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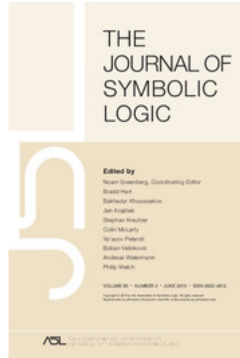

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NONEXISTENCE OF UNIVERSAL ORDERS IN MANY CARDINALS

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Abstract. Our theme is that not every interesting question in set theory is independent of ZFC. We give an example of a first order theory T with countable $D(T)$ which cannot have a universal model at $\aleph_1$ without CH; *we prove in ZFC a covering theorem from the hypothesis of the existence of a universal model for some theory*; and we prove—again in ZFC—that for a large class of cardinals there is no universal linear order (e.g. in every regular $\aleph_1 < \lambda < 2^{\aleph_0}$). In fact, what we show is that if there is a universal linear order at a regular λ and its existence is not a result of a trivial cardinal arithmetical reason, then λ "resembles" $\aleph_1$—a cardinal for which the consistency of having a universal order is known. As for singular cardinals, we show that for many singular cardinals, if they are not strong limits then they have no universal linear order. As a result of the nonexistence of a universal linear order, we show the nonexistence of universal models for all theories possessing the strict order property (for example, ordered fields and groups, Boolean algebras, p-adic rings and fields, partial orders, models of PA and so on).

§0. Introduction.

General description. This paper consists entirely of proofs in ZFC. We can even dare to recommend reading it to anybody who is interested in linear orders or partial orders in themselves, and to whom axiomatic set theory and model theory are of less interest. Such a reader should, though, consult the Appendix to this paper, or a standard textbook like [CK], for the notion of "elementary submodel", and confine his reading to §§3, 4 and 5.

The general problem addressed in this paper is the computation of the universal spectrum of a theory (or a class of models), namely the class of cardinals in which the theory (the class) has a universal model. (A definition of "universal model" is found below.) As the universal spectrum of a theory usually depends on cardinal arithmetic, and even on the particular universe of set theory in which a given cardinal arithmetic holds (see below), the problem of determining the universal spectrum of a theory must be rephrased as: under which cardinal arithmetical assumptions *can* a given theory (class) possess a universal model in a given cardinality λ?

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All results in this paper are various *negative* answers in ZFC to this question, namely theorems of the form "if $C(\lambda)$ (some cardinal arithmetic condition on a cardinal λ), then there is *no* universal model of T at cardinality λ". In general, it is harder to prove such theorems when the cardinal λ in question is singular. Such theorems are first proved for the case where T is the theory of linear orders, and then are shown to hold also for a larger class of theories, including the theory of Boolean algebras, the theory of ordered fields, the theory of partial orders, and others.

Background and detailed content. A universal model at power λ, for a class of models K, is a model $M \in K$ of cardinality λ with the property that for all $N \in K$ such that $|N| \leq \lambda$ there is an embedding of N into M. At this point let us clarify what "embedding" means in this paper. If $K = \mathrm{MOD}(T)$ is the class of models of a first order theory T, then "embedding" should be understood as "elementary embedding" when T is complete, and "universal" is with respect to elementary embeddings; when T is not complete (e.g. the theory of linear orders, the theory of graphs or the theory of Boolean algebras), "embedding" is an ordinary embedding, namely a 1-1 function which preserves all relations and operations, and "universal" is with respect to ordinary embeddings. This distinction is necessary, because there are theories for which universal models in the sense of an ordinary embedding exist, whereas universal models in the sense of an elementary embedding do not exist (see the Appendix for such an example).

Although the notion "universal model" is older then its relative, "saturated model", and arises more often and more naturally in other branches of mathematics, it has won less attention, perhaps because answers to questions involving the former notion were harder to get. As one example of a contribution to the theory of universal models we can quote [GrSh 174], in which it is shown that the class of locally finite groups has a universal model in any strong limit of cofinality $\aleph_0$ above a compact cardinal. The class $\{\lambda\colon T$ has a saturated model of cardinality $\lambda\}$ has been characterized for a first order theory T (see the situation, with history, in VIII.4 of [Sh-a] or [Sh-c]).

Saturated models are universal, and their existence is known at cardinals λ such that $\lambda = \lambda^{<\lambda} > |T|$ or just $\lambda = \lambda^{<\lambda} \geq |D(T)|$ ($D(T)$ is defined below) for every T; furthermore, when $\lambda = 2^{<\lambda}$, essentially the same proof gives a "special model", which is also universal (for these results see [CK]). Therefore the problem of the existence of a universal model for a first order theory remains unsettled in classical model theory only for cardinals $\lambda < 2^{<\lambda}$.

The consistency of *not* having universal models at such λ's for all theories which do not have to have one at every infinite power is very easy (see the Appendix). In the other direction, the second author proved in [Sh 100] the consistency of the existence of a universal order at $\aleph_1$ with the negation of CH, and, in [Sh 175] and [Sh 175a], he proved the consistency of the existence of universal graph at λ, if there is a κ such that $\kappa = \kappa^{<\kappa} < \lambda < 2^\kappa = \mathrm{cf}(2^\kappa)$. One could expect at that point to prove that every theory T which has no trivial reason for not having a universal model at $\aleph_1 < 2^{\aleph_0}$, can have one. (By a "trivial reason" we mean an uncountable $D(T)$. $D(T)$ is the set of all complete n-types over the empty set, $n < \omega$; it is known and easy to prove that if $D(T)$ is uncountable, it is of size $2^{\aleph_0}$; and every type in $D(T)$ must be realized in a universal model.) But this is not the case. In §1 we show that there is a first order theory T with $|D(T)| = \aleph_0$ (which is even $\aleph_0$-categorical) which has a universal model in $\aleph_1$ iff CH.

An attempt to characterize the class of theories for which it is *consistent* to have a universal model at $\aleph_1 < 2^{\aleph_0}$ was done by Mekler. Continuing [Sh 175], he has shown in [M] that it is consistent with the negation of CH that every universal theory of relational structures with the joint embedding property and amalgamation for $P^-(3)$-diagrams and only finitely many isomorphism types at every finite power, has a universal model at $\aleph_1$. He has also shown, continuing [Sh 175a], that it is consistent with $\kappa^{<\kappa} = \kappa < \mathrm{cf}(2^\kappa) < \lambda < 2^\kappa$ that every 4-amalgamation class, which in every finite power has only finitely many isomorphism types, has a universal structure in power λ.

In §2 we prove a covering theorem which shows, as one corollary, that if $2^{\aleph_0} = \aleph_{\omega_1}$, then there are no universal models for non-ω-stable theories in every regular uncountable λ below the continuum.

In §3 we prove in ZFC several nonexistence theorems for universal linear orders in regular cardinals. We show that there can be a universal linear order at a regular cardinal λ only if $\lambda = \lambda^{<\lambda}$ or if $\lambda = \mu^+$ and $2^{<\mu} \leq \lambda$. In §4 we prove nonexistence theorems for universal linear orders in singular cardinals. For example, if μ is not a strong limit and is not a fixed point of the $\aleph$ function, then there is no universal linear order in μ.

In §5 we reduce the existence of a universal linear order in cardinality λ to the existence of a universal model for any theory possessing the strict order property. Thus the nonexistence theorems from §§3 and 4 which were proven for linear orders are shown to hold for a large collection of theories.

The combined results from §§3, 4 and 5 show that it is impossible to generalize [Sh 100] in the same fashion that [Sh 175a] and [M] generalize [Sh 175]: While the proof of the consistency of having a universal graph in $\aleph_2 < 2^{\aleph_0}$ generalizes the proof for the case $\aleph_1 < 2^{\aleph_0}$, the consistency of universal linear order is true for the former case and is *false* for the latter. This points out an interesting difference between the theory of order and the theory of graphs.

The second author is interested in the classification of unstable theories (see [Sh 93]) with respect to the problem of determining the stability spectrum of a theory T (namely the class $K_T = \{\lambda: T \text{ has a universal model at } \lambda\}$). There are several more results which were obtained, in addition to what is published here: the main one is a satisfactory distinction between superstable and stable non-superstable theories (see [KjSh 447]).

Notation and terminology. By "order" we mean a linear order. $|M|$ denotes the universe of a model M and $\|M\|$ denotes its cardinality. For a set of cardinals X, acc X is the set of accumulation points of X, and nacc X is $X \setminus \mathrm{acc}\, X$.

§1. A theory without universal models in $\aleph_1 < 2^{\aleph_0}$. We present a theory T. In the language $L(T)$ there are two n-ary relation symbols, $R_n(\cdots)$ and $P_n(\cdots)$ for every natural number $n \geq 2$. T has no constants or function symbols. The axioms of T are:

1. The sentences saying that P_n and R_n are invariant under permutation of arguments and that $P_n(x_1, \dots, x_n)$ and $R_n(x_1, \dots, x_n)$ do not hold if $x_i = x_j$ for some $1 \leq i < j \leq n$, for all $n \geq 2$.

