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NON-REFLECTION OF THE BAD SET FOR $\check{I}_\theta[\lambda]$ AND pcf

S. SHELAH[1,2,*]

[1]Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem, 91904, Israel

[2]Department of Mathematics, Rutgers, The State University of New Jersey,
110 Frelinghuysen Road, Piscataway, NJ 08854-8019, USA
e-mail: shelah@math.huji.ac.il
url: http://shelah.logic.at

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Abstract. We reconsider here the following related pcf questions and make some advances:

(Q1) concerning the ideal $\check{I}_\kappa[\lambda]$ how much reflection do we have for the bad set $S^{\mathrm{bd}}_{\lambda,\kappa} \subseteqq \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ assuming it is well defined (for transparency only)?

(Q2) are there somewhat free black boxes?

The advances in (Q2) will be used in subsequent for constructions of Abelian groups and modules.

0. Introduction

0(A). Background. On $\check{I}_\theta[\lambda]$ for $\lambda > \theta$ regular see (Definition 0.12(3) and) [9], [8], [13]. So we know that in many cases there is a set $S^{\mathrm{bd}}_{\lambda,\theta} \subseteqq S^\lambda_\theta := \big\{\delta < \lambda : \mathrm{cf}(\delta) = \theta\big\}$ such that dual $\big(\check{I}_\theta[\lambda]\big) = \mathscr{D}_\lambda + (S^\lambda_\theta \backslash S^{\mathrm{bd}}_{\lambda,\theta})$ and so $S^{\mathrm{bd}}_{\lambda,\theta}$ is unique (0.12(4)) modulo the club filter, $\mathscr{D}_\lambda$; for definitions see §(0C).

We know that consistently, starting with a supercompact we can force that; e.g. GCH and $S^{\mathrm{bd}}_{\aleph_{\omega+1},\aleph_n}$ (0.12(4)) is stationary for $n = 1$ but we do not know it for $n > 1$. Still this set reflects in no $\aleph_n$, however we use G.C.H. or just $\aleph_n > 2^{\aleph_0}$. More generally, if μ is strong limit of cofinality $\aleph_0$ and $S = S^{\mathrm{bd}}_{\mu^+,\aleph_1}$ we do not know if S can reflect in stationarily many δ's of cofinality $\aleph_n > \aleph_1$ when $\aleph_n \leqq 2^{\aleph_0}$. Similarly for μ strong limit of cofinality $\kappa < \mu$ (see 0.1, 0.2).

By [13, §1] for regular λ, κ such that $\lambda > \kappa^+$ there is $S \in \check{I}_\kappa[\lambda]$ which is stationary, in fact reflect in stationarily many $\delta < \lambda$ of cofinality, e.g.

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$\kappa^{+n} < \lambda$ for $n \geqq 1$. Related subsets are the good/bad/chaotic sets of scales $(\langle f_\alpha : \alpha < \lambda\rangle,\ f_\alpha \in {}^\kappa\mu)$, see [5, Ch. II], [11], [18] and 0.18 here.

The proof in [5, Ch. IX, §2] of $\mathrm{pp}(\aleph_\omega) < \aleph_{\omega_4}$ in particular continue these ideas.

Recall that if $\bar f = \langle f_\alpha : \alpha < \lambda\rangle$ is $<_J$-increasing, $<_J$-cofinal in $\prod\limits_{i<\kappa} \lambda_i$, $\lambda_i = \mathrm{cf}(\lambda_i) > \theta \geqq \kappa^+$ then $S^{\mathrm{gd}}_\theta(\bar f) := \big\{\delta < \lambda : \mathrm{cf}(\delta) = \theta$ and $\bar f \restriction \delta$ is flat (see 0.18)$\big\}$ has complement orthogonal to $\check I_\theta[\lambda]$ modulo the non-stationary ideal (i.e. has a non-stationary intersection with any $A \in \check I_\theta[\lambda]$).

Combining the proofs of [13, §1] and [5, Ch. IX, §2] it follows that $S^{\mathrm{gd}}_\theta(\bar f) = S^\lambda_\theta \mod D_\lambda$ when $\theta = \kappa^{+n}$, $n \geqq 4$ but we have not look at it. It was pointed out by Sharon–Viale [3, footnote 5], that is followed by Abraham–Magidor [1, 2.12, 2.19] which contains a representation of pcf theory. We made this work after learning Kojman–Milovich–Spadaro [2].

We start by continuing [13, §1], [5, Ch. IX,§2], to re-examine some of those problems; see §(0B). More specifically, we shed some light on question (Q1) in 0.1, 0.2 proved in §(1A).

What about (Q2)? This was a central issue of [18] which deals with one dimensional black boxes. The n-dimensional black boxes are from [17]. See more applications to Abelian groups and modules in Göbel–Shelah [19], Göbel–Shelah–Strüngman [22], Göbel–Herden–Shelah [21]; and lately [23], which relies on the results here; see 0.6, 0.4, 0.7 which are proved in §(1B).

Much earlier Solovay proved that above a compact cardinal, the singular cardinal hypothesis holds; it follows that the so called strong hypothesis $\big(\mu > \mathrm{cf}(\mu) \Rightarrow \mathrm{pp}(\mu) = \mu^+\big)$ holds; so pcf becomes trivial. Moreover, by [5, Ch. II] if $\mathrm{pp}_J(\mu) > \lambda = \mathrm{cf}(\lambda) > \mu > \mathrm{cf}(\mu) = \kappa$ (where $J \supseteq [\kappa]^{<\kappa}$ is an ideal on κ) then there is a sequence $\langle f_\alpha : \alpha < \lambda\rangle$ with $f_\alpha \in {}^\kappa\mu$ which is $<_J$-increasing and is μ^+-free even as a sequence, so $\bar f \restriction \delta$ is flat when $\kappa < \mathrm{cf}(\delta) < \mu$ (i.e. the good set of $\bar f, \mathrm{gd}(\bar f)$ is large).

But if $\kappa = \mathrm{cf}(\mu) < \mu$, the consistency result on $\check I_{\kappa^+}[\mu^+]$ from [9] can be strengthened; we know that consistently there are strong reflection properties, say if GCH, consistently the case of Chang conjecture holds from $(\aleph_{\omega+1}, \aleph_\omega) \to (\aleph_1, \aleph_1)$, by Levinski–Magidor–Shelah [10] and $(\aleph_{\omega+\omega+1}, \aleph_{\omega+\omega}) \to (\aleph_{\omega+1}, \aleph_\omega)$. We can manipulate 2^κ for κ regular.

0(B). Results. What do we accomplish here? First, assume $\lambda > \kappa > \aleph_0$ and for transparency assume $S^{\mathrm{bd}}_{\lambda,\kappa}$ is well defined. How much can it reflect? Assume $\lambda = \mu^+$, $\mathrm{cf}(\mu) = \kappa$, μ strong limit. We knew that [9] if, e.g. $\theta = (2^\kappa)^{+n+1}$ then $S^{\mathrm{bd}}_{\lambda,\kappa}$ does not reflect in S^λ_θ. Here 0.2 gives more: assuming $(\forall n)(2^{\kappa^{+n}} < \lambda)$ we have, e.g. for $n \geqq 2$, $m \geqq n+2$: if $S^{\mathrm{bd}}_{\lambda,\kappa}$ reflects in $S^\lambda_{\kappa^{+n}}$ this reflection does not reflect in $S^\lambda_{\kappa^{+m}}$; moreover does not reflect in any $S^\lambda_{\theta^+}$, $\theta \in \mathrm{Reg} \cap \lambda \backslash \kappa^{+n+2}$. See more in 0.2.

Second, turning to "if $\bar f$ is $<_J$-increasing cofinal in $\prod\limits_{i<\kappa} \lambda_i/J$ and $i < \kappa \Rightarrow \lambda_i = \mathrm{cf}\,(\lambda_i) > \kappa$; how large is $S^{\mathrm{gd}}_\theta[\bar f, J]$"? We knew $S^{\mathrm{gd}}_\theta[\bar f, J]$ is large; here we prove in 0.1(1) that: if $\theta \in \big[\kappa^{+4}, \kappa^{+\,\mathrm{comp}\,(J)}\big)$, $(\forall i)(\theta < \lambda_i)$ and θ is regular $< \lambda$ then $S^{\mathrm{gd}}_\theta[\bar f, J]$ contains S^λ_θ (modulo the club filter of course). Hence, e.g. $\bar f$ is $(\theta^{+\,\mathrm{comp}\,(J)}, \theta^{+4}, J)$-free when $\kappa \leqq \theta$, $\theta^{+\,\mathrm{comp}\,(J)} < \min\{\lambda_i : i < \kappa\}$, see Definition 1.10(2). So if $\lambda_\ell = \mathrm{pp}\,(\mu_\ell) > \mu^+_\ell$, $\mu_\ell > \aleph_0 = \mathrm{cf}\,(\mu_\ell)$ for $\ell = 1, 2$ and $\mu_1^{+4} \leqq \lambda_1 < \lambda_2$ then $(\lambda_2, \mu_2) \nrightarrow (\lambda_1, \mu_1)$.

