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Some compact logics—Results in ZFC

By Alan H. Mekler and Saharon Shelah*

Dedicated to the memory of Alan by his friend, Saharon†

1. Preliminaries

While first-order logic has many nice properties, it lacks expressive power. On the other hand, second-order logic is so strong that it fails to have nice model-theoretic properties such as compactness. It is desirable to find natural logics which are stronger than first-order logic, but which still satisfy the compactness theorem. Particularly attractive are those logics that allow quantification over natural algebraic objects. One of the most natural choices is to quantify over automorphisms of a structure (or isomorphisms between substructures). Generally compactness fails badly [16], but if we restrict ourselves to certain concrete classes, then we may be able to retain compactness. In this article we will show that if we enrich first-order logic by allowing quantification over isomorphisms between definable ordered fields, the resulting logic, $L(Q_{\mathrm{Of}})$, is fully compact. In this logic we can give standard compactness proofs of various results. For example, to prove that there exist arbitrarily large, rigid, real closed fields, fix a cardinal κ and form the $L(Q_{\mathrm{Of}})$ theory in the language of ordered fields together with κ constants, which says that the constants are pairwise distinct and the field is a real closed field, which is rigid. (To say the field is rigid we use the expressive power of $L(Q_{\mathrm{Of}})$ to say that any automorphism is the identity.) This theory is consistent, as the reals can be expanded to form a model of any finite subset of the theory. But a model of the theory must have cardinality at least κ. (Since we do not have the downward Löwenheim–Skolem theorem, we cannot assert that there is a model of cardinality κ.)

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In [10] and [8] the compactness of two interesting logics is established under certain set-theoretic hypotheses. The logics are those obtained from first-order logic by the addition of quantifiers, which range over automorphisms either of definable Boolean algebras or of definable ordered fields. Instead of dealing with the weaker version of automorphisms, one can also deal with a quantifier that says that two Boolean algebras are isomorphic or that two ordered fields are isomorphic. The key step in proving these results lies in establishing the following theorems (by definable we shall mean definable with parameters).

THEOREM 1.1. *Suppose that* λ *is a regular cardinal and both* $\diamondsuit(\lambda)$ *and* $\diamondsuit(\{\alpha < \lambda^+ : cf\alpha = \lambda\})$ *hold. If* T *is any consistent theory and* $|T| < \lambda$, *then there is a model* M *of* T *of cardinality* λ^+ *with the following properties:*

(i) *If* B *is a Boolean algebra definable in* M, *then every automorphism of* B *is definable.*

(ii) M *is* λ*-saturated.*

(iii) *Every nonalgebraic type of cardinality* $< \lambda$ *is realized in* M *by* λ^+ *elements.*

THEOREM 1.2. *Suppose that* λ *is a regular cardinal and both* $\diamondsuit(\lambda)$ *and* $\diamondsuit(\{\alpha < \lambda^+ : cf\alpha = \lambda\})$ *hold. If* T *is any consistent theory and* $|T| < \lambda$, *then there is a model* M *of* T *of cardinality* λ^+ *with the following properties:*

(i) *If* F *is an ordered field definable in* M, *then every automorphism of* F *is definable and every isomorphism between definable ordered fields is definable.*

(ii) M *is* λ*-saturated.*

(iii) *Every nonalgebraic type of cardinality* $< \lambda$ *is realized in* M *by* λ^+ *elements.*

(iv) *Every definable dense linear order is not the union of* λ *nowhere-dense sets.*

These theorems are proved in [10] (Theorem 1.1 is proved from GCH, the generalized continuum hypothesis, in [10], §9), although there is not an explicit statement of them there. In order to show the desired compactness results (from the assumption that there are unboundedly many cardinals λ as in the theorem statements) it is enough to use parts (i). However in our work on Boolean algebras we will need the more exact information above. Notice how the compactness of the various languages follows from these results. Since the idea is the same in all cases, just consider the case of Boolean algebras.

First we will describe the logic $L(Q_{\mathrm{Ba}})$. We add second-order variables (to range over automorphisms of Boolean algebras) and a quantifier Q_{Ba}, whose intended interpretation is that there is an automorphism of the Boolean algebra. More formally, if $\Theta(f)$, $\phi(x), \psi(x,y), \rho(x,y)$ are formulas (where f is a

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second-order variable, x and y are first-order variables and the formulas may have other variables), then

$$Q_{\mathrm{Ba}}f\bigl(\phi(x),\psi(x,y)\bigr)\Theta(f)$$

is a formula. In a model M the tuple $(\phi(x),\psi(x,y))$ *defines* a Boolean algebra (where parameters from M replace the hidden free variables of $\phi(x),\psi(x,y)$) if $\psi(x,y)$ defines a partial order $<$ on $B=\{a\in M:\ M\models\phi[a]\}$ so that $(B;<)$ is a Boolean algebra. A model M satisfies the formula $Q_{\mathrm{Ba}}f(\phi(x),\psi(x,y))\Theta(f)$ (where parameters from M have been substituted for the free variables) if whenever $(\phi(x),\psi(x,y))$ defines a Boolean algebra $(B;<)$, there is an automorphism f of $(B;<)$ such that $M\models\Theta(f)$. (It is easy to extend the treatment to look at Boolean algebras that are definable on equivalence classes, but we will avoid the extra complication.) We can give a more colloquial description of the quantifier Q_{Ba} by saying the interpretation of Q_{Ba} is that "$Q_{\mathrm{Ba}}f(B)\ldots$" holds if there is an automorphism of the Boolean algebra B so that .... We will describe some of the other logics we deal with in this looser manner. For example, we will want to consider the quantifier Q_{Of}, where $Q_{\mathrm{Of}}f(F_1,F_2)\ldots$ holds if there is an isomorphism f from the ordered field F_1 to the ordered field F_2 such that ....

The proof of compactness for $L(Q_{\mathrm{Ba}})$ follows easily from Theorem 1.1. By expanding the language,[1] we can assume that there is a ternary relation $\mathrm{R}(*,*,*)$ so that the theory says that any first-order definable function is definable by R and one parameter. By the ordinary compactness theorem, if we are given a consistent theory in this logic, then there is a model of the theory, where all the sentences of the theory hold if we replace automorphisms by definable automorphisms in the interpretation of Q_{Ba}, since quantification over definable automorphisms can be replaced by first-order quantification. Then we can apply the theorem to get a new model elementarily equivalent to the one given by the compactness theorem in which definable automorphisms and automorphisms are the same.

In what follows we will make two assumptions about all of our theories. First we will assume that all definable partial functions are in fact defined by a fixed formula (by varying the parameters). Second we will always assume that the language is countable, except for the constant symbols.

In this article we will attempt to get compactness results without recourse to $\diamondsuit$, i.e., all our results will be in ZFC, Zermelo–Fraenkel set theory. We will

[1]More exactly, for every model M, define a model M^* with universe $M\cup\{f:f$ a partial function from $|M|$ to $|M|\}$ with the relations of M and the unary predicate P, $P^{M'}=|M|$, and the ternary predicate R, $R=\{(f,a,b):f\in {}^{M}M,\ a\in M, b=f(a)\}$. We shall similarly transform a theory T to T' and consider automorphisms only of structures $\subseteq P$.

get the full result for the language, where we quantify over automorphisms (isomorphisms) of ordered fields in Theorem 6.4. Unfortunately we are not able to show that the language with quantification over automorphisms of Boolean algebras is compact, but will have to settle for a close relative of that logic. This is Theorem 5.1. In Section 4 we prove that models can be constructed in which all relevant automorphisms are somewhat definable: Theorem 4.1, Lemma 4.8 for Boolean algebras (BA), Theorem 4.13 for ordered fields.

The reader may wonder why these results are being proved now, about 10 years after the results that preceded them. The key technical innovation that has made these results possible is the discovery of $\diamondsuit$-like principles which are true in ZFC. These principles, which go under the common name of the Black Box, allow one to prove, with greater effort, many of the results that were previously known to follow from $\diamondsuit$ (see the discussion in [15] for more details). There have been previous applications of the Black Box to abelian groups, modules and BA's—often building objects with specified endomorphism rings. This application goes deeper both in the sense that the proof is more involved and the result is more surprising. The investigation is continued in [12], [13].

In this article we will also give a new proof of the compactness of another logic—the one that is obtained when a quantifier Q_{Brch} is added to first-order logic, which says that a level tree (definitions will be given later) has an infinite branch. This logic was previously shown to be compact—in fact it was the first logic shown in ZFC to be compact that is stronger than first-order logic on countable structures—but our proof will yield a somewhat stronger result and provide a nice illustration of one of our methods. (The first logic stronger than first-order logic shown to be compact was the logic which expresses that a linear order has cofinality greater than ω_1; cf. [11].) This logic, $L(Q_{\mathrm{Brch}})$, has been used by Fuchs and Shelah [3] to prove the existence of nonstandard uniserial modules over (some) valuation domains. The proof uses the compactness of the tree logic to transfer results proved using $\diamondsuit$ to ZFC results. Eklof [2] has given an explicit version of this transfer method and was able to show that it settles other questions that had been raised. (Osofsky [6], [7] has found ZFC constructions that avoid using the model theory.)

Theorems 3.1 and 3.2 contain parallel results for BA's and fields. They assert the existence of a theory (of sets) T_1 such that, in each model M_1 of T_1, $P(M_1)$ is a model M of the first-order theory T; hence for every BA (resp. field) defined in M, every automorphism of the BA (resp. field) that is definable in M_1 is definable in M. Moreover each such M_1 has an elementary extension, one of whose elements is a pseudofinite set a with the universe of M_1 contained in a and with $t(a/M_1)$ definable over the empty set. This result depends on the earlier proof of our main result assuming $\diamondsuit$ and absoluteness. Theorem 4.1 uses the Black Box to construct a model $\mathfrak{C}$ of T_1. We show that for any automorphism f of a Boolean algebra $B = P(\mathfrak{C})$ there is a pseudofinite set c such that, for any atom $b \in B$, $f(b)$ is definable from b and c. Theorem 4.13 is an analogous but stronger result for fields showing that, for any b, $f(b)$ is definable from b and c. In Lemma 4.7 we extend this pointwise definability by constructing a pseudofinite partition of atoms of the Boolean algebra (resp. the elements of the field) such that f is definable on each member of the partition. In Theorem 5.1 for Boolean algebras and Theorem 6.4 for fields, this local definability is extended to global definability.

1.1. *Outline of proof.* We want to build a model A of a consistent $L(Q_{\mathrm{Of}})$ theory T, which has only definable isomorphisms between definable ordered fields. By the ordinary compactness theorem, there is a nonstandard model $\mathfrak{C}$ of an expansion of a weak set theory (say, ZFC$^-$), which satisfies that there is a model A_1 of T. So A_1 would be a model of T if the interpretation of the quantifier Q_{Of} were taken to range over isomorphisms internal to $\mathfrak{C}$. We can arrange that A_1 will be the domain of a unary predicate P. Then our goal is to build our nonstandard model $\mathfrak{C}$ of a weak set theory in such a way that every external isomorphism between definable ordered subfields of $P(\mathfrak{C})$ is internal, i.e., definable in $\mathfrak{C}$.