2. For each n the sentence saying that there are no $2n - 1$ distinct elements, $x_1,\ldots,x_n, y_1,\ldots,y_{n-1}$ such that $P_n(x_1,\ldots,x_n)$ and, for all $1 \le i \le n$, $R_n(y_1,\ldots,y_{n-1},x_i)$.

1.1. *Fact.* (1) *There are only finitely many quantifier-free n-types of T for every finite n.*

(2) *T has the joint embedding property and the amalgamation property.*

Proof. (1) is obvious. Suppose that M and N are two models of T which agree on their intersection. As T is universal, the intersection is also a model of T. Define a model M' such that $|M'| = |M| \cup |N|$ and such that $P_n^{M'}$ and $R_n^{M'}$ are equal, respectively, to $P_n^M \cup P_n^N$ and $R_n^M \cup R_n^N$. Suppose to the contrary that M' does not satisfy T. So, for some n there are $a_1,\ldots,a_n, b_1,\ldots,b_{n-1}$ which realize the forbidden type. Certainly,

$$\{a_1,\ldots,a_n,b_1,\ldots,b_{n-1}\} \not\subseteq M,$$

as $M \models T$, and

$$\{a_1,\ldots,a_n,b_1,\ldots,b_{n-1}\} \not\subseteq N,$$

as $N \models T$. So either

(a) there is some $a_i \notin M$ and $a_j \notin N$, or

(b) there is some $a_i \notin M$ and $b_j \notin N$, or

(c) there is some $b_i \notin M$ and $b_j \notin N$.

If (a) holds this contradicts $P(a_1,\ldots,a_n)$; if (b) holds this contradicts $R(b_1,\ldots,b_{n-1},a_i)$; and if (c) holds this contradicts $R(b_1,\ldots,b_{n-1},a_1)$. ⊣

1.2. *Fact. T has a universal homogeneous model M at* $\aleph_0$.

This should be well known, but for completeness of the presentation we give a proof.

Proof. Construct an increasing sequence of finite models M_n:

1. $M_n \models T$ and M_n is finite.
2. $M_n \subset M_{n+1}$.
3. In $M_{n+1} \backslash M_n$ all quantifier-free types (of T) over M_n are realized.

As T has only finitely many quantifier free types over every finite set, and because of Fact 1.1, this construction is possible. The model $M = \bigcup M_n$ clearly satisfies T.

Suppose that h is a finite embedding from any other model N into M and that $a \in N \backslash \mathrm{dom}(h)$. There is some n_0 such that $\mathrm{ran}(h) \subseteq M_{n_0}$. In M_{n_0+1} there is some b such that its relational type over $\mathrm{ran}(h)$ (in M) equals the relational type of a over $\mathrm{dom}(h)$ (in N). Set $h' = h \cup \langle a, b\rangle$ to obtain an embedding with a in its domain. By this observation it is immediate that every countable model of T is embeddable into M. Hence the universality of M in $\aleph_0$.

As there are no unary relation symbols in T, any $h = \{\langle a_1, a_2\rangle\}$, where $a_1, a_2 \in M$, is an embedding. Suppose that h is a finite embedding, $\mathrm{dom}(h), \mathrm{ran}(h) \subseteq M$, and that $b \in M \backslash \mathrm{ran}(h)$. Pick, as before, some $a \in M \backslash \mathrm{dom}(h)$ such that its relational type over $\mathrm{dom}(h)$ equals the relational type of b over $\mathrm{ran}(h)$, and extend h to include b in its range. These observations show that for every two sequences $\bar{a}, \bar{b} \in M^n$ there is an automorphism f of M with $f(\bar{a}) = \bar{b}$. Hence M is homogeneous. ⊣

Denote by T_1 the theory $\mathrm{Th}(M)$, the theory of the model M. Then

1.3. *Fact.* T_1 *is a complete theory extending T, which admits elimination of quantifiers and is* $\aleph_0$*-categorical.*

Proof. Clearly, every simple existential formula is equivalent to a quantifier-free formula in T_1. Hence the elimination of quantifiers. By Fact 1.1 there are only finitely many n-types of T over the empty set. Therefore T is $\aleph_0$-categorical (see [CK] for details). ⊣

1.4. *Fact.* (1) *For every infinite model $M \models T$ there is a model M' such that $M \subseteq M' \models T_1$ and $\|M\| = \|M'\|$.*

(2) *T and T_1 have the same spectrum of universal models; namely, for every cardinal λ, T has a universal model in λ (with respect to ordinary embeddings) iff T_1 has a universal model in λ (with respect to elementary embeddings).*

Proof. (1) follows by the compactness theorem, as every finite submodel of M (even countable) satisfies T, and is therefore embeddable into the countable model of T_1. For (2), we may forget about finite cardinals, as neither of the two theories has universal finite models (in fact T_1 has no finite models at all). Suppose first that $M \models T$ is universal for T in power λ. Then by (1) there is some $M' \supseteq M$, a model of T_1 of the same cardinality. Let $N \models T_1$ be arbitrary of power λ. As $N \models T$, there is an embedding $h: N \to M$. h is also an embedding into M'. As T_1 has elimination of quantifiers, h is an elementary embedding of N into M'. So M' is a universal model of T_1 in λ. Conversely, suppose that $M \models T_1$ is universal for T_1 in power λ. In particular, $M \models T$. Let $N \models T$ be arbitrary of power λ. By (1) there is some $N' \supseteq N$ of cardinality λ, with $N' \models T_1$. Let $h: N' \to M$ be an elementary embedding. Then $h \restriction N$ is an embedding of N into M. So M itself is universal for T. ⊣

1.5. Remarks. 1. T does not satisfy the 3-amalgamation property, as we can see by a simple example.

2. Also T_1 has the joint embedding property.

1.6. Theorem. *T has a universal model in $\aleph_1$ iff $\aleph_1 = 2^{\aleph_0}$.*

Proof. If $\aleph_1 = 2^{\aleph_0}$, then all countable theories have universal models in $\aleph_1$. We proceed now to prove that $2^{\aleph_0} > \aleph_1$ implies that T has no universal model at $\aleph_1$. Suppose to the contrary that CH fails, but that M is a universal model at $\aleph_1$. Without loss of generality, $|M| = \omega_1$. We define now $2^{\aleph_0}$ models of T: for each $\eta \in {}^{\omega}2$ let M_η be a model with universe ω_1 such that

(a) $P_n^{M_\eta} = [\omega_1]^n$ iff $R_n^{M_\eta} = \emptyset$ iff $\eta(n) = 0$, and

(b) $R_n^{M_\eta} = [\omega_1]^n$ iff $P_n^{M_\eta} = \emptyset$ iff $\eta(n) = 1$.

For each η the model M_η trivially satisfies T.

As M is universal, we can choose for each η an embedding $h_\eta: M_\eta \to M$. Let M_η^* be the model obtained from M by enriching it with the relations of M_η and the function h_η. Let C_η be the closed unbounded set $\{\delta \in \omega_1: M_\eta^* \restriction \delta \prec M_\eta^*\}$, and let $\delta_\eta \in C_\eta$.

As we have $2^{\aleph_0}$ η's, by the pigeonhole principle there are more than $\aleph_1$ sequences $\langle \eta_i: i < i(*)\rangle$ such that, for all $i < i(*)$, $\delta_{\eta_i} = \delta_0$. As there are only $\aleph_1$ possible values to $h_{\eta_i}(\delta_0)$, we may assume that, for all $i < i(*)$, $h_{\eta_i}(\delta_0) = \gamma_0$ for some fixed γ_0, and that $\eta_i \restriction 2$ is fixed. (Note that by elementarity, and as h is one-to-one, $\gamma_0 \geq \delta_0$). Now pick $i < j < i(*)$. There exists an $n > 1$ with $\eta_i(n) \neq \eta_j(n)$, and we assume by symmetry $\eta_i(n) = 0$. This means that every n-tuple of distinct members of the range of h_{η_i} satisfies the relation P_n^M, while every n-tuple of distinct members of the range of h_{η_j} satisfies the relation R_n^M. We intend now to derive a contradiction by constructing the forbidden type inside M. Pick any $n-1$ points $b_1, \ldots, b_{n-1} \in \delta_0$ in the range of h_{η_j}. Notice that $M \models R_n(b_1, \ldots, b_{n-1}, \gamma_0)$. Work now in $M_{\eta_i}^*$.

$M_{\eta_i}^* \models \exists(x) R_n(b_1, \ldots, b_{n-1}, h_{\eta_i}(x))$, as δ_0 witnesses this.

So by elementarity there is such a c_1 below δ_0 with $a_1 = h_{\eta_i}(c_1)$ also below δ_0.

$M_{\eta_i}^* \models \exists(x \neq c_1) R_n(b_1, \ldots, b_{n-1}, h_{\eta_i}(x))$, as δ_0 witnesses this.