But this is not enough to prove what we need for Q2, i.e. 0.4 which is (θ_2, θ_1)-freeness; (the problem being for $\langle \delta_i : i < \theta\rangle$ increasing continuous, for i of cofinality $\leqq \kappa$) but 1.11 tells us more, in particular, enough for Theorem 0.4.

More specifically, we shall show (the proofs are given later, the definitions appear in §(0C) and 1.10 below):

THEOREM 0.1. *Assume $\lambda > \sigma > \partial > \theta^+ > \theta > \aleph_0$ are regular.*

1) *Some $S \in \check{I}_\theta[\lambda]$ reflect in every $\delta \in S^\lambda_\sigma$, see Definition 0.14(1).*

2) *Moreover, if $\delta \in S^\lambda_\sigma$ then $\big\{\delta_1 < \delta : \mathrm{cf}\,(\delta_1) = \partial$ and S reflects in $\delta_1\big\}$ is a stationary subset of δ.*

3) *Moreover, for any $(\partial, \theta, <\sigma)$-system $\bar{\mathscr{P}}^*$, see Definition 0.9, for any ordinal $\delta \in S^\lambda_\sigma$, for any increasing continuous sequence $\langle \delta_i : i < \sigma\rangle$ of ordinals with limit δ (clearly exists) for some $S_1 \in \check{I}^{\mathrm{ac}}_\partial\langle \sigma, \sigma\rangle$, see Definition 0.13(2) we have:*

$$(*)\quad \begin{cases} \text{if } j \in S^\sigma_\partial \backslash S_1 \text{ then there is } S_2 \in I^{\mathrm{cg}}_\theta(\bar{\mathscr{P}}^*) \text{ such that for some increasing} \\ \text{continuous sequence } \langle i_\varepsilon : \varepsilon < \partial\rangle \text{ with limit } j \text{ we have } \varepsilon \in S^\partial_\theta \backslash S_2 \Rightarrow \delta_{i_\varepsilon} \\ \in \mathrm{good}''_\theta(\bar{\mathscr{P}}). \end{cases}$$

With stronger assumptions on cardinal arithmetic we get more:

THEOREM 0.2. *Assume $\lambda > \theta^{+\omega}$ and λ, θ are regular uncountable and $2^{\theta^{+n}} < \lambda$ for every n.*

1) *If $S^{\mathrm{bd}}_{\lambda,\theta}$ is (well defined and) stationary then there are n and stationary $S \subseteq S^\lambda_{\theta^{+n}}$ which reflects in no ordinal δ of cofinality $\in [\theta, \theta^{+\omega})$.*

2) *There is $S \in \check{I}_\theta[\lambda]$ such that for every $n \geqq 2$, either $S_1 = S^\lambda_{\theta^{+n}} \cap \mathrm{refl}\,(\lambda \backslash S)$ is not stationary (in λ) or S_1 is stationary but is the union of $\leqq 2^{\theta^{+n}}$ sets each of which reflect in no δ of cofinality $\in [\theta^{n+2}, \theta^{+\omega})$.*

3) *In part (2) in the second possibility some stationary $S_2 \subseteq S_1 (\subseteq S^\lambda_{\theta^{+n}})$ either reflect in no ordinal of cofinality $< \theta^{+\omega}$ or $S_3 = \big\{\delta \in S^\lambda_{\theta^{+n+1}} : S_2 \cap \delta$ is stationary in $\delta\big\}$ is a stationary subset of $S^\lambda_{\theta^{+n+1}}$ which reflect in no $\delta < \lambda$ of cofinality $< \theta^{+\omega}$.*

4) *If $S^\lambda_\theta \notin \check{I}_\theta[\lambda]$ and $m \geqq 2$ then there is $n \in \{m, m+1\}$ and stationary $S \subseteq S^\lambda_\theta$ such that S reflects in no $\delta < \lambda$ of cofinality $\in [\theta^{+n+1}, \theta^{+\omega})$.*

In [18] we consider another version of freeness, note that being (θ, σ)-free follows from θ-free and is stronger than stable in every $\kappa \in [\sigma, \theta)$. We do not get it fully but enough to get "quite free $\mathbf{k}$-combinatorial parameters" which is enough for applications in [23].

REMARK 0.3. 1) Recall that for regular $\partial > \aleph_0$, $\mu \in \mathbf{C}_\partial$ means just that μ is strong limit singular of cofinality ∂.

2) For $\partial = \aleph_0$ the class $\mathbf{C}_\partial$ is almost equal to (and is contained in) the class $\{\mu : \mu > \aleph_0$ strong limit of cofinality $\aleph_0\}$, more specifically, the difference does not reflect in any singular cardinal.

3) Having two possibilities in 0.4, make us prefer the non-tree version of the black box (see [23]).

THEOREM 0.4. *Assume $\sigma < \kappa$ are regular, $\mu \in \mathbf{C}_\kappa$, i.e. μ is strong limit singular of cofinality κ. At least one of the following holds*:

(A) *there is a μ^+-free $\mathscr{F} \subseteqq {}^\kappa\mu$ of cardinality $\lambda := 2^\mu$, this is called "μ has a 1-solution"*

(B) *$\lambda = 2^\mu$ is regular and there is a $(\lambda, \mu, \sigma, \kappa) - 5$-solution, see Definition 0.6.*

CLAIM 0.5. *If $\mu > \kappa = \mathrm{cf}(\mu) > \sigma = \mathrm{cf}(\sigma)$ and we let $\lambda = \mu^+$ then there is $\bar\eta$ satisfying clauses* (a)–(f) *of Definition 0.6.*

DEFINITION 0.6. Assume $\mu \in \mathbf{C}_\kappa$, $\lambda = 2^\mu = \mathrm{cf}(\lambda)$, $\sigma = \mathrm{cf}(\sigma) < \kappa$; we say $\mathbf{x}$ is a $(\lambda, \mu, \kappa, \sigma) - 5$-solution when it consists of:

(a) $\bar\eta = \langle \eta_\delta : \delta \in S\rangle$,

(b) $S \subseteqq S^\lambda_\sigma$ is stationary in λ (and $\in \check{I}_\sigma[\lambda]$),

(c) $\eta_\delta := \big\langle \alpha_{\delta,i,j} : (i,j) \in \sigma \times \kappa\big\rangle$ and $\langle \alpha_{\delta,i,0} : i < \sigma\rangle$ is increasing with limit δ and $\alpha_{\delta,i,j} \in [\alpha_{\delta,i,0}, \alpha_{\delta,i,0} + \mu)$ increasing with j and $\alpha_{\delta,i,0} + \mu \leqq \alpha_{\delta,i+1,0}$; and let $C_\delta = \big\{\alpha_{\delta,i,j} : (i,j) \in \sigma \times \kappa\big\}$,

(d) [treeness] if $\alpha_{\delta_1,i_1,j_1} = \alpha_{\delta_2,i_2,j_2}$ then $(i_1, j_1) = (i_2, j_2)$ and $i < i_1 \wedge j < j_2$ $\Rightarrow \alpha_{\delta_1,i,j} = \alpha_{\delta_2,i,j}$,

(e) [freeness] $\bar\eta$ is $(\theta^{+\kappa+1}, \theta^{+4}, J_*)$-free, see 1.10(4) when $\kappa \leqq \theta < \mu$ and $J_* = J^{\mathrm{bd}}_{\sigma\times\kappa} = \{u \subseteqq \sigma \times \kappa$: for some $(i_*, j_*) \in \sigma \times \kappa$ we have $u \subseteqq \big\{(i,j) \in \sigma \times \kappa : i < i_*$ or $j < j_*\big\}$,

(f) [freeness] $\bar\eta$ is (κ^+, J_*)-free,

(g) [black box] for every $\chi < \mu$ and $\bar F = \langle F_\delta : \delta \in S\rangle$ such that $F_\delta : {}^{(C_\delta)}\delta \to \chi$ there is $\bar\alpha = \langle \alpha_\delta : \delta \in S\rangle \in {}^S\chi$ such that $(\forall \eta \in {}^\lambda\lambda)(\exists^{\mathrm{stat}}\delta \in S)\big(\alpha_\delta = F(\eta \restriction C_\delta)\big)$, e.g.

(g)$'$ for every relational vocabulary τ of cardinality $< \mu$ there is a sequence $\bar M = \langle M_\delta \in S\rangle$, M_δ a τ-model with universe $C_\delta := \mathrm{Rang}(\eta_\delta) = \{\alpha_{\delta,i,j} : i < \sigma,\ j < \kappa\}$ such that for every τ-model M with universe λ we have $(\exists^{\mathrm{stat}}\delta \in S)(M_\delta = M \restriction C_\delta)$.

