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ON UNIVERSAL GRAPHS WITHOUT INSTANCES OF CH*

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We first prove the consistency of: there is a universal graph of power $\aleph_1 < 2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$. The consistency of the non-existence of a universal graph of power $\aleph_1$ is trivial. Add $\aleph_2$ Cohen generic reals. We then show that we can have $2^{\aleph_0} = \aleph_2 < 2^{\aleph_1}$, and get similar results for other cardinals.

1. A universal graph in $\aleph_1 < 2^{\aleph_0}$

In this section we shall concentrate on the simplest case. Notice that a graph G is just a pair (A, R), A a set, R a symmetric and reflexive two place relation on it.

1.1. Theorem. *Suppose $V \models 2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$. Then for some forcing notion P of power $\aleph_2$, in V^P, $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ and there is a universal graph of power $\aleph_1$.*

Proof. The proof is broken to stages.

1.2. Preliminary Forcing. *For some forcing notion P, $|P| = \aleph_2$, P satisfying the $\aleph_2$-c.c. is proper and in $V_a = V^P$, there is a set $\mathfrak{A} = \{A_\alpha : \alpha \in \omega_2\}$ of $\aleph_2$ subsets of ω_1, such that:*

(a) *$A \neq B \in \mathfrak{A}$ implies $A \cap B$ is finite.*

(b) *Each $A \in \mathfrak{A}$ is a stationary subset of ω_1.*

Note that $\aleph_\alpha^{V_a} = \aleph_\alpha$ for every α, $(2^{\aleph_\alpha})^{V_a} = (2^{\aleph_\alpha})^V + \aleph_2$.

Proof. First we can force stationary $B_\alpha \subseteq \omega_1$ $(\alpha < \omega_2)$ such that $B_\alpha \cap B_\beta$ is countable for $\alpha \neq \beta$ (e.g. force $\diamondsuit_{\aleph_1}$). Next use a forcing of Baumgartner on $\langle B_\alpha : \alpha < \omega_2 \rangle$:

(A) A condition is a finite set of atomic conditions with no three contradicting.

(B) An atomic condition is:

(I) $[i \in \mathbf{A}_\alpha]$ where $i \in B_\alpha$, $\alpha < \omega$, or

(II) $\mathbf{A}_\alpha \cap \mathbf{A}_\beta \subseteq w$ where w is finite, $\alpha \neq \beta$.

(C) Three atomic conditions are contradictory if they have the form $i \in \mathbf{A}_\alpha$, $i \in \mathbf{A}_\beta$, $\mathbf{A}_\alpha \cap \mathbf{A}_\beta \subseteq w$ where $i \notin w$.

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1.3. General Description. We shall define a finite support iteration of forcing notions saisfying the c.c.c. $\bar{Q}=\langle P_\alpha, \mathbf{Q}_\alpha : \alpha<\omega_2\rangle$. Q_0 forces a graph $(\omega_1, \mathbf{R}_0)$, which shall be a universal graph of power $\aleph_1$. We shall define the $\mathbf{Q}_\alpha$ by induction on α (together with some auxilliary things), and will have to prove that it satisfies the c.c.c.

In stage $\alpha>0$ we will have a P_α-name $\mathbf{R}_\alpha$ so that $\Vdash_{P_\alpha}$ "$(\omega_1, \mathbf{R}_\alpha)$ is a graph" and in $V_a^{P_\alpha}$, $\mathbf{Q}_\alpha$ will force an embedding $\mathbf{f}_\alpha$ of the graph $(\omega_1, \mathbf{R}_\alpha)$ into the graph $(\omega_1, \mathbf{R}_0)$. It is known that we can take care that every P_{ω_2}-name of a graph on ω_1, appears as $(\omega_1, \mathbf{R}_\alpha)$ for some $\alpha<\omega_2$.

The problem is, of course, that the various $\mathbf{f}_\alpha$ may give contradicting demands on $(\omega_1, \mathbf{R}_0)$. In order to avoid this as much as possible we shall make the $\mathbf{f}_\alpha$'s such that for $\beta<\alpha$ $(\operatorname{Rang} \mathbf{f}_\alpha)\cap(\operatorname{Rang} \mathbf{f}_\beta)$ is finite. It is reasonable to demand that "$\operatorname{Rang} \mathbf{f}_\alpha \subseteq A_\alpha$".

1.4. The full inductive definition. We let

$$Q_0=\{(w, r):\ w \text{ a finite subset of } \omega_1,\ r \text{ a reflexive symmetric two-place relation on } w\}.$$

The order on Q_0 is: $q_1\le q_2$ iff q_1 is a submodel of q_2.

Now F will be a function which for each finite support iteration $\bar{Q}^\gamma=\langle P_\alpha, \mathbf{Q}_\alpha : \alpha<\gamma\rangle$, $F(\bar{Q}^\gamma)$ is a P_γ-name of a graph $(\omega_1, \mathbf{R})$. Let λ be a large enough regular cardinal.

Now for each $\alpha>0$, we let $(\omega_1, \mathbf{R}_\alpha)=F(\langle P_\beta, \mathbf{Q}_\beta : \beta<\alpha\rangle)$, and we shall define $\langle N_{\alpha,i} : i<\omega_1\rangle$, and $\mathbf{Q}_\alpha$.

First let $\langle N_{\alpha,i} : i<\omega_1\rangle$ be an increasing continuous sequence of countable elementary submodels of $(H(\lambda)^{V_\alpha}, \in)$ such that

$$\langle P_\beta, \mathbf{Q}_\beta : \beta<\alpha\rangle, \qquad (\omega_1, \mathbf{R}_\alpha), \qquad \langle A_\beta : \beta<\omega_2\rangle \text{ belong to } N_{\alpha,0},$$

(hence $A_\alpha\in N_{\alpha,0}$), and $\alpha+1\subseteq\bigcup_{i<\omega_1} N_{\alpha,i}$. Note this is done in V_a, so $\langle N_{\alpha,i} : i<\omega\rangle\in V_a$ and even $\langle\langle N_{\beta,i} : i<\omega_1\rangle : \beta\le\alpha\rangle\in V_a$.

Note that $\xi_\alpha(i)\overset{\text{def}}{=}N_{\alpha,i}\cap\omega_1$ is always a limit ordinal and $\langle\xi_\alpha(i) : i<\omega_1\rangle$ is increasing continuous. As A_α is a stationary subset of ω_1, $\{i : i=\xi_\alpha(i)\in A_\alpha\}$ is stationary. So w.l.o.g. $\xi_\alpha(i)\in A_\alpha$ for every non-limit $i<\omega_1$. We let $A'_\alpha=\{\xi_\alpha(i+1) : i<\omega_1\}$ and note that $A'_\alpha\in V_a$.

Now we come to the main point: defining Q_α (in $V_a^{P_\gamma}$):

(A) A member of $\mathbf{Q}_\alpha$ will consist of finitely many atomic conditions (see B) with no two of them explicitly contradictory (see (C)).

(B) There are two kinds of atomic conditions:

(I) $\mathbf{f}_\alpha(i)=j$ where $i<j, j\in A'_\alpha$, and $|A'_\alpha\cap(i, j)|<\aleph_0$ (or if you want, the sequence $\langle\alpha, 0, i, j\rangle$, is a condition).

(II) $i\notin\operatorname{Range} \mathbf{f}_\alpha$.

(C) We shall have to say when two atomic conditions are explicitly contradictory; this occurs just in one of the following three cases:

(α) *One-to-one*: $\mathbf{f}_\alpha(i_1)=j_1$ and $\mathbf{f}_\alpha(i_2)=j_2$ when $i_1\neq i_2, j_1=j_2$ or $i_1=i_2, j_1\neq j_2$.

