APPENDIX

On a Question of Grinblat

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We prove the consistency of: there is a $\varkappa$-complete ideal on $\varkappa$ for some uncountable $\varkappa < 2^{\aleph_0}$ such that the Boolean algebra $\mathcal{P}(\varkappa)/I$ is σ-centered and there are (uncountable) Q-sets of reals.

In set theoretic language, Grinblat has been asking for some time

A.1. PROBLEM. Is it consistent with ZFC that:

(a) there is an $\aleph_1$-complete ideal I on some $\varkappa < 2^{\aleph_0}$ such that $\mathcal{P}(\varkappa)/I$ is σ-centered (see below, and also Section 6.1 of this monograph)?
(b) there is a Q-set?

We answer positively.

REMARK. Of course, $\alpha < \varkappa \Rightarrow \{\alpha\} \in I$.

A.2. CLAIM. *Assume that $\varkappa < \chi = \chi^{\varkappa}$ and $\varkappa$ is measurable.*
Then for some c.c.c. forcing notion $\mathbb{P}$ of cardinality χ we have in $\mathbf{V}^{\mathbb{P}}$:

(i) $2^{\aleph_0} = \chi$;
(ii) $\mathrm{MA}_{<\varkappa,<\mathrm{cf}(\chi)}$($\sigma$-centered) *holds* (*i.e.*, MA *for σ-centered forcing notions of cardinality $< \varkappa$ and $< \mathrm{cf}(\chi)$ dense open subsets*); *hence*
(ii)$^-$ *every uncountable set of reals of cardinality $< \varkappa$ is a Q-set*;
(iii) *assume D is a normal ultrafilter on $\varkappa$ and I is the dual ideal. The Boolean algebra $\mathcal{P}(\varkappa)/I$ is σ-centered, i.e., $\mathcal{P}(\varkappa)\backslash I^{\mathbf{V}^{\mathbb{P}}}$ is the union of countably many filters where*

$$I^{\mathbf{V}^{\mathbb{P}}} = \{A \in \mathbf{V}^P : A \subseteq \varkappa \text{ and } A \text{ is included in some member of } I\};$$

note that $I^{\mathbf{V}^{\mathbb{P}}}$ is a normal ideal on $\varkappa$.

A.3. REMARK. Why "hence (ii)$^-$" in (ii)?
Because of the following result.

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A.4. Claim. *For $X \subseteq Y \subseteq {}^{\omega}2$ the natural forcing $\mathbb{Q} = \mathbb{Q}_{X,Y}$ (defined below) satisfies*

(i) *it adds subtrees $\underset{\sim}{T_n} \subseteq {}^{\omega>}2$ for $n < \omega$ such that $\bigcup_{n<\omega} \lim(\underset{\sim}{T_n}) \cap Y = X$;*

(ii) *$\mathbb{Q}$ is σ-centered of cardinality $\leq |Y| + \aleph_0$;*

(iii) *if we can find a directed $G \subseteq \mathbb{Q}$ intersecting $|Y| + \aleph_0$ dense subsets $\mathcal{J}^{\ell}_{\eta,n}$ ($\eta \in Y \backslash X$), $\mathcal{J}_m$ ($m < \omega$), I_η (for $\eta \in X$) (defined below), then there is $\langle T_n : n < \omega\rangle$ as above.*

A.5. Definition. 1) For $X \subseteq Y \subseteq {}^{\omega}2$ let $\mathbb{Q} = \mathbb{Q}_{X,Y}$ be defined as follows:

(A)

$$\mathbb{Q} = \Big\{ p = (\bar{t}, \bar{u}) : \text{for some } n = n(p) < \omega \text{ we have:}$$

$$\begin{aligned} &\text{(a)} \quad \bar{t} = \langle t_\ell : \ell < n\rangle, \text{ each } t_\ell \text{ has the form } T \cap {}^{m_\ell(p)\geq}2, \\ &\qquad T \text{ a subtree}^3 \text{ of } {}^{\omega>}2 \text{ with } m_\ell(p) < \omega, \\ &\text{(b)} \quad \bar{u} = \langle u_\ell : \ell < n(p)\rangle, u_\ell \subseteq X \text{ is finite} \\ &\text{(c)} \quad \text{if } \ell < n(p),\ n \in u_\ell, \text{ then } \eta \restriction m_\ell(p) \in t_\ell \Big\}. \end{aligned}$$