So by elementarity we can find c_2 and a_2 below δ_0. We proceed by induction, each time picking a_{i+1} different from all the previous a's. So when $i = n$ we have constructed the forbidden type, as $a_1, \dots, a_n$, being in the range of h_{η_i}, satisfy the relation P_n^M. This contradicts $M \models T$. ⊣

The above proof tells us a bit more than is stated in Theorem 1.6: what was actually done, was to construct $2^{\aleph_0}$ models of T, each of size $\aleph_1$, such that no $2^{\aleph_0}$ can be embedded into a single model of T. But this construction uses no special feature of $\aleph_1$, and the models as defined above can be defined in any cardinality. Let us state the following.

1.7. THEOREM. *Let T be the theory in 1.6. If $\aleph_0 < \lambda = \mathrm{cf}\,\lambda < 2^{\aleph_0}$ and $\mu < 2^{\aleph_0}$ are cardinals, then for every family $\{M_i : i < \mu\}$ of models of T, each M_i of cardinality λ, there is a model M of T which cannot be embedded into any M_i in the family.*

PROOF. Suppose such a family is given. As in the previous proof, there are $2^{\aleph_0}$ trivial models of T, each with universe λ. Suppose that each of them is embedded into some member of the family. As $\mu < 2^{\aleph_0}$, there must be a fixed member $M_{i(*)}$ of the family into which more than λ such models are embedded. The contradiction now follows as above. ⊣

§2. A covering theorem.

We prove here a theorem that as one consequence puts a restriction on the cofinality of 2^λ, provided there is a universal model for a suitable theory in some cardinality $\kappa \in (\lambda, 2^\lambda)$.

2.1. THEOREM. *Let T be a first-order theory, and let $\lambda < \kappa < \mu$ be cardinals. Suppose that T has a universal model at κ and that there is a model M of T, $|M| = \mu$, with a subset $A \subseteq \mu$, $|A| = \lambda$, such that $|S(A)| \geq \mu$; namely, there are μ complete 1-types over A. Then there is a family $\langle B_i : i < \kappa^\lambda \rangle \subseteq [\mu]^\kappa$ which covers $[\mu]^\kappa$; namely, for every $C \in [\mu]^\kappa$ there is an $i < \kappa$ such that $C \subseteq B_i$.*

As corollaries we get

2.2. COROLLARY. *If $\mathrm{cf}(2^\lambda) \leq \kappa < 2^\lambda$ and T is a first order theory possessing the independence property, then T has no universal model at κ.*

2.3. COROLLARY. *Suppose that $2^{\aleph_0} = \aleph_{\omega_1}$. Then all theories unstable in $\aleph_0$ (e.g. the theory of graphs, the theory of linear order, and so on) do not have a universal model in any cardinal $\kappa \in [\aleph_1, \aleph_{\omega_1})$.*

PROOF OF COROLLARY 2.2 FROM THEOREM 2.1. If T possesses the independence property, then there is a model of T in which 2^λ types over a set of size λ are realized. By Theorem 2.1, if there were a universal model for T at κ, then there would be a covering family of $[2^\lambda]^\kappa$ of size 2^λ; but as $\mathrm{cf}(2^\lambda) \leq \kappa$, this is clearly impossible. ⊣

PROOF OF COROLLARY 2.3 FROM THEOREM 2.1. If a countable theory T is not stable at $\aleph_0$, then T has a model in which $2^{\aleph_0}$ complete 1-types over a countable set are realized. So a universal model at $\kappa \in [\omega_1, 2^{\aleph_0})$ would imply, by Theorem 2.1, that there is a covering family of $[2^{\aleph_0}]^\kappa$ of size $2^{\aleph_0}$, which is impossible as $\mathrm{cf}(2^{\aleph_0}) \leq \kappa$. ⊣

PROOF OF THEOREM 2.1. Let U be a universal model for T with universe κ, and let M be a model of T with a subset $A \subset |M|$ of size λ with $\langle p_i \in S(M) : i < \mu \rangle$ a sequence of distinct complete types over A. Without loss of generality M is of size μ and in M all p_i's are realized. We can further assume (by enumerating $|M|$) that $|M| = \mu + \mu$, that p_i is realized by the element i, and that $A = \{\alpha : \mu \leq \alpha < \mu + \lambda\}$. For each submodel N, $A \subseteq N \prec M$, of size κ pick an embedding $h_N : N \to U$. For each function $f : A \to \kappa$ let C_f be the set of submodels $\{N \subseteq M : |N| \leq \kappa,\ A \subseteq N,$ and $h_N \restriction A = f\}$.

2.4. *Claim. For each* $f \in \kappa^A$, $|\bigcup C_f \cap \mu| \le \kappa$.

Proof. Enumerate all members of C_f in a sequence $\langle N_\alpha : \alpha < \alpha(*)\rangle$ and define a function $g: \bigcup C_f \cap \mu \to \kappa$ by induction on α as follows:

$$g \restriction \left(N_\alpha \setminus \bigcup_{\beta<\alpha} N_\beta\right) = h_{N_\beta} \restriction \left(N_\alpha \setminus \bigcup_{\beta<\alpha} N_\beta\right).$$

We are done if g is a 1-1 function. This follows from

2.5. *Fact. If* $i \in N_\alpha$, $j \in N_\beta$, *and* $i < j < \mu$, *then* $h_{N_\alpha}(i) \ne h_{N_\beta}(j)$.

Proof of Fact 2.5. As both embeddings agree on the image of A and i and j realize different types over A, the fact is immediate.

2.6. *Claim. The family* $\{(\bigcup C_f) \cap \mu : f \in {}^A\kappa\}$ *is a covering family of* $[\mu]^\kappa$ *of size* λ^κ.

Proof of Claim 2.6. Clearly the size of the family is as stated. Let $B \in [\mu]^\kappa$ be any set. Then it is a subset of some elementary submodel $N \subseteq M$ which contains A as a subset. So it is a subset of $\bigcup C_{f_N \restriction A} \cap \mu$. ⊣

§3. Nonexistence of universal linear orders. In this section we prove some nonexistence theorem for universal linear orders in regular cardinals. We start by showing that there is no universal linear order in a regular cardinal λ if $\aleph_1 < \lambda < 2^{\aleph_0}$. We shall generalize this for more regular cardinals later in this section. The combinatorial tool which enables us to prove these theorems is the guessing of clubs which was introduced in [Sh-e] and can be found also in [Sh-g] (which will, presumably, be available sooner). Proofs of the relevant combinatorial principles are repeated in the Appendix to this paper for the reader's convenience.

3.1. Definitions. 1. If C is a set of ordinals and δ an ordinal, we denote by δ_C^s the element $\min\{C \setminus (\delta+1)\}$, when it exists.

2. A *cut* D of a linear order O is pair $\langle D_1, D_2\rangle$ such that D_1 is an initial segment of O, namely $D_1 \subseteq |O|$ and $y < x \in D \Rightarrow y \in D_1$, while D_2 is an end segment, namely $D_2 \subseteq |O|$ and $y > x \in D_2 \Rightarrow y \in D_2$, with $D_1 \cap D_2 = \varnothing$ and $D_1 \cup D_2 = |O|$. If $O_1 \subseteq O_2$ are linear orders, then an element $x \in O_2 \setminus O_1$ *realizes* a cut D of O_1 if $D_1 = \{y \in O_1 : y < x\}$.

3. Let $O = \bigcup_{j<\lambda} O_j$ be an increasing continuous union of linear orders, let $\delta \in \lambda$ be limit, and let $C \subseteq \delta$ be unbounded in δ. Let $x \in (O \setminus \bigcup_{j<\delta} O_j)$. Define $\mathrm{Inv}_O(C, \delta, x)$, *the invariance of x in O with respect to C*, as $\{\alpha \in C : \exists y \in O_{\alpha_C^s}$ such that y and x realize the same cut of $O_\alpha\}$. Note that this definition is applicable also to cuts (rather than only to elements).

4. A κ-*scale* for λ is a sequence $\bar{C} = \langle c_\delta : \delta \in S\rangle$, where $S = \langle \alpha < \lambda : \mathrm{cf}\, \alpha = \mathrm{cf}\, \kappa\}$ and for every $\delta \in S$, c_δ is a club of δ of order type κ; $c_\delta = \langle \alpha_j^\delta : j < \kappa\rangle$ is an increasing enumeration of c_δ. If $O = \bigcup_{i<\lambda} O_i$ is a linear order represented as a continuous increasing union of smaller orders, and $\bar{C}$ is a κ-scale for some $\kappa < \lambda$, let

$$\mathrm{INV}(O, \bar{C}) \stackrel{\mathrm{def}}{=} \{X \subseteq \kappa : (\exists \delta \in S)(\exists x \notin O_\delta)\mathrm{Inv}(c_\delta, \delta, x) = \{\alpha_j^\delta : j \in X\}\}.$$

So $\mathrm{INV}(O, \bar{C})$ is the set of all subsets of κ which are obtained (when identifying κ with c_δ) as an invariance of some element in O with respect to some c_δ in the scale.