(β) *Embedding*: $\mathbf{f}_\alpha(i_1)=j_1$ and $\mathbf{f}_\alpha(i_2)=j_2$ when $V_a^{P_\alpha}\vDash$"$i_1\boldsymbol{R}_\alpha i_2\equiv\neg j_1\boldsymbol{R}_0 j_2$".

(γ) *Range*: $\mathbf{f}_\alpha(i)=j$ and $j\notin \operatorname{Rang}\mathbf{f}_\alpha$.

The order is inclusion.

Explanations. $A'_\alpha\cap(i,j)$ should be finite in order that $\boldsymbol{Q}_\alpha$ satisfies the c.c.c. Each $i<\omega_1$ should have only countably many possible images. Why in (B)(I), $j\in A'_\alpha$? for reasons similar to those in the club method (see [1]).

1.5. $\boldsymbol{Q}_\alpha$ gives an embedding. We want to prove (in $V_a^{P_\alpha}$) that $\Vdash_{Q_\alpha}$ "(ω_1, R_α) is embeddable into (ω_1, R_0)". We have a natural name for exemplifying this: $\mathbf{f}_\alpha$ (defined by $f_\alpha(i)=j$ iff $[\mathbf{f}_\alpha(i)=j]$ belongs to the generic subset of Q_α). It is (forced to be) an embedding by 1.4(C)(β). But we should still prove that for every $i<\omega_1$, $\Vdash_{Q\alpha}$ "$i\in\operatorname{Dom}\mathbf{f}_\alpha$". This is equivalent to proving that for every $q\in Q_\alpha$ for some $j, q\cup\{[\mathbf{f}_\alpha(i)=j]\}\in Q_\alpha$ (assuming q itself has no such member). By 1.4(B)(I) we have countably many candidates:

$$B=\{j\in A'_\alpha : j>i, (i,j)\cap A'_\alpha \text{ is finite}\}.$$

Now 1.4(C)(α) disqualifies finitely many of them: those j s.t. $(\exists i_1\neq i)[\mathbf{f}_\alpha(i_1)=j]\in q$, and also 1.4(C)($\gamma$) disqualifies finitely many j's: those j s.t. $[f\notin\operatorname{Rang}\mathbf{f}_\alpha]\in q$. What about 1.4(C)($\beta$)? As $B\in V_a$ (as $A'_\alpha\in V_a$, $\langle N_{\alpha,i}: i<\omega_1\rangle\in V_a$), by the definition of Q_0, infinitely many $j\in B$ satisfy this so we finish the proof of 1.5.

Now the rest of the proof is dedicated to proving that $\boldsymbol{Q}_\alpha$ satisfies the c.c.c., or what is equivalent, that $P_{\alpha+1}$ satisfies the c.c.c. For this we shall derive more detailed information on $\bar{Q}^{\alpha+1}$ (using the fact that all $\boldsymbol{Q}_\beta$, $\beta<\alpha$, were defined in a way similar to that of $\boldsymbol{Q}_\alpha$).

1.6. Nice dense subsets of $P_{\alpha+1}$. Remember that

$$P_\beta=\{p : p \text{ a finite function with domain} \subseteq\beta \text{ and } p(\gamma) \text{ a } P_\gamma\text{-name of a member of } \boldsymbol{Q}_\gamma, \text{ for } \gamma\in\operatorname{Dom} p\}$$

(and for $\gamma\notin\operatorname{Dom} p$ we let $p(\gamma)=\emptyset$). Let

$$D^0_\beta=\{p\in P_\beta : \text{for each } \gamma\in\operatorname{Dom} p,\ p(\gamma) \text{ is an actual finite set of atomic conditions}\}.$$

Note that not every function p with domain a finite subset of β, $p(\gamma)$ a finite set of atomic conditions of the forms mentioned in 1.4(B), is in D^0_β, we need $p\restriction\gamma\Vdash_{P_\gamma}$ "$p(\gamma)\in\boldsymbol{Q}_\gamma$" for each $\gamma\in\operatorname{Dom} p$.

1.7. Definition. $D^1_\beta=\{p:p$ is a finite function with domain $\subseteq\beta$ and for $\gamma\in \mathrm{Dom}\,p$, $p(\gamma)$ satisfies the demand for $p(\gamma)\in \boldsymbol{Q}_\gamma$ in 1.4(A), (B), (C) except that "there are no two atomic conditions in $p(\gamma)$ which are explicitly contradictory by 1.4(C)(β)"$\}$.

For $p\in D^1_\beta$, $\gamma\notin \mathrm{Dom}\,p$, let $p(\gamma)=\emptyset$.

We define an order on D^1_β:

$$p\leqslant r \quad \text{iff} \quad \text{for every } \gamma \quad p(\gamma)\subseteq r(\gamma).$$

1.8. Fact. (1) *D^0_β is a dense subset of P_β.*

(2) *On $D^1_\beta\cap P_\beta$ the orders of P_β and of D^1_β coincide.*

(3) *For $p\in D^1_\beta$, $p\in D^0_\beta$ iff for every $\gamma\in\mathrm{Dom}\,p$ and $[f_\gamma(i_1)=j_1]$, $[f_\gamma(i_2)=j_2]$ in $p(\gamma)$*

$$p\restriction\gamma\Vdash_{P_\gamma}\text{“}i_1\boldsymbol{R}_\gamma i_2 \text{ iff } j_1\boldsymbol{R}_0 j_2\text{”}$$

(prove $p\restriction\gamma\in P_\gamma$ by induction).

(4) *If $p\in D^1_\beta$, $w\subseteq\mathrm{Dom}\,p$ then $p\restriction w\in D^1_\beta$.*

1.9. Fact. *If $p^1,p^2\in D^0_\beta$, and for every $\gamma\in\mathrm{Dom}\,p^1\cup\mathrm{Dom}\,p^2$, $p^1(\gamma)\subseteq p^2(\gamma)$ or $p^2(\gamma)\subseteq p^1(\gamma)$, then $p^1\vee p^2\in D^0_\beta$ where $(p^1\vee p^2)(\gamma)=p^1(\gamma)\cup p^2(\gamma)$ for $\gamma\in \mathrm{Dom}\,p^1\cup\mathrm{Dom}\,p^2$.*

Now we continue with

1.10. Definition. *For $\gamma\leqslant\alpha$, $q\in\boldsymbol{Q}_\gamma$, and $\delta<\omega_1$ we let, if $\gamma>0$*

$$q^{[\delta]}=\{[f_\gamma(i)=j]:[f_\gamma(i)=j]\in q \text{ and for some } \varepsilon<\omega_1,\ j<\xi_\gamma(\varepsilon)\leqslant\delta\}$$
$$\cup\{[j\notin\mathrm{Rang}\,f_\gamma]:[j\notin\mathrm{Rang}\,f_\gamma]\in q \text{ and for some } \varepsilon<\omega_1,\ j<\xi_\gamma(\varepsilon)\leqslant\delta\}$$
$$q^{(\delta)}=q^{[\delta]}\cup\{[j\notin\mathrm{Rang}\,f_\gamma]:[j\notin\mathrm{Rang}\,f_\gamma]\in q\}$$

If $\gamma=0$, $q^{(\delta)}=q^{[\delta]}=q\restriction\delta$, i.e., if $q=(w,r)$, then

$$q^{[\delta]}=(w\cap\delta, r\restriction(w\cap\delta)).$$

1.11. Definition. (1) For $p\in P_\gamma$, and limit $\delta<\omega_1$, let $p^{[\delta]}$ be a function with domain $\mathrm{Dom}\,p$ and $p^{[\delta]}(\gamma)=(p(\gamma))^{[\delta]}$.