The construction of $\mathfrak{C}$ is a typical construction with a prediction principle, in this case the Black Box, where we kill isomorphisms that are not pointwise definable over a set, which is internally finite (or, synonymously, *pseudofinite*). A predicted isomorphism is killed by adding an element that has no suitable candidate for its image. One common problem that is faced in such constructions is the question "How do we ensure that no possible image of such an element exists?" To do this we need to omit some types of elements. Much is known about omitting a type of size λ in models of power λ and even λ^+. But if, say, $2^\lambda > \lambda^{++}$, we cannot omit a dense set of types of power λ. Without instances of GCH we are thus reduced to omitting small types, which is much harder. To omit the small types we will use techniques that originated in "Classification Theory". In the construction we will see for some cardinal θ that the type of any element does not split over a set of cardinality less than θ (see also precise definitions below). This is analogous to saying the model is θ-stable (of course we are working in a very nonstable context). The element we add will have the property that its image (if one existed) would split over every set of cardinality $< \theta$.

The final problem is to go from pointwise definability to definability. The first ingredient is a general fact about $\aleph_0$-saturated models of set theory. We will show for any isomorphism f that there are a large (internal) set A and a pseudofinite sequence of one-to-one functions $(f_i\colon i < k^*)$, which cover $f \restriction A$ in the sense that for every $a \in A$ there is an i so that $f_i(a) = f(a)$. Using this sequence of functions we can then define f on a large subset of A. Finally, using the algebraic structure extends the definition to the entire ordered field.

In this article we will need to use the following principle: In order to have the cleanest possible statement of our results (and to conform to the notation in [15]) we will state our results using slightly nonstandard notation. To obtain the structure $\mathrm{H}_\chi(\lambda)$ we first begin with a set of ordered urelements of order type λ and then form the least set containing each urelement and closed under formation of sets of size less than χ. When we refer to $\mathrm{H}_\chi(\lambda)$, by λ we will mean the urelements and not the ordinals. In practice we believe that, given the context, there will be no confusion.

THEOREM 1.3. *Suppose that* $\lambda = \mu^+$, $\mu = \kappa^\theta = 2^\kappa$, χ *is a regular cardinal,* κ *is a strong limit cardinal,* $\theta < \chi < \kappa$, $\kappa > cf\kappa = \theta \geq \aleph_0$ *and* $S \subseteq \{\delta < \lambda\colon cf\delta = \theta\}$ *is stationary. Let* ρ *be some cardinal greater than* λ. *Then there are* $W = \{(\overline{M}^\alpha, \eta^\alpha)\colon \alpha < \alpha(*)\}$ *(actually a sequence),*[2] *a function* $\zeta : \alpha(*) \to S$ *and* $(C_\delta\colon \delta \in S)$ *such that:*

(a1) $\overline{M}^\alpha = (M_i^\alpha\colon i \leq \theta)$ *is an increasing continuous elementary chain, each* M_i^α *is a model belonging to* $\mathrm{H}_\chi(\lambda)$ *(and so necessarily has cardinality less than* χ*),* $M_i^\alpha \cap \chi$ *is an ordinal,* $\eta^\alpha \in {}^\theta\lambda$ *is increasing with limit* $\zeta(\alpha) \in S$, *for* $i < \theta$, $\eta^\alpha \restriction i \in M_{i+1}^\alpha$, $M_i^\alpha \in \mathrm{H}_\chi(\eta^\alpha(i))$ *and* $(M_j^\alpha\colon j \leq i) \in M_{i+1}^\alpha$.

(a2) *For any set* $X \subseteq \lambda$ *there is an* α *so that* $M_\theta^\alpha \equiv_{\lambda \cap M_\theta^\alpha} (\mathrm{H}(\rho), \in, <, X)$, *where* $<$ *is a well ordering of* $\mathrm{H}(\rho)$ *and* $M \equiv_A N$ *means that* $(M, a)_{a\in A}$ *and* $(N, a)_{a\in A}$ *are elementarily equivalent.*

(b0) *If* $\alpha \neq \beta$, *then* $\eta^\alpha \neq \eta^\beta$.

(b1) *If* $\{\eta^\alpha \restriction i\colon i < \theta\} \subseteq M_\theta^\beta$ *and* $\alpha \neq \beta$, *then* $\zeta(\alpha) < \zeta(\beta)$.

(b2) *If* $\eta^\alpha \restriction (j + 1) \in M_\theta^\beta$, *then* $M_j^\alpha \in M_\theta^\beta$.

(c2) $\overline{C} = (C_\delta\colon \delta \in S)$ *is such that each* C_δ *is a club subset of* δ *of the order type* θ.

(c3) *Let* $C_\delta = \{\gamma_{\delta,i}\colon i < \theta\}$ *be an increasing enumeration. For each* $\alpha < \alpha(*)$ *there is* $((\gamma_{\alpha,i}^-, \gamma_{\alpha,i}^+)\colon i < \theta)$ *such that* $\gamma_{\alpha,i}^- \in M_{i+1}^\alpha$, $M_{i+1}^\alpha \cap \lambda \subseteq \gamma_{\alpha,i}^+$, $\gamma_{\zeta(\alpha),i} < \gamma_{\alpha,i}^- < \gamma_{\alpha,i}^+ < \gamma_{\zeta(\alpha),i+1}$; *and if* $\zeta(\alpha) = \zeta(\beta)$ *and* $\alpha \neq \beta$, *then for every large enough* $i < \theta$, $[\gamma_{\alpha,i}^-, \gamma_{\alpha,i}^+) \cap [\gamma_{\beta,i}^-, \gamma_{\beta,i}^+) = \emptyset$. *Furthermore, for all* i, *the sequence* $(\gamma_{\alpha,j}^-\colon j < i)$ *is in* M_{i+1}^α.

This principle, which is one of the Black Box principles, is a form of $\diamondsuit$, which is a theorem of ZFC. This particular principle is proved in [15], III, 6.13(2). The numbering here is chosen to correspond with the numbering there. Roughly speaking, clauses (a1) and (a2) say that there is a

[2]In our case, $\alpha(*) = \lambda$ is fine.

family of elementary substructures, which predict every subset of λ as it sits in $H(\lambda)$. (We will freely talk about a countable elementary substructure *predicting* isomorphisms and the like.) The existence of such a family would be trivial if we allowed all elementary substructures of cardinality less than χ. The rest of the clauses say that the structures are sufficiently disjoint that we can use the information they provide without (too much) conflict.

The reader who wants to follow the main line of the arguments without getting involved (initially) in the complexities of the Black Box can substitute $\diamondsuit(\lambda)$ for the Black Box. Our proof of the compactness of $L(Q_{\mathrm{Of}})$ does not depend on Theorem 1.2, so even this simplification gives a new proof of the consistency of the compactness of $L(Q_{\mathrm{Of}})$. Our work on Boolean algebras does require Theorem 1.1.

2. Nonsplitting extensions

In this section θ will be a fixed regular cardinal. Our treatment is self-contained, but the reader can look at [14], I, §2, pp. 9-17, VII, §4, pp. 426-431, for more details.

Definition. If M is a model and $X, Y, Z \subseteq M$, then X/Y *does not split over* Z if and only if, for every finite $d \subseteq Y$, the type of X over d (denoted by either $\mathrm{tp}(X/d)$ or X/d) depends only on the type of d over Z.

We will use two constructions to guarantee that types will not split over small sets. The first is obvious by definition. (The type of A/B is *definable over* C if for any tuple $\bar{a} \in A$ and formula $\phi(\bar{x}, \bar{y})$ there is a formula $\psi(\bar{y})$ with parameters from C so that, for any $\bar{b}$, $\phi(\bar{a}, \bar{b})$ if and only if $\psi(\bar{b})$.) The second is by averaging an ultrafilter.

PROPOSITION 2.1. *If X/Y is definable over Z, then X/Y does not split over Z.*

Definition. Suppose that M is a model and $X, Y \subseteq M$. Let D be an ultrafilter on X^α. Then the $\mathrm{Av}(X, D, Y)$ (read the (D)-average type that X^α realizes over Y) is the type p over Y defined by the following: for $\bar{y} \subseteq Y$, $\phi(\bar{z}, \bar{y}) \in p$ if and only if $\{\bar{x} \in X^\alpha : \phi(\bar{x}, \bar{y}) \text{ holds}\} \in D$. We will omit Y if it is clear from the context. Similarly we will omit α and the "bar" for singletons, i.e., the case where $\alpha = 1$.

The following two propositions are clear from the definitions above.

PROPOSITION 2.2. *If $\bar{a}$ realizes $\mathrm{Av}(X, D, Y)$, then $\bar{a}/Y$ does not split over X. Also, if there is Z such that for $\bar{b} \in X$ the type of $\bar{b}/Y$ does not split over Z, then $\mathrm{Av}(X, D, Y)$ does not split over Z.*

PROPOSITION 2.3. (i) *Suppose that* A/B *does not split over* D, $B \subseteq C$ *and* $C/B \cup A$ *does not split over* $D \cup A$. *Then* $A \cup C/B$ *does not split over* D.

(ii) *Suppose that* $(A_i\colon i < \delta)$ *is an increasing chain and, for all* i, A_i/B *does not split over* C. *Then* $\bigcup_{i<\delta} A_i/B$ *does not split over* C.

(iii) X/Y *does not split over* Z *if and only if* $X/\operatorname{dcl}(Y \cup Z)$ *does not split over* Z. *Here* $\operatorname{dcl}(Y \cup Z)$ *denotes the definable closure of* $Y \cup Z$.

(iv) *If* X/Y *does not split over* Z *and* $Z \subseteq W$, *then* X/Y *does not split over* W.

Definition. Suppose that $M_1 \prec M_2$ are models. Define $M_1 \prec_\theta^\otimes M_2$ if for every $X \subseteq M_2$ of cardinality less than θ there is $Y \subseteq M_1$ of cardinality less than θ so that X/M_1 does not split over Y. (If θ is regular, then we only need to consider the case where X is finite.)

PROPOSITION 2.4. *Assume that* θ *is a regular cardinal* (*needed for* (2) *only*) *and that all models are models of some fixed theory with Skolem functions* (*although this is needed for* (3) *only*).

(1) $\prec_\theta^\otimes$ *is transitive and, for all* M, $M \prec_\theta^\otimes M$.

(2) *If* $(M_i\colon i < \delta)$ *is a* $\prec_\theta^\otimes$*-increasing chain, then, for all* i, $M_i \prec_\theta^\otimes \bigcup_{j<\delta} M_j$.

(3) *Suppose that* M_2 *is generated by* $M_1 \cup N_2$ *and* $N_1 = M_1 \cap N_2$. (*Recall the Skolem functions.*) *If* $|N_1| < \theta$ *and* N_2/M_1 *does not split over* N_1, *then* $M_1 \prec_\theta^\otimes M_2$.

An immediate consequence of these propositions is the following proposition:

PROPOSITION 2.5. *Suppose that* $M \prec_\theta^\otimes N$. *Then there is a* θ*-saturated model* M_1 *such that* $N \prec M_1$ *and* $M \prec_\theta^\otimes M_1$.

Proof. By the propositions it is enough to show that, given a set X of cardinality $< \theta$ and a type p over X, we can find a realization a of that type so that a/N does not split over X. Since we have Skolem functions, every finite subset of p is realized by an element, whose type over N does not split over X, namely an element of the Skolem hull of X. We can thus take a to realize an average type of these elements. □

3. Building new theories

The models we will eventually build will be particular nonstandard models of an enriched version of ZFC$^-$. (Recall that ZFC$^-$ is ZFC without the power set axiom and is true in the sets of hereditary cardinality $< \kappa$ for any regular uncountable cardinal κ.) The following two theorems state that appropriate theories exist.