3.2. *Claim. Suppose* $h: O_1 \to O_2$ *is an embedding of linear orders*, $\|O_1\| = \|O_2\| = \lambda = \mathrm{cf}\,\lambda > \aleph_0$. *Then for any representations* $O_i = \bigcup_{j<\lambda} O_j^i$, $i = 1, 2$, *the union increasing and continuous and each* $|O_j^i| < \lambda$, *there exists a club* $E \subseteq \lambda$ *such that, for* *any $\delta < \lambda$ and $C \subseteq \delta$ a club of δ which satisfies $C \subseteq E$, we have*

$$(*) \qquad (\forall x \in O_1 \backslash O_\delta)(\mathrm{Inv}_{O_1}(C, \delta, x) = \mathrm{Inv}_{O_2}(C, \delta, h(x)).$$

Proof of the Claim. Without loss of generality we may assume that $|O_1| = |O_2| = \lambda$. Define the model $M = \langle \lambda, <_{O_1}, <_{O_2}, \in, h \rangle$. Let $E = \{\delta < \lambda: M \restriction \delta \prec M$ and, for all $\delta \in E$, $\delta = \bigcup_{j<\delta} O^i_j$, $i = 1, 2\}$. Let $x \in (O_1 \backslash O^1_\delta)$. Note that by elementarity $h(x) \in (\lambda \backslash \delta)$. Suppose first that α belongs to the left-hand side of the equality in $(*)$, and let $y \in [\alpha, \alpha^s_C)$ demonstrate this. So x and y realize the same cut of $O_1 \restriction \alpha$. As h is an embedding, $h(x)$ and $h(y)$ satisfy the same cut of $h''(O_1 \restriction \alpha)$ (which equals, by elementarity, $(h''(O_1)) \restriction \alpha$). If $h(x)$ and $h(y)$ satisfy also the same cut of $O_2 \restriction \alpha$ we are done, but the problem is, of course, that h is not necessarily onto. Otherwise we suppose without loss of generality that $h(y) < h(x)$ and that there is an element $z \in O_2 \restriction \alpha$ such that $h(y) <_{O_2} z <_{O_2} h(x)$. Define in M the set $D = \{t$: there is no q such that $z \leq_{O_2} h(q) \leq_{O_2} t\}$. D is definable in M with parameters in $M \restriction \alpha^s$. By elementarity the definition is absolute between M and $M \restriction \alpha^s$; that is, $D \cap \alpha^s$ is the same as D interpreted in $M \restriction \alpha^s$. D is a cut of $O_2 \restriction \alpha$. Let D' be $D \cap \alpha$; D' is definable in $M \restriction \alpha^s$.

3.2.1. *Subclaim. $h(x)$ satisfies the cut D' determined by D.*

Proof of the Subclaim. Let $z <_{O_2} \beta <_{O_2} h(x)$. As there are no points in the range of $h \restriction \alpha$ between z and $h(x)$, there are certainly none between z and β. So $\beta \in D'$. Conversely, suppose $h(x) <_{O_2} \beta$. Then M satisfies that there is an image under h (namely $h(x)$) between z and β. By elementarity there is such an image $h(x')$ where $x' \in \alpha$. So $\beta \notin D'$.

As D' is a cut of $O' \restriction \alpha$ definable in $M \restriction \alpha^s$ which is realized by $h(x)$, elementarity assures us that it is realized by some $y' \in [\alpha, \alpha^s_C)$. So α belongs to the right-hand side of $(*)$.

Assume that α belongs to the right-hand side of $(*)$. Then there is an element $y \in [\alpha, \alpha^s)$ which satisfies the same cut of $O_2 \restriction \alpha$ as $h(x)$. If $y = h(y')$ for some y', we are done. Else, we note that the cut of $O_2 \restriction \alpha$ which y determines is definable in $M \restriction \alpha^s$. Now clearly $h(x)$ and y satisfy the same cut of $O_2 \restriction \alpha$. By elementarity there is an element y' such that $h(y')$ satisfies the same cut as y, therefore as $h(x)$. In other words, α belongs to the left-hand side of $(*)$. ⊣

3.3. *Fact. If O is an order with universe λ and $\bar{C}$ is a κ-scale, then $|\mathrm{INV}(O, \bar{C})| \leq \lambda$.*

Proof. Trivial.

3.4. Lemma (the construction lemma). *If $\lambda < 2^{\aleph_0}$ is a regular uncountable cardinal, $\bar{C}$ is an ω-scale, and $A \subseteq \omega$ is given, then there is an order O with universe λ, $O = \bigcup_{i<\lambda} O_i$, an increasing continuous union of smaller orders, such that, for every $\delta < \lambda$ with* $\mathrm{cf}\, \delta = \aleph_0$,

$$\mathrm{Inv}(c_\delta, \delta, \delta) = \{\alpha^\delta_n : n \in A\}.$$

Proof. We define by induction on $0 < \alpha < \lambda$ an order O_α with the properties listed below. We denote by Q the order of the rationals. If $O_1 \subseteq O_2$ are linear orders, D a cut of O_1 and D' a cut of O_2, we say that D' *extends* D if $D'_1 \restriction |O_1| = D_1$ and $D'_2 \restriction |O_1| = D_2$. Also note that if $O_1 \subseteq O_2$ are linear orders and D^1 is a cut of O_1 which is not realized in O_2, then it corresponds naturally to a cut D^2 of O_2. In such a case we say that D^1 is (really) also a cut of O_2.

1. O_α has universe $|O_\alpha| \in \lambda$.

2. If $\beta + 1 = \alpha$ and $x \in (O_\alpha \backslash O_\beta)$, then $\{y \in O_\alpha \backslash O_\beta$: x and y satisfy the same cut of $O_\beta\}$ has order type Q.

3. If $\alpha < \beta < \gamma$, γ is a successor, and there is a cut D of O_α which is realized by an element of O_β but is not realized by no element of O_ν for $\alpha < \nu < \beta$, then there is a cut D' of O_β, which extends D, which is realized in O_γ but is not realized in O_ν for all $\beta < \nu < \gamma$. Also for every successor α there is a cut of O_0 which is realized in O_α but not in O_β for every $\beta < \alpha$.

4. If α is limit then $O_\alpha = \bigcup_{\beta < \alpha} O_\beta$.

5. If $\operatorname{cf} \delta = \aleph_0$ and, for all $\beta \in C_\alpha$, $|O_\beta| = \beta$, then $\operatorname{Inv}_{O_{\delta+1}}(c_\delta, \delta, \delta) = \{\alpha_n^\delta : n \in A\}$.

There should be no problem taking care of 1–4. Assume that the conditions of 5 are satisfied. We wish to define the order $O_{\alpha+1}$. Let $C_\alpha = \langle \beta_n : n < \omega \rangle$. By induction on $A = \langle a_n : n < \omega \rangle$ define an increasing sequence of cuts $\langle D_{\alpha_n} : n \in \omega \rangle$ such that D_{a_n} is a cut of O_{a_n} which is realized for the first time in $O_{a_{n+1}}$. Demand 3 enables this. In $O_{\alpha+1}$ let α satisfy $\bigcup D_n$ to get 5. ⊣

We are almost ready to prove the nonexistence of a universal order in a regular λ, $\aleph_1 < \lambda < 2^{\aleph_0}$. We recall from [Sh-e, Chapter III.7.8] (see also [Sh-g]) the existence of "club guessing sequences":

3.5. *Fact. If $\lambda > \aleph_1$ is regular, then there is a sequence $\bar{C} = \langle c_\delta : \delta < \lambda, \operatorname{cf} \delta = \aleph_0 \rangle$, such that $c_\delta \subseteq \delta$ is a club of δ of order type ω_0, with the property that for every club $E \subseteq \mu$ the set $S_E = \{\delta < \lambda : \operatorname{cf} \delta = \aleph_0$ and $c_\delta \subseteq E\}$ is stationary ($\bar{C}$ is called a "club guessing sequence").* ⊣

A proof of this fact is found in the Appendix.