(2) We define $p^{(\delta)}$ similarly. We can make those definitions even for $p\in D^1_\beta$.

1.12. Fact. (1) *For any $\gamma>0$ limit δ and $q\in Q_\gamma$, $q^{[\delta]}=\emptyset$ or for some ε, $q^{[\delta]}=q^{[\xi_\gamma(\varepsilon)]}$.*

(2) *If $p\in D^0_{\beta+1}$, $\varepsilon<\omega_1$, then $(p^{[\xi_\beta(\varepsilon)]}\restriction|N_{\beta,\varepsilon}|)\in N_{\beta,\varepsilon}$.*

(3) *If $p\in D^0_\beta$, $\delta<\omega_1$, then $p^{[\delta]}\in D^1_\beta$ and $p^{(\delta)}\in D^1_\beta$.*

(4) *$p^{[\delta]}\leqslant p^{(\delta)}\leqslant p$ (in D^1_β).*

(5) *If $p\in D^0_\beta$, $p\leqslant r\in D^1_\beta$, $r=r^{(\delta)}$, $r^{[\delta]}\leqslant p$, then $r\in D^0_\beta$.*

1.13. Definition. Let $D_\beta=\{p\in D^0_\beta:$ for every limit $\delta<\omega_1$, $p^{[\delta]}\in D^0_\beta\}$.

1.14. The Crucial Claim. *D_β is a dense subset of P_β (for $\beta \leqslant \alpha+1$).*

Proof. We prove this by induction on β. For $\beta=0$ there is nothing to prove, the case β is limit is trivial as we are dealing with a finite support iteration. Also the case $\beta=1$ is clear (as $\boldsymbol{Q}_0$ is so simple). Hence w.l.o.l. $\beta=\gamma+1$, $\gamma>0$.

So suppose $p\in P_\beta$ and we shall find $p'\geqslant p$, $p'\in D_\beta$. First by Fact 1.8(1) there is $p_1\geqslant p$, $p_1\in D^0_\beta$, second, by the induction hypothesis, there is $r\in D_\gamma$, $r\geqslant p_1\restriction\gamma$. As $p_1\in D^0_\beta$, by 1.12(1), there are n, $0=\delta_0<\delta_1<\cdots<\delta_n<\omega_1$, $\delta_0,\ldots,\delta_n\in\{\xi_\gamma(i):i<\omega_1\}\cup\{0\}$ such that for every limit δ, for some $l\leqslant n$, $\delta\geqslant\delta_l$ and $p_1(\gamma)^{[\delta]}=p_1(\gamma)^{[\delta_l]}$. We now define by induction on $l\leqslant n+1$, $r_l\in D_\gamma$ such that:

$$(*)\qquad r_0=r,\ r_{l+1}\geqslant r_l,\quad\text{and}\quad r^{[\delta_l]}_{l+1}\Vdash_{P_\gamma}\text{“}p_1(\gamma)^{[\delta_l]}\in\boldsymbol{Q}_\gamma\text{”}.$$

If we succeed we shall finish to prove the main claim: $p_1\vee r_{n+1}$ is as required: for each limit δ choose l as above (i.e., $\delta\geqslant\delta_l$, $p_1(\gamma)^{[\delta]}=p_1(\gamma)^{[\delta_l]}$), so $r^{[\delta_l]}_{n+1}\Vdash_{P_\gamma}\text{“}p_1(\gamma)^{[\delta_l]}\in\boldsymbol{Q}_\gamma\text{”}$ but $r^{[\delta_l]}_{n+1}\leqslant r^{[\delta]}_{n+1}$ (see end of 1.4), hence

$$r^{[\delta]}_{n+1}\Vdash_{P_\gamma}\text{“}p_1(\gamma)^{[\delta]}=p_1(\gamma)^{[\delta_l]}\in\boldsymbol{Q}_\gamma\text{”},$$

but $r_{n+1}\in D_\gamma$, hence $r^{[\delta]}_{n+1}\in D^0_\gamma$, hence $r^{[\delta]}_{n+1}\vee p_1^{[\delta]}\in D^0_{\gamma+1}$, as required. So we have proved $r_{n+1}\vee p_1\in D_{\gamma+1}$.

So for proving the crucial claim we just have to do induction step in proving $(*)$: assume r_l is as required and we shall define r_{l+1}. This is the heart of the matter. For $l=0$ there is nothing to prove as $p_1(\gamma)^{[0]}=\emptyset$. So let $l>0$.

Let ε be such that $\xi_\gamma(\varepsilon)=\delta_l$. So $r_l^{[\delta_l]}\restriction|N_{\gamma,\varepsilon}|$ belong to $N_{\gamma,\varepsilon}$ (see 1.12(2)) though it does not necessarily belong to D^0_γ. Also $p_1((\gamma)^{[\delta_l]}\in N_{\gamma,\varepsilon}$. Let

$$I=\{r\in P_\gamma: r\in D_\gamma,\ \textit{and}\ \text{either } r\Vdash_{P_\gamma}\text{“}p_1(\gamma)^{[\delta_l]}\text{ satisfies 1.4(C)}(\beta)\text{”}$$
$$\text{or } r\Vdash_{P_\gamma}\text{“}p_1(\gamma)^{[\delta_l]}\text{ fail 1.4(C)}(\beta)\text{”}\}$$

Clearly I is a dense subset of P_γ, and also clearly $I\in N_{\gamma,\varepsilon}$. As P_γ satisfies the c.c.c., $I\cap N_{\gamma,\varepsilon}$ is a predense subset of P_γ. Hence there is $r^0\in I\cap N_{\gamma,\varepsilon}$ compatible with r_l, hence there is $r_{l+1}\in D_\gamma$, $r_l\leqslant r_{l+1}$ and $r^0\leqslant r_{l+1}$. By the definition of I, as $r_l\leqslant r_{l+1}$, $r^0\Vdash\text{“}p_1(\gamma)^{[\delta_l]}$ satisfies 1.4(C)(β)”. Hence by 1.8(3) $r^0\Vdash_{P_\gamma}\text{“}p_1(\gamma)^{[\delta_l]}\in\boldsymbol{Q}_\gamma\text{”}$, but $r^0\leqslant r^{[\delta_l]}_{l+1}$, so we finish.

1.15. Main Lemma. *$P_{\alpha+1}$ satisfies the c.c.c.*

Proof. Let $p_i\in P_{\alpha+1}$ for $i<\omega_1$, and for $i\neq j$, p_i, p_j are not compatible, and we shall eventually derive a contradiction. Clearly we can replace $\langle p_i:i<\omega_1\rangle$ by $\langle p_i':i<\omega_1\rangle$ if $p_i'\geqslant p_i$, and by $\langle p_i:i\in A\rangle$ (if $A\subseteq\omega_1$, $|A|=\aleph_1$). We shall use this freely.

W.l.o.g. for every i:

(a) $p_i\in D_{\alpha+1}$.

(b) $0\in\operatorname{Dom}p_i$.

(c) If $\beta\neq\gamma\in\operatorname{Dom}p_i$, $j\in A'_\beta\cap A'_\gamma$, then j belongs to the universe of $p_i(0)$.

(d) If $[j\notin\operatorname{Rang}f_\beta]\in p_i(\beta)$ or $[f_\beta(\varepsilon)=j]\in p_i(\beta)$ for some β or ε, then j belongs to the universe of $p_i(0)$.

(e) If $[f_\beta(\varepsilon)=j]\in p_i(\beta)$ and $j_1\in A'_\beta$, $\varepsilon<j_1<j$, then j_1 belongs to the universe of $p_i(0)$.