THEOREM 3.1. *Suppose that T is a theory in a language that is countable, except for constant symbols. Suppose further that P_0 is a unary predicate so that, in every model M of T, every definable automorphism of a definable atomic Boolean algebra $\subseteq P_0^M$ is definable by a fixed formula (together with some parameters). Then there is T_1, an expansion of* ZFC$^-$ *in a language that is countable, except for constant symbols with a unary predicate P_0, so that if M_1 is a model of T_1, then $P_0(M_1)$ is a model M of T (when restricted to the right vocabulary). The following are then satisfied to be true in M_1:*

(i) *Any automorphism of a definable (in M) atomic Boolean algebra contained in $P_0(M)$, which is definable in M_1, is definable in M.*

(ii) *M_1 (which is a model of* ZFC$^-$*) satisfies, for some regular cardinal μ (of M_1), $|M| = \mu^+$, where M is μ-saturated and every nonalgebraic type (in the language of M) of cardinality $< \mu$ is realized in M by μ^+ elements of M.*

(iii) *$M \in M_1$.*

(iv) *M_1 satisfies the separation scheme for all formulas (not just those of the language of set theory).*

(v) *M_1 has Skolem functions.*

(vi) *For any M_1 there is an elementary extension N_1 so that the universe of M_1 is contained in N_1 in a pseudofinite set (i.e., one which is finite in N_1), whose type over the universe of M_1 is definable over the empty set.*

Proof. We first consider a special case. Suppose that there is a cardinal λ greater than the cardinality of T satisfying the hypotheses of Theorem 1.1 (i.e., both $\diamondsuit(\lambda)$ and $\diamondsuit(\{\alpha < \lambda^+ \colon cf\alpha = \lambda\})$ hold). Then we could choose κ, a regular cardinal greater than λ^+. Our model M_1 will be taken to be a suitable expansion of $\mathrm{H}(\kappa)$, where the interpretation of the unary predicate P_0 is the model M guaranteed by Theorem 1.1 and $\mu = \lambda$, $\mu^+ = \lambda^+$. Since any formula in the enriched language is equivalent in M_1 to a formula of set theory together with parameters from M_1, our model M_1 will also satisfy condition (iv).

What remains is to ensure clause (v) and that appropriate elementary extensions always exist, i.e., clause (vi). To achieve this we will expand the language by induction on n. Let L_0 be the language consisting of the language of T, $\{P\}$ and Skolem functions, and let M_1^0 be any expansion by Skolem functions of the structure on $\mathrm{H}(\kappa)$ described above. Fix an index set I and an ultrafilter D on I so that there is an $a \in \mathrm{H}(\kappa)^I/D$ such that $\mathrm{H}(\kappa)^I/D \models$ "a is finite" and, for all $b \in \mathrm{H}(\kappa)$, $\mathrm{H}(\kappa)^I/D \models b \in a$. Let $N_1^0 = {M_1^0}^I/D$.

Then for every formula $\phi(y, x_1, \ldots, x_n)$ of L_0 that does not involve constants, add a new n-ary relation R_ϕ. Let $L_{0.5}$ be the language containing all the R_ϕ. Let $M_1^{0.5}$ be the $L_{0.5}$ structure with universe $H(\kappa)$ obtained by letting, for all $b_1, \ldots, b_n$, $M_1^{0.5} \models R_\phi[b_1, \ldots, b_n]$ if and only if $N_1^0 \models \phi[a, b_1, \ldots, b_n]$. Let L_1 be an extension of $L_{0.5}$ by Skolem functions and let M_1^1 be an expansion of $M_1^{0.5}$ also by Skolem functions. Condition (iv) still holds as it holds for any expansion of $(H(\kappa), \in)$. We now let $N_1^1 = {M_1^1}^I/D$ and continue as before.

Let $L = \bigcup_{n<\omega} L_n$ and $M_1 = \bigcup_{n<\omega} M_1^n$ (i.e., the least common expansion; the universe stays the same). Let T_1 be the theory of M_1. As we have already argued, T_1 has properties (i)–(iv). It remains to see that any model of T_1 has the desired extension property. First we consider M_1 and let $N_1 = {M_1}^I/D$. Then the type of a over M_1 is definable over the empty set by means of the relations R_ϕ, which we have added. Since T_1 has Skolem functions, for any model A_1 of T_1 there will be an extension B_1 of A_1 generated by A_1 and an element realizing the definable type over A_1.

In the general case, where we may not have the necessary hypotheses of Theorem 1.1, we can force with a notion of forcing that adds no new subsets of $|T|$ to get some λ satisfying the hypotheses of Theorem 1.1 (alternately we can use $L[A]$, where A is a large enough set of ordinals). Since the desired theory will exist in an extension, it already must (as it can be coded by a subset of $|T|$) exist in the ground model. $\square$

Later on we will be juggling many different models of set theory. These include the ones given by the Black Box and the nonstandard ones, which are models of T_1. When we want to refer to notions in models of T_1, we will use words like "pseudofinite" to refer to sets that are satisfied to be finite in the model of T_1.

In the same way we proved the last theorem we can show the following theorem:

THEOREM 3.2. *Suppose that T is a theory in a language that has a unary predicate P_0 and is countable, except for constant symbols, so that in every model M of T, every definable isomorphism between definable ordered fields $\subseteq P_0^M$ is definable by a fixed formula (together with some parameters). Then there is T_1, an expansion of ZFC$^-$ in a language that is countable, except for constant symbols with a unary predicate P_0, so that if M_1 is a model of T_1, then $P_0(M_1)$ is a model M of T. The following are then satisfied to be true in M_1:*

(i) *Any isomorphism of definable (in M) ordered fields contained in P_0, which is definable in M_1, is definable in M.*

(ii) $M \in M_1$.

(iii) *M_1 satisfies the separation scheme for all formulas (not just those of the language of set theory).*

(iv) *M_1 has Skolem functions.*

(v) *For any M_1 there is an elementary extension N_1 so that the universe of M_1 is contained in N_1 in a pseudofinite set (i.e., one which is finite in N_1), whose type over the universe of M_1 is definable over the empty set.*

Since we have no internal saturation conditions, this theorem can be proved without recourse to Theorem 1.2 (see the next section for an example of a similar construction).

3.1. *A digression.* The method of expanding the language to get extensions, which realize a definable type, is quite powerful in itself. We can use the method to give a new proof of the compactness of a logic that extends first-order logic and is stronger even for countable structures. This subsection is not needed in the rest of the paper.

LEMMA 3.3. *Suppose that N is a model. Then there is a consistent expansion of N to a model of a theory T_1 with Skolem functions so that for every model M of T_1 there is an a so that a/M is definable over the empty set. And in $M(a)$ (the model generated by M and $\{a\}$), for every definable directed partial ordering $<$ of M without the last element there is an element greater than any element of M in the domain of $<$. Furthermore the cardinality of the language of T_1 is no greater than that of N plus $\aleph_0$.*

Proof. Fix a model N. Choose κ and an ultrafilter D so that for every directed partial ordering $<$ of N without the last element there is an element of N^κ/D that is greater than every element of N (e.g., let $\kappa = |N|$ and D be any regular ultrafilter on κ). Fix an element $a \in N^\kappa/D \setminus N$. Abusing notation, we will let $a : \kappa \to N$ be a function representing the element a. The new language is defined by induction on ω. Let $N = N_0$. There are three tasks, and so we will divide the construction of N_{n+1} into three cases.

If $n \equiv 0 \bmod 3$, expand N_n to N_{n+1} by adding Skolem functions.

If $n \equiv 1 \bmod 3$, add a k-ary relation R_ϕ for every formula of arity $k+1$ and let $\mathrm{R}_\phi(b_0, \dots, b_{k-1})$ hold if and only if $\phi(b_0, \dots, b_{k-1}, a)$ holds in ${N_n}^\kappa/D$.

If $n \equiv 2 \bmod 3$, we ensure that there is an upper bound to every definable directed partial order without a last element in N_n. For each $k+2$-ary formula $\phi(x_0, \dots, x_{k-1}, y, z)$ we will add a $k+1$-ary function f_ϕ so that for all $\bar{b}$, if $\phi(\bar{b}, y, z)$ defines a directed partial order without a last element, then $f_\phi(\bar{b}, a)$ is greater in that partial order than any element of N. Notice that there is something to do here, since we must define f_ϕ on N and then extend to N^κ/D using the ultraproduct. For each such $\bar{b}$ choose a function $c : \kappa \to N$ so that c/D is an upper bound (in the partial order) to all the elements of N. Now choose f_ϕ so that $f_\phi(\bar{b}, a(i)) = c(i)$. Let T_1 be the theory of the expanded model.

Suppose now that M is a model of T_1. The type we want is the type p defined by $\phi(c_1, \ldots, c_{k-1}, x) \in p$ if and only if $M \models \mathrm{R}_\phi(c_1, \ldots, c_n)$. □

In the process of building the new theory, we made some choices of language. But these choices can be made uniformly for all models. We will in the sequel assume that such a uniform choice has been made.

From this lemma we can prove a stronger version of a theorem from [9], which says that the language allowing quantification over branches of level trees is compact. A *tree* is a partial order in which the predecessors of any element are totally ordered. A *level tree* is a tree together with a ranking function to a directed set. More exactly a *level tree* is a model $(A : U, V, <_1, <_2, \mathrm{R})$, where

(1) A is the union of U and V;

(2) $<_1$ is a partial order of U such that, for every $u \in U$, $\{y \in U : y <_1 u\}$ is totally ordered by $<_1$;

(3) $<_2$ is a directed partial order on V with no last element;

(4) R is a function from U to V, which is strictly order preserving.

The definition here is slightly more general than in [9]. There the levels were required to be linearly ordered. Also what we have called a "level tree" is simply called a "tree" in [9]. A *branch* b of a level tree is a maximal linearly ordered subset of U such that $\{\mathrm{R}(t) : t \in b\}$ is unbounded in V. We will refer to U as the tree and V as the levels. For $t \in U$ the *level* of t is $\mathrm{R}(t)$.

A tuple of formulas (which may use parameters from M),

$$\big(\phi_1(x), \phi_2(x), \psi_1(x, y), \psi_2(x, y), \rho(x, y, z)\big),$$

defines a level tree in a model M if

$$\big(\phi_1(M) \cup \phi_2(M); \phi_1(M), \phi_2(M), \psi_1(x, y)^M, \psi_2(x, y)^M, \rho(x, y, z)^M\big)$$

is a level tree. (There is no difficulty in extending the treatment to all level trees, which are definable by the use of equivalence relations.)

Given the definition of a level tree, we now define an extension of first-order logic by adding second-order variables (to range over branches of level trees) and a quantifier Q_{Brch} such that

$$Q_{\mathrm{Brch}} b\big(\phi_1(x), \phi_2(x), \psi_1(x, y), \psi_2(x, y), \rho(x, y, z)\big)\Theta(b).$$

This says that if $(\phi_1(x), \phi_2(x), \psi_1(x, y), \psi_2(x, y), \rho(x, y, z))$ defines a level tree, then there is a branch b of the level tree such that $\Theta(b)$ holds.

In [9] it is shown that (a first-order version of) this logic was compact. This is the first language to be shown (in ZFC) to be fully compact and stronger than first-order logic for countable structures. In [9] the models are obtained at successors of regular cardinals.