3.6. Theorem. *If $\aleph_1 < \lambda = \operatorname{cf} \lambda < 2^{\aleph_0}$, then there is no universal order in cardinality λ.*

Proof. Suppose to the contrary that UO is a universal order in cardinality λ. Without loss of generality, $|UO| = \lambda$. Fix some club guessing sequence $\bar{C} = \langle c_\delta : \delta < \lambda, \operatorname{cf} \delta = \aleph_0 \rangle$. This is known to exist by the previous fact. As $|\operatorname{INV}(UO, \bar{C})| \leq \lambda$, there is some $A \subseteq \omega$, $A \notin \operatorname{INV}(UO, \bar{C})$. Use the construction lemma to get an order M with universe λ and with the property that, for every $\delta < \lambda$, $\operatorname{cf} \delta = \aleph_0$ implies that $\operatorname{Inv}_M(c_\delta, \delta, \delta) = \{\alpha_n^\delta : n \in A\}$. Let $h: M \to UO$ be an embedding. Let E_h be the club given by 3.2. As $\bar{C}$ guesses clubs, there is some $\delta(*)$ with $c_{\delta(*)} \subseteq E_h$. Therefore, by 3.2,

$$\operatorname{Inv}_M(c_{\delta(*)}, \delta(*), \delta(*)) = \operatorname{Inv}_{UO}(c_{\delta(*)}, \delta(*), h(\delta(*))).$$

But $\operatorname{Inv}_M(\gamma_{\delta(*)}, \delta(*), \delta(*)) = \{\alpha_n^\delta : n \in A\}$. This means that $A \in \operatorname{INV}(UO, \bar{C})$, a contradiction to the choice of $A \notin \operatorname{INV}(UO, \bar{C})$. ⊣

We wish now to generalize Theorem 3.6 by replacing ω_0 by a more general κ. As the proof of 3.6 made use of both club guessing and the construction lemma, we should see what remains true of these two facts for $\kappa > \aleph_0$. The proof of the construction lemma does not work when we replace $\aleph_0$ by some other cardinal. We need some extra machinery to handle the limit points below κ.

3.7. Lemma (the second construction lemma). *Suppose $\kappa < \lambda = \operatorname{cf} \lambda$ are cardinals, $2^\kappa \geq \lambda$, and that there is a stationary $S \subseteq \lambda$ and sequences $\langle c_\delta : \delta \in S \rangle$ and $\langle P_\alpha : \alpha < \lambda \rangle$ which satisfy*:

(1) $\operatorname{otp} c_\delta = \kappa$ *and* $\sup c_\delta = \delta$;

(2) $P_\alpha \subseteq \mathscr{P}(\alpha)$ *and* $|P_\alpha| < \lambda$;

(3) *if* $\alpha \in \operatorname{nacc} c_\delta$ *then* $c_\delta \cap \alpha \in \bigcup_{\beta < \alpha} P_\beta$.

Then, when given such sequences and a closed $A \subseteq \operatorname{Lim} \kappa$, there is a linear order O with universe λ with the property that, for every $\delta \in S$, $\operatorname{Inv}(c_\delta, \delta, \delta) = A_\delta$, where A_δ is the subset of c_δ which is isomorphic to A.

PROOF. We pick some linear order L of cardinality smaller than λ which has at least λ cuts. We assume, without loss of generality, that $P_\alpha \subseteq P_\beta$ whenever $\alpha < \beta$, that, for limit α, $P_\alpha = \bigcup_{\beta < \alpha} P_\beta$, and that if $\alpha \in \operatorname{nacc} c_\delta$ then $A_\delta \cap \alpha \in P_\alpha$. Next we construct by induction on $\alpha < \lambda$ an order O_α and a partial function F that meet the following demands:

(1) the universe of O_α is an ordinal below λ.

(2) $\alpha < \beta \Rightarrow O_\alpha \subseteq O_\beta$, and if α is limit, then $O_\alpha = \bigcup_{\beta < \alpha} O_\beta$.

(3) If $x \in O_\beta \setminus O_\alpha$, then the order type of $\{y \in O_\beta \colon x$ and y satisfy the same cut of $O_\alpha\}$ contains L as a suborder. Also, if α is a successor, then there is an element in O_α which satisfies a new cut of O_0.

(4) If $\alpha < \beta < \gamma$ and γ is a successor, and if D is a cut of O_α which is realized in O_β but not in an earlier stage, then there is a cut D' of O_β which extends D and is realized in O_γ but is not realized in O_δ for any $\delta < \gamma$.

(5) F is a partial function, $\operatorname{dom} F \subseteq S \times (\lambda \setminus \operatorname{Lim} \lambda)$. A pair $\langle \delta, \alpha \rangle \in \operatorname{dom} F$ iff $\alpha < \delta$ and $\varnothing \neq A_\delta \cap \alpha \in P_\alpha \setminus P_{\alpha - 1}$. $F(\delta, \alpha)$ is a pair $\langle \beta(\delta, \alpha), D(\delta, \alpha) \rangle$, where $\beta < \alpha$ and D is a cut of O_β which is realized in O_α. If β is not a limit of A_δ, then D is not realized in O_γ for any $\gamma < \alpha$. $F(\delta, \alpha)$ depends only on $A_\delta \cap \alpha$; namely, if $A_{\delta_1} \cap \alpha = A_{\delta_2} \cap \alpha$ then $F(\delta_1, \alpha) = F(\delta_2, \alpha)$. If $\alpha < \gamma$ and $F(\delta, \alpha)$ and $F(\delta, \gamma)$ are both defined, then $\beta(\delta, \alpha) < \beta(\delta, \gamma)$ and $D(\delta, \gamma)$ extends $D(\delta, \alpha)$.

(6) If $\delta \in S$ then $\operatorname{Inv}_{O_{\delta+1}}(c_\delta, \delta, \delta) = A_\delta$.

As O_0 we pick L. When α is limit, we define O_α as the union of previous orders. When α is a successor we add less than λ elements to take care of demands (3) and (4). If $A_\delta \cap \alpha \in P_\alpha \setminus \alpha - 1$ we must define $F(\delta, \alpha)$. If $A_\delta \cap \alpha$ contains exactly one member, let $\beta(\delta, \alpha) = 0$ and as $D(\delta, \alpha)$ pick (by (3)) a cut of O_0 which is realized in O_α but not in O_γ for any $\gamma < \alpha$. In case the order type of $\{\gamma < \alpha \colon F(\delta, \gamma)$ is defined$\}$ is limit, we let $D(\delta, \alpha) = \bigcup_{\gamma < \alpha} D(\delta, \gamma)$ and $\beta(\delta, \alpha) = \bigcup_{\gamma < \alpha} \beta(\delta, \gamma)$. Note that $\beta(\delta, \alpha) < \alpha$, because it is limit. Add more elements to O_α to realize D. Since $|P_\alpha| < \lambda$, this requirement of addition of elements is satisfied by adding less than λ new elements. In case there is a last $\gamma < \alpha$ for which $F(\delta, \gamma)$ is defined, let this γ be $\beta(\delta, \alpha)$ and pick (by (4)) a cut D of O_γ which extends $D(\delta, \gamma)$ and is realized in O_α, but is not realized earlier, as $D(\delta, \alpha)$. When $\alpha = \delta + 1$ and $\delta \in S$, let the element δ realize in O_α the cut $\bigcup_{\gamma < \alpha} F(\delta, \gamma)$.

Having added less than λ new elements, we fulfill demand (1). (2) and (3) are obvious, and (4) and (5) have been taken care of.

Claim. Demand (6) *holds.*

Proof. Suppose that $\delta \in S$. We show by induction that for every $x \in A_\delta$, for every $y \leq x$ in c_δ there is some $\gamma < y^s_{c_\delta}$ which satisfies the cut of δ over O_y iff $y \in A_\delta$. Suppose x is the first member of A_δ. Then the first γ for which $F(\delta, \gamma)$ is defined satisfies $x < \gamma < x^s_{cd}$ by the assumptions on $\langle P_\alpha \colon \alpha < \lambda \rangle$. $F(\delta, \gamma)$ is a cut of O_0 which is realized in O_γ but not before. If $y \in c_\delta \cap x$, as x is a limit of c_δ, we have $y^s_{cd} < x$. The cut of δ over O_0 is $F(\delta, \gamma)$, and the cut of δ over O_y extends this cut. As $F(\delta, \alpha)$ is not realized by the stage O_y, certainly the cut of δ over O_y is not realized by this stage either. So $y \notin \operatorname{Inv}(C_\delta, \delta, \delta)$. As the cut of δ over O_0 is not realized in O_x, it is really also a cut of O_x. This cut is realized in O_γ, where $\gamma < x^s_{\gamma d}$. So, by definition, $x \in \operatorname{Inv}(c_\delta, \delta, \delta)$.