(f) If j belongs to the universe of $p_i(0)$ and $\beta\in\operatorname{Dom} p_i$, *then* $[j\notin \operatorname{Rang} f_\beta]\in p_i(\beta)$ or $(\exists\varepsilon)([f_\beta(\varepsilon)=j]\in p_i(\beta))$.

We can easily find $i(*)<j(*)$ such that, for some δ,

$$p_{i(*)}^{[\delta]}=p_{i(*)},\ p_{j(*)}^{[\delta]}\restriction(\operatorname{Dom} p_{i(*)})\leqslant p_{i(*)}.$$

By 1.9, $p_{i(*)}\vee p_{j(*)}^{[\delta]}\in D^0_{\alpha+1}$. Let $w=\{j:\text{for some } \gamma, i, [f_\gamma(i)=j]\in p_{j(*)}(\gamma)\}$, and $(w-\delta)\cup\{\delta\}=\{\delta(l):l<k\}$. So $\delta(l)$ is increasing. $\delta(0)=\delta$.

We shall define by induction on $l\leqslant k$, r_l such that:

(A) $r_l\in D^0_{\alpha+1}$, r_l is increasing, and $r_l\geqslant p_{i(*)}$.

(B) $r_l=r_l^{[\delta(l)]}$.

(C) $p_{j(*)}^{[\delta(l)]}\leqslant r_l$.

Clearly if we succeed, then r_k is $\geqslant p_{i(*)}, p_{j(*)}$ (and $r_k\in D^0_{\alpha+1}\subseteq P_{\alpha+1}$), so $p_{i(*)}, p_{j(*)}$ are compatible, so we prove the c.c.c.

Case I: $l=0$. We have already said that $\delta(0)=\delta$, and $p_{i(*)}\vee p_{j(*)}^{[\delta]}\in D^0_{\alpha+1}$.

Case II. Defining for $l+1$, assuming r_l is defined (satisfying (A), (B), (C)) there are finitely many $\beta\in\operatorname{Dom} p_{j(*)}$ such that for some j $[f_\beta(j)=\delta(l)]\in p_{j(*)}(\beta)$. Let those β's be $\beta_{l,0}>\cdots>\beta_{l,m(l)-1}$ and let $j_{l,m}$ be such that $[f_{\beta_{l,m}}(j_{l,m})=\delta(l)]\in p_{j(*)}(\beta)$. Note that $j_{l,m}$ is unique (by 1.4(C)(β)), and $j_{l,m}<\delta(l)$. Let $\xi_{\beta_{l,m}}(\varepsilon_{l,m})=\delta(l)$ (there is a unique such $\varepsilon_{l,m}$).

What are the requirements on r_{l+1}? By (C) we inherit some 'soft' demands: atomic conditions of type II, and $p_{j(*)}(0)^{[\delta(l+1)]}$, but also hard ones: the $[f_{\beta_{l,m}}(j_{l,m})=\delta(l)]$. Our strategy will be as follows: we extend r_l to r'_l preserving it 'below $\delta(l)$', so that we 'know'

$$\boldsymbol{R}_{\beta_{l,m}}\restriction\{\varepsilon:(\exists j)([f_{\beta_{l,m}}(\varepsilon)=j]\in r'_l)\}\cup\{j_{l,m}\}.$$

Now we add the 'hard' atomic conditions and corresponding condition to the Q_0-coordinate so that 1.4(C)(β) is satisfied, then we take care of the 'soft' demands.

Let $\delta(l)=\xi_{\beta_{l,m}}(\varepsilon_{l,m})$. So now we define by induction on $m\leqslant m(l)$ a condition $r_{l,m}$ such that:

(α) $r_{l,0}=r_l$, $r_{l,m+1}\geqslant r_{l,m}$, and $r_{l,m}\in D^0_{\alpha+1}$.

(β) $r_{l,m}=r_{l,m}^{[\delta(l)]}$.

(γ) $r_{l,m+1}\restriction\beta_{l,m}$ forces the value of $\boldsymbol{R}_{\beta_{l,m}}\restriction w_{l,m}$ where

$$w_{l,m}=\{i:(\exists j)([f_{\beta_{l,m}}(i)=j]\in r_{l,m}\}\cup\{j_{l,m}\}.$$

(δ) $r_{l,m}\restriction(\beta_{l,m},\alpha+1)=r_{l,m+1}\restriction[\beta_{l,m},\alpha+1)$.

For $m=0$, $r_{l,0}=r_l$ is as required. Suppose $r_{l,m}$ is defined as required, $m<m(l)$. Let

$$I_{l,m}=\{p\in P_{\beta_{l,m}}:p\in D_{\beta_{l,m}},\ p \text{ forces a value to } \boldsymbol{R}_{\beta_{l,m}}\restriction w_{l,m} \text{ and either } p\geqslant r_{l,m}\restriction|N_{\beta_{l,m},\varepsilon_{l,m}}| \text{ or } p \text{ has no extension satisfying this}\}.$$

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Clearly $I_{l,m}$ is a dense subset of $P_{\beta_{l,m}}$. Also $I_{l,m} \in N_{\beta_{l,m}\varepsilon_{l,m}}$ (as the relevant parameters belong to it: $w_{l,m}$ as a finite set of ordinals $<\delta(l) = \xi_{\beta_{l,m}}(\varepsilon_{l,m})$). As $P_{\beta_{l,m}}$ satisfies the c.c.c. (as $\beta_{l,m} < \alpha$), $I_{l,m} \cap N_{\beta_{l,m}\varepsilon_{l,m}}$ is a predense subset of $P_{\beta_{l,m}}$, hence there is $p = p^{l,m} \in I_{l,m} \cap N_{\beta_{l,m}\varepsilon_{l,m}}$ which is compatible with $r_{l,m}$: clearly $p \geqslant r_{l,m} \restriction |N_{\beta_{l,m}\varepsilon_{l,m}}|$.

So p, $r_{l,m} \restriction \beta_{l,m}$ has a common upper bound, hence a common upper bound in $D_{\beta_{l,m}}$, hence has one $r'_{l,m+1} \in D_{\beta_{l,m}}$, $(r'_{l,m+1})^{[\delta(l)]} = r'_{l,m+1}$. Now $r_{l,m+1} \overset{\text{def}}{=} r_{l,m} \vee r'_{l,m+1}$ is as required.

So $r_{l,m(l)}$ is defined and we can define r_{l+1}: its domain is $\operatorname{Dom} r_{l,m(l)} \cup \{\beta_{l,m} : m < m(l)\}$.

For $\gamma > 0$, $r_{l+1}(\gamma) = r_{l,m(l)}(\gamma) \cup p_{j(*)}(\gamma)^{[\delta(l+1)]}$.

For $\gamma = 0$, $r_{l+1}(0)$ is a model extending $r_{l,m(l)}(0)$, $p_{j(*)}(0)^{[\delta(l+1)]}$. Its universe is their union, and

$$(*)\qquad \text{if } m < m(l),\quad [f_{\beta_{l,m}}(i) = \delta(l)] \in p_{j(*)}(\beta_{l,m})$$
$$[f_{\beta_{l,m}}(i_1) = j_1] \in r_{l,m(l)}(\beta_{l,m}),$$
$$\text{then } r_{l+1}(0) \vDash j_1 R_0 \delta(l) \quad \text{iff} \quad r_{l,m(l)} \Vdash i_1 \boldsymbol{R}_{\beta_{l,m}} i.$$

We should now only clarify why is this legitimate.

