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A TWO-CARDINAL THEOREM AND A COMBINATORIAL THEOREM

SAHARON SHELAH[1]

ABSTRACT. We prove a new two-cardinal theorem, e.g. $(\aleph_\omega, \aleph_0) \to (2^{\aleph_0}, \aleph_0)$. For this we prove a combinatorial theorem.

This is a sequel of Shelah **[S1]**, and solves the main problem there. This problem also appears in Chang and Keisler **[CK]** and Friedman [**Fr**, Problem 30]. Our result is:

THEOREM 1. (A) *If for every $n < \omega$, the first order theory T has a model type $(\aleph_{\alpha+n}, \aleph_\alpha)$ then whenever $|T| \leqslant \mu < \lambda < \operatorname{Ded}^* \mu$, T has a model type (λ, μ).*

(B) *If $\aleph_{\alpha+\omega} < \operatorname{Ded}^* \aleph_\alpha$ then $(\aleph_{\alpha+\omega}, \aleph_\alpha)$ is $\aleph_\alpha$-compact and is complete.*

REMARK. $\operatorname{Ded}^* \mu$ is the first cardinal χ such that no tree with $\leqslant \mu$ nodes has $\geqslant \chi$ branches of the same height. Note that $\operatorname{Ded}^* \aleph_0 = (2^{\aleph_0})^+$, for every λ $\lambda^+ < \operatorname{Ded}^* \lambda \leqslant (2^\lambda)^+$, and it is consistent with ZFC that $\operatorname{Ded}^* \aleph_1 \leqslant 2^{\aleph_1}$.

This leads to many conjectures whose difficulty is not known to me; a sample is:

CONJECTURE 2. (A) $(\aleph_{\alpha+\omega+\omega}, \aleph_{\alpha+\omega}, \aleph_\alpha) \to (\lambda, \mu, \chi)$ *whenever* $\chi < \mu < \lambda < \operatorname{Ded}^* \chi$.

(B) *If a countable theory T has a λ-like model, λ a limit cardinal, and $|T| \leqslant \mu < \lambda_1 < \operatorname{Ded}^* \mu$, λ_1 a singular cardinal then T has a λ_1-like model. If λ is M_ω-Mahlo weakly inaccessible cardinal, we can remove the singularity of λ_1.*

(C) *If $\psi \in L_{\omega_1,\omega}$ has a model of cardinality $\aleph_{\omega_1}$, then ψ has a model of cardinality $2^{\aleph_0}$.*

NOTATION. Let I denote a well-ordered set. A (λ,n)-box B is $\prod_{l<n} I_l$ where I_l has order type λ; λ, μ, χ denote infinite cardinals, elements of boxes will be denoted by η, τ, v, and $\eta = \langle \eta(0), \dots, \eta(n-1) \rangle$. For a (λ,n)-box B, and $\eta_l \in B$ $(l < n)$ we say $\langle \eta_0, \dots, \eta_{n-1} \rangle$ is proper for B if $k \neq l < n \Rightarrow \eta_k(l) < \eta_l(l)$.

Let $\lambda^{+0} = \lambda$, $\lambda^{+(k+1)} = (\lambda^{+k})^+ =$ the successor of λ^{+k}.

A B-indexed set is $\{a_\eta : \eta \in B\}$ such that $\eta \neq \tau \to a_\eta \neq a_\tau$. Under those conditions $\langle a_{\eta_0}, \dots \rangle$ is proper, iff $\langle \eta_0, \dots \rangle$ is proper. A (λ, n)-indexed set is a B-indexed set for some (λ, n)-box B.

LEMMA 3. *Suppose that $f_\alpha : (\lambda^+)^2 \to \lambda$ for each $\alpha < \lambda$. Then there exist*

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$s, t < \lambda^+$ such that for each $\alpha < \lambda$ there exist a, b, c, d for which $s < a < b < \lambda^+$, $t < c < d < \lambda^+$ and $f_\alpha(s,t) = f_\alpha(a,d) = f_\alpha(b,c)$.