THEOREM 3.4. *The logic $L(Q_{\text{Brch}})$ is compact. Furthermore every consistent theory T has a model in all uncountable cardinals $\kappa > |T|$.*

Proof. By expanding the language, we can assume that any model of T admits elimination of quantifiers. (I.e., add new relations for each formula and the appropriate defining axioms.)

For each finite $S \subseteq T$ choose a model M_S of S. For each S choose a cardinal μ so that $M_S \in \mathrm{H}(\mu)$. Let N_S be the model $(\mathrm{H}(\mu^+), M_S, \in)$, where M_S is the interpretation of a new unary predicate P and the language of N_S includes the language of T (with the correct restriction to M_S). By expanding the structure N_S, we can assume that the theory T_S of N_S satisfies the conclusion of Lemma 3.3. Furthermore we note that T_S satisfies two additional properties. If a formula defines a branch in a definable level tree contained in the domain of P, then this branch is an element of the model N_S. As well N_S satisfies the property that $P(N_S)$ is a model of S.

Now let D be an ultrafilter on the finite subsets of T such that, for all finite S, $\{S_1 \colon S \subseteq S_1\} \in D$. Finally let T_1 be the (first-order) theory of $\prod N_S/D$. If N is any model of T_1, then for any sentence $\phi \in T$, N satisfies "$P(N)$ satisfies ϕ". If we can arrange that the only branches of an $L(Q_{\text{Brch}})$-definable (in the language of T) level tree of $P(N)$ are first-order definable in N, then $P(N)$ satisfying an $L(Q_{\text{Brch}})$-formula will be the same in N and the real world. Before constructing this model, let us note that our task is a bit easier than it might seem.

CLAIM 3.5. *Suppose that N is a model of T_1 and every branch of a first-order definable (in the language of T) level tree of $P(N)$ is first-order definable in N. Then every branch of an $L(Q_{\text{Brch}})$-definable (in the language of T) level tree of $P(N)$ is first-order definable in N.*

Proof of the claim. Since we have quantifier elimination, we can prove by induction on the construction of formulas that satisfaction is the same in N and the real world, and so the quantifier elimination holds for $P(N)$. In other words, any $L(Q_{\text{Brch}})$-definable level tree is first-order definable. □

It remains to do the construction and prove that it works. To begin let N_0 be any model of T_1 of cardinality κ. Let $\mu \leq \kappa$ be any regular cardinal. We will construct an increasing elementary chain of models N_α for $\alpha < \mu$ by induction. At limit ordinals we will take unions. If N_α has been defined, let $N_{\alpha+1} = N_\alpha(a_\alpha)$, where a_α is as guaranteed by Lemma 3.3. Now let $N = \bigcup_{\alpha<\mu} N_\alpha$.

SUBCLAIM 3.6. *Suppose that X is any subset of N that is definable by parameters. Then, for all α, $X \cap N_\alpha$ is definable in N_α.*

Proof of the subclaim. Suppose not and let β be the least ordinal greater than α so that $X \cap N_\beta$ is definable in N_β. Such an ordinal must exist, since for sufficiently large β the parameters necessary to define X are in N_β. Similarly there is a γ such that $\beta = \gamma+1$. Since N_β is the Skolem hull of $N_\gamma \cup \{a_\gamma\}$, there are $\bar{b} \in N_\gamma$ and a formula $\phi(x, \bar{y}, z)$ so that $X \cap N_\beta$ is defined by $\phi(x, \bar{b}, a_\gamma)$. But by the definability of the type of a_γ over N_γ there is a formula $\psi(x, \bar{y})$ so that, for all $a, \bar{c} \in N_\gamma$, $N_\beta \models \psi(a, \bar{c})$ if and only if $\phi(a, \bar{c}, a_\gamma)$. Hence $\psi(x, \bar{b})$ defines $X \cap N_\gamma$ in N_γ. □

It remains to see that every branch of a definable level tree is definable. Suppose that $(A; U, V, <_1, <_2, \mathrm{R})$ is a definable level tree. Without loss of generality we can assume it is definable over the empty set. Let B be a branch. For any $\alpha < \mu$ there is a $c \in V$ so that, for all $d \in V \cap N_\alpha$, $d <_2 c$. Since the levels of B are unbounded in V, $B \cap N_\alpha$ is not cofinal in B. Hence there is a $b \in B$ so that $B \cap N_\alpha \subseteq \{a \in U\colon\ a <_1 b\}$. By the subclaim, $B \cap N_\alpha$ is definable in N_α.

Since μ has uncountable cofinality, by Fodor's lemma there is an $\alpha < \mu$ so that, for unboundedly many (and hence all) $\gamma < \mu$, $B \cap N_\gamma$ is definable by a formula with parameters from N_α. Fix a formula $\phi(x)$ with parameters from N_α, which defines $B \cap N_\alpha$. Then $\phi(x)$ defines B. To see this consider any $\gamma < \alpha$ and a formula $\psi(x)$ with parameters from N_α, which define $B \cap N_\gamma$. Since the N_α satisfy "for all x, $\phi(x)$ if and only if $\psi(x)$" and since $N_\alpha \prec N_\gamma$, $\phi(x)$ also defines $B \cap N_\gamma$. □

Remark. The compactness result above is optimal as far as the cardinality of the model is concerned. Any countable level tree has a branch, and so there is no countable model that is $L(Q_{\mathrm{Brch}})$-equivalent to an Aronszajn tree. By the famous theorem of Lindstrom [4], this result is the best that can be obtained for any logic, since any compact logic, which is at least as powerful as first-order logic and has countable models for all sentences, is in fact the first-order logic. The existence of a compact logic, such that every consistent countable theory has a model in all uncountable cardinals, was first proved by Shelah [11], who showed that first-order logic is compact if we add a quantifier Q^{cf}, which says of a linear order that its cofinality is ω. (Lindstrom's theorem and the logic $L(Q^{\mathrm{cf}})$ are also discussed in [1].) The logic $L(Q_{\mathrm{Brch}})$ has the advantage of being stronger than first-order logic even for countable models (cf. [9]).

Notice in the proof above that in any definable level tree the directed set of the levels has cofinality μ. Since we can obtain any uncountable cofinality, this is also the best possible result. Also in the theorem above we can demand just $\kappa \geq |T| + \aleph_1$.

4. The models

For the purposes of this section let T and T_1 be theories, as defined above in Theorems 3.1 or 3.2 (for Boolean algebras or ordered fields). Here we will build models of our theory T_1. The cases of Boolean algebras and ordered fields are similar, but there are enough differences that they have to be treated separately. We shall deal with Boolean algebras first.

We want to approximate the goal of having every automorphism of every definable atomic Boolean algebra in the domain of P be definable. In this section we will find that they are definable in a weak sense. In order to spare ourselves some notational complications we will make a simplifying assumption and prove a weaker result. It should be apparent at the end how to prove the same result for every definable atomic Boolean algebra in the domain of P.

Assumption. T is the theory of an atomic Boolean algebra on P with some additional structure.

THEOREM 4.1. *There is a model $\mathfrak{C}$ of T_1 so that if $B = P(\mathfrak{C})$ and f is any automorphism of B as a Boolean algebra, then there is a pseudofinite set c such that for any atom $b \in B$, $f(b)$ is definable from b and elements of c.*

Proof. We will use the notation from the Black Box. In particular we will use an ordered set of urelements of order type λ. We can assume that μ is larger than the cardinality of the language (including the constants). We shall build a chain of structures $(\mathfrak{C}_\varepsilon : \varepsilon < \lambda)$ such that the universe of $\mathfrak{C}_\varepsilon$ will be an ordinal $< \lambda$ and the universe of $\mathfrak{C} = \bigcup_{\varepsilon<\lambda} \mathfrak{C}_\varepsilon$ will be λ (we can specify in a definable way what the universe of $\mathfrak{C}_\varepsilon$ is, e.g., $\mu(1+\alpha)$ is alright). We choose $\mathfrak{C}_\varepsilon$ by induction on ε. Let $B_\varepsilon = P(\mathfrak{C}_\varepsilon)$. We will view the $(\overline{M}^{\alpha}, \eta^\alpha) \in W$ in the Black Box as predicting a sequence of models of T_1 and an automorphism of the Boolean algebra. (See the following paragraphs for more details on what we mean by predicting.)

The construction will be done so that if $\varepsilon \notin S$ and $\varepsilon < \zeta$, then $\mathfrak{C}_\varepsilon \prec^{\otimes}_{\theta} \mathfrak{C}_\zeta$. (We will make further demands later.) The model $\mathfrak{C}$ will be $\bigcup_{\varepsilon<\lambda} \mathfrak{C}_\varepsilon$. When we are done, if f is an automorphism of the Boolean algebra, then we can choose $(\overline{M}^{\alpha}, \eta^\alpha) \in W$ to code, in a definable way, f and the sequence $(\mathfrak{C}_\varepsilon : \varepsilon < \lambda)$.

The limit stages of the construction are determined. The successor stage when $\varepsilon \notin S$ is simple. We construct $\mathfrak{C}_{\varepsilon+1}$ so that $\mathfrak{C}_\varepsilon \prec^{\otimes}_{\theta} \mathfrak{C}_{\varepsilon+1}$, $\mathfrak{C}_{\varepsilon+1}$ is θ-saturated and there is a pseudofinite set in $\mathfrak{C}_{\varepsilon+1}$, which contains $\mathfrak{C}_\varepsilon$. By the construction of the theory T_1 there is c, a pseudofinite set, which contains $\mathfrak{C}_\varepsilon$ such that the type of c over $\mathfrak{C}_\varepsilon$ is definable over the empty set. Hence $\mathfrak{C}_\varepsilon \prec^{\otimes}_{\theta} \mathfrak{C}_\varepsilon(c)$. By Proposition 2.5 there is $\mathfrak{C}_{\varepsilon+1}$, which is θ-saturated so that $\mathfrak{C}_\varepsilon(c) \prec^{\otimes}_{\theta} \mathfrak{C}_{\varepsilon+1}$. Finally, by transitivity (Proposition 2.3), $\mathfrak{C}_\varepsilon \prec^{\otimes}_{\theta} \mathfrak{C}_{\varepsilon+1}$.

The difficult case occurs when $\varepsilon \in S$; rename ε by δ. Consider α so that $\zeta(\alpha) = \delta$. We are interested mainly in α's which satisfy:

(*) The model M^α_θ "thinks" it is of the form $(\mathrm{H}(\rho), \in, <, X)$ and, by our coding, yields (or predicts) a sequence of structures $(\mathfrak{D}_\nu \colon \nu < \lambda)$ and a function f_α from $\mathfrak{D} = \bigcup_{\nu<\lambda} \mathfrak{D}_\nu$ to itself. (Of course all the urelements in $M^\alpha_\theta \cap \lambda$ will have order type less than δ.)

At the moment we will only need to use the function predicted by M^α_θ.

We will say an *obstruction occurs at* α if ((*) holds and) we can make the following choices: If possible, choose $N_\alpha \subseteq \mathfrak{C}_\delta$ so that $N_\alpha \subseteq M^\alpha_0$ of cardinality less than θ, and a sequence of atoms $(a^\alpha_i \colon i \in C_\delta)$ so that (naturally $a^\alpha_i \in M^\alpha_{i+1}$) $a^\alpha_i/\mathfrak{C}_{\gamma^-_{\alpha,i}}$ does not split over N_α and $a^\alpha_i \in [\gamma^-_{\alpha,i}, \gamma^+_{\alpha,i})$ and so that $f_\alpha(a^\alpha_i)$ is not definable over a and parameters from $\mathfrak{C}_{\gamma^-_{\alpha,i}}$. At ordinals where an obstruction occurs we will take action to stop the f_α from extending to an automorphism of $B = B^{\mathfrak{C}}$.