Point (i). Why $r_{l+1}(0)$ is well defined? I.e., a priory we may have two conflicting demands on the truth value of $j_1 R_0 j_2$. We have three sources of such demands: $r_{l,m(l)}(0)$, $p_{j(*)}(0)^{[\delta(l+1)]}$, and $(*)$. The first two do not contradict as $r_{l,m(l)}(0) = r_{l,m(l)}^{[\delta(l)]}$ whereas $p_{j(*)}(0)^{[\delta(l)]} \subseteq r_l(0) \subseteq r_{l,m(l)}(0)$. Also $(*)$ cannot contradict $r_{l,m(l)}(0)$ as $r_{l,m(l)}(0) = r_{l,m(l)}(0)^{[\delta(l)]}$. So what about a contradiction between $(*)$ and $p_{j(*)}(0)^{[\delta(l+1)]}$? This is possible only if (with $(*)$ notation) j_1 is in the universe of $p_{j(*)}(0)^{[\delta(l+1)]}$, hence of $p_{j(*)}(0)^{[\delta(l)]}$, by (d) also $\delta(l)$ belongs to the universe of $p_{j(*)}(0)$, so by (f)) as $p_{j(*)}^{[\delta(l)]} \leqslant r_{l,m(l)}$, $j_1 < \delta(l) = \xi_{\beta_{l,m}}(\varepsilon_{l,m})$, clearly $[f_{\beta_{l,m}}(i_1) = j_1] \in p_{j(*)}(\beta_{l,m})$, so no contradictions arise.

We still have the possibility of a contradiction between two instances of $(*)$. But for $m(1) \neq m(2)$, $A_{\beta_{l,m(1)}} \cap A_{\beta_{l,m(2)}}$ is included in the universe of $p_{j(*)}(0)$ (by (c)), so this is impossible too.

Point (ii). Why $r_{l+1}(\beta)$ $(\beta > 0)$ satisfies 1.4(C)(α)? (one-to-oneness)? The only problem possible is $\beta = \beta_{l,m}$, and for some j, $[f_{\beta_{l,m}}(j_{l,m}) = j] \in r_{l,m(l)}$, but as $p_{j(*)}$ satisfies (e) and (b) and $p_{j(*)}^{[\delta(l)]} \leqslant r_{l,m(l)}$ this is impossible. (Note that for $k_1 < k_2$, necessarily $\beta_{l,k_1} \notin N_{\beta_{l,k_2}\varepsilon_{l,k_2}}$, hence $r_{l,m}(\beta_{l,k}) = r_l(\beta_{l,k})$.)

Point (iii). Why $r_{l+1}(\beta)$ $(\beta > 0)$ satisfies 1.4(C)(β)? All our constructions of $r_{l,m}$, r_{l+1} were done to satisfy this.

Point (iv). Why $r_{l+1}(\beta)$ $(\beta > 0)$ satisfies 1.4(C)(γ)? Trivial.

Point (v). Why $p_{j(*)}^{[\delta(l+1)]} \leqslant r_{l+1}$, $r_{l+1} = r_{l+1}^{[\delta(l+1)]}$? Trivial too.

2. Generalizations

In 2.1 we use a forcing of Baumgartner: there may be $2^{\aleph_1}$ subsets of $\aleph_1$ each of power $\aleph_1$, with finite intersection of any two. In 2.2 we prove a slight strengthening of Baumgartner [5, 6.1]. In 2.4 we prove for any $\kappa=\kappa^{<\kappa}$ we can have 2^κ arbitrary large and there is a universal graph in every $\lambda\in[\kappa,2^\kappa)$.

2.1. Theorem. (1) *In* 1.1 *we can get such a model with* $2^{\aleph_0}=\aleph_2$, $2^{\aleph_1}$ *arbitrary.*
(2) *Really also any value for* $2^{\aleph_0}$ *is O.K.*

Proof. Starting with V satisfying CH, $2^{\aleph_1}$ arbitrary, the only difficult point in the proof of 1.1 is that we have too many ($2^{\aleph_1}$ which maybe $>\aleph_2$) graphs on ω_1. However we do not need to explicitly force an embedding of any such graph into (ω_1,R_0). It is enough if in each $V_a^{P_\alpha}$ ($\alpha<\omega_2$) we can find $\aleph_2$ such graphs $(\omega_1,R_{\alpha,\gamma})$ ($\gamma<\omega_2$) so that any other one can be embedded into one of them, and take care that any such $R_{\alpha,\gamma}$ appear (up to isomorphism) among the $\boldsymbol{R}_\beta$'s. Why is this possible? In $V_a^{P_\alpha}$, $2^{\aleph_0}=\aleph_2$, hence there is an $\aleph_1$-saturated graph of power $\aleph_2$, say (ω_2,R_α^*). Clearly any graph of power $\leqslant\aleph_1$ (in $V_a^{P_\alpha}$) can be embedded into (ω_2,R_α^*), hence into some $(\beta,R_\alpha^*\restriction\beta)$. So $\{(\beta,R_\alpha^*\restriction\beta):\omega_1\leqslant\beta<\omega_2\}$ is a list of graphs of power $\aleph_1$ as required.

(2) We use that, if P satisfies the c.c.c., then any set of ordinals in V^P is a subset of some set of ordinals in V with the same power. If $\mathrm{cf}(2^{\aleph_0})=\aleph_1$, the iteration will have length $(2^{\aleph_0})^{V^P}\times\omega_2$ (ordinal multiplication).

2.2. Lemma. *Suppose* $\mu>\kappa$ *are regular. Then we can define a forcing notion R, and P-names* $\mathbf{A}_\alpha$ ($\alpha<\mu$) *such that*

(a) $\Vdash_P$ "$\mathbf{A}_\alpha\subseteq\mu$".

(b) $\Vdash_P$ "*for* $\alpha\neq\beta$, $\mathbf{A}_\alpha\cap\mathbf{A}_\beta$ *has power* $<\kappa$".

(c) $\Vdash_P$ "*for any* α, *and any* $\delta<\mu$ *of cofinality* $>\kappa$, $\{\gamma<\delta:\mathrm{cf}\,\gamma=\kappa,\gamma\in\mathbf{A}_\alpha\}$ *is a stationary subset of* δ".

(d) *P is* κ*-complete,* $|P|\leqslant\mu^{<\mu}$.

(e) *If* $\lambda<\mu$ *is regular,* $\lambda^{<\lambda}=\lambda$, *then* λ^+ *is not collapsed by P, moreover any set* $A\in V^P$ *of ordinals of power* $\leqslant\lambda$ *is included in some* $B\in V^P$, *of power* $\leqslant\lambda$.

Proof. We shall first define atomic conditions of various kinds:

Kind 0: $i\in\mathbf{A}_\alpha$ ($i<\mu,\alpha<\mu$).

Kind λ ($\kappa\leqslant\lambda\leqslant\mu$): $\mathbf{A}_\alpha\cap\mathbf{A}_\beta\subseteq W$ where $W\subseteq\mu$ is a set of power $<\lambda$, or $i\in\mathbf{A}_i$. So p is of kind λ, $0<\lambda\leqslant\lambda_1$, implies p is of kind λ_1.

Now for a set p of atomic conditions let p/λ be the set of atomic conditions in p which are of kind zero or kind λ. Let for $\lambda_1<\lambda_2$, $p\restriction(\lambda_1,\lambda_2)=p/\lambda_2-p/\lambda_1$.

Now we define the forcing notion:

$P=\{p:p$ is a set of atomic conditions, $|p/\lambda|<\lambda$ whenever $\kappa\leqslant\lambda\leqslant\mu$, λ regular, and p contains no three contradicting conditions$\}$

where three atomic conditions are contradictory if they have the form $i\in\mathbf{A}_\alpha$, $i\in\mathbf{A}_\beta$, $\mathbf{A}_\alpha\cap\mathbf{A}_\beta\subseteq W$ where $i\notin W$.