(B) The order is natural: $p \leq q$ iff $\{n(p) \leq n(q)$ and $\bigwedge_{\ell<n(p)}[t^p_\ell \subseteq t^q_\ell$ and $m_\ell(p) \leq m_\ell(q)$ and $t^p_\ell = t^q_\ell \cap {}^{m_\ell(p)\geq}2]$ and $\bigwedge_{\ell<n(p)} u^p_\ell \subseteq u^q_\ell\}$.

2) For $\eta \in Y \backslash X$ and $m < \omega$ let $\mathcal{J}_m = \{p \in \mathbb{Q} : m < n(p)$ and $\ell < m \Rightarrow m \leq m_\ell(p)\}$ and $\mathcal{I}^\ell_\eta = \{p \in \mathbb{Q} : \ell < n(p)$ and $\eta \restriction m_\ell(p) \notin t^p_\ell\}$. For $\eta \in X$ let $\mathcal{I}_\eta$ be $\{p \in \mathbb{Q} : \eta \in u^p_\ell$ for some $\ell < n(p)\}$. Clearly, these sets are dense and open.

Proof of A.4. Let $\underset{\sim}{T_\ell}[G] = \bigcup\{t^p_\ell : p \in \underset{\sim}{G}\}$. So any directed $G \subseteq \mathbb{Q}$ meeting all the $\mathcal{I}^\ell_\eta$ (for $\eta \in Y \backslash X$), I_η (for $\eta \in X$), and $\mathcal{J}_m (m < \omega)$ gives $\underset{\sim}{T_\ell}[G]$ as required. $\square$

Proof of A.2. Let $\mathbb{P} = \mathbb{P}_\chi$, where $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}_j} : i \leq \chi, j < \chi\rangle$, be an FS iteration such that in $\mathbf{V}^{\mathbb{P}_i}, \underset{\sim}{\mathbb{Q}_j}$ is a σ-centered forcing notion of cardinality $< \varkappa$ and its set of elements is an ordinal $< \varkappa$ (not just a $\mathbb{P}_j$-name of such an ordinal), and any such forcing notion appears unboundedly often even χ times, more exactly, if $i_0 < \chi$, $\underset{\sim}{\mathbb{Q}}$ is a $\mathbb{P}_{i_0}$-name of a forcing notion with a set of elements (forced to be) $\subset \underset{\sim}{\alpha_{\mathbb{Q}}} < \varkappa$, then for χ many (hence unboundedly many) ordinals $j \in (i, \chi)$ we have: $\Vdash_{\mathbb{P}_j}$ "if $\underset{\sim}{\mathbb{Q}}$ is σ-centered, then $\underset{\sim}{\mathbb{Q}} \cong \underset{\sim}{\mathbb{Q}_j}$". Also as usual, $\mathbb{P}_i \subseteq \mathcal{H}_{<\aleph_1}(\varkappa + i)$, i.e., the members are sets which are hereditarily countable over the set of ordinals $< \varkappa + i$ (see [S4, Ch. III] or just use $\mathbb{P}_{1,i}$ as below).

As each $\underset{\sim}{\mathbb{Q}_j}$ is σ-center (in $\mathbf{V}^{\mathbb{P}_j}$) there is $\underset{\sim}{\bar{f}} = \langle \underset{\sim}{f_j} : j < \chi\rangle$ such that $\Vdash_{\mathbb{P}_j}$ "$\underset{\sim}{f_j}$ is a function from $\underset{\sim}{\mathbb{Q}_j}$ to ω such that each $\{p \in \underset{\sim}{\mathbb{Q}_j} : \underset{\sim}{f_j}(p) = n\}$ is directed". We can explicate further. Letting the universe of $\underset{\sim}{Q_i}$ be α_i for any $i < \chi$ and $\alpha, \beta < \alpha_i$ let $\langle g^i_{\alpha,\beta,n} : n < \omega\rangle$ be a maximal antichain of $\mathbb{P}_{1,i}$ such that $g^i_{\alpha,\beta} \Vdash_{\mathbb{P}_i}$ "$\alpha \leq_{\underset{\sim}{\mathbb{Q}_i}} \beta$