Notice that the N_α are contained in $M^\alpha_0 \subseteq M^\alpha_\theta$.

Suppose that an obstruction occurs at α. Let X_α be the set of finite joins of the $\{a^\alpha_i \colon i \in C_\delta\}$. In the obvious way, X_α can be identified with the set of finite subsets of $Y_\alpha = \{a^\alpha_i \colon i \in C_\delta\}$. Fix U_α as an ultrafilter on X_α so that, for all $x \in X_\alpha$, $\{y \in X_\alpha \colon x \subseteq y\} \in U_\alpha$. Now define by induction on $\mathrm{Ob}(\delta) = \{\alpha \colon \zeta(\alpha) = \delta$ and an obstruction occurs at $\alpha\}$ an element x_α so that x_α realizes the U_α average type of X_α over $\mathfrak{C}_\delta \cup \{x_\beta \colon \beta \in \mathrm{Ob}(\delta), \beta < \alpha\}$. Then $\mathfrak{C}_{\delta+1}$ is the Skolem hull of $\mathfrak{C}_\delta \cup \{x_\alpha \colon \alpha \in \mathrm{Ob}(\delta)\}$.

We now want to verify the inductive hypothesis and give a stronger property, which we will use later in the proof. The key is the following claim:

CLAIM 4.2. *Suppose that $\alpha_0, \ldots, \alpha_{n-1} \in \mathrm{Ob}(\delta)$. Then, for all but a bounded set of $\gamma < \delta$, $(\bigcup_{k<n} Y_{\alpha_k})/\mathfrak{C}_\gamma$ does not split over $\bigcup_{k<n} N_{\alpha_k} \cup \bigcup_{k<n}(\mathfrak{C}_\gamma \cap Y_{\alpha_k})$.*

Proof of the claim. Suppose that γ is large enough that, for all $m \neq k < n$, $[\gamma^-_{\alpha_m,i}, \gamma^+_{\alpha_m,i}) \cap [\gamma^-_{\alpha_k,i}, \gamma^+_{\alpha_k,i}) = \emptyset$, whenever $\gamma^-_{\alpha_m,i} \geq \gamma$ (recall clause (c3) of the Black Box). It is enough to show by induction on $\gamma \leq \sigma < \delta$ that $(\bigcup_{k<n} Y_{\alpha_k}) \cap \mathfrak{C}_\sigma$ has the desired property. For $\sigma = \gamma$ there is nothing to prove. In the inductive proof we only need to look at a place where the set increases. By the hypothesis on γ we can suppose the result is true up to $\sigma = \gamma^-_{\alpha_k,i}$ and try to prove the result for $\sigma = \gamma^+_{\alpha_k,i}$ (since new elements are added only in these intervals). The new element added is $a^{\alpha_k}_i$. Denote this element by a. By hypothesis, $a/\mathfrak{C}_{\gamma^-_{\alpha_k,i}}$ does not split over N_{α_k} and so also not over $\bigcup_{k<n} N_{\alpha_k} \cup \bigcup_{k<n}(\mathfrak{C}_\gamma \cap Y_{\alpha_k})$. Now we can apply the induction hypothesis and Proposition 2.3.

Notice that the X_α are contained in the definable closure of Y_α and vice versa. Consequently we also have $(\bigcup_{k<n} X_{\alpha_k})/\mathfrak{C}_\gamma$ not splitting over $\bigcup_{k<n} N_{\alpha_k} \cup \bigcup_{k<n}(\mathfrak{C}_\gamma \cap Y_{\alpha_k})$. We can immediately verify the induction hypothesis that if $\gamma < \delta$, $\gamma \notin S$, then $\mathfrak{C}_\gamma \prec^{\otimes}_{\theta} \mathfrak{C}_{\delta+1}$. It is enough to verify for $\alpha_0, \dots, \alpha_{n-1}$ and sufficiently large β that $(x_{\alpha_0}, \dots, x_{\alpha_{n-1}})/\mathfrak{C}_\beta$ does not split over $\bigcup_{k<n} N_{\alpha_k} \cup \bigcup_{k<n}(\mathfrak{C}_\beta \cap Y_{\alpha_k})$ (a set of size $< \theta$). But this sequence realizes the ultrafilter average of $X_{\alpha_0} \times \cdots \times X_{\alpha_{n-1}}$. Hence we are done by Proposition 2.2. □

This completes the construction. Before continuing with the proof of Theorem 4.1, notice that we claim the following:

CLAIM 4.3. (1) *For all $\alpha \in \mathrm{Ob}(\delta)$ and $D \subseteq \mathfrak{C}_{\delta+1}$, if $|D| < \theta$ then, for all but a bounded set of $i < \theta$, $D/\mathfrak{C}_{\gamma^+_{\alpha,i}}$ does not split over $\mathfrak{C}_{\gamma^-_{\alpha,i}} \cup \{a^\alpha_i\}$. Moreover, for all but a bounded set of $i < \theta$, $D/\mathfrak{C}_{\gamma^+_{\alpha,i}}$ does not split over a subset of $\mathfrak{C}_{\gamma^-_{\alpha,i}} \cup \{a^\alpha_i\}$ of size $< \theta$.*

(2) *For every subset D of $\mathfrak{C}_{\delta+1}$ of cardinality $< \theta$ there are a subset w of* $\mathrm{Ob}(\delta)$ *of cardinality $< \theta$ and a subset Z of $\mathfrak{C}_\delta$ of cardinality $< \theta$ such that the type of D over $\mathfrak{C}_\delta$ does not split over $Z \cup \bigcup_{j\in w} Y_j$.*

(3) *In (2), for every large enough i, for every $\alpha \in w$ the type of D over $\mathfrak{C}_{\gamma^+_{\alpha,i}}$ does not split over $Z \cup \bigcup\{Y_j \cap \mathfrak{C}_{\gamma^-_{\alpha,i}} : j \in w\} \cup \{a_{\alpha,i}\}$.*

(4) *In (2) and (3) allow $D \subseteq \mathfrak{C}$.* □

We now have to verify that $\mathfrak{C}$ has the desired properties. Assume that f is an automorphism of $B = B^{\mathfrak{C}}$. We must show that the following claim holds:

CLAIM 4.4. *There is a γ so that, for all atoms a, $f(a)$ is definable with parameters from $\mathfrak{C}_\gamma$ and a.*

Proof of the claim. Assume that f is a counterexample. For every $\gamma \notin S$ choose an atom a_γ, which witnesses the claim is false with respect to $\mathfrak{C}_\gamma$. Since $\{\delta < \lambda\colon\ \delta \notin S, cf\delta \geq \theta\}$ is stationary, there is a set N of cardinality less than θ so that, for a stationary set of γ, $a_\gamma/\mathfrak{C}_\gamma$ does not split over N. In fact (since (for all $\alpha < \lambda)\alpha^\theta < \lambda$ as $\lambda = \mu^+$, $\mu^{<\theta} = \mu$), for all but a bounded set of γ we can use the same N. Let X code the sequence $(\mathfrak{C}_\gamma\colon\ \gamma < \lambda)$ and the function f. Then, by the previous discussion, $(\mathrm{H}(\rho), \in, <, X)$ satisfies "there exists an $N \subseteq \mathfrak{C}$ so that $|N| < \theta$ and, for all but a bounded set of ordinals γ, there is an atom z so that $z/\mathfrak{C}_\gamma$ does not split over N". Choose α so that

$$M^\alpha_\theta \equiv_{M^\alpha_\theta \cap \lambda} \big(\mathrm{H}(\rho), \in, <, X\big).$$

It is now easy to verify that an obstruction occurs at α. Let $\delta = \zeta(\alpha)$. In this case, f_α is the restriction of f. We use the notation of the construction. By the construction there is $D \subseteq \mathfrak{C}_{\delta+1}$, $|D| < \theta$ such that the type of $f(a_\alpha)$

over $\mathfrak{C}_{\delta+1}$ does not split over D. Apply Claim 4.3 above, parts (2) and (3), and get Z, w and $i^* < \theta$ (the i^* is just explicating the "for every large enough i to" for every $i \in [i^*, \theta)$"). Let $D^* = Z \cup \bigcup\{Y_j \cap C_{\gamma^-_{\alpha,i}} : j \in w\}$; thus for every $i \in [i^*, \theta)$ we have

(1) $f(a_\alpha)/\mathfrak{C}_{\delta+1}$ does not fork over $D(\subseteq \mathfrak{C}_{\delta+1})$;
(2) $D/\mathfrak{C}_{\gamma^+_{\alpha,i}}$ does not split over $D^* \cup a_i$;
(3) $D^* \cup a_{\alpha,i} \subseteq \mathfrak{C}_{\gamma^+_{\alpha,i}} \subseteq \mathfrak{C}_{\delta+1}$.

By the basic properties of nonsplitting, $f(x_\alpha)/\mathfrak{C}_{\gamma^+_{\alpha,i}}$ does not split over $D^* \cup \{a^\alpha_i\}$. And note that we have : $D^* \cup N_\alpha \subseteq \mathfrak{C}_{\gamma^-_{\alpha,i}}$ and $|D^* \cup N_\alpha| < \theta$, where the N_α come from the construction.

An important point is that, by elementariness for all ordinals $\tau \in M^\alpha_i \cap \lambda$ and atoms $a \in M^\alpha_i$, there is an ordinal β so that $\tau < \beta \in M^\alpha_i \cap \lambda$, $a \in B_\beta$, $\mathfrak{C}_\beta$ is θ-saturated (just take $cf\beta \geq \theta$) and f is an automorphism of B_β. Choose such a $\beta \in M^\alpha_{i+1}$ with respect to a^α_i and $\gamma^-_{\alpha,i}$.

Since $f(a^\alpha_i)$ is not definable from a^α_i and parameters from $\mathfrak{C}_{\gamma^-_{\alpha,i}}$ and since $\mathfrak{C}_\beta$ is θ-saturated, there is a $b \neq f(a^\alpha_i)$ realizing the same type over $D^* \cup \{a^\alpha_i\} \cup \{f(a^\alpha_j)\colon\ j < i\}$ with $b \in B_\beta$. Now for any atom $c \in B_\beta$ we have (by the definition of x_α) $c \leq x_\alpha$ if and only if $c = a^\alpha_j$ for some $j \leq i$. Since this property is preserved by f, we have $f(a^\alpha_i) \leq f(x_\alpha)$ and $b \not\leq f(x_\alpha)$. However $\beta < \gamma^+_{\alpha,i}$ and $f(x_\alpha)/\mathfrak{C}_{\gamma^+_{\alpha,i}}$ does not split over $\{a^\alpha_i\} \cup D^*$. So we have arrived at a contradiction. □

In the proof above, if we take θ to be uncountable, then we can strengthen the theorem (although we will not have any current use for the stronger form).