The order on P is defined by:

$$p \leq q \quad \textit{iff} \quad p \subseteq q \textit{ and } [\mathbf{A}_\alpha \cap \mathbf{A}_\beta \subseteq W_1] \in q - p \text{ with } |W_1| \geq \kappa \textit{ implies} \text{ that } W_1 \text{ includes } \bigcup\{W : |W| \leq |W_1|, \text{ and for some } \alpha_1, \beta_1\ [\mathbf{A}_{\alpha_1} \cap \mathbf{A}_{\beta_1} \subseteq W] \in p\}.$$

So now we shall prove that P satisfies the requirement. The P-name $\mathbf{A}_\alpha$ is of course $\{i<\mu : [i \in \mathbf{A}_\alpha]$ belong to some condition in the generic set$\}$.

Requirement a: $\Vdash_P$ "$\mathbf{A}_\alpha \subseteq \mu$". Trivial.

Requirement b: For $\alpha \neq \beta < \mu$, $\Vdash_P$ "$|\mathbf{A}_\alpha \cap A_\beta| < \kappa$". It is enough to prove that for any $p \in P$ there is a set $W \in V$ of power $<\kappa$ and q, $p<q \in P$ such that $q \Vdash$"$\mathbf{A}_\alpha \cap \mathbf{A}_\beta \subseteq W$". Let $W = \{i<\mu :$ the atomic conditions $[i \in \mathbf{A}_\alpha]$ and $[i \in \mathbf{A}_\beta]$ belong to $p\}$.

Now W has power $<\kappa$, as the number of atomic conditions of the zero kind which belong to p is $<\kappa$. By the definition of P, $q = p \cup \{\mathbf{A}_\alpha \cap \mathbf{A}_\beta \subseteq W\}$ belongs to p, and it is not hard to see that $q \Vdash$"$\mathbf{A}_\alpha \cap \mathbf{A}_\beta \subseteq W$" (in fact, equality is forced).

Requirement c: Use the proof of (e) for $\lambda = \kappa$, in its notation, if $\delta \in N_0$, then letting $i = \sup(N_\lambda \cap \delta)$, clearly cf $i = \lambda$, $p^* \Vdash_p$ "$i \in \boldsymbol{C}$ is a club of δ" for every $\boldsymbol{C} \in N_\lambda$, and $p^* \leq p^* \cup \{[i \in \mathbf{A}_\alpha]\} \in P$.

Requirement d: R is κ-complete, and satisfies the $(\mu^{<\kappa})^+$-c.c., $|P| \leq \mu^{<\mu}$. Trivial.

Requirement e: This is the main point. We prove a kind of properness. Let χ be a large enough regular cardinal, $<^*$ a well-ordering of $H(\chi)$. By (d) w.l.o.g. $\lambda \geq \kappa$.

Suppose $p \in P$, $\boldsymbol{B}$ a P-name, $p \Vdash_P$ "$|\boldsymbol{B}| \leq \lambda$, $\boldsymbol{B} \subseteq$ord" so for some P-names of ordinals $\boldsymbol{\beta}_i$ $(i<\lambda)$, $p \Vdash \boldsymbol{B} \subseteq \{\boldsymbol{\beta}_i : i<\lambda\}$. Suppose $\langle N_i : i<\lambda \rangle$ is an increasing sequence of elementary submodels of $(H(\chi), \in, <^*)$ each of power λ, $\langle N_j : j \leq i \rangle \in N_{i+1}$, and every subset of N_i of power $<\lambda$ belongs, to N_i. Lastly assume that $\boldsymbol{B}, \boldsymbol{\beta}_i \in N_0$. Let h_0, h_1 be the $<^*$-first pair of functions from λ to λ such that $i \neq 0 \Rightarrow h_l(i) < i$ and for every $i_0, i_1 < \lambda$, for some $i<\lambda$, $h_0(i) = i_0 < i$, $h_1(i) = i_1 < i$; so $h_0, h_1 \in N_0$. Let $\langle \langle \boldsymbol{\gamma}^i_j, q^i_j, r^i_j \rangle : j<\lambda \rangle$ be the $<^*$-first list of the triples $\langle \boldsymbol{\gamma}, q, r \rangle \in N_i$, $\boldsymbol{\gamma}$ a P-name of an ordinal, $q, r \in P$, $q \leq r$ and $r = r/\lambda$.

Now we define by induction on $i<\lambda$, conditions p_i, q_i, p'_i, such that

(A) $p = p_0 = q_0$, for $j<i$, $p_j \leq p'_j \leq p_i$, and $p_i \leq q_i$ and $p_i/\lambda = p_0/\lambda$.

(B) $p_i, q_i \in N_{i+1}$, $p'_i = p_i \cup \{\mathbf{A}_\alpha \cap \mathbf{A}_\beta \subseteq |N_{i+1}| \cap \text{ord} : \alpha, \beta \in N_{i+1}\}$.

(C) For $i>0$, q_i is the $<^*$-first member of P satisfying:

(C1) $q_i \geq p'_j$ for $j<i$.

(C2) If there is q_i, $q_i \geq p'_j$ for $j<i$, q^i is $\geq r$ for some $r \geq q^{h_1(i)}_{h_2(i)}$, r isomorphic to $r^{h_1(i)}_{h_2(i)}/\lambda$ above $q^{h_1(i)}_{h_2(i)}$ and q_i forces a value for $\boldsymbol{\gamma}^{h_1(i)}_{h_2(i)}$, *then* q_i is like that, $q_i \in N_{i+1}$.

(D) p_i is $p_0 \cup (q_i \restriction (\lambda, \infty])$.

It is easy to see that this can be done.

Let $p^* = \bigcup_{i<\lambda} p_i$, and by (A), $p \leq p^* \in P$. Let $N_\lambda = \bigcup_{i<\lambda} N_i$, clearly (e) follows from:

2.3. Fact. *p^* is (N_λ, P)-generic, i.e., for every P-name $\boldsymbol{\beta}$ of an ordinal which belongs to N_λ, $p^* \Vdash_P$ "$\beta \in N_\lambda$".*

Proof. If not, for some q^*, $p^* \leq q^* \in P$, $q^* \Vdash$"$\boldsymbol{\beta} = \beta^*$" and $\beta^* \notin N_\lambda$. Let $r^+ \in N_\lambda$ be isomorphic to q^*/λ over $(q^* \cap N_\lambda) \overset{\text{def}}{=} q^+$. So for some $i_0 < \lambda$, r^+, $q^+ \in N_{i_0}$, and w.l.o.g. $\boldsymbol{\beta} \in N_{i_0}$, hence

$$(\boldsymbol{\beta}, q^+, r^+) = (\boldsymbol{\gamma}^{i_0}_{i_1}, \boldsymbol{q}^{i_0}_{i_1}, r^{i_0}_{i_1}/\lambda) \quad \text{for some } i_1 < \lambda.$$

Now for some $i > i_0$, i_1, $h_0(i) = i_0$, $h_1(i) = i_1$. Clearly by (C2), $q_i \geq q^+$ and q_i/λ, q^+/λ are isomorphic above q^+ (remember $q^+ = q^{i_0}_{i_1}$), as q^* exemplified the hypothesis. We shall prove that q^*, q_i are compatible thus deriving a contradiction (as $q_i \Vdash$"$\boldsymbol{\gamma}^{i_0}_{i_1} \in N$").