DISCUSSION 0.7. 1) It may be helpful to use this to prove results by cases. First, find a proof using a 1-solution, that is with μ^+-freeness using (A) of 0.4 or at least θ_*-free, $\mathscr{F} \subseteq {}^\kappa\mu$, $|\mathscr{F}| = 2^\mu$, θ_* large enough so in [23] terms using $\mathbf{x}$ with $\mathbf{k}_\mathbf{x} = 1$. Second, use n cases of a 5-solution (see 0.4(B) and Definition 0.6) so have $\mathbf{x} = \mathbf{x}_0 \times \mathbf{x}_1 \times \ldots \times \mathbf{x}_n$, $\mathbf{x}_\ell$ is as above so have enough cases of $(\theta^\kappa, \theta^{+4})$-freeness. This is done in [23] which uses Theorem 0.4.

2) We may use a different division to cases then 0.4, dividing case (B) as in [18]. Let $\Upsilon = \min\{\partial : 2^\partial > 2^\mu\}$; and ask whether $\Upsilon = \lambda$ or $\Upsilon < \lambda$.

2A) If $\Upsilon = \lambda$ then $\lambda = \lambda^{<\lambda}$ hence we have better statements on λ, e.g. if λ is a successor cardinal then we have $\diamondsuit_{S^\lambda_{\aleph_0}}$ or $\diamondsuit_{S^\lambda_{\aleph_1}}$ by [20].

2B) If $\Upsilon < \lambda$, by [18, §2], we can construct a (one dimensional) black box for Υ by [18, §2].

0(C). Quoting definitions. We try to make this work reasonably self-contained.

NOTATION 0.8. 1) For regular uncountable cardinal λ let $\mathscr{D}_\lambda$ be the filter generated by the clubs of λ.

2) $\mathscr{H}(\chi)$ is the set of x with transitive closure of cardinality $< \chi$.

3) Let $<^*_\chi$ denote a well ordering of $\mathscr{H}(\chi)$.

4) For regular κ and cardinal (or ordinal) $\lambda > \kappa$ let $S^\lambda_\kappa = \big\{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\big\}$.

5) For an ideal J on κ let $\mathrm{comp}(J)$ be $\max\{\theta : J$ is θ-complete$\}$, it is well defined.

DEFINITION 0.9. 1) We say $\bar{\mathscr{P}}$ is a $(\partial, \theta, < \mu)$-system when:

(a) $\theta \leqq \partial$ and ∂ is regular uncountable, usually θ is regular,

(b) $\bar{\mathscr{P}} = \langle \mathscr{P}_\alpha : \alpha < \partial\rangle$,

(c) if $a \in \mathscr{P}_\alpha$ then $a \subseteq \alpha$ and $|a| < \theta$,

(d) $\beta \in a \in \mathscr{P}_\alpha \Rightarrow a \cap \beta \in \mathscr{P}_\beta$,

(e) $\mathscr{P}_\alpha$ has cardinality $< \mu$.

2) If $\mu = \partial$ we may write (∂, θ)-system. Instead "$< \mu^+$" we may write μ. If $\mathscr{P}_\alpha = \{a_\alpha\}$ for $\alpha < \partial$ so $\bar{\mathscr{P}}$ a $(\partial, < \theta, 1)$-system, and we may write $\bar{a} = \langle a_\alpha : \alpha < \partial\rangle$ instead of $\bar{\mathscr{P}}$. Instead of θ we may write $\leqq \partial$ when $\theta = \partial^+$.

3) We say $\bar{\mathscr{P}}$ is closed when each $a \in \mathscr{P}_\alpha$ is a closed subset of α.

REMARK 0.10. Concerning Definition 0.9(1) note that we allow $\mu > \partial$; in fact, this case was used in [5, Ch. II], in proving: if $\lambda = \mathrm{tcf}\Big(\prod\limits_{i<\kappa} \lambda_i, <_J\Big)$, $\lambda_i = \mathrm{cf}(\lambda_i) > \kappa$ and $\mu = \lim_J\langle\lambda_i : i < \kappa\rangle < \lambda_* = \mathrm{cf}(\lambda_*) < \lambda$ then there are $\lambda^*_i = \mathrm{cf}(\lambda^*_i) < \lambda_i$ with $\mu = \lim_J\langle\lambda^*_i : i < \kappa\rangle$ such that $\lambda_* = \mathrm{tcf}\Big(\prod\limits_{i<\kappa} \lambda^*_i, <_J\Big)$ exemplified by some μ^+-free $\langle f_\alpha : \alpha < \lambda_*\rangle$.

FACT 0.11. For every regular θ and stationary $S \subseteq \{\delta < \theta^+ : \mathrm{cf}(\delta) < \theta\}$ there is a $(\theta^+, \theta, 1)$-system, moreover there is $\bar{a}$ satisfying:

(a) $\bar{a} = \langle a_\alpha : \alpha < \theta^+ \rangle$,

(b) $a_\alpha \subseteq \alpha$,

(c) $|a_\alpha| < \theta$,

(d) $\beta \in a_\alpha \Rightarrow a_\beta = a_\alpha \cap \beta$,

(e) if E is a club of θ^+ and $\zeta < \theta$ then there is α such that $a_\alpha \subseteq E \wedge \alpha = \sup(a_\alpha) \wedge \mathrm{otp}(a_\alpha) = \zeta$,

(f) if E is a club of θ^+ and $\zeta < \theta$, then for some $\delta \in S \cap E$ we have $a_\delta \subseteq E \wedge \delta = \sup(a_\delta)$ and ζ divides $\mathrm{otp}(a_\delta)$.

PROOF. See [5, Ch. III] + correction in [6]. As of guessing clubs for clause (f), it is like [13, §1]. We just are more explicit in what we get. □

Recall ([9] = [8],[13, §1]) (there we vary θ).

DEFINITION 0.12. Let $\lambda > \theta$ with λ regular.

1) For a $(\lambda, \theta, < \mu)$-system $\bar{\mathscr{P}} = \langle \mathscr{P}_\alpha : \alpha < \lambda \rangle$ let

- $\mathrm{good}'_{<\theta}(\bar{\mathscr{P}}) = \{\delta < \lambda : \delta$ is a limit ordinal of cofinality $< \theta$ and there is an unbounded $u \subseteq \delta$ of order type $< \delta$ such that $\alpha \in u \Rightarrow u \cap \alpha \in \mathscr{P}_\alpha\}$,
- $\mathrm{good}''_{<\theta}(\bar{\mathscr{P}})$ is defined similarly but $\mathrm{otp}(u) = \mathrm{cf}(\delta)$.

2A) For a $(\lambda, \theta, < \mu)$-system $\bar{\mathscr{P}}$, we define $\mathrm{good}'_{\leq\theta}(\bar{\mathscr{P}})$, $\mathrm{good}''_{\leq\theta}(\bar{\mathscr{P}})$ naturally; we defined $\mathrm{good}'_{=\theta}(\bar{\mathscr{P}})$, $\mathrm{good}''_{=\theta}(\bar{\mathscr{P}})$ similarly but demand $\mathrm{cf}(\delta) = \theta$.

3) $\check{I}_\theta[\lambda]$ is the set of $S \subseteq S^\lambda_\theta := \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$ such that for some $(\lambda, \theta, 1)$-system $\bar{a}$ and club E of λ we have $S \cap E \subseteq \mathrm{good}'_\theta(\bar{a})$, equivalently for some $(\lambda, < \theta, 1)$-system $\bar{a}$ and club E of λ, $S \cap E \subseteq \mathrm{good}''_\theta(\bar{a})$; equivalently, we may use $\bar{\mathscr{P}}$ a $(\lambda, \lambda, < \lambda)$-system or $(\lambda, \theta, < \lambda)$-system; abusing notation for $S \subseteq \lambda$, $S \in \check{I}_\theta[\lambda]$ means $S \cap S^\lambda_\theta \in \check{I}_\theta[\lambda]$; the "equivalently" holds by [13, §1] or see [7].

4) If $\check{I}_\theta[\lambda] =$ (the non-stationary ideal on S^λ_θ) $+ S_*$ then we call S_* the good set on λ for cofinality θ; it will be denoted $S^{\mathrm{gd}}_{\lambda,\theta}$; its complement $S^{\mathrm{bd}}_{\lambda,\theta} := S^\lambda_\theta \backslash S_*$ is called the bad set; of course, as only $S_*/\mathscr{D}_\lambda$ is unique this notation pedentically is not justified.

5) Let $\check{I}^\perp_\kappa[\lambda] = \{S \subseteq S^\lambda_\kappa$: if $S_1 \in \check{I}_\kappa[\lambda]$ then $S_1 \cap S$ is not stationary (in λ)$\}$.

6) Let $\check{I}[\lambda] = \{S \subseteq \lambda$: if $\theta = \mathrm{cf}(\theta) < \lambda$ then $S \cap S^\lambda_\theta \in \check{I}_\theta[\lambda]\}$.

DEFINITION 0.13. Let $\lambda > \theta$ be regular.