THEOREM 4.5. *In the theorem above, if θ is uncountable, then there are a finite set of formulas L' and a pseudofinite set c so that, for every atom $b \in B$, $f(b)$ is L'-definable over $\{b\} \cup c$.*

Proof. The argument so far has constructed a model in which every automorphism of B is pointwise definable on the atoms over some $\mathfrak{C}_\gamma$ (i.e., for every atom $b \in B$, $f(b)$ is definable from b and parameters from $\mathfrak{C}_\gamma$). In the construction of the model we have every $\mathfrak{C}_\gamma$ contained in some pseudofinite set, so we have come a long way toward our goal. To prove the theorem it remains to show that we can restrict ourselves to a finite sublanguage. (Since all the interpretations of the constants will be contained in $\mathfrak{C}_1$, we can ignore them.) Choose $\mathfrak{C}_\gamma$ so that f and f^{-1} are pointwise definable over $\mathfrak{C}_\gamma$. Let c be a pseudofinite set containing $\mathfrak{C}_\gamma$. We can assume that f permutes the atoms of B_γ.

Let the language L be the union of an increasing chain of finite sublanguages $(L_n\colon\ n < \omega)$. Assume by way of contradiction that, for all n, f is not pointwise definable on the atoms over any pseudofinite set (and hence not over any $\mathfrak{C}_\alpha$) using formulas from L_n. Choose a sequence d_n of atoms so that, for all n, $e_n = f(d_n)$ is not L_n-definable over $\{d_n\} \cup c$ and both d_{n+1} and e_{n+1} are not definable over $\{d_k\colon k \leq n\} \cup \{e_k\colon k \leq n\} \cup c$. Furthermore d_{n+1} should not be L_n-definable over $\{d_k\colon k \leq n\} \cup c$. The choice of d_n is possible by hypothesis, since only a pseudofinite set of possibilities has been eliminated from the choice.

Let $\bar{d}$ and $\bar{e}$ realize the average type (modulo some ultrafilter) over $\mathfrak{C}_\gamma \cup \{c\}$ of $\{(d_k\colon k \leq n), (e_k\colon k \leq n)\colon n < \omega\}$. These are pseudofinite sequences, which have $(d_n\colon n < \omega)$ and $(e_n\colon n < \omega)$ as initial segments. Say that $\bar{d} = (d_i\colon i < n^*)$ for some nonstandard natural number n^*. Now let $x \in B$ be the join of $\{d_i\colon i < n^*\}$. (This join exists, since $\bar{d}$ is a pseudofinite sequence.) For every $i < n^*$ there is a (standard) n_i so that $f(d_i)$ is L_{n_i}-definable over $\{d_i\} \cup \mathfrak{C}_\gamma$ and d_i is L_{n_i}-definable over $\{f(d_i)\} \cup \mathfrak{C}_\gamma$. Since $\mathfrak{C}$ is θ-saturated, the coinitiality of $n^* \setminus \omega$ is greater than ω. There is thus some n such that $\{i\colon n_i = n\}$ is coinitial. (Notice that, for nonstandard i, there is no connection between e_i and $f(d_i)$.)

Choose k so that the formulas in L_n have at most k free variables. Let Z be the set of subsets Y of c of size k so that, for all $i < n^*$, neither d_i nor e_i is L_n-definable from $Y \cup \{d_j, e_j\colon j < i\}$. By the choice of $\bar{d}$ and $\bar{e}$ every subset of $\mathfrak{C}_\gamma$ of size k is an element of Z.

Consider $\omega > m > n+1$. We claim there is no atom $y < f(x)$ such that d_m and y are L_n-interdefinable over elements of Z. Suppose that there is one and $y = f(d_i)$. Then i is nonstandard, since by the choice of Z, $y \neq e_j$ for all j. Since d_i is definable from $\{f(d_i)\} \cup c$, it is definable from $\{d_m\} \cup c$. This contradicts the choice of the sequence. We can finally get our contradiction. For $i < n^*$ we have $i < \omega$ if and only if for all $j > n+1$, the following holds: there is an atom $y < f(x)$ so that d_j and y are L_n-interdefinable over elements of Z implies $i < j$. □

We will want to work a bit harder and show that the automorphisms in the model above are actually definable on a large set. To this end we prove an easy graph-theoretic lemma.

LEMMA 4.6. *Suppose that G is a graph and there is $0 < k < \omega$ so that the valence of each vertex is at most k. Then there is a partition of (the set of nodes of) G into k^2 pieces $A_0, \ldots, A_{k^2-1}$ such that, for any i and any node v, v is adjacent to at most one element of A_i. Furthermore, if λ is an uncountable cardinal, then each A_i can be chosen to meet any λ sets of cardinality λ.*

Proof. Apply Zorn's lemma to get a sequence $\langle A_0, \ldots, A_{k^2-1}\rangle$ of pairwise-disjoint sets of nodes such that, for $i < k^2$ and node v, v is adjacent to at most one member of A_i and $\bigcup_{i<k^2} A_i$ maximal [under those constraints]. Suppose that there is a v, which is not in any of the A_i. Since v is not in any of the A_i, for each i there are u_i adjacent to v and $w_i \neq u_i$ adjacent to v so that $w_i \in A_i$. If no such u_i, w_i existed, we could extend the partition by adding v to A_i. But as the valency of every vertex of G is $\leq k$, there are at most $k(k-1) < k^2$ such pairs.

As for the second statement, since the valence is finite, every connected component is countable. Hence we can partition the connected components and then put them together to get a partition meeting every one of the λ sets. □

Actually for infinite graphs we can get a sharper bound. Given G we form an associated graph by joining vertices if they have a common neighbor. This gives a graph, whose valence is at most $k^2 - k$. We want to vertex-color this new graph. Obviously (see the proof above) it can be vertex-colored in $k^2 + 1 - k$ colors. In fact, by a theorem of Brooks ([5], Thm. 6.5.1), the result can be sharpened further. In our work we will only need that the coloring be finite, so these sharpenings need not concern us.

We will want to form in $\mathfrak{C}$ a graph, whose vertices are the atoms of B with a pseudofinite bound so that any atom b is adjacent to $f(b)$. This is easy to do in the case where all the definitions of $f(b)$ from $\{b\} \cup c$ use only a finite sublanguage. In the general case (i.e., when $\operatorname{cf}\theta$ may be ω) we have to cover the possible definitions by a pseudofinite set.

LEMMA 4.7. *Continue with the notation of the proof. Then there are a pseudofinite set D, a pseudofinite natural number k^* and a set of tuples Z of length at most k^* so that for every formula $\phi(\bar{x}, \bar{y})$ there is $d \in D$, such that for all $\bar{a} \in c$ and $\bar{b} \in B$, the tuple $(d, \bar{a}, \bar{b}) \in Z$ if and only if $\phi(\bar{a}, \bar{b})$.*

Proof. Since $\mathfrak{C}$ is $\aleph_0$-saturated, the lemma just says that a certain type is consistent. Now $B \in \mathfrak{C}$, and $\mathfrak{C}$ satisfies the separation scheme for all formulas (not just those of set theory). Hence, for any formula ϕ, $\{(\bar{a}, \bar{b}) \colon \bar{a} \in c, \bar{b} \in B \text{ and } \phi(\bar{a}, \bar{b})\}$ exists in $\mathfrak{C}$. □

Fix such sets D and Z for c. Say that an atom a is *D, Z-definable from* b over c if there are $d \in D$ and a tuple (perhaps of nonstandard length) $\bar{x} \in c$ so that a is the unique atom of B such that $(d, \bar{x}, b, a) \in Z$. We say that a and b are *D, Z-interdefinable* over c if a is D, Z-definable from b over c and b is D, Z-definable from a over c. Notice (and this is the content of the last lemma) that if a is definable from b over c, then it is D, Z-definable from b over c.

We continue now with the notation of the theorem and the model we have built. Suppose that f is an automorphism, which is pointwise definable over a pseudofinite set c. Define the *c-graph* to be the graph whose vertices are the atoms of B, where a, b are adjacent if a and b are D, Z-interdefinable over c. Since D, k^* and c are pseudofinite, this is a graph whose valency is bounded by some pseudofinite number. So in $\mathfrak{C}$ we will be able to apply Lemma 4.6. We will say that $(A_i \colon i < n^*)$ is a *good partition* of the c-graph if it is an element of $\mathfrak{C}$, which partitions the atoms of B into a pseudofinite number of pieces and, for any i and any a, a is adjacent to at most one element of A_i. If $(A_i \colon i < n^*)$ is a good partition, then for all i, j let $f_{i,j}$ be the partial function from A_i to A_j. Define this by letting $f_{i,j}(a)$ be the unique element of A_j if any so that a is adjacent to $f_{i,j}(a)$. Otherwise let $f_{i,j}(a)$ be undefined.

We have proved the following lemma:

LEMMA 4.8. *Use the notation above. For all $a \in A_i$ there is a unique j so that $f(a) = f_{i,j}(a)$.* □

The proof of the last lemma applies in a more general setting. Since we will want to use it later, we will formulate a more general result.

LEMMA 4.9. *Suppose that $\mathfrak{C}$ is an $\aleph_0$-saturated model of an expansion of ZFC^-, which satisfies the separation scheme for all formulas. If A and B are sets in $\mathfrak{C}$ and f is a bijection from A to B such that f and f^{-1} are pointwise definable over some pseudofinite set, then there are (in $\mathfrak{C}$) a partition of A into pseudofinitely many sets $(A_i \colon i < k^*)$ and a pseudofinite collection $(f_{i,j} \colon i, j < k^*)$ of partial one-to-one functions such that, for all i, the domain of $f_{i,j}$ is contained in A_i and, for all $a \in A_i$, there exists j so that $f_{i,j}(a) = f(a)$. Moreover, if P in $\mathfrak{C}$ is a family of $|A|$ (in $\mathfrak{C}$'s sense) subsets of A of cardinality $|A|$, then for every $i < k^*$ and $A^* \in P$, it can be demanded that $|A_i \cap A'| = |A|$ (in $\mathfrak{C}$'s sense).* □

LEMMA 4.10. *Use the notation and assumptions above. For any $i < k^*$, $f \restriction A_i$ is definable.*

Proof. For each $\gamma < \lambda$ such that $\gamma \notin S$ and $\mathrm{cf}\,\gamma \geq \theta$ choose y_γ so that y_γ is the join of a pseudofinite set of atoms contained in A_i and containing $A_i \cap B_\gamma$.

CLAIM 4.11. *For all $a \in A_i$, such that $a \leq y_\gamma$, $f(a)$ is definable from $f(y_\gamma)$ and the set $\{f_{i,j} \colon j < n^*\}$. In particular this claim applies to all $a \in A_i \cap B_\gamma$.*

Proof of the claim. First note that $f(a) < f(y_\gamma)$, and so there is a j such that $f_{i,j}(a) < f(y_\gamma)$. Suppose that there is a $k \neq j$ so that $f_{i,k}(a) < f(y_\gamma)$. Choose a $b \in A_i$ (not neccessarily in B_γ) so that $f(b) = f_{i,k}(a)$ (such a b must exist, since every atom below y_γ is in A_i). Since $f_{i,k}$ is a partial one-to-one function, however, we have $a = b$. But $f(a) = f_{i,j}(a) \neq f_{i,k}(a)$. This is a contradiction. Thus $f \restriction (A_i \cap B_\gamma)$ is defined by "$b = f(a)$ if there exists a $g \in \{f_{i,j}\colon j < n^*\}$ so that $b = g(a)$ and $g(a) < f(y_\gamma)$".

