$ by fixing the ordinal β and demanding only $(A \backslash \bigcup_{j<i} Y_j^+ \backslash Z) \cap (h^2_\beta)^{-1}(\{\beta\})$ has cardinality μ. But we do not seem to gain anything.

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