1) Let $I^{\mathrm{cg}}_\theta[\lambda, \mu]$ be the set of $S \subseteq S^\lambda_\theta$ such that (cg stands for club guessing) there is no $(\lambda, \theta, < \mu)$-system $\bar{\mathscr{P}}$ witnessing $S \in \left(I^{\mathrm{cg}}_\theta[\lambda, \mu]\right)^+$ which means $S \subseteq S^\lambda_\theta \wedge S \notin I^{\mathrm{cg}}_\theta(\bar{\mathscr{P}})$ that is:

$(*)_1$ $\bar{\mathscr{P}} = \langle \mathscr{P}_\alpha : \alpha < \lambda \rangle$ is a $(\lambda, \theta, < \mu)$-system,

$(*)_2$ for $\bar{\mathscr{P}}, \lambda$ as above let $I^{\rm cg}_\theta(\bar{\mathscr{P}})$ be the set of $S \subseteq S^\lambda_\theta$ such that for some club E of λ for no $\delta \in S$ and $a \in \mathscr{P}_\delta$ do we have $a \subseteq E \wedge \sup(a) = \delta$.

1A) We define $I^{\rm dg}_\theta[\lambda, \mu]$, $I^{\rm dg}_\theta(\bar{\mathscr{P}})$ similarly except that in $(*)_2$ we demand only $a \in \mathscr{P}_{<\lambda}$.

2) Assume $\lambda = {\rm cf}(\lambda) \geqq \theta = {\rm cf}(\theta)$, $\lambda \geqq \mu$ and $\mu^+ \geqq \theta$. Let $\check{I}^{\rm ac}_\theta\langle\lambda,\mu\rangle$ be the set of $S \subseteq S^\lambda_\theta$ such that there are $\chi > \lambda + \mu$ and $x \in \mathscr{H}(\chi)$ for which there is no sequence $\bar{N} = \langle N_\varepsilon : \varepsilon < \theta\rangle$ satisfying:

(a) $N_\varepsilon \prec \big(\mathscr{H}(\chi), \theta, <^*_\chi\big)$,
(b) $\langle N_\zeta : \zeta < \theta\rangle$ is increasing continuous,
(c) $\langle N_\zeta : \zeta \leqq \varepsilon\rangle \in N_{\varepsilon+1}$,
(d) $\|N_\varepsilon\| < \mu$ and $N_\varepsilon \cap \mu$ is an ordinal,
(e) $\{x, \lambda, \mu, \theta\} \in N_0$,
(f) $\cup\{N_\varepsilon \cap \lambda : \varepsilon < \theta\} \in S$.

Definition 0.14. For λ regular uncountable and unbounded $S \subseteq \lambda$ let ${\rm refl}(S) = \big\{\delta < \lambda : {\rm cf}(\delta) > \aleph_0$ and S reflects in $\delta\big\}$ where "S reflects in δ" means $S \cap \delta$ is a stationary subset of δ.

2) We say $S \subseteq \lambda$ reflects in S^λ_θ if $\{\delta \in S^\lambda_\theta : S \cap \delta$ is stationary in $\delta\}$ is a stationary subset of λ. We may replace S^λ_θ by any stationary subset of λ.

Definition 0.15. For a regular cardinal ∂, let $\mathbf{C}_\partial$ be the class of strong limit singular cardinals μ of cofinality ∂ such that ${\rm pp}^*(\mu) =^+ 2^\mu$.

Discussion 0.16. 1) For the equivalence of the two versions in Definition 0.12(3), see [13, §1].

2) When does $S^{\rm gd}_{\lambda,\theta}$ exist? See [9] = [8], $S^{\rm gd}_{\lambda,\theta}$ exists under quite weak cardinal arithmetic assumptions (much weaker than GCH).

3) Trivially, if $\alpha < \lambda \Rightarrow |\alpha|^{<\theta} < \lambda$ then $S^{\rm bd}_{\lambda,\theta} = \emptyset$.

4) It is proved there for λ, e.g. successor of strong limit singular μ and $\theta \in \big({\rm cf}(\mu), \mu\big)$ that $S^{\rm bd}_{\lambda,\theta}$ exists and does not reflect in cofinality $(2^\theta)^+$ and in cofinality ∂ when $(\forall\alpha < \partial)\big[|\alpha|^\theta < \partial\big]$.

5) Also it is proved ([5, Ch. II]) that if λ is a successor of regular $\aleph_0 < \theta = {\rm cf}(\theta)$ and $\theta^+ < \lambda$ then $S^{\rm bd}_{\lambda,\theta}$ is $\emptyset$ (i.e. not stationary), see 0.17(1).

Fact 0.17. 1) Assume λ is regular and $\lambda = {\rm cf}(\lambda) > \mu$ and $\lambda = \mu^+ \wedge \mu = {\rm cf}(\mu)$, then $\theta = {\rm cf}(\theta) < \mu \Rightarrow S^\lambda_\theta \in \check{I}_\theta[\lambda]$, moreover, there is a closed $(\lambda, \mu, <\lambda)$-system $\bar{\mathscr{P}}$ such that $\delta < \lambda \wedge {\rm cf}(\delta) < \mu \Rightarrow (\exists a \in \mathscr{P}_\delta)\big(\sup(a) = \delta \wedge {\rm otp}(a) = {\rm cf}(\delta)\big)$.

1A) In part (1) instead of "$\lambda = \mu^+ \wedge \mu = {\rm cf}(\mu)$" we can demand $\alpha < \lambda \Rightarrow {\rm cf}\big([\alpha]^{<\mu}, \subseteq\big) < \lambda$.

2) $\check{I}^{\rm ac}_\theta\langle\lambda,\mu\rangle \cap \check{I}_\theta[\lambda]$ is the non-stationary ideal when defined.

3) If $\lambda > \theta^+$ and λ, θ are regular and $S \in \check{I}_\theta[\lambda]$ is stationary, then there is a $(\lambda, \leqq \theta, < \lambda)$-system $\bar{\mathscr{P}}$ such that $S \notin I^{\rm cg}_\theta(\bar{\mathscr{P}})$ and $\alpha < \lambda \wedge a \in \mathscr{P}_\alpha \Rightarrow \operatorname{otp}(a) = \theta$.

Proof. 1) By [12, §4] or [5, Ch. III] as corrected in [6].

1A) By Dzamonja–Shelah [15].

2) See 1.3.

3) By part (1) the proof of "club guessing", see [5, Ch. III], i.e. let $\bar{\mathscr{P}} = \langle \mathscr{P}_\alpha : \alpha < \lambda \rangle$ be a $(\lambda, \theta, < \lambda)$-system such that $S \subseteqq \operatorname{good}_\theta(\bar{\mathscr{P}})$. Without loss of generality $\mathscr{P}_\alpha$ is increasing with α and shows that for some club E of λ the sequence $\bar{\mathscr{P}}_E = \langle \mathscr{P}_{E,\alpha} : \alpha < \lambda \rangle$ is as required where $g\ell(a, E) := \big\{ \sup(\alpha \cap E) : \alpha \in a \big\}$ and $\mathscr{P}_{E,\alpha} := \big\{ g\ell(a, E) : a \in \mathscr{P}_\beta$ for some $\beta \in \big[\sup(E \cap \alpha), \min\big(E \backslash (\alpha + 1)\big)\big]\big\}$. $\square$

In §(1B) we shall use [5, Ch. II].

Definition 0.18. Let $\bar{f}$ be $<_J$-increasing in ${}^\kappa\mathrm{Ord}$, J an ideal on I.

1) We say $\bar{f}$ is flat in δ or $\delta \in S_{\rm gd}[\bar{f}, J] = S^{\rm gd}_J[\bar{f}]$ when $\delta \leqq \ell g(\bar{f})$, $\operatorname{cf}(\delta) > \kappa$ and there is a $<_J$-club g to $\bar{f} \restriction \delta$ such that $(\forall i < \kappa)\big(\operatorname{cf}\big(g(i)\big) = \operatorname{cf}(\delta)\big)$, equivalently there are increasing sequences $\langle \alpha_{i,\varepsilon} : \varepsilon < \operatorname{cf}(\delta) \rangle$ for $i < \kappa$ such that $(\forall \alpha < \delta)\big(\exists \varepsilon < \operatorname{cf}(\delta)\big)\big(f_\alpha <_J \langle \alpha_{i,\varepsilon} : i < \kappa \rangle\big)$ and $\big(\forall \varepsilon < \operatorname{cf}(\delta)\big)$ $(\exists \alpha < \delta)\big(\langle \alpha_{i,\varepsilon} : i < \kappa \rangle <_J f_\alpha\big)$.

2) We say δ is strongly chaotic for $\bar{f}$ or $\delta \in S_{\rm sch}[\bar{f}, J] = S^{\rm sch}_J[\bar{f}]$ when there is a sequence $\langle u_i : i < \kappa \rangle$, $u_i \subseteqq \mathrm{Ord}$, $|u_i| \leqq \kappa$ and $(\forall \alpha < \delta)\Big(\exists g \in \prod\limits_i u_i\Big)$ $(\exists \beta < \delta)(f_\alpha <_J g <_J f_\beta)$.