PROOF. For every $u < \lambda^+$ let $t_u < \lambda^+$ be such that whenever $\alpha < \lambda$ and $t \geqslant t_u$, then

$$|\{v < \lambda^+ : f_\alpha(u,t) = f_\alpha(u,v)\}| = \lambda^+.$$

Let $X_u = \{(\alpha,\beta): f_\alpha(u,t) = \beta \text{ for some } t \geqslant t_u\}$. Now let $s < \lambda^+$ be such that whenever $u \geqslant s$ and $(\alpha,\beta) \in X_u$, then

$$|\{v < \lambda^+ : (\alpha,\beta) \in X_v\}| = \lambda^+.$$

Let $t = t_s$. It is clear that this s and t work.

LEMMA 4. *Let $f: A^n \to \lambda$ where A is any $(\lambda^+, n+1)$-indexed set and $k < n$. Then there is a (λ,n)-indexed set $A^* \subset A$ such that:*

(*) *For any proper sequence $\langle a_0, \ldots, a_{n-1}\rangle$ from A^* there is a proper sequence $\langle b_0, \ldots, b_n\rangle$ from A such that*

$$f(a_0, \ldots, a_{n-1}) = f(b_0, \ldots, b_{n-1}) = f(b_0, \ldots, b_{k-1}, b_n, b_{k+1}, \ldots, b_{n-1}).$$

PROOF. Let A be a B-indexed set where $B = \prod_{l<n+1} I_l$ and $A = \{a_\eta : \eta \in B\}$. For notational simplicity let $k = n-1$ and each $I_l = \lambda^+$. Now we define $\langle s_\alpha : \alpha < \lambda\rangle$ and $\langle t_\alpha : \alpha < \lambda\rangle$ by induction on α such that:

(i) $s_\alpha, t_\alpha < \lambda^+$.

(ii) $\langle s_\alpha : \alpha < \lambda\rangle$ and $\langle t_\alpha : \alpha < \lambda\rangle$ are increasing.

(iii) Whenever $\eta_0, \ldots, \eta_{n-2} \in B$ and $\tau \in \lambda^{n-1}$ are such that for each $i, l < n-1$ there is $\beta < \alpha$ such that $\eta_i(l) < \lambda$, $\eta_i(n-1) = s_\beta$ and $\eta_i(n) = t_\beta$, *then* there are a, b, c, d such that $s_\alpha < a < b < \lambda^+$, $t_\alpha < c < d < \lambda^+$ and

$$\begin{aligned} f(a_{\eta_0}, \ldots, a_{\eta_{n-2}}, a_{\tau ^\frown \langle s_\alpha, t_\alpha\rangle}) &= f(a_{\eta_0}, \ldots, a_{\eta_{n-2}}, a_{\tau ^\frown \langle a,d\rangle}) \\ &= f(a_{\eta_0}, \ldots, a_{\eta_{n-2}}, a_{\tau ^\frown \langle b,c\rangle}). \end{aligned}$$

Suppose we have defined s_β and t_β for $\beta < \alpha$. For each $\eta_0, \ldots, \eta_{n-2}, \tau$ satisfying the conditions of (iii), there is a function $g: (\lambda^+)^2 \to \lambda$ defined by

$$g(x,y) = f(a_{\eta_0}, \ldots, a_{\eta_{n-2}}, a_{\tau ^\frown \langle x,y\rangle}).$$

There are $\leqslant \lambda$ such functions g. So we can apply Lemma 3 to get s_α, t_α such that $s_\alpha > t_\beta$ and $t_\alpha > t_\beta$ for each $\beta < \alpha$, and for each such g there are a, b, c, d such that $s_\alpha < a < b < \lambda^+$, $t_\alpha < c < d < \lambda^+$ and $g(s_\alpha, t_\alpha) = g(a, d) = g(b,c)$.

Now we define the (λ,n)-indexed set A^*. For each $\tau \in \lambda^{n-1}$ let $b_{\tau ^\frown \langle\alpha\rangle} = a_{\tau ^\frown \langle s_\alpha, t_\alpha\rangle}$. Then let $A^* = \{b_\eta : \eta \in \lambda^n\}$. Now it is easy to check that (*) holds.