Suppose the contradiction involves $[\varepsilon \in \mathbf{A}_\alpha]$, $[\varepsilon \in \mathbf{A}_\beta]$, $[\mathbf{A}_\alpha \cap \mathbf{A}_\beta \subseteq W]$. If $\varepsilon \notin N_\lambda$, then necessarily $[\varepsilon \in \mathbf{A}_\alpha]$, $[\varepsilon \in \mathbf{A}_\beta]$ belong to q^*, hence $[\mathbf{A}_\alpha \cap \mathbf{A}_\beta \subseteq W] \in q_i - q^*$, hence $|W| < \lambda$ and $\alpha, \beta \in N_\lambda$, hence for some j,

$$[\mathbf{A}_\alpha \cap \mathbf{A}_\beta \subseteq |N_j| \cap \text{ord}] \in p_j \leq p^*,$$

so we get a contradiction in q^*. So $\varepsilon \in N_\lambda$, hence the isomorphism from r to q^*/λ maps ε to itself.

Suppose $[\mathbf{A}_\alpha \cap \mathbf{A}_\beta \subseteq W] \in q_i - q^*$. If $|W| \geq \lambda$, then $[\mathbf{A}_\alpha \cap \mathbf{A}_\beta \subseteq W] \in q_i \leq p_i \leq p^* \leq q^*$, contradiction, so $|W| < \lambda$, hence $[\mathbf{A}_\alpha \cap \mathbf{A}_\beta \subseteq W] \in q_i/\lambda \in N_\lambda$, but $|q_i/\lambda| < \lambda$, hence $q_i/\lambda \subseteq N_\lambda$, hence $[\mathbf{A}_\alpha \cap \mathbf{A}_\beta \subseteq W] \in N_\lambda$, hence $\alpha, \beta \in N_\lambda$, $W \in N_\lambda$, $W \subseteq N_\lambda$. As $\varepsilon, \alpha, \beta \in N_\lambda$

$$[\varepsilon \in \mathbf{A}_\alpha] \in q^* \Rightarrow [\varepsilon \in \mathbf{A}_\alpha] \in q^+ \subseteq q_i,$$
$$[\varepsilon \in \mathbf{A}_\beta] \in q^* \Rightarrow [\varepsilon \in \mathbf{A}_\beta] \in q^+ \subseteq q_i,$$

so the contradiction arises already in q_i, contradiction. So $[\mathbf{A}_\alpha \cap \mathbf{A}_\beta \subseteq W] \in q^*$.

Suppose $\alpha, \beta \in N_\lambda$, then again $[\varepsilon \in \mathbf{A}_\alpha]$, $[\varepsilon \in \mathbf{A}_\beta] \in q_i$, hence $[\mathbf{A}_\alpha \cap \mathbf{A}_\beta \subseteq W] \in q^* - q_i$ and $\varepsilon, \alpha, \beta \in N_{i+1}$, hence $[\mathbf{A}_\alpha \cap \mathbf{A}_\beta \subseteq W] \notin p_i$. If $|W| \geq \lambda$, by the definition of p_i' and of $p_i' \leq q^*$, clearly $|N_{i+1}| \cap \text{ord} \subseteq W$, so $\varepsilon \in W$ and no contradiction arises. So $|W| < \lambda$, hence $[\mathbf{A}_\alpha \cap \mathbf{A}_\beta \subseteq W] \notin N_\lambda$ and for some W', $W' \cap N_\lambda = W \cap N_\lambda$ and $[\mathbf{A}_\alpha \cap \mathbf{A}_\beta \subseteq W'] \in q_i$ (see (C2)), hence $\varepsilon \in W'$, but $W' \in N_\lambda$, hence

$$[\mathbf{A}_\alpha \cap \mathbf{A}_\beta \subseteq |N_j| \cap \mu] \in p^* \quad \text{for some } j.$$

If $W \subseteq N_\lambda$, then (as $|W| < \lambda$) $W \in N_\lambda$, hence $[\mathbf{A}_\alpha \cap \mathbf{A}_\beta \subseteq W] \in q^+ \subseteq q_i$, contradiction. So $W \nsubseteq N_\lambda$, so $[\mathbf{A}_\alpha \cap \mathbf{A}_\beta \subseteq W] \notin p^*$, but by the definition of the p_i' for some $B \subseteq N_\lambda$, $[\mathbf{A}_\alpha \cap \mathbf{A}_\beta \subseteq B] \in p^*$ and $|B| < \lambda$, hence $W \cap B \in N_\lambda$, $|W \cap B| < \lambda$, so there are

$$[\mathbf{A}_\alpha \cap \mathbf{A}_\beta \subseteq B'], [\mathbf{A}_\alpha \cap \mathbf{A}_\beta \subseteq W'] \in q_i,$$

$B' \cap W' = B \cap W$ and again we get a contradiction.

So $\{\alpha, \beta\} \nsubseteq N_\lambda$, on the other hand $\alpha, \beta \notin N_\lambda$ implies that all the three atomic conditions causing the contradiction are in q^*. So w.l.o.g. $\alpha \in N_\lambda$, $\beta \notin N_\lambda$, hence $[\mathbf{A}_\alpha \cap \mathbf{A}_\beta \subseteq W]$, $[\varepsilon \in \mathbf{A}_\beta] \in q^* - q_i$, so necessarily $[\varepsilon \in \mathbf{A}_\alpha] \in q_i - q^*$.

So there are β_1, W_1 such that $[\varepsilon \in \mathbf{A}_{\beta_1}]$, $[\mathbf{A}_\alpha \cap A_{\beta_1} \subseteq W_1] \in q_i$, and $\varepsilon \notin W_1$, so we get a contradiction in q_i.

2.4. Remark. (1) So if (in 2.2) $V \models \text{GCH}$, then P does not collapse any cardinal, $\Vdash_P$ "$2^\kappa = \mu$, for $\lambda < \kappa : 2^\lambda = \lambda^+$, for $\lambda > \kappa : 2^\lambda = \lambda^+ + \mu$".

(2) From the proof it is clear that the product of R with Easton forcing (see [2]) again does not collapse cardinals (if $V \models \text{GCH}$), hence we get quite arbitrary functions $2^{\aleph_\alpha}$ ($\aleph_\alpha$ regular).

(3) Does 2.2(e) hold for $\lambda = \lambda^{<\kappa} = \text{cf}\,\lambda$? We have not looked into the matter.

2.5. Theorem. *Suppose $V \models \text{GCH}$, $\kappa < \lambda < \mu$ are regular, $(\forall \theta < \lambda)\ \theta^{<\kappa} < \lambda$. Then for some κ-complete forcing notion P of power μ,*

$\Vdash_P$ "$2^\kappa = \mu$ *and* $2^\lambda = \lambda^+ + \mu$ *for* $\lambda \geq \kappa$, *and there is a graph* $(\mu, \boldsymbol{R}_0)$ *such that* $(\lambda, \boldsymbol{R}_0 \restriction \lambda)$ *is a universal graph.*"

Proof. We repeat the proof of 1.1, with some changes which we explain below, in particular everywhere 'finite' is replaced by 'of power $<\kappa$'.

1.2. We use here 2.2.

1.3. The iteration will be ($<\kappa$)-support iteration of κ-complete forcing notions satisfying the κ^+-chain condition. In stage α we will have a pair $(\mu, \boldsymbol{R}_\alpha)$, $\boldsymbol{R}_\alpha$ a P_α-name of reflexive symmetric relation on μ.