2A) We say δ is chaotic for $\bar{f}$ or $\delta \in S^{\rm ch}_J[f] = S_{\rm ch}[\bar{f}, J]$ when there is $\bar{u}$ as above such that for every $\alpha < \delta$ for some $\beta \in (\alpha, \delta)$ the set $A_{\alpha,\beta} = A_{\alpha,\beta}[\bar{u}, \bar{f}]$ belongs to J^+ where $A_{\alpha,\beta} = \big\{ i < \kappa : \min\big(u_i \cup \{\infty\} \backslash f_\alpha(i)\big) < \min\big(u_i \cup \{\infty\} \backslash f_\beta(i)\big)\big\}$.

2B) We define $S^{\rm sch}_\theta[\bar{f}, J] = S^{\rm sch}_{J,\theta}[\bar{f}]$, $S^{\rm ch}_\theta[\bar{f}, J] = S^{\rm ch}_{J,\theta}[\bar{f}]$ similarly but restricting ourselves to δ of cofinality θ.

3) We say δ is bad for $\bar{f}$ or $\delta \in S_{\rm bd}[\bar{f}, J] = S^{\rm bd}_J[\bar{f}]$ when $\delta \leqq \ell g(\bar{f})$, $\operatorname{cf}(\delta) > \kappa$ and $\bar{f} \restriction \delta$ has $<_J$-club g but is not flat.

Claim 0.19. *Let J, $\bar{f}$ be as in 0.18.*

1) *If $\delta \leqq \ell g(\bar{f})$ and $\operatorname{cf}(\delta) > \kappa^+$ then δ satisfies exactly one of good, bad or chaotic.*

2) *In other words $\big\{ \delta : \delta \leqq \ell g(\bar{f})$ and $\operatorname{cf}(\delta) > \kappa^+ \big\}$ is included in the disjoint union of $S_{\rm gd}[\bar{f}], S_{\rm bd}[\bar{f}], S_{\rm ch}[\bar{f}]$.*

Proof. By [5, Ch. II, §2]. $\square$

CLAIM 0.20. *Let $\bar f$, J, κ be as in 0.18 and $\lambda = \ell g(\bar f)$.*

1) *If $\delta \in S^{\rm ch}_J[\bar f]$ then for some club e of δ, we have $\alpha \in e \wedge {\rm cf}\,(\alpha) > \kappa \Rightarrow \alpha \in S^{\rm ch}_J[\bar f]$.*

1A) *Similarly for $S_{\rm sch}[\bar f]$.*

2) *If $\delta \in S^{\rm gd}_J[\bar f]$ then for some club e of δ we have $\alpha \in e \wedge {\rm cf}\,(\alpha) > \kappa \Rightarrow \alpha \in S^{\rm gd}_J[\bar f]$.*

3) *If $\delta \leqq \lambda$, ${\rm cf}\,(\delta) \in S_{\rm bd}[\bar f]$ then ${\rm cf}\,(\delta) \geqq \kappa^{+\,{\rm comp}\,(J)+1}$.*

4) *If $\delta \in S^{\{d}_J[\bar f]$, g an $<_J$-club of $\bar f \restriction \delta$, $\sigma = {\rm cf}\,(\sigma)$ and $\big\{i < \kappa : {\rm cf}\,\big(g(i)\big) > \sigma\big\} \in J^+$ then $\big\{\delta_1 : {\rm cf}\,(\delta_1)$ but $\delta_1 \notin S^{\{\delta}_J[\bar f]\big\} \cap S^\delta_J$ is not a stationary subset of δ.*

PROOF. 1) By [11].

2) Should be clear.

By [5, Ch. I].

CLAIM 0.21. *Assume $(\lambda, \bar\lambda, J, \kappa)$ is a* pcf *case, $\bar f$ a witness for it, see Definition 1.6. If $\kappa < \sigma < \min\{\lambda_i : i < \kappa\}$ or just $\kappa < \sigma < \lim-\inf_J(\bar\lambda)$ and $S \in \check{I}_\sigma[\lambda]$ then $E \cap S \subseteqq S_{\rm gd}[\bar f]$ for some club E of λ.*

1. On systems

1(A). Existence of large members of $\check{I}_\theta[\lambda]$.

CLAIM 1.1. *Assume $\lambda > \aleph_1$ is regular and $M_* \prec \big(\mathscr{H}(\lambda), \in\big)$ has cardinality $< \lambda$ and $\{\lambda, \theta\} \subseteqq M_*$ and $M_* \cap \lambda \in \lambda$. Then we can find a pair $(E, \bar{\mathscr{P}})$ which is (λ, M_*)-suitable, which means:*

$\boxplus$ (a) *E is a club of λ; we may add $\alpha \in E \wedge \alpha > \sup\,(\alpha \cap E) \Rightarrow {\rm cf}\,(\alpha) = \aleph_0$,*

(b) *$\bar{\mathscr{P}} = \langle \mathscr{P}_\alpha : \alpha < \lambda\rangle$ is a $(\lambda, \lambda, < \lambda)$-system and $\theta = {\rm cf}\,(\theta) < \lambda \cap M_* \Rightarrow {\rm good}''_\theta(\bar{\mathscr{P}}) \supseteqq S^\lambda_\theta \backslash E$,*

(c) *if $\sigma > \partial$ are regular $\in \lambda \cap M_*$ and $\bar{\mathscr{P}}^* = \langle \mathscr{P}^*_\alpha : \alpha < \partial\rangle \in M_*$ is a $(\partial, \partial, < \sigma)$-system and $\langle \delta_i : i < \sigma\rangle$ is an increasing continuous sequence of members from E, then there are f, e such that:*

(α) *e is a club of ∂,*

(β) *f is an increasing continuous function from ∂ into $\{\delta_i : i < \sigma\}$,*

(γ) *if $\varepsilon < \partial$, $a \in \mathscr{P}^*_\varepsilon$ and $a \subseteqq e$ then $\big\{f(\xi) : \xi \in a$ and ${\rm otp}\,(a \cap \xi)$ is a successor ordinal$\big\} \in \mathscr{P}_{f(\varepsilon+1)}$,*

(c)$^+$ *like* (c) *but we replace* (γ) *by*

(γ)$^+$ *if $\varepsilon < \partial$, $a \in \mathscr{P}^*_\varepsilon$ and $a \subseteqq e$ and $\big\langle \gamma_\iota : \iota < {\rm otp}\,(a)\big\rangle$ list a in increasing order then in addition to the conclusion of* (γ),

- *we can choose $\beta_\iota \in [\gamma_\iota, \gamma_{\iota+1})$ for $\iota < {\rm otp}\,(a)$ such that $\{\beta_j : j \leqq \iota\} \in \mathscr{P}_{\beta_{\iota+1}}$ for every $\iota < {\rm otp}\,(a)$,*

• *if* a *has no last member then* $\sup(a) \in \mathrm{good}''_\theta(\bar{\mathscr{P}})$,

(d) *if* $\langle \delta_i : i < \sigma\rangle$ *is an increasing continuous sequence of members of* E *and* $\sigma > \partial > \theta$ *are regular* $\in \lambda \cap M_*$ *and* $\bar{\mathscr{P}}^* = \langle \mathscr{P}^*_\varepsilon : \varepsilon < \partial\rangle \in M_*$ *is a* $(\partial, \leqq \theta, < \sigma)$*-system then for some* e, f *satisfying clauses* (α), (β), (γ), $(\gamma)^+$ *we have*

(δ) *the following set belongs to* $I^{\mathrm{dg}}_\theta(\bar{\mathscr{P}}^*)$, *recalling* 0.13(1A) $\big\{\zeta \in S^\partial_\theta$: *there is no* $a \subseteqq e$, $a \in \mathscr{P}^*_{<\partial}$ *such that* $a \subseteqq \zeta = \sup(a)$ *and* $\mathrm{otp}(a) = \theta\big\}$,

(ε) *the following set belongs to* $\check{I}^{\mathrm{ac}}_\partial\langle \sigma, \sigma\rangle$, *see Definition* 0.13(2) $\big\{i \in S^\sigma_\partial$: *there are no* e, f *satisfying* $\sup(e) = i$ *and clauses* (α), (β), (γ), $(\gamma)^+$, (δ) *above*$\big\}$.

Remark 1.2. 1) Note that for $\mathrm{good}''_\theta(\bar{\mathscr{P}})$, only $\big\langle \mathscr{P}_\alpha \cap [\alpha]^{<\theta} : \alpha < \lambda\big\rangle$ matters.

2) For $\bar{M}$ as in $\odot_1$ in the proof and $\alpha < \lambda$ essentially $\bar{\mathscr{P}}$ satisfies the conclusion with M_* replaced by M_α; the essentially because we should ignore the ordinals $\leqq \alpha$, i.e. in clauses (c), $(\mathrm{c})^+$, (d) demand $\delta_0 > \alpha$.

Proof. Let $\chi > \lambda$ and let $\bar{M}$ be such that:

$\odot_1$ (a) $\bar{M} = \langle M_\alpha : \alpha < \lambda\rangle$ be a $\prec$-increasing continuous sequence,
(b) $M_\alpha \prec \big(\mathscr{H}(\chi), \in, <^*_\chi\big)$,
(c) $\|M_\alpha\| < \lambda$,
(d) $\bar{M} \restriction (\alpha + 1) \in M_{\alpha+1}$,
(e) $M_\alpha \cap \lambda \in \lambda$ for every $\alpha < \lambda$,
(f) if $\alpha < \lambda$ is non-limit, then $M_\alpha \cap \lambda$ has cofinality $\aleph_0$,
(g) $M_* \in M_0$ hence $M_* \subseteqq M_0$.