Choose N_γ of cardinality $< \theta$ so the type of $f(y_\gamma), (f_{i,j}\colon j < n^*)/\mathfrak{C}_\gamma$ does not split over N_γ. Notice that $f \restriction (A_i \cap B_\gamma)$ is definable as a disjunction of types over N_γ, namely the types satisfied by the pair $(a, f(a))$. By Fodor's lemma and cardinal arithmetic, there are an N and a stationary set, where all of the $N_\gamma = N$ and all of the definitions as a disjunction of types coincide. Hence we have $f \restriction A_i$ defined as a disjunction of types over a, N, set of cardinality $< \theta$. We now want to improve the definability to definability by a formula. We show that, for all γ, $f \restriction (A_i \cap B_\gamma)$ is defined by a formula with parameters from N. This will suffice, since some choice of parameters and formula will work for unboundedly many (and hence all) γ.

Suppose that $f \restriction (A_i \cap B_\gamma)$ is not definable by a formula with parameters from N. Consider the following type in variables x_1, x_2, z_1, z_2:

> $\{\phi(x_1, x_2)$ if and only if $\phi(z_1, z_2)\colon\ \phi$ a formula with parameters from $N\} \cup \{x_1, z_1 \leq y_\gamma\} \cup \{\psi(x_1, x_2), \neg\psi(z_1, z_2)\}$,
> where $\psi(u, v)$ is a formula saying "there exists $g \in \{f_{i,j}\colon j < n^*\}$ so that $u = g(v)$ and $g(u) < f(y_\gamma)$".

This type is consistent, because by hypothesis it is finitely satisfiable in $A_i \cap B_\gamma$. As $\mathfrak{C}$ is θ-saturated, there are a_1, a_2, b_1, b_2 realizing this type. Since $a_1, b_1 \leq y_\gamma$, $\psi(a_1, a_2)$ implies that $f(a_1) = a_2$ and, similarly, $\neg\psi(b_1, b_2)$ implies that $f(b_1) \neq b_2$. On the other hand, a_1, a_2 and b_1, b_2 realize the same type over N. This contradicts the choice of N and finishes the proof of Lemma 4.10 and Theorem 4.1. □

The information we obtain above is something, but what we really want is a definablility requirement on every automorphism of every definable Boolean algebra. It is easy to modify the proof so that we can get the stronger result. Namely, in the application of the Black Box, we attempt to predict the definition of a definable Boolean algebra in the scope of P as well as the construction of the model $\mathfrak{C}$ and an automorphism. With this change the proof goes as before. So the following stronger theorem is true:

THEOREM 4.12. *There is a model $\mathfrak{C}$ of T_1 so that if $B \subseteq P(\mathfrak{C})$ is a definable atomic Boolean algebra and f is any automorphism of B (as a Boolean algebra), then there is a pseudofinite set c such that, for any atom $b \in B$, $f(b)$ is definable from $\{b\} \cup c$.* □

As before, if θ is taken to be uncountable, then we can find a finite sublanguage to use for all of the definitions.

The proof of the analogue of Theorem 4.1 for ordered fields is quite similar. Let T_1 denote the theory constructed in Theorem 3.2.

THEOREM 4.13. *There is a model $\mathfrak{C}$ of T_1 so that if $F_1, F_2 \subseteq P(\mathfrak{C})$ are definable ordered fields and f is any isomorphism from F_1 to F_2, then there is a pseudofinite set c so that, for all $b \in F_1$, $f(b)$ is definable from $\{b\} \cup c$.*

Proof. To simplify the proof we will assume that $F = P(\mathfrak{C})$ is an ordered field and f is an automorphism of F. The construction is similar to the one for Boolean algebras, although we will distinguish between the case where $\theta = \omega$ and the case in which θ is uncountable. We will need to take a little more care in the case of uncountable cofinalities. The difference between the case of ordered fields (dealt with now) and Boolean algebras occurs at stages $\delta \in S$. Again we predict the sequence of structures $(\mathfrak{C}_i : i < \lambda)$ and an automorphism f_α of the ordered field F. We say that *an obstruction occurs at α* if we can make the choices below. (The intuition behind the definition of an obstruction is that there are infinitesimals of arbitrarily high order for which the automorphism is not definable.) There are two cases, the one where $\theta = \omega$ and the second where θ is uncountable.

Since it is somewhat simpler, we will consider the first case where $\theta = \omega$. Suppose that an obstruction occurs at α. By this we mean that we can choose $N_\alpha \in M_0^\alpha$ such that, for each $i < \omega$, we can choose $a_i^\alpha \in [\gamma_{\alpha,i}^-, \gamma_{\alpha,i}^+)$ so that $a_i^\alpha/\mathfrak{C}_{\gamma_{\alpha,i}^-}$ does not split over N_α, $a_i^\alpha > 0$, the a_i^α make the same cut in $F_{\gamma_{\alpha,i}^-}$ as 0 does and $f(a_i^\alpha)$ is not definable over a_i^α and parameters from $\mathfrak{C}_{\gamma_{\alpha,i}^-}$.

Then we let $x_i^\alpha = \sum_{j \le i} a_i^\alpha$. We choose x_α so that they realize the average type over some nonprincipal ultrafilter of $\{x_i^\alpha : i < \omega\}$. As before, we can show that both the inductive hypothesis and Claim 4.3 are satisfied. Notice in the construction that $f_\alpha(x_i^\alpha)$ is not definable over $\{a_i^\alpha\} \cup \mathfrak{C}_{\gamma_{\alpha,i}^-}$, since $x_i^\alpha = x_{i-1}^\alpha + a_i^\alpha$ and $f(x_i^\alpha) = f(x_{i-1}^\alpha) + f(a_i^\alpha)$ (here we conventionally let $x_i^\alpha = 0$ when it is undefined) and each M_i^α is closed under f_α. Notice as well that, for all i, $x_i^\alpha < x_\alpha$ and the x_i^α and x_α make the same cut in $\mathfrak{C}_{\gamma_{\alpha,i}^+}$.

Suppose now that f is an automorphism, which is not pointwise definable over any $\mathfrak{C}_\gamma$. Since F is an ordered field, this is equivalent to saying that f is not definable on any interval. Also, since $\mathfrak{C}_\gamma$ is contained in a pseudofinite set (in $\mathfrak{C}$), there is a positive interval, which makes the same cut in the pseudofinite set (and hence in F_γ) that 0 does. We can thus choose the a_γ in this interval so that $f(a_\gamma)$ is not definable over $\mathfrak{C}_\gamma \cup \{a_\gamma\}$. Arguing as before, we can find a set $N \subseteq \mathfrak{C}_\gamma$ of cardinality less than θ such that, for all but boundedly many γ, the a_γ can be chosen so that $a_\gamma/\mathfrak{C}_\gamma$ does not split over N. Still arguing as before, we get an ordinal α such that an obstruction occurs at α. Consider now $f(x_\alpha)$, i and a finite set $N_\alpha \subseteq \mathfrak{C}_{\gamma_{\alpha,i}^-}$ so that $f(x_\alpha)/\mathfrak{C}_{\gamma_{\alpha,i}^+}$ does not split over $N_\alpha \cup \{a_i^\alpha\}$. Since $f(x_i^\alpha)$ is not definable over $\{a_i^\alpha\} \cup \mathfrak{C}_{\gamma_{\alpha,i}}$, there is a $b > f(x_i^\alpha)$, $b \in F_{\gamma_{\alpha,i}^+}$, which realizes the same type over $N_\alpha \cup \{a_i^\alpha\}$. (Otherwise $f(x_i^\alpha)$ would be the rightmost element satisfying some formula with parameters in this set.) However $f(x_i^\alpha) < f(x_\alpha) < b$, contradicting the choice of N_α.

We now consider the second case where θ is uncountable. Here we have to do something different at the limit ordinals i. If i is a limit ordinal, we choose $x_i^\alpha \in [\gamma_{\alpha,i}^-, \gamma_{\alpha,i}^+) \cap M_{i+1}^\alpha$ to realize the average type of the $\{x_j^\alpha : j < i\}$. For the successor case we let $x_{i+1}^\alpha = x_i^\alpha + a_{i+1}^\alpha$ (where the a_j^α are chosen as before). In this construction, for all γ and all i, $x_i^\alpha / \mathfrak{C}_\gamma$ does not split over $N_\alpha \cup \{x_j^\alpha : \gamma_{\alpha,j}^- < \gamma\}$. This is enough to verify the inductive hypothesis and Claim 4.3 if we restrict the statement to a successor ordinal i. The rest of the proof can be finished as above. □

To apply the theorem we explicate the explanation in the preliminaries.

Observation 4.14. For proving the compactness of $L(Q_{\mathrm{Of}})$ (or $L(Q_{\mathrm{BA}})$, etc.), we only have to do the following:

(**) Let T be a first-order theory T and let P and R be unary predicates. Then for every model M of T and an automorphism f of a definable ordinal field (or Boolean algebra, etc.) $\subseteq P^M$ definable (with parameter), for some $c \in M$, $R(x, y, c)$ defines f.

Further, T has a model M^* such that every automorphism f of a definable ordered field (or Boolean algebra, etc.) $\subseteq P^M$ is defined in M^* by $R(x, y, c)$ for some $c \in M$.

Assume (**) and let T_0 be a given theory in the stronger logic; without loss of generality, all formulas are equivalent to relations. For simplicity ignore function symbols.

For every model M let M' be the model with the universe $|M| \cup \{f : f \text{ a partial function from } |M| \text{ to } |M|\}$, and let the relations be those of M, $P^{M'} = |M|$, $\mathrm{R}^{M'} = \{(f, a, b) : a, b \in M,\ f \text{ a partial function from } |M| \text{ to } |M| \text{ and } f(a) = b\}$. There is a parallel definition of T' from T and, if we intersect with first-order logic, we get a theory to which it suffices to apply (**). This proves the observation.

5. Augmented Boolean algebras

For Boolean algebras we do not have the full result we would like, but we can define the notion of augmented Boolean algebras and then prove the compactness theorem for quantification over automorphisms of these structures.

Definition. An *augmented Boolean algebra* is a structure $(B, \leq, I, P)$ in which $(B, \leq)$ is an atomic Boolean algebra; I is an ideal of B containing all the atoms; $P \subseteq B$, $|P| > 1$, for $x \neq y \in P$ the symmetric difference of x and

y is not in I; and for all atoms $x \neq y$ there is a $z \in P$ such that either $x \leq z$ and $y \not\leq z$ or vice versa.

Notice that if we know the restriction of an automorphism f of an augmented Boolean algebra to P then we can recover f. Since for any atom x, $f(x)$ is the unique z, such that for all $y \in P$, $z \leq f(y)$ if and only if $x \leq y$ and the action of f on the atoms determines its action on the whole Boolean algebra.

Let Q_{Aug} be the quantifier, whose interpretation "$Q_{\text{Aug}} f\,(B_1, B_2)\ \ldots$" holds if there is an isomorphism f of the augmented Boolean algebras B_1, B_2 so that ....

THEOREM 5.1. *The logic $L(Q_{\text{Aug}})$ is compact.*

Proof. By Theorem 3.1 (and Lemma 4.9) it is enough to show that if T is a theory saying that every automorphism of a definable Boolean algebra $\subseteq P$ is definable by a fixed formula and T_1 and $\mathfrak{C}$ are as above, then every automorphism f of a definable augmented Boolean algebra $(B, \leq, I, P)$ of $P(\mathfrak{C})$ is definable. We work for the moment in $\mathfrak{C}$. First note that I contains the pseudofinite sets, so $\mathfrak{C}$ thinks that the cardinality of B is some μ^* and, for every $x, y \in P$, the symmetric difference of x and y contains μ^* atoms (since $\mathfrak{C}$ thinks that B is μ^* saturated).