In the case when x is a successor of A_δ, denote by z its predecessor in A_δ. The minimal γ above x for which $F(\delta,\gamma)$ is defined is smaller than $X^s_{\gamma_\delta}$, and $\beta(\delta,\gamma)$ is in the interval $(z, z^s_{c_d})$. The same argument as in the previous case shows that, for every $y \in (z,x]$, $y \in \operatorname{Inv}(\gamma_\delta,\delta,\delta)$ iff $y \in A_\delta$. When x is a limit of A_δ, by the induction hypothesis, for every $y < x$ the required holds. As for x itself, if γ is the minimal above x for which $F(\delta,\gamma)$ is defined, then $\gamma < x^s_{c_\delta}$ and $F(\delta,\gamma)$ is realized in O_γ. Therefore $x \in \operatorname{Inv}(c_\delta,\delta,\delta)$. ⊣

By [Sh 420] we know:

3.8. *Fact. If κ is a cardinal and $\kappa^+ < \lambda = \operatorname{cf}\lambda$, then there is a stationary set S and sequences $\langle c_\delta: \delta \in S\rangle$ and $\langle P_\alpha: \alpha \in \lambda\rangle$ as in the assumptions of* 3.7. ⊣

What is still lacking is the appropriate club guessing sequence, the existence of which we now quote from [Sh-g]:

3.9. *Fact. If κ is a cardinal, $\kappa^+ < \lambda = \operatorname{cf}\lambda$, and there is a stationary set $S \subseteq \lambda$ and sequences $\langle c_\delta: \delta \in S\rangle$ and $\langle P_\alpha: \alpha < \lambda\rangle$ as in* 3.7, *then there are such with the additional property that for every club E the set $S_E = \{\delta: \delta \in S \wedge c_\delta \subseteq E\}$ is stationary.* ⊣

3.10. THEOREM. Suppose $\lambda = \operatorname{cf}\lambda$ *and there is some cardinal κ such that $\kappa^+ < \lambda < 2^\kappa$. Then there is no universal linear order in cardinality λ.* ⊣

PROOF. Suppose O is any order of cardinality λ, and assume without loss of generality that its universe is λ. Pick a stationary set S and sequences as in 3.7, with the property that $\bar{C} = \langle c_\delta: \delta \in S\rangle$ guesses clubs. Pick a closed set $A \subseteq \operatorname{Lim}\kappa$ which is not in $\operatorname{INV}(O,\bar{C})$. Use 3.7 to construct an order O' with universe λ and the property that for every $\delta \in S$, $\operatorname{Inv}_{O'}(c_\delta,\delta,\delta) \simeq A$. If O' were embedded into O, some c_δ would guess the club of the embedding, which would lead to a contradiction. So O' is not embeddable into O, and therefore there is no universal linear order in λ. ⊣

§4. Singular cardinals. We shall now state a theorem which concerns the nonexistence of universal linear orders in singular cardinals. Let us first note the following well-known fact.

4.1. *Fact. If μ is a strong limit, then for every first order theory T such that $|T| < \mu$ there is a special model of size μ, and therefore also a universal model in μ.* ⊣

For the definition of special model see the Appendix. A special model is universal. For more details see [CK, p. 217].

This means that for nonexistence of universal models we must look at singulars which are not strong limits. We will see at the end of this section that if, e.g., $\aleph_\omega$ is not a strong limit, then there is no universal linear order at $\aleph_\omega$.

We recall a definition from [Sh-g, 355,5].

4.2. DEFINITION. $\operatorname{cov}(\lambda,\mu,\theta,\sigma)$ is the minimal size of a family $A \subseteq [\lambda]^{<\mu}$ which satisfies the requirement that for all $X \in [\lambda]^{<\theta}$ there are less than σ members of A whose union covers X.

4.3. THEOREM. *Suppose $\theta = \operatorname{cf}\theta < \theta^+ < \kappa$ are regular cardinals, $\kappa < \mu$ and there is a binary tree $T \subseteq {}^{<\theta}2$ of size $<\kappa$ with $> \mu^* := \operatorname{cov}(\mu,\kappa^+,\kappa^+,\kappa)$ branches of length θ. Then*

$(*)_{\mu,\kappa}$ *There is no linear order of size μ which is universal for linear orders of size κ (namely, such that every linear order of size κ is embedded in it).*

PROOF. Let $\bar{A} = \langle A_i: i < \mu^*\rangle \subseteq [\mu]^{<\kappa^+}$ demonstrate the definition of μ^*. Without loss of generality, $|A_i| = \kappa$ for all i. Suppose to the contrary that there is an order $UO = \langle \mu, <_{UO} \rangle$ into which every order of size κ is embedded. Let M_i be $UO \upharpoonright A_i$ for every $i < \mu^*$. Then every M_i is isomorphic to some M_i' with universe κ, and for every order O of size κ there is a set $J \subseteq \mu^*$, $|J| < \kappa$, such that O is embedded into $\bigcup_{i \in J} M_i$.

We fix a club guessing sequence $\bar{C} = \langle c_\delta : \delta < \kappa \text{ and } \mathrm{cf}\,\delta = \theta \rangle$ and an increasing continuous sequence $\langle P_\alpha : \alpha < \kappa \rangle$ such that P_α is a family of subsets of α, $|P_\alpha| < \kappa$, and, for all $\alpha < \mu$, if $\delta \in S$ and $\alpha \in \mathrm{nacc}\, c_\delta$, then $(c_\delta \cap \alpha) \in P_\alpha$. For the existence of these, see [Sh 420] or 3.9 above.

For each $\delta \in S$ enumerate c_δ as $\langle \alpha_i^\delta : i < \theta \rangle$ in an increasing continuous fashion. Now T can be viewed as T_δ, a tree of subsets of c_δ. Under the assumptions we already have, it is no loss of generality to assume that for every $\alpha \in \kappa$, if $\alpha \in \mathrm{nacc}(c_\delta)$, then $T_\delta \cap \mathscr{P}(\alpha) \subseteq P_\alpha$. The reason is that there are θ possibilities for the unique i such that $\alpha = \alpha_i^\delta$, and for each such possibility there are $<\kappa$ subsets in $T \cap \mathscr{P}(i)$; so we can add all the required sets into P_α without changing the fact that $|P_\alpha| < \kappa$.

So by now we have the assumptions of 3.7. Using it we construct a linear order O on κ, with $A \subseteq \kappa$ not in $\{\mathrm{Inv}_{M_i'}(c_\delta, \delta, x) : i < \mu^*, x \in \kappa\}$.

Suppose now that there is an embedding $h: O \to UO$. The image of h is covered by $\bigcup\{O_i : j \in J\}$ for some J of size $<\kappa$. Let $S_j = \{x \in \kappa : h(x) \in M_j\}$. Then there is some j_0 such that $S_{j_0} \notin \overline{\mathrm{id}}^a(\bar{C})$. (The latter is the ideal of nonguessing, namely $X \in \overline{\mathrm{id}}^a(\bar{C})$ iff there is a club E such that $\forall(\delta \in S \cap X)(c_\delta \not\subseteq E)$. This ideal is clearly κ-complete.)

Let O' be $O \upharpoonright S_{j_0}$. Let $O_i' = O' \upharpoonright i$ for $i < \kappa$. Then this is a presentation of L' as an increasing continuous union of small orders. By 3.2 and the fact that the identity map embeds O' into O, almost everywhere the invariance with relation to O is the same as with relation to O'. So we can get again the same contradiction as in previous proofs by inspecting the embedding $h \upharpoonright O'$. ⊣

We wish now to obtain the same results using more concrete assumptions. We first review some facts concerning covering numbers.

Recall the following well-known fact (see, for example, [Sh-g 355,5]).

4.4. *Fact. If* $\delta < \kappa = \mathrm{cf}\,\kappa < \mu = \aleph_\delta$, *then* $\mathrm{cov}(\mu, \kappa^+, \kappa^+, \kappa) = \mu$.

Proof. By induction on χ, a cardinal with $\kappa < \chi \le \mu$.

(a) $\chi = \theta^+$. For every $\alpha < \chi$ fix a family $P_\alpha \subseteq [\alpha]^\kappa$ with the property that for every set $A \in [\alpha]^\kappa$ there is a set $X \subseteq P_\alpha$ with $|X| < \kappa$ and $A \subseteq \bigcup X$. Let P be the union of the P_α for $\alpha < \chi$. The size of P is clearly χ, and clearly for every set $A \subseteq \chi$ of size κ there is a covering of A by less than κ members of P.

(b) $\chi = \aleph_\beta$ is a limit cardinal. As $\mu = \aleph_\delta$ with $\delta < \kappa$, certainly $\beta < \kappa$. Let $\langle \chi_i : i < \mathrm{cf}\,\beta \rangle$ be increasing and unbounded below χ. Let P_i demonstrate that

$$\mathrm{cov}(\chi_i, \kappa^+, \kappa^+, \kappa) = \chi_i,$$

and let $P = \bigcup_i P_i$. Then $|P| = \chi$. If $A \in [\chi]^\kappa$, cover $A \cap \chi_i$ by less than κ members of P. Thus to cover A we need less than κ members of P.

4.5. *Improved Fact. If μ is a fixed point of the first order (i.e. $\lambda = \aleph_\lambda$), but not of the second order, i.e.* $|\{\lambda < \mu : \lambda = \aleph_\lambda\}| = \sigma < \mu$, *and* $\sigma + \mathrm{cf}\,\mu < \mu$, *then* $\mathrm{cov}(\mu, \kappa^+, \kappa^+, \kappa) = \mu$.