Let $E = \{\alpha : M_\alpha \cap \lambda = \alpha\}$. Clearly E is a club of λ, hence clause (a) of $\boxplus$ holds.

Let $\bar{\mathscr{P}} = \langle \mathscr{P}_\alpha : \alpha < \lambda\rangle$ be defined by:

$$\odot_2 \begin{cases} \mathscr{P}_\alpha = \{a \in M_{\alpha+1} : a \subseteqq \alpha \text{ so } |a| < \lambda \text{ and } \beta \in a \Rightarrow a \cap \beta \in M_{\beta+1}\} \\ \text{so } \bar{\mathscr{P}} = \langle \mathscr{P}_\alpha : \alpha < \lambda\rangle \text{ is a } (\lambda, \lambda, < \lambda)\text{-system, moreover,} \\ \boxplus(\mathrm{b}) \text{ holds.} \end{cases}$$

[Why does $\boxplus$(b) hold? Let $\delta \in S^\lambda_\theta \backslash E$ be a limit ordinal, so for some $\alpha < \delta$ we have $\delta \in M_\alpha$ hence there is an unbounded (and even closed) subset a of δ in M_α of order type $\mathrm{cf}(\delta)$ so $\beta \in (a \backslash \alpha) \Rightarrow (a \backslash \alpha) \cap \beta \in M_\alpha \subseteqq M_\beta \Rightarrow (a \backslash \alpha) \cap \beta \in M_\beta$. So indeed $\mathrm{good}''_\theta(\bar{\mathscr{P}}) \supseteqq S^\lambda_\theta \backslash E$.]

So we arrive to the main point, that is to prove clauses (c), $(\mathrm{c})^+$ and later comment on its relative (d). So let $\partial < \sigma \in M_* \cap \lambda$ be regular and $\bar{\mathscr{P}}^* \in M_*$ be a $(\partial, \partial, < \sigma)$-system and let $\bar{\delta} = \langle \delta_i : i < \sigma\rangle$ be an increasing continuous sequence of ordinals from E and let $\delta_\sigma := \cup\{\delta_i : i < \sigma\}$ so also $\langle \delta_i : i \leqq \sigma\rangle$ is an increasing continuous sequence of ordinals from E.

We choose N_ε by induction on $\varepsilon \leqq \partial$ such that:

$\odot_3$ (a) $N_\varepsilon \prec \big(\mathscr{H}(\chi), \in, <^*_\chi\big)$,
(b) $\|N_\varepsilon\| < \sigma$,
(c) $\langle N_\xi : \xi \leqq \zeta\rangle \in N_\varepsilon$ when $\zeta < \varepsilon$,
(d) $\langle N_\zeta : \zeta \leqq \varepsilon\rangle$ is $\prec$-increasing continuous,
(e) λ, σ, ∂, θ, E, $\bar{M}$, $\bar{\delta}$ and $\mathscr{P}^*$ belongs to N_ε,
(f) $\partial + 1 \subseteqq N_\varepsilon$ moreover (follows if $\sigma = \partial^+$) $N_\varepsilon \cap \sigma \in (\partial, \sigma)$.

This is easy. Let $i(\varepsilon) := N_\varepsilon \cap \sigma$ for $\varepsilon \leqq \partial$, hence $i(\varepsilon) < \sigma$ is increasing continuous with ε. So $\delta_{i(\varepsilon)}$ is an ordinal $\in E \subseteqq \lambda$ hence $M_{\delta_{i(\varepsilon)}}$ is well defined and $\delta_{i(\varepsilon)} \in M_{\delta_{i(\varepsilon)}+1}$, also $\big\langle \delta_{i(\varepsilon)} : \varepsilon < \partial\big\rangle$ is increasing continuous with limit $\delta_{i(\partial)}$. For $\varepsilon = \partial$ clearly $\mathrm{cf}\big(\delta_{i(\varepsilon)}\big) = \mathrm{cf}\big(\delta_{i(\partial)}\big) = \mathrm{cf}\,(\partial) = \partial$ hence

$\oplus_1$ (a) there is a club C of $\delta_{i(\partial)}$ of order type $\mathrm{cf}\big(\delta_{i(\partial)}\big) = \partial$,
(b) necessarily $C \in \mathscr{H}(\chi)$ and without loss of generality $C \in M_{\delta_{i(\partial)}+1}$,
(c) let g be the unique increasing continuous function from ∂ onto C, so necessarily $g \in M_{\delta_{i(\partial)}+1}$,
(d) let $e = \big\{\varepsilon < \partial : \delta_{i(\varepsilon)} \in C$, moreover $\varepsilon = \mathrm{otp}\big(C \cap \delta_{i(\varepsilon)}\big)$ and, actually follows, $\delta_{i(\varepsilon)} = g(\varepsilon)\big\}$,
(e) let $f : \partial \to \sigma$ be defined by $f(\varepsilon) = \delta_{i(\varepsilon)}$.

Now C is a club of ∂ and both $\big\langle g(\varepsilon) : \varepsilon < \partial\big\rangle$ and $\big\langle \delta_{i(\varepsilon)} : \varepsilon < \partial\big\rangle$ are increasing continuous sequences of ordinals with limit $\delta_{i(\partial)}$, so clearly

$\oplus_2$ e is a club of ∂.

So concerning clause (c) (of $\boxplus$) it suffices to prove that the pair (f, e) we have just chosen is as required there. Now obviously e, f satisfy sub-clauses (α), (β) of (c). What about sub-clause (γ) of clause (c) and subclause $(\gamma)^+$ of clause $(c)^+$?

Clearly

$\oplus_3$ $f \restriction e = g \restriction e$, see the definition of e.

Now we shall prove

$\oplus_4$ if $\varepsilon < \partial$ and $a \in \mathscr{P}^*_\varepsilon$ satisfies $a \subseteqq e$, then $\big\{g(\zeta) : \zeta \in a\big\} \in M_{f(\varepsilon+1)}$.

The proof of $\oplus_4$ is done in $(*)_{4.1} - (*)_{4.7}$.

Note

$(*)_{4.1}$ $\mathscr{P}^*_\varepsilon \subseteqq N_0 \cap M_0 \subseteqq N_{\varepsilon+1} \cap M_{\delta(\partial)+1} \subseteqq N_{\varepsilon+1} \cap M_{\delta_\sigma}$.

[For the first inclusion, obviously $\bar{\mathscr{P}}^* \in M_*$, $\partial = \ell g(\bar{\mathscr{P}}^*) \in M_* \cap \lambda$ but $M_* \cap \lambda \subseteqq M_0 \cap \lambda \in \lambda$ hence $\partial \subseteqq M_0$ so together $\mathscr{P}^*_\varepsilon \in M_0$. Now $|\mathscr{P}^*_\varepsilon| < \sigma < \lambda$ and $\sigma \in M_* \cap \lambda \subseteqq M_0 \cap \lambda \in \lambda$ so $\mathscr{P}^*_\varepsilon \subseteqq M_0 \subseteqq M_{\delta_{i(\varepsilon)}} \subseteqq M_{i(\partial)} \subseteqq M_{\delta_\sigma}$. Also $\bar{\mathscr{P}}^* \in N_0$ and $\varepsilon, \partial \in N_\varepsilon$ and $|\mathscr{P}^*_\varepsilon| + \partial < \sigma$ and by $\odot_3(f)$ we have $N_\varepsilon \cap \sigma \in \sigma$ hence $\mathscr{P}^*_\varepsilon \subseteqq N_\varepsilon$, so together we are done. The other inclusions are immediate as $\bar{N}$ is $\subseteqq$-increasing by $\odot_3$(d) and $\bar{M}$ is $\subseteqq$-increasing by $\odot_1(a)$.]

Also

$(*)_{4.2}$ $\{g(\zeta) : \zeta \in a\} \in M_{\delta_{i(\partial)+1}} \prec M_{\delta_\sigma}$.

[As a and g belong to this model; why? For a because $a \in \mathscr{P}^*_\varepsilon$, see the assumption of $\oplus_4$ and $\mathscr{P}^*_\varepsilon \subseteq M_0 \subseteq M_{\delta_{i(\partial)}} \subseteq M_{\delta_{i(\partial)+1}}$ by $(*)_{4.1}$. For g, by the choice of C and g, see $\oplus_1$(a), (b), (c).]

$(*)_{4.3}$ $\{g(\zeta) : \zeta \in a\} = \{(f \restriction \varepsilon)(\zeta) : \zeta \in a\} \in N_{\varepsilon+1}$.