1.4. In Q_0's definition $|w| < \kappa$ and $\langle N_{\alpha,i} : i < \mu_\alpha \rangle$ is an increasing continuous sequence of elementary submodels of $(H(\lambda)^{V_\alpha}, \in)$, $\|N_{\alpha,i}\| = (\kappa + |i|)^{<\kappa}$, $\alpha \subseteq N_{\alpha,i}$ and so $a \in N_{\alpha,i} \wedge |a| \leq \|N_{\alpha,i}\| \Rightarrow a \subseteq N_{\alpha,i}$, also $a \in N_{\alpha,i+1}$, $|a| < \kappa \Rightarrow a \in N_{\alpha,i+1}$. Again we assume $\xi_\alpha(i+1) \in A_\alpha$, $\text{cf}(\xi_\alpha(i+1)) = \kappa$: but now $\xi_\alpha(i) = \text{Min}\{\xi < \mu : \xi \notin N_{\alpha,i}\}$.

In (A) member of $\mathbf{Q}_\alpha$ has power $<\kappa$ and replace (B)(I) by: $f_\alpha(i) = j$ where $i < j$, $j \in A'_\alpha$, and the order type of $A'_\alpha \cap j$ is $>\kappa i$ but $<\kappa(i+1)$.

1.6. Note that D^0_β is closed under directed union of $<\kappa$ conditions.

1.7, 1.13. Similarly for D^1_β, D_β.

1.14. (Crucial Lemma) In the induction there is a new case: limit ordinal of cofinality $<\kappa$ (when $\kappa > \aleph_0$) but then use κ-completeness.

1.15. So we suppose $\langle p_i : i < \kappa^+ \rangle$ is a sequence of conditions in $P_{\alpha+1}$.

As maybe $\kappa^+ < \mu_\alpha$, we can demand less on $i(*) < j(*)$: if $\beta \in \text{Dom}\, p_{i(*)} \cap \text{Dom}\, p_{j(*)}$, then $p_{i(*)}(\beta) \cup p_{j(*)}(\beta)$ belongs to D^1_β. We let

$$w = \{j : \text{for some } \gamma,\ i[f_\gamma(i) = j] \in p_{j(*)}(\gamma) \cup p_{i(*)}(\gamma)\}$$

and let $w = \{\delta(l) : l < k\}$ where here $k < \kappa$ (so may be infinite), $\delta(l)$ increasing. The induction hypothesis is now

(A) $r_l \in D^0_{\alpha+1}$ r_l increasing,

(B) $r_l = r_l^{[\delta(l)]}$,

(C) $p_{i(*)}^{[\delta(l)]} \leq r_l$ and $p_{j(*)}^{[\delta(l)]} \leq r_l$,

(D) $p^{[\delta(l)]} \subseteq r_l$ where $p^{[\delta(l)]} = \{[j \notin \operatorname{Rang} f_\alpha]$: for some $\beta \neq \alpha$, $j \in A'_\alpha \cap A'_\beta$ and

$$(\exists i, \delta)([f_\alpha(i) = \delta] \in p_{i(*)}(\beta) \cup p_{j(*)}(\beta))$$

and

$$[\exists i, \delta)([f_\beta(i) = \delta] \in p_{i(*)}(\beta) \cup p_{j(*)}(\beta))$$

but $j < \delta(l)$ and

$$\neg(\exists i)([f_\beta(i) = j] \in p_{i(*)}(\beta) \cup p_{j(*)}(\beta))\}.$$

In the definition by induction on $l \leq k$, we have a new case: l limit and we let

$$r_l = \bigcup_{m<l} r_m \cup p_{i(*)}^{[\delta(l)]} \cup p_{j(*)}^{[\delta(l)]}$$

(note that $\bigcup_{m<l} r_m \in D_{\alpha+1}^0$ trivially (as r_m is increasing) and $r_l(\beta) - \bigcup_{m<l} r_m(\beta)$ is empty).

The first case, $l = 0$ is now trivial.

So we have to deal only with case II: $l+1$.

If $\kappa > \omega$ we have the following new problem. We cannot use downward induction $\{\beta_{l,m} : m < m(l)\}$ as maybe $m(l) \geq \omega$. However if we can define by induction on $m \leq m(l)$, $r_{l,m} = r_{l+m}^{[\delta(l)]} \in D_{\alpha+1}^0$ increasing, $r_{l,m+1} \restriction \beta_{l,m}$ forces a value

$$\boldsymbol{R}_{\beta_{l,m}} \restriction \{i : (\exists j)[f_{\beta_{l,m}}(i) = j] \in r_{l,m}\}.$$

Now $r_{l,m(l)}$ is not as we want but $r_{l,m(l)}$ forces a value of

$$\boldsymbol{R}_{\beta_{l,m}} \restriction \{i : (\exists j)[f_{\beta_{l,m}}(i) = j] \in r_l\}.$$

Letting $r_l^0 = r_l$, $r_l^1 = r_{l,m(l)}$ and we can continue similarly to define r_l^2, r_3^3, and their union serves the function served previously by $r_{l,m(l)}$, so we shall ignore this point.

Another easy new point is the care for (D).

Now we come to a more serious problem: defining r_{l+1}. If only one of

$$(\exists i, m)([f_{\beta_{l,m}}(i) = \delta(l)] \in p_{i(*)}(\beta_{l,m})),$$

$$(\exists i, m)([f_{\beta_{l,m}}(i) = \delta(l)] \in p_{j(*)}(\beta_{l,m}))$$

holds, this is just as in the proof of 1.14. However (as maybe $\kappa^+ < \mu$) maybe both hold. But the additional condition (D) solves this.

Now in addition to 1.15 we should prove that P_δ satisfies the κ^+-c.c. when $\operatorname{cf} \delta < \kappa$, but the proof is similar.

2.4. Discussion (1) Can we make the function 2^λ as we like? Clearly if we want to have $(\lambda, R_0 \restriction \lambda)$ universal for each $\lambda < \lambda_0$, then there is no restriction on 2^λ $(\lambda_0 \leq \lambda)$, and in fact also on 2^λ $(\lambda^+ \geq \lambda_0)$. But if we want to get more (e.g., $2^{\aleph_l} = \aleph_{n+l}$ for $l \leq n$, and we want a universal graph in each $\aleph_l$, $l \leq 2^n$ it seems a downward induction is in order, and things become more complicated for infinitely many cardinals (but such problems were overcome). We have not looked into the matter.

(2) Can we prove parallel theorems for other first-order theorems T? Or even for other reasonable classes, e.g. the class of locally finite groups? (See Grossberg and Shelah [6].)

A trivial restriction is that if T has a universal model of power λ, then $|D(T)| \leq \lambda$, so e.g. Piano arithmetic has no universal model in $\aleph_1$ if $\aleph_1 < 2^{\aleph_0}$.

Now in defining Q_0 we can replace $(w, r) \in Q_0$ by a model generted by $\{i : i \in w\}$, and in fact demand that only $(w, r) \upharpoonright (A_i \cap w)$ are defined (for $\alpha < \mu$). Moreover in 2.2 (and 1.2, and first stage of 2.3) we can get

(*) $\mathfrak{A} = \{A_\alpha \cap A_\beta : \alpha \neq \beta < \mu\}$ forms a tree, i.e., if $a, b \in \mathfrak{A}$ then $a \cap b$ is an initial segment of a (and of b).

This is done by a minor change in P (then every $p \in P$, for the α's it forces, specific members describe such situations.

So in the end we have to find a model M, extending M_α with universe A_α, where M_α, M_β agree on $A_\alpha \cap A_\beta$.

This applies to many classes, but e.g. not to the class of linear order (which was treated in [4]).

Question. Is there a countable T which does not have a universal model in λ^+ whenever $\lambda^+ < 2^\lambda$?

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