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[3] T is nonempty, $\nu \triangleleft \eta$, and $\eta \in T \Rightarrow \nu \in T$ and no $\eta \in T$ is $\triangleleft$-maximal.

iff $\mathbf{t}^i_{\alpha,\beta,n}$ = truth". So $\langle\langle(t^i_{\alpha,\beta,n}, g^i_{\alpha,\beta,n}) : n < \omega, \alpha, \beta < \alpha_i\rangle : i < \chi\rangle$ gives all the information on $\bar{\mathbb{Q}}$.

Therefore $\mathbb{P}_{1,i} =: \{p \in \mathbb{P}_i :$ if $j \in$ Dom(p), then $p \restriction j$ forces a value to $\underset{\sim}{f_j}(p(j))$ and a value to $p(j)\}$ is a dense subset of $\mathbb{P}_i$; recall that the set of members of $\underset{\sim}{\mathbb{Q}_j}$ is an ordinal $< \varkappa$. Now clearly clauses (i) and (ii) in Claim A.2 hold in $\mathbf{V}^{\mathbb{P}}$. As D is a normal ultrafilter on $\varkappa$ there is a transitive class M such that $M^\varkappa \subseteq M$ and there is an elementary embedding $\mathbf{j}$ from $\mathbf{V}$ to M with critical ordinal $\varkappa$ such that $D = \{A : A \in \mathbf{V}, A \subseteq \varkappa$ and $\varkappa \in \mathbf{j}(A)\}$. Let $\mathbf{j}(\bar{\mathbb{Q}})$ be $\bar{\mathbb{Q}}' = \langle \mathbb{P}'_i, \underset{\sim}{\mathbb{Q}'_j} : i \leq \mathbf{j}(\varkappa), j < \mathbf{j}(\chi)\rangle$, and let $\underset{\sim}{\bar{f}'} = \mathbf{j}(\underset{\sim}{\bar{f}'}) = \langle \underset{\sim}{f'_j} : j < \mathbf{j}(\varkappa)\rangle$; so M "thinks" that $(\bar{\mathbb{Q}}', \underset{\sim}{\bar{f}'})$ satisfies all the properties listed above, hence in $\mathbf{V}$ it satisfies all of those properties replacing $\varkappa, \chi$ by $\mathbf{j}(\varkappa), \mathbf{j}(\chi)$.

Let $\mathbb{P}^* = \{\mathbf{j}(p) : p \in \mathbb{P}'_\chi\}$; so it is well known that $\mathbb{P}^* \lessdot \mathbb{P}'_{\mathbf{j}(\chi)}, \mathbb{P}^*$ is isomorphic to $\mathbb{P}_\chi$, all those forcing notions satisfy the c.c.c., and (some complete subalgebra of) the completion of the Boolean algebra corresponding to $\mathbb{P}'_{\mathbf{j}(\chi)}/\mathbb{P}^*$ is isomorphic to $\mathcal{P}(\varkappa)/I^{\mathbf{V}^{\mathbb{P}}}$. Therefore it is enough to prove that $\Vdash_{P^*}$ "$\mathbb{P}'_{\mathbf{j}(\chi)}/\mathbb{P}^*$ is σ-centered" (in $\mathbf{V}^{\mathbb{P}^*}$, which is the same as $\mathbf{V}^{\mathbb{P}_\chi}$). Note also that $\mathbb{P}'_{\mathbf{j}(\chi)}$, hence $\mathbb{P}'_{\mathbf{j}(\chi)}/\mathbb{P}^*$, has cardinality $\leq |\mathbb{P}_{1,\chi}|^\varkappa = \chi^\varkappa = \chi$.

Now the point is that we shall prove below that

⊠ we can reorder the iteration $\bar{\mathbb{Q}}'$: first do $\langle \mathbb{Q}_{\mathbf{j}(j)} : j < \chi\rangle$ and then the rest, and in this reordering, each $\mathbb{Q}_j$ is still σ-centered and $\bar{\mathbb{Q}}'$ is FS iteration.