So by Lemmas 4.9 and 4.10 we can find a set of atoms A such that $f \restriction A$ is definable and, for every x and y, A contains some element in the symmetric difference. Now we can define $f(y)$ as the unique $z \in B$ so that, for all $a \in A$, $a \leq y$ if and only if $f(a) \leq z$. Clearly $f(y)$ has this property. Suppose for the moment that there is some $z \neq f(y)$ that also has this property. Since f is an automorphism, there is an $x \in P$ such that $f(x) = z$. Now choose $a \in A$ in the symmetric difference of x and y. Then $f(a)$ is in exactly one of $f(y)$ and $f(x)$. □

6. Ordered fields

Here we will prove the compactness of the quantifier Q_{Of}. Let Q_{Of} be the quantifier, whose interpretation "$Q_{\text{Of}} f\,(F_1, F_2)\ \ldots$" holds if there is an isomorphism f of the ordered fields F_1, F_2 so that .... We will use various facts about dense linear orders. A subset of a dense linear order is *somewhere dense* if it is dense in some nonempty open interval. A subset, which is not somewhere dense, is *nowhere dense*. The first few properties are standard and follow easily from the fact that a finite union of nowhere-dense sets is nowhere dense.

PROPOSITION 6.1. *If a somewhere-dense set is divided into finitely many pieces, one of the pieces is somewhere dense.* □

PROPOSITION 6.2. *If $\{f_k\colon k \in K\}$ is a finite set of partial functions defined on a somewhere-dense set A, then there is a somewhere-dense subset $A' \subseteq A$ so that either each f_k is total on A' or no element of the domain of f_k is in A'.* □

We can define an equivalence relation on functions to a linearly ordered set by $f \equiv g$ if, for every nonempty open interval I, the symmetric difference of $f^{-1}(I)$ and $g^{-1}(I)$ is nowhere dense.

PROPOSITION 6.3. *Suppose that $\mathcal{F}$ is a finite collection of functions from a somewhere-dense set A to a linearly ordered set. Then there are $\mathcal{I}$, a collection of disjoint intervals (not necessarily open), and a somewhere-dense set $A' \subseteq A$ so that, for all $f, g \in \mathcal{F}$, either $f \restriction A' \equiv g \restriction A'$ or there are disjoint intervals $I, J \in \mathcal{I}$, such $f(A') \subseteq I$ and $g(A') \subseteq J$.*

Proof. The proof is by induction on the cardinality of $\mathcal{F}$. Suppose that $\mathcal{F} = \mathcal{G} \cup \{g\}$, $\mathcal{J}$ is a set of intervals and $A'' \subseteq A$ is a somewhere-dense set satisfying the conclusion of the theorem with respect to $\mathcal{G}$. If there are some $f \in \mathcal{G}$ and A''', a somewhere-dense subset of A'' such that $f \restriction A''' \equiv g \restriction A'''$, then we can choose a somewhere-dense $A' \subseteq A'''$ such that, for all $I \in \mathcal{J}$, we have $(f \restriction A')^{-1}(I) = (g \restriction A')^{-1}(I)$. Then A' and $\mathcal{J}$ are as required in the theorem. We can thus assume that no such f and A''' exist.

Let $\{f_0, \ldots, f_n\}$ enumerate a set of $\equiv$ class representatives of $\mathcal{G}$ and let $\mathcal{J} = \{J_0, \ldots, J_n\}$, where $f_k(A'') \subseteq J_k$ for all k (so for $\ell < m \leq n$, $J_\ell \cap J_m = \emptyset$). By induction on k we will define a descending sequence A_k of somewhere-dense subsets of A'' and intervals $I_k \subseteq J_k$ with the property that $f_k(A_k) \subseteq I_k$ and $g(A_k)$ is disjoint from I_k. Let $A_{-1} = A''$. Consider any k and suppose that A_{k-1} has been defined. If possible, choose an interval $J'_k \subseteq J_k$ so that the symmetric difference of $(g \restriction A_{k-1})^{-1}(J'_k)$ and $(f_k \restriction A_{k-1})^{-1}(J'_k)$ is somewhere dense. There are two possibilities. If $(f_k \restriction A_{k-1})^{-1}(J'_k) \setminus (g \restriction A_{k-1})^{-1}(J'_k)$ is somewhere dense, then let $A_k = (f_k \restriction A_{k-1})^{-1}(J'_k) \setminus (g \restriction A_{k-1})^{-1}(J'_k)$ and let $I_k = J'_k$. Otherwise we can choose I_k as a subinterval of $J_k \setminus J'_k$ and A_k as a somewhere-dense subset of $(g \restriction A_{k-1})^{-1}(J'_k)$ so that $f_k(A_k) \subseteq I_k$. If there were no respective J'_k, then we would have $f_k \restriction A_{k-1} \equiv g \restriction A_{k-1}$, contrary to our assumption. Given A_n and $I_0, \ldots, I_n$, we can choose $A_{n+1} \subseteq A_n$ to be somewhere dense and an interval I_{n+1} disjoint from I_k, for all k, so that $g(A_{n+1}) \subseteq I_{n+1}$. Finally we let $A' = A_{n+1} \cap \bigcap_{f \in \mathcal{G}} f^{-1}(\bigcup_{k \leq n} I_k)$ and $\mathcal{I} = \{I_0, \ldots, I_n\}$. □

THEOREM 6.4. *The logic $L(Q_{\mathrm{Of}})$ is compact.*

Proof. We work in the model $\mathfrak{C}$ constructed before (suffice by Theorem 3.2 (and Lemma 4.9)). Suppose we have two definable ordered fields F_1 and F_2 contained in $P(\mathfrak{C})$ and an isomorphism f between them. Since f and f^{-1} are locally definable over a pseudofinite set, there are $(A_i\colon i < k^*)$ and $(f_{i,j}\colon i,j < k^*)$, as in Lemma 4.9. Since $\mathfrak{C}$ satisfies the fact that k^* is finite, there is some i so that the A_i are somewhere dense. Fix such an i and for notational simplicity drop the subscript i. Hence f_j denotes $f_{i,j}$. By restricting ourselves to a somewhere-dense subset (and perhaps eliminating some of the f_j), we can assume that each f_j is a total function (we work in $\mathfrak{C}$ to make this choice) (use Proposition 6.2). Again choosing a somewhere-dense subset, we can find a somewhere-dense set A and a collection $\mathcal{I}$ of disjoint intervals of F_2 so that the conclusion of Proposition 6.3 is satisfied. Since each f_j is a one-to-one function, we can assume that all the intervals in $\mathcal{I}$ are open. Choose now an interval J contained in F_1 so that A is dense in J.

Next choose $a_1 < a_2 \in J$ and an interval $I \in \mathcal{I}$ so that $f(a_1), f(a_2) \in I$. To see that such objects exist, first choose any $a_1 \in A \cap J$. There is a unique interval $I \in \mathcal{I}$ such that $f(a_1) \in I$. Choose $b \in I$ so that $f(a_1) < b$ and let a_2 be any element of $J \cap A$ such that $a_1 < a_2 < f^{-1}(b)$. Of course this definition of a_1 and a_2 cannot be made in $\mathfrak{C}$, but the triple (a_1, a_2, I) exists in $\mathfrak{C}$. Without loss of generality we can assume that $A \subseteq (a_1, a_2) = J$. Consider now $\mathcal{F} = \{f_j\colon f_j(A) \subseteq I\}$. This is an equivalence class of functions and, for every $a \in A$, there is some $g \in \mathcal{F}$ so that $g(a) = f(a)$.

The crucial fact is that, for all $b \in J$ and $g \in \mathcal{F}$, $B = \{a \in A\colon b < a \text{ and } g(a) < f(b)\}$ is nowhere dense. Assume not. Since $\mathcal{F}$ is an equivalence class, for all $h \in \mathcal{F}$, $\{a \in B\colon h(a) \geq f(b)\}$ is a nowhere-dense set. But since $\mathcal{F}$ is a pseudofinite set, $\{a \in B\colon \text{for some } h \in \mathcal{F}, h(a) \geq f(b)\}$ is nowhere dense. So there is some $a \in A$ such that $b < a$ and, for all $g \in \mathcal{F}$, $g(a) < f(b)$. This gives a contradiction, since there is some g such that $f(a) = g(a)$. Similarly $\{a \in A\colon a < b \text{ and } g(a) > f(b)\}$ is nowhere dense.

With this fact in hand we can define f on J by $f(b)$ being the greatest x. Hence, for all $g \in \mathcal{F}$, both $\{a \in A\colon b < a \text{ and } g(a) < x\}$ and $\{a \in A\colon a < b \text{ and } g(a) > x\}$ are nowhere dense. Since an isomorphism between ordered fields is definable if and only if it is definable on an interval, we are done proving that the logic $L(Q_{\mathrm{Of}})$ is compact. □

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REFERENCES

[1] J. BARWISE and S. FEFERMAN, eds., *Model-Theoretic Logics*, Springer–Verlag, New York, 1985.

[2] P.C. Eklof, A transfer theorem for nonstandard uniserials, Proc. A.M.S. **114** (1992), 593-600.
[3] L. Fuchs and S. Shelah, Kaplansky problem on valuation rings, Proc. A.M.S. **105** (1989), 25-30.
[4] P. Lindstrom, On extensions of elementary logic, Theoria **32**(1969), 1-11.
[5] O. Ore, *The Four-Color Problem*, Academic Press, New York, 1967.
[6] B.L. Osofsky, Construction of nonstandard uniserial modules over valuation domains, Bull. A.M.S. **25** (1991), 89-97.
[7] ———, Construction of nonstandard uniserial modules over valuation domains, Azumaya algebras, actions and modules, Cont. Math. **124** (1992), 151-164.
[8] M. Rubin and S. Shelah, On the elementary equivalence of automorphism groups of Boolean algebras, downward Skolem–Löwenheim theorems and compactness of related quantifiers, J. Symb. Logic **45** (1980), 265-283.
[9] S. Shelah, Models with second order properties, II. On trees with no undefinable branches, Ann. of Math. Logic **14** (1978), 73-87.
[10] ———, Models with second order properties, IV. A general method and eliminating Diamonds, Ann. Pure and Appl. Logic **25** (1983), 183-212.
[11] ———, Generalized quantifiers and compact logic, Trans. A.M.S. **204** (1975), 379-419.
[12] ———, Compact Logics in ZFC: complete embeddings of atomless Boolean algebras, to appear in *Non Structure Theory*, Oxford University Press.
[13] ———, Compactness in ZFC of the quantifiers "complete embedding of Boolean rings", Publ. Sh. 482., in preparation.
[14] ———, Classification theory and the number of non-isomorphic models, Studies in Logic and the Foundation of Math. **92**, North Holland, 1978, 542 + xvi; 1990, 705 + xxxiv.
[15] ———, *Non Structure Theory*, to be published by Oxford University Press.
[16] ———, Refuting Ehrenfeucht Conjecture on rigid models, Proc. of Symp. in memory of A. Robinson, Yale, 1975, A special volume in Israel J. of Math. **25** (1976), 273-286.

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