THEOREM 5. *Let $f_l: (\lambda^{+n})^l \to \lambda$ whenever $0 < l \leqslant n$, and let $h: (n+1) \to n$ be such that $h(l) < l$ whenever $0 < l \leqslant n$. Then there are distinct $a_0, \ldots, a_n < \lambda^{+n}$ such that*

$$f_l(a_0, \ldots, a_{l-1}) = f_l(a_0, \ldots, a_{h(l)-1}, a_l, a_{h(l)+1}, \ldots, a_{l-1})$$

whenever $0 < l \leqslant n$.

PROOF. We let λ^{+n} be $(\lambda^{+n}, n+1)$-indexed. Now we prove the theorem by induction on n. For $n = 0$ there is nothing to prove. For $n+1$ we use Lemma 4, and the induction hypothesis on A^*.

PROOF OF THEOREM 1. Clear from [**S1**, §3] and Lemma 4.

REMARKS. (1) The following theorem is clear.

THEOREM . *If every finite subset of T has, for each n, a model of type* $(\lambda_m, \ldots, \lambda_0)$ *where* $\lambda_0^{+n} < \lambda_1$, $[(\lambda_{l+1})^{(\lambda_l^{+n})}]^{+n} < \lambda_{l+2}$ $(l < n-1)$ *and* $|T| \leqslant \mu_0 \leqslant \mu_1 \leqslant \cdots \leqslant \mu_m < \mathrm{Ded}^* \mu_0$ *then T has a model of type* $(\mu_m, \ldots, \mu_0)$. *(The parallel theorem in* [**S1**] *was noted by Papageorgiou.)*

(2) We can prove the main theorem of [**S1**] in a way similar to the proof here.

(3) In the notation of Shelah [**S2**, §3], we have proved in Lemma 4 that for $m = 2^n$, $r < \omega$, $\lambda^{+m} \xrightarrow{wt} (n)^r_\lambda$. This answers positively question 3 from [**S2**]. But it is still unknown whether we have the best results.

(4) Halperin and Levi [**H Le**] used an indiscernibility similar to the one used in [**S1**], and Halperin and Lauchli proved the necessary combinatorial theorem. We have not succeeded in generalizing their proof. However, we can use our method to prove a weaker variant of their theorem, which is sufficient to prove that if $T \subset T_1$, $|T_1| = \aleph_0$, T is complete and there are $> \aleph_0$ complete $L(T)$-types consistent with T, $\lambda > \aleph_0$ then T has $\geqslant \min\{2^\lambda, 2^{2^{\aleph_0}}\}$ nonisomorphic $L(T)$-reducts of models of T_1. (This will appear in [**S3**].)

(5) To see the connection note that by [**S3**] it follows by Theorem 5 that

THEOREM . *Let* $f_l \colon (\lambda^{+n})^l \to \lambda$ *whenever* $0 < l \leqslant n$, $n = 2^m - 1$. *Then there are distinct* $a_\eta < \lambda^{+n}$ *for* $\eta \in 2^m$ *(i.e.* η *is a sequence of ones and zeros of length* m*) such that: if* $k < m$, $0 < l \leqslant n$, $\tau_1, \ldots, \tau_l$ *are distinct members of* 2^k, *and for* $1 \leqslant i \leqslant l$, $0 \leqslant j \leqslant 1$, $\eta_i^j \in 2^m$ *and* τ_i *is an initial segment of* η_i^j *then* $f_l(\eta_1^0, \ldots, \eta_l^0) = f_l(\eta_1^1, \ldots, \eta_l^1)$.

(6) In Lemma 4 and Theorem 5 we can consider μ such functions, provided that $\mu \leqslant x$, $\lambda^\mu = \lambda$, resp.

(7) We can use only a (B, n)-indexed set for a fixed n in (4), but then in Remark (3) m will become bigger.

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INSTITUTE OF MATHEMATICS, THE HEBREW UNIVERSITY, JERUSALEM, ISRAEL