[The equality holds by $\oplus_3$ as $a \subseteq e \wedge a \subseteq \varepsilon$ by the assumptions of $\oplus_4$. Why the membership "$\in N_{\varepsilon+1}$" holds? On the one hand $a \subseteq \varepsilon$, $a \in \mathscr{P}^*_\varepsilon$ hence by $(*)_{4.1}$ also $a \in N_{\varepsilon+1}$. On the other hand $f \restriction \varepsilon \in N_{\varepsilon+1} \prec N_\partial$ because $\langle N_\zeta : \zeta \leqq \varepsilon\rangle \in N_{\varepsilon+1}$ by $\odot_3$(c) hence $\langle i(\zeta) : \zeta \leqq \varepsilon\rangle \in N_{\varepsilon+1}$ by the choice $i(\zeta) = \sup(N_\zeta \cap \sigma)$ after $\odot_3$ and $\bar\delta \in N_0$ by $\odot_3(e)$ hence $\langle \delta_{i(\zeta)} : \zeta \leqq \varepsilon\rangle \in N_{\varepsilon+1}$ so $f \restriction (\varepsilon+1) \in N_{\varepsilon+1}$ by $\oplus_1(e)$.]

As $\bar\delta \in N_0 \prec N_{i(\partial)}$ by $\odot_3(e)$ we have $\bar\delta = \langle \delta_i : i \leqq \sigma\rangle \in N_0 \prec N_{\varepsilon+1}$ so necessarily $\delta_\sigma \in N_0 \prec N_{\varepsilon+1}$ and recalling $\bar M \in N_0$ by $\odot_3(e)$ it follows that $M_{\delta_\sigma} = \cup\{M_\alpha : \alpha < \delta_\sigma\} \in N_{\varepsilon+1}$ and $\bar M \restriction \delta_\sigma \in N_{\varepsilon+1} \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ hence

$(*)_{4.4}$ $M_{\delta_\sigma} \cap N_{\varepsilon+1} \subseteq M_{\sup(N_{\varepsilon+1} \cap \delta_{i(\sigma)})}$

but (by $(*)_{4.2} + (*)_{4.3}$)

$(*)_{4.5}$ $\{g(\zeta) : \zeta \in a\} \in M_{\delta_\sigma} \cap N_{\varepsilon+1}$.

Now as $\bar M, \bar\delta \in N_0$ and $\sigma \in N_0$ by $\odot_3(e)$, clearly $M_{\delta_\sigma} \in N_0$ and as $N_{\varepsilon+1} \cap \sigma = i(\varepsilon+1)$ by the choice of $i(\varepsilon+1)$ after $\odot_3$ and $\|N_{\varepsilon+1}\| < \sigma$ by $\odot_3$(b) clearly

$(*)_{4.6}$ $N_{\varepsilon+1} \cap M_{\delta_\sigma} \subseteq M_{\delta_{i(\varepsilon+1)}}$.

But $f(\varepsilon+1) = \delta_{i(\varepsilon+1)}$ by $\oplus_1(e)$ hence by $(*)_{4.5} + (*)_{4.6}$ we have

$(*)_{4.7}$ $\{g(\zeta) : \zeta \in a\} \in M_{f(\varepsilon+1)}$.

So we have proved $\oplus_4$.

$$\oplus_5 \quad \begin{cases} \text{if } \varepsilon < \partial,\ a \in \mathscr{P}^*_\varepsilon,\ a \subseteq e \text{ and } \xi \in a \wedge (a \cap \xi \text{ has a last member) then} \\ \{g(\zeta) : \zeta \in a \cap \xi\} \in M_{f(\xi)}. \end{cases}$$

[Let $\zeta(*) = \max(a \cap \xi)$, it is well defined by the assumption on ξ. But $\bar{\mathscr{P}}^*$ is a $(\partial, \partial, < \sigma)$-system by the assumption of clause (c) (so of clause $(c)^+$) of $\boxplus$, hence by clause (d) of Definition 0.9(1) we have $a \cap \zeta(*) \in \mathscr{P}^*_{\zeta(*)}$ and, of course, $a \cap \zeta(*) \subseteq e$ hence we can apply $\oplus_4$ with $(\zeta(*), a \cap \zeta(*))$ here standing for (ε, a) there, so we can deduce $\{g(\zeta) : \zeta \in a \cap \zeta(*)\} \in M_{f(\zeta(*)+1)}$. But $\zeta(*)+1 \leqq \xi$ hence $f(\zeta(*)+1) \leqq f(\xi)$ hence $M_{f(\zeta(*)+1)} \subseteq M_{f(\xi)}$. So

$\{g(\zeta) : \zeta \in a \cap \zeta(*)\} \in M_{f(\xi)}$, hence by the obvious closure properties of $M_{f(\xi)} \cap [f(\xi)]^{\leqq\theta}$ also $\{g(\zeta) : \zeta \in a \cap \xi\} \in M_{f(\xi)}$.]

$$\oplus_6 \begin{cases}\text{if } \varepsilon < \partial,\ a \in \mathscr{P}^*_\varepsilon \text{ and } a \subseteqq e \text{ then the set } b = \{f(\zeta) : \zeta \in a \text{ and} \\ \text{otp}\,(a \cap \zeta) \text{ is a successor ordinal}\} \text{ belongs to } \mathscr{P}_{f(\varepsilon+1)}.\end{cases}$$

[By $\oplus_4 + \oplus_5$, the definition of $\mathscr{P}_{f(\varepsilon+1)}$ in $\odot_2$ and the obvious closure properties of each M_α.]

So we are done proving clause (c)(γ) of $\boxplus$ hence clause (c). Clause $(c)^+(\gamma)^+$ is proved similarly. Say let h_α be chosen by induction on $\alpha \leqq \lambda$ such that $\langle h_\beta : \beta \leqq \alpha\rangle$ is $\subseteqq$-increasing continuous and h_α is a one-to-one function from M_α onto some ordinal $\gamma < \alpha$ and h_α is $\leqq^*_\chi$-minimal under those restrictions; now $\langle h(f \restriction (a \cap \zeta)) : \zeta \in q\rangle$ will be as required.

We are left with proving clause (d) of $\boxplus$, let $x = \{\lambda, \sigma, \partial, \theta, \bar{\mathscr{P}}^*, E, \bar{M}\}$ and let $S_1 = \{j \in S^\sigma_\partial$: there is $\bar{N}$ as in $\odot_3$ such that $j = \sup(\cup\{N_\varepsilon : \varepsilon < \partial\} \cap \sigma)\}$. Now by the definition 0.13(2) of $\check{I}^{\rm ac}_\partial\langle\sigma,\sigma\rangle$ we know that $S^\sigma_\theta \backslash S_1 \in \check{I}^{\rm ac}_\partial\langle\sigma,\sigma\rangle$.

Next, for each $j \in S_1$ let $\langle N_\varepsilon : \varepsilon < \partial\rangle$ witness that $j \in S_1$. Now choose C, g, e, f as in $\oplus_1$. So by the definition of $I^{\rm dg}_\theta(\bar{\mathscr{P}}^*)$ in 0.13(1A) the set $S^\partial_\theta \backslash S_2$ belongs to $I^{\rm dg}_\theta(\bar{\mathscr{P}}^*)$ where $S_2 = \{\zeta \in S^\partial_\theta$: there is $a \in \mathscr{P}^*_{<\partial}$ such that otp$\,(a) = \theta$, sup$\,(a) = \zeta$ and $a \subseteqq e$ hence $\zeta \in e\}$.

For each $\zeta \in S$, let $a \in \mathscr{P}^*_{<\partial}$ witness $\zeta \in S_2$, as in the proof of clause (c)(γ) we get that $\zeta \in \text{good}''_\theta(\bar{\mathscr{P}})$. Clearly this suffices for proving clauses (d)(δ), (ε). $\square$

Claim 1.3. *Let $\sigma > \partial > \theta$.*

1) *$S^\sigma_\partial \notin \check{I}^{\rm ac}_\partial\langle\sigma,\sigma\rangle$ moreover $\check{I}^{\rm ac}_\partial\langle\sigma,\sigma\rangle$ is a normal ideal on S^σ_∂.*

2) *If $S_1 \in \check{I}_\theta[\sigma]$ and $S_2 \in \check{I}^{\rm ac}_\theta\langle\sigma,\partial\rangle$ then $S_1 \cap S_2$ is non-stationary.*

Remark 1.4. If $\sigma = \partial^+$, see 0.17.

Proof. 1) Easy.

2) Let $\bar{\mathscr{P}}' = \langle\mathscr{P}'_\varepsilon : \varepsilon < \sigma\rangle$ be a $(\sigma, \partial, <\sigma)$-system witnessing $S_1 \in \check{I}_\theta[\sigma]$.