Proof. Suppose that $\kappa < \chi < \mu$ and $\mathrm{cf}\,\chi = \kappa$. By the assumptions, $\chi \ne \aleph_\chi$, say $\chi = \aleph_\delta$. By [Sh 400, §2], $\mathrm{pp}(\chi) < \aleph_{|\delta|^{+4}} < \mu$ (what precisely is pp does not matter here). By [Sh-g 355, 5.4], $\mathrm{pp}(\chi) \geq \mathrm{cov}(\chi, \kappa^{=}, \kappa^+, \kappa)$. So $\mathrm{cov}(\chi, \kappa^+, \kappa^+, \kappa) < \mu$. As χ is arbitrarily large below μ, we are done. ⊣

We see that we can have arbitrarily large κ below a singular μ with $\mu = \mathrm{cov}(\mu, \kappa^+, \kappa^+, \kappa)$, when μ is a limit which is not a second order fixed point of the $\aleph$ function. But for applying Theorem 4.3 we need also a binary tree of height and size $<\mu$ with $>\mu$ branches. This happens if there is some $\sigma < \mu$ with $2^{<\sigma} < \mu$ and $2^\sigma > \mu$. So we can state

4.6. COROLLARY. *If μ is a singular cardinal which is not a second order fixed point, and there is some $\sigma < \mu$ such that $2^{<\sigma} < \mu < 2^\sigma$, then there is no universal linear order of power μ.*

PROOF. Let T be the tree ${}^{<\sigma}2$. By the fact on covering numbers, pick $\sigma < \sigma^+ < \kappa$ such that $\mathrm{cov}(\mu, \kappa^+, \kappa^+, \kappa) = \mu$, and apply Theorem 4.3.

As for no μ with $\mathrm{cf}\,\mu = \aleph_0$ is there a $\sigma < \mu$ with $2^\sigma = \mu$, we can weaken the assumptions to get

4.7. COROLLARY. *If $\aleph_\mu > \mu$ or μ is not a second order fixed point, $\mathrm{cf}\,\mu = \aleph_0$ or $2^{<\mathrm{cf}\,\mu} < \mu$, and $\mu \neq 2^{<\mu}$, then there is no universal linear order at μ.* ⊣

§5. Generalizations. In this section we prove that if there is no universal linear order in a cardinal λ then there is no universal model in λ for any countable theory T possessing the strict order property (e.g. there is no universal Boolean algebra in λ). This means that all the nonexistence theorems in §§3 and 4 hold for a large class of theories.

5.1. DEFINITION. 1. A formula $\varphi(\bar{x}; \bar{y})$ has the *strict order property* if for every n there are $\bar{a}_l$ ($l < n$) such that, for any $k, l < n$,

$$\models (\exists x)[\neg\varphi(\bar{x}; \bar{a}_k) \wedge \varphi(\bar{x}; \bar{a}_l)] \Leftrightarrow k < l.$$

2. A theory T has the *strict order property* if some formula $\varphi(\bar{x}; \bar{y})$ has the strict order property.

This definition appears in [Sh-a, p. 68], or [Sh-c, p. 69]. Every unstable theory possesses the strict order property or the independence property (or both). For details see [Sh-a] or [Sh-c].

5.2. *Fact.* 1. *Suppose T has the strict order property, with $\varphi(\bar{x}; \bar{y})$ witnessing this, and let $M \models T$. Then φ defines a partial order $P_M = \langle |M|^n, \leq_\varphi \rangle$, where $n = \lg(\bar{y})$, the length of $\bar{y}$, the order being given by $\bar{y}_1 \leq_\varphi \bar{y}_2 \Leftrightarrow M \models \varphi(\bar{x}; \bar{y}_1) \rightarrow \varphi(\bar{x}; \bar{y}_2)$. In this order there are arbitrarily long chains.*

2. *If $h\colon M_1 \rightarrow M_2$ is an embedding between two models of T which preserves φ, then $h'\colon P_{M_1} \rightarrow P_{M_2}$ is an embedding of partial orders, where $h'(x_1, \ldots, x_n) = (h(x_1), \ldots, h(x_n))$.*

Proof. Immediate from the definition. ⊣

5.3. LEMMA. *Suppose T has the strict order property, with $\varphi(\bar{x}; \bar{y})$ witnessing this. If L is a given linear order, then there is a model M of T such that L is isomorphic to a suborder of P_M and $\|M\| = |L| + \aleph_0$.*

PROOF. As there are arbitrarily long chains with respect to $\leq_\varphi$, this is an immediate corollary of the compactness theorem. ⊣

5.4. LEMMA. *If there is a partial order of size λ with the property that every linear order of size λ can be embedded into it, then there is a universal linear order in power λ.*

PROOF. Suppose that $P = \langle |P|, \leq \rangle$ is a partial order of size λ with this property. Divide by the equivalence relation $x \sim y \Leftrightarrow x \leq y \wedge y \leq x$, to obtain P', a partial order in the strict sense. There is some linear order $<$ on $|P'|$ which extends $\leq$. Let $UO = \langle |P'|, < \rangle$. Let L be any linear order, and let $h: L \to P$ be an embedding. Whenever $x \neq y$ are elements of L, $h(x) \nsim h(y)$ in P. Therefore $h': L \to P'$ defined by $h'(x) = [h(x)]$ is still an embedding. h' is an embedding of L into UO. So UO is universal. ⊣

5.5. THEOREM. *Suppose that T has the strict order property, λ is a cardinal, and T has a universal model (with respect to elementary embeddings) in cardinality λ. Then there exists a universal linear order in cardinality λ.*

PROOF. By 5.4 it is enough to show that there is a partial order of size λ which is universal for linear orders, namely that every linear order of the same size is embedded into it. Let M be a universal model of T, and let φ witness the strict order property. We check that P_M is a partial order universal for linear orders. Suppose that L is some linear order of size λ. By 5.3, L is isomorphic to some suborder of P_{M_L}. Pick an elementary embedding $h: M_L \to M$. In particular, h preserves φ. So, by 5.2, there is an embedding of P_{M_L} into P_M. The restriction of this embedding to (the isomorphic copy of) L is the required embedding. ⊣

5.6. REMARK. If there is a quantifier-free formula in T which defines a partial order on models of T with arbitrarily long chains, then also the existence of a universal model of T in power λ in the sense of *ordinary embedding* implies the existence of a universal linear order in λ.

5.7. CONCLUSIONS. Under the hypotheses of 3.6, there are no universal models in λ for the following theories:

Partial orders (ordinary embeddings).
Boolean algebras (ordinary embeddings).
Lattices (ordinary embeddings).
Ordered fields (ordinary embeddings).
Ordered groups (ordinary embeddings).
Number theory (elementary embeddings).
The theory of p-adic rings (elementary embeddings).

PROOF. All these theories have the strict order property, and most have a definable order via a quantifier-free formula. ⊣

Appendix. We review here several notions and definitions from set theory and model theory.

Set theory. A set of ordinals C is *closed* if $\sup(C \cap \alpha) = \alpha$ implies $\alpha \in C$. A *club* of a cardinal λ is a closed unbounded set of λ. When λ is an uncountable cardinal, the intersection of two clubs contains a club. For details see any standard textbook, such as [Le].

Model theory. A model M is a *submodel* of N if $|M| \subseteq |N|$ (the universe of M is a subset of the universe of N) and every relation or function of M is the restriction of the respective relation or function of N. A model M is an *elementary submodel* of N if it is a submodel of N, and for every formula φ with parameters from M, $M \models \varphi \Leftrightarrow N \models \varphi$. An embedding of models $h: M \to N$ is an *elementary embedding* if its image is an elementary submodel. A model M is *λ-saturated* if for every set

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$A \subseteq |M|$ with $|A| < \lambda$ and a type p over A, p is realized in M. A model M is *saturated* if it is $\|M\|$-saturated. A model M is *special* if there is an increasing sequence of elementary submodels $\langle M_\lambda : \lambda < \|M\|$ is a cardinal$\rangle$, $M = \bigcup_\lambda M_\lambda$, and every M_λ is λ^+-saturated.

A formula $\varphi(\bar{x};\bar{y})$ has the *independence property* if for every $n < \omega$ there are sequences $\bar{a}_l$ $(l < n)$ such that, for every $w \subseteq n, \models (\exists \bar{x}) \bigwedge_{l<n} \varphi(\bar{x};\bar{a}_l)^{\text{if} l \in w}$. A first order theory T has the *independence property* if some formula $\varphi(\bar{x};\bar{y})$ has the independence property.