Note first that proving ⊠ suffices since

(a) the limit of FS iteration of σ-centered forcing notion each of cardinality $\leq 2^{\aleph_0}$ and of length $< (2^{\aleph_0})^+$ (in $\mathbf{V}^{\mathbb{P}_{1,\chi}}$!) is σ-centered;
(b) $\mathbb{P}'_{j(\varkappa)}/\mathbb{P}^*$ is in the universe $\mathbf{V}^{\mathbb{P}^*}$, the limit of such iteration.

They are easy.

Second, this reordering is possible. [Why? The set of elements of $\underset{\sim}{\mathbb{Q}_j}$ is $\alpha_{\underset{\sim}{\mathbb{Q}_j}}$, $\alpha_{\underset{\sim}{\mathbb{Q}_j}}$ is a $\mathbb{P}'_j$-name of an ordinal $< \varkappa$, and $\mathbb{P}'_j$ satisfies the c.c.c.; hence for some $\alpha^*_j < \varkappa$ we have $\Vdash_{P^*_{1,j}}$ "$\alpha^*_{\mathbb{Q}_j} \leq \alpha^*_j$" so the quasiorder $\underset{\sim}{\mathbb{Q}_j}$ is a subset of $\alpha^*_j \times \alpha^*_j$. For any $\beta, \gamma < \alpha^*$, there is a maximal antichain $\mathcal{I}_{j,\beta,\gamma}$ of $\mathbb{P}^*_{1,j}$ of conditions forcing "$\underset{\sim}{\mathbb{Q}_j} \models \beta \leq \gamma$, so $\beta \in \underset{\sim}{\mathbb{Q}_j}$, $\gamma \in \underset{\sim}{\mathbb{Q}_j}$" or forcing its negation. We choose $\bar{\mathcal{I}} = \langle \mathcal{I}_{j,\beta,\gamma} : j < \chi, \beta, \gamma < \alpha^*_j\rangle$. Let $A_j = \bigcup_{\beta,\gamma<\alpha^*_j} \bigcup_{p\in\mathcal{I}_{j,\beta,\gamma}}$ Dom(p), so $|A_j| < \varkappa$; all this for every $j < \chi$. Let $A \subseteq \chi$ be called $\bar{\mathbb{Q}}$-closed if $(\forall j \in A)(A_j \subseteq A)$ and clearly for a permutation π of χ, changing the order of the iteration to $\underset{\sim}{Q_{\pi(0)}}, Q_{\pi(1)}, \ldots$ is OK *provided that* $i \in A_j \Rightarrow \pi(i) \in \pi(j)$. Now in M we can compute $\mathbf{j}(\bar{\mathcal{I}})$, hence $\langle A^M_i : i < \mathbf{j}(\chi)\rangle$; now easily $A^M_{\mathbf{j}(j)} = \{\mathbf{j}(i) : i \in A_j\}$ as $|A_j| < \varkappa$. Now in $\mathbf{V}$ all the relevant properties of $\bar{\mathbb{Q}}'$ are preserved. In $\mathbf{V}$ let $A = \{\mathbf{j}(i) : i < \chi\} \subseteq j(\chi)$, which is $\bar{\mathbb{Q}}'$-closed by the form of $A^M_{\mathbf{j}(j)}$. Hence the reordering suggested above ($\pi(\mathbf{j}(i)) = i$ for $i \in A$ and $\pi(j) = \chi + \text{otp}\{\varepsilon : \varepsilon < i, \varepsilon \notin A\}$) is OK.]

So we are done. □

A.6. REMARKS. 1) If $\varkappa < \lambda = \mathrm{cf}(\lambda) \leq \varkappa$ we can even get $\mathrm{MA}_{<\varkappa,<\lambda}$ in A.2; just use iteration of length $\chi \times \lambda$.

2) If D is a normal ultrafilter on $[\theta]^{<\varkappa}$, to which $\mathcal{A} = \{A_\alpha : \alpha < \theta\}$ belongs, $A_\alpha \in [\alpha]^{<\varkappa}$, then in $\mathbf{V}^{\mathbb{P}}$, D generates a normal filter D' with $\mathcal{P}(\mathcal{A})/D$ σ-centered.