A theory with universal models with respect to ordinary but not elementary embeddings. Let T be the theory of the following model M: the universe is divided into two infinite parts, the domain of the unary predicate P and its complement. The domain of P is the set $\langle x_i : i < \omega + \omega\rangle$. There are two binary relations, R_1 and R_2. For every pair $\langle \alpha, \beta\rangle$ of ordinals below $\omega + \omega$ there is a unique element $y \in M$ such that $\neg P(y)$ (think of y as an ordered pair) which satisfies $R_1(y, x_\alpha) \wedge R_2(y, x_\beta)$ iff $\alpha < \beta < \omega$ or $\alpha < \omega \leq \beta$ or $\alpha \geq \omega$ and $\beta \geq \omega$. So for the elements above ω in the P part all possible ordered pairs exist, while those below ω are linearly ordered by the existence of ordered pairs. Let

$$\varphi(x_1; x_2) = P(x_1) \wedge P(x_2) \wedge (\exists y)(\neg P(y) \wedge R_1(y, x_1) \wedge R_2(y, x_2)).$$

This formula witnesses that T has the strict order property. By Theorems 3.5 and 5.5, T has no universal model with respect to elementary embeddings in any regular λ, $\aleph_1 < \lambda < 2^{\aleph_0}$. But with respect to ordinary embeddings T has a universal model in every infinite cardinality λ: let M be any model of T of cardinality λ in which there is a set $X = \{x_i : i < \lambda\}$ of elements in the domain of P for which all possible ordered pairs exist. If M' is any other model of cardinality λ, and h is a 1-1 function which maps the domain of P in M' into X, then h can be completed to an embedding of M' into M. ⊣

It is worth noting that the same method of coding the relations in special elements and adding an "absorbing part" can be applied to, say, the theory of the rationals with addition, or the theory in §1. The resulting theory will always have a universal model in $\aleph_1$ with respect to ordinary embeddings, but will have a universal model with respect to elementary embeddings iff CH.[1]

The consistency of not having universal models. Let us state it here, for the sake of those who read several times that it is easy but are still interested in the details:

Fact. If λ is regular, $V \models GCH$, for simplicity, and P is a Cohen forcing which adds λ^{++} Cohen subsets to λ, then in V^P there is no universal graph (linear order, model of a complete first order T which is unstable in λ) in power λ^+.

Proof. Suppose to the contrary that there is a universal graph G^* of power λ^+. We may assume, without loss of generality, that its universe $|G^*|$ equals λ^+. As G is an object of size λ^+, it is in some intermediate universe V', $V \subseteq V' \subseteq V^P$, such that there are λ^{++} Cohen subsets outside of V'. So, without loss of generality, $G \in V$. Let G be the graph with universe λ^+ defined as follows: fix a 1-1 enumeration $\langle A_\alpha : \lambda \leq \alpha < \lambda^+\rangle$ of λ^+ Cohen subsets of λ. A pair α, β is joined by an edge iff $\beta \geq \lambda$

[1] We thank the referee for pointing this out to us.

and $\alpha \in A_\beta$, or $\alpha \geq \lambda$ and $\beta \in A_\alpha$. By the universality of G^*, there should exist an embedding $h: G \to G^*$. Consider $h \restriction \lambda$. This is an object of size λ, and therefore is in some V', $V \subseteq V' \subseteq V^P$, where in V' there are at most λ of the Cohen subsets. For every $y \in G^*$, the set $\{x \in \lambda: h(x)$ is joined by an edge to $y\}$ is in V'. Pick an $\alpha \geq \lambda$ such that A_α is not in V' and set $y = h(\alpha)$ to get a contradiction. ⊣

The same proof is adaptable to the other cases.

Combinatorics. *Proof of Fact* 3.5. Let S_0 be $\{\delta < \lambda: \operatorname{cf} \delta = \aleph_0\}$. Suppose to the contrary that for every sequence $\bar{C}$ as above there is a club $C \subseteq \lambda$ such that, for every $\delta \in S_0 \cap C$, $c_\delta \nsubseteq C$. By induction on $\beta < \aleph_1$ we construct $\bar{C}_\beta$ and C_β, as follows. $\bar{C}_\beta = \langle c^\beta_\delta: \delta \in S_0 \rangle$ is such that, for every $\delta \in C_\beta \cap S_0$, c^β_δ is a club of δ of order type ω, and $c^\beta_\delta \nsubseteq C_\beta$. Furthermore, letting $c^\beta_\delta = \langle \alpha^{\beta,\delta}_n: n < \omega \rangle$, where $\alpha^{\beta,\delta}_n \leq \alpha^{\beta,\delta}_m$ if $n < m$, we demand that $\alpha^{\beta+1,\delta}_n = \sup\{\alpha^{\beta,\delta}_n \cap C_\beta\}$ if this intersection is nonempty, and $\alpha^{\beta+1,\delta}_n = 0$ otherwise. When β is limit, we demand that $\alpha^{\beta,\delta}_n = \min\{\alpha^{\gamma,\delta}_n$ for $\gamma < \beta\}$. Let $\bar{C}_0$ be arbitrary. At the induction step pick C_β which demonstrates that $\bar{C}_\beta$ is not as required by the fact and define each $c^{\beta+1}_d$ by the demand above. Note that for club many δ's the resulting $c^{\beta+1}_\delta$ is cofinal in δ, so without loss of generality this is so for every $\delta \in C'_{\beta+1}$.

It is straightforward to verify that, for all $\beta < \gamma < \omega_1$ and $\delta \in S_0$, the following hold:

1. For all $\delta \in S_0 \cap C_\beta$, $c^{\beta,\delta} \subseteq \delta$ is a club of δ of order type ω.
2. For all $\delta \in S_0$, $c^\beta_\delta \backslash \{0\} \subseteq C_{\beta+1}$.
3. For all $\delta \in S_0 \cap C_\beta$, $c^\beta_\delta \nsubseteq C_\beta$.
4. $\alpha^{\gamma,\delta}_n \leq \alpha^{\beta,\delta}_n$.

Let $C = \bigcap_{\beta < \omega_1} C_\beta$. Pick $\delta_0 \in C \cap S_0$. Then $C \cap \delta$ is unbounded in δ and of order type ω. Furthermore, for every $\beta < \omega_1$, $c^\beta_{\delta_0} \nsubseteq C_\beta$.

But, on the other hand, there is a β_0 such that, for all $\beta_0 < \beta < \gamma$, $c^\beta_{\delta_0} = c^\gamma_{\delta_0}$, because of fact 4. This is a contradiction. ⊣

Now for the case of uncountable cofinality. We prove

3.9. *Fact. If* $\operatorname{cf} \kappa = \kappa < \kappa^+ < \lambda = \operatorname{cf} \lambda$, *then there is a sequence* $\bar{C} = \langle c_\delta: \delta \in S^\lambda_\kappa \rangle$, S^λ_κ *being the set of members of* λ *with cofinality* κ, *where* c_δ *is a club of* δ *of order type* κ *with the property that for every club* $E \subseteq \lambda$ *the set* $S_E = \{\delta \in S^\lambda_\kappa: c_\delta \subseteq E\}$ *is stationary. Furthermore, if the assumptions of* 3.7 *hold for some sequence, then we can make* $\bar{C}$ *satisfy these assumptions too.*

Proof. This proof is actually simpler than that of the previous fact. Start with any sequence $\bar{C}_0 = \langle c^0_\delta: \delta \in S^\lambda_\kappa \rangle$. By induction on $i < \kappa^+$ define $\bar{C}_i$ as follows: if $\bar{C}_i$ has the property of guessing clubs, we stop. Otherwise there is a club E_i such that E_i is not guessed stationarily often. This means there is a club C_i such that $\delta \in C_i$ implies that $c_{i_\delta} \nsubseteq E_i$. We may assume that $C_i = E_i$. Let $c^{i+1}_\delta = c^i_\delta \cap E_i$. If i is limit, $c^i_\lambda = \bigcap_{j<i} c^j_\delta$. Suppose the induction goes on for κ^+ steps. Let $E = \bigcap E_i$. For every $\delta \in E \cap S^\lambda_\kappa$, $E \cap C^0_\delta$ is a club of δ. Therefore, for stationarily many points δ, c^{κ^+} is a club of δ; say this holds for all $\delta \in S \subseteq S^\lambda_\kappa$. Clearly, for every δ there is an i such that, for every $i < j$, $C^i_\delta = c^j_\delta$, because the size of c^0_δ is κ and κ^+ is regular. But, on the other hand, for every $\delta \in S$ and $C < \kappa^+$, $c^i_\delta \nsubseteq E_i$, while $c^{i+1}_\delta \subseteq E_i$, a contradiction. Thus the induction stops before κ^+, and the resulting sequence guesses clubs.

What about the additional properties required in 3.7? If there is a sequence on S^λ_κ which satisfies these properties, let $\bar{C}_0$ in the proof be that sequence. Notice that the operation of intersecting the c_δ with a club E preserves the property: suppose $\delta_1 < \delta_2$ and $\delta_1 \in \operatorname{acc}(\gamma_{d_2}) \cap E$. Then clearly $\delta_1 \in \operatorname{acc}(c_{\delta_2})$. Therefore, by the square property, $c_{\delta_1} = \gamma_{\delta_2} \cap \delta_1$. Intersecting both sides of the equation with E yields that $c_{\delta_1} \cap E = c_{\delta_2} \cap E \cap \delta_1$. Therefore the proof is complete. ⊣

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