Now instead of choosing N_ε for $\varepsilon \leqq \partial$ we choose N_ε and $\bar{N}_\varepsilon$ by induction on $\varepsilon < \sigma$ such that:

$\oplus$(A) (a) $N_\varepsilon \prec (\mathscr{H}(\chi), \in, <^*_\chi)$,

(b) $\|N_\varepsilon\| < \sigma$ and $N_\varepsilon \cap \sigma \in \sigma$,

(c) $\langle N_\zeta : \zeta \leqq \xi\rangle \in N_\varepsilon$ for $\xi < \varepsilon$,

(B) (a) $\bar{N}_\varepsilon = \langle N_{\varepsilon,a} : a \in \mathscr{P}'_\varepsilon\rangle$,

(b) $N_{\varepsilon,a} \prec (\mathscr{H}(\chi), \in, <^*_\chi)$,

(c) $\|N_{\varepsilon,a}\| < \partial$,

(d) if $a \in \mathscr{P}'_\varepsilon$ then $\langle N_{\xi,a\cap\xi} : \xi \in a \cup \{\varepsilon\}\rangle$ is $\prec$-increasing and $\xi \in a \cup \{\zeta\} \wedge \xi = \sup(a \cap \xi) \Rightarrow N_{\xi,a\cap\xi} = \cup\{N_{\zeta,a\cap\zeta} : \zeta \in a\}$ and $\xi \in a \Rightarrow a \cap \xi \in N_{\varepsilon,a}$,

(e) E, $\bar{M}$, $\bar{\delta}$, σ, $\bar{\mathscr{P}}^*$ and $\bar{\mathscr{P}}'$ belongs to $N_{\varepsilon,a}$,

(f) $\langle N_{\zeta,b} : \zeta \leqq \xi,\ b \in \mathscr{P}'_\zeta\rangle$ and $\langle N_\zeta : \zeta \leqq \xi\rangle$ belongs to $N_{\varepsilon,a}$ and to N_ε when $\xi < \varepsilon_* < \sigma$,

(g) $\partial \cap N_{\xi,a} \in \partial$.

The rest should be clear. $\square$

Proof of 0.1. 1) As ∂, θ are regular cardinals and $\partial > \theta^+$ let $\bar{\mathscr{P}}^* := \langle \mathscr{P}^*_\alpha : \alpha < \partial\rangle$ be a $(\partial, \leqq \theta, < \partial)$-system satisfying $S^\sigma_\theta \notin I^{\rm cg}_\theta(\bar{\mathscr{P}}^*)$, see 0.13(1), 0.17(3). Let χ, M_* be as in 1.1 for our λ such that $\bar{\mathscr{P}}^* \in M_*$. Let $E, \bar{\mathscr{P}}$ be as constructed in 1.1 for our λ, M_* and recall $\alpha \in \mathrm{nacc}(E) \Rightarrow \mathrm{cf}(\alpha) = \aleph_0$. So if $\delta \in E \cap S^\lambda_\sigma$ then $\delta \in \mathrm{acc}(E)$ and so there is an increasing continuous sequence $\langle \delta_i : i < \sigma\rangle$ of members of E with limit δ; hence by clauses $(\mathrm{c})^+(\gamma)$ we have $(\exists^{\rm stat} i < \delta)\big[i \in \mathrm{good}''_\theta(\bar{\mathscr{P}})\big]$.

As we have started with any $\delta \in E \cap S^\lambda_\theta$ clearly $\mathrm{good}''_\theta(\bar{\mathscr{P}})$ reflects in any $\delta \in E \cap S^\lambda_\sigma$, but $\mathrm{good}''_\theta(\bar{\mathscr{P}}) \in \check{I}_\theta[\lambda]$. Now by $\boxplus$(b) of 1.1 $\delta \in S^\lambda_\theta \backslash E \Rightarrow \delta \in \mathrm{good}''_\theta(\bar{\mathscr{P}})$ so $\mathrm{good}''_\theta(\bar{\mathscr{P}}) \in \check{I}_\theta[\lambda]$ is as required.

2) Same proof.

3) Similarly using clause (d)(ε) of 1.1. $\square$

Proof of 0.2. 1) Let χ, λ, M_* be as the assumption of 1.1 such that in addition $2^{\theta^{+n}} < M_* \cap \lambda$ for every n. Let E and $\bar{\mathscr{P}} = \langle \mathscr{P}_\alpha : \alpha < \lambda\rangle$ be as in the conclusion of 1.1.

Recalling Definition 0.12(2A), let $S_* = \mathrm{good}''_\theta(\bar{\mathscr{P}}) \subseteqq S^\lambda_\theta$, so obviously $S_* \in \check{I}_\theta[\lambda]$ and for every n let $S_n = \big\{\delta : \mathrm{cf}(\delta) = \theta^{+n}$ and $n = 0 \Rightarrow \delta \notin S_*$ and $[n \geqq 1 \Rightarrow \delta \cap S^\lambda_\theta \backslash S_*$ is a stationary subset of $\delta]\big\}$.

Note that by the assumption of part of the theorem

$\boxplus_1$ S_0 is a stationary subset of λ.

For $n \geqq 1$ and $\delta \in S_n$ we choose $\big\langle \gamma_{\delta,\varepsilon} : \varepsilon < \mathrm{cf}(\delta)\big\rangle$, an increasing continuous sequence with limit δ and let $s_\delta = \big\{\varepsilon < \mathrm{cf}(\delta) : \mathrm{cf}(\varepsilon) = \theta$ and $\gamma_{\delta,\varepsilon} \notin S_*\big\}$, so as $\delta \in S_n$ necessary s_δ is a stationary subset of θ^{+n}.

For every stationary $s \subseteqq S^{\theta^{+n}}_\theta$ let $S_{n,s} = \{\delta \in S_n : s_\delta = s\}$, the sequence $\langle S_{n,s} : s \subseteqq S^{\theta^{+n}}_\theta$ is stationary$\rangle$ is a partition of S_n and for some club $E_{n,s} \subseteqq E$ of λ we have $[S_{n,s} \cap E_{n,s} = \emptyset \Leftrightarrow S_{n,s}$ is not stationary$]$ for every such s.

Let $E_* = \cap\{E_{n,s} : n \geqq 1$ and $s \subseteqq \theta^{+n}$ is stationary$\}$, so as we are assuming $2^{\theta^{+n}} < \lambda$, clearly E_* is a club of λ.

Clearly if "$n \geqq 2 \wedge (s \subseteqq S^{\theta^{+n}}_\theta$ stationary$) \Rightarrow S_{n,s} \subseteqq \lambda$ is not stationary" then for some $k < m$, $S = S_k$ satisfy the desired conclusion. So assume that $n \geqq 2$ and $s \subseteqq \theta^{+n}$ is stationary and $S_{n,s}$ is stationary. If $S_{n,s}$ reflects in no $S^\lambda_{\theta^{+m}}$, $m > n$ we are done, and also if $\mathrm{refl}(S_{n,s}) \cap S^\lambda_{\theta^{+n+1}}$ is stationary but reflect in no $S^\lambda_{\theta^{+m}}$, $m > n+1$, we are done.

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Hence it suffices to prove

$$\boxplus_2 \quad \begin{cases} \text{if } n \geqq 2, \quad s \subseteq S^{\theta^{+n}}_\theta \text{ is stationary and } S_{n,s} \subseteq \lambda \text{ is stationary,} \\ m \geqq n+2 \text{ then } S_{n,s} \text{ does not reflect in any } \delta_* \in S^\lambda_{\theta^{+m}} \cap \operatorname{acc}(E_*). \end{cases}$$

Toward this let $\sigma = \theta^{+m}$ and $\bar{\delta} = \langle \delta_i : i < \sigma \rangle$ be an increasing continuous sequence of ordinals from E_* with limit $\delta_{i(\sigma)} := \delta_*$. As $s \subseteq S^{\theta^{+n}}_\theta$ is stationary and $n \geqq 2$, let $\partial = \theta^{+n}$ by 0.11, 0.17(3) there is $\bar{\mathscr{P}}^* = \langle \mathscr{P}^*_\zeta : \zeta < \partial \rangle$ a (∂, θ)-system such that $s \notin I^{\rm cg}_\theta(\bar{\mathscr{P}}^*)$.

Note that $\bar{\mathscr{P}}^* \in M_*$ because $2^{\theta^{+n}} < \lambda$ and $M_* \cap \lambda$. So our $\bar{\mathscr{P}}$ satisfies the conclusion of 1.1, so $\boxplus$ holds indeed hence we are done.

2), 3), 4) The proof is really included in the proof of part (1). $\square$

Remark 1.5. In the proof of 1.1, for regular $\kappa \in (\theta, \lambda)$ and s a stationary subset of S^κ_θ we can let $S_{\kappa,s} = \big\{ \delta \in S^\lambda_\kappa$: for some increasing continuous sequence $\langle \alpha_i : i < \kappa \rangle$ of ordinals with limit δ, the set $\{i \in S^\kappa_\theta : i \in s$ iff $\alpha_i \in S_*\}$ is not stationary$\big\}$. Let $E_{\kappa,s}$ be a club of λ, disjoint to $S_{\kappa,s}$ if $S_{\kappa,s}$ is not stationary. Let $\kappa_* < \lambda$ and $E_* = \cap \big\{ E_{\kappa,s} : \kappa \in (\theta, \kappa_*)$ is regular and $s \subseteq \kappa \big\}$. We can then continue as above.

1(B). Quite free witnesses of pcf-cases exist.

Definition 1.6. 1) We say $(\lambda, \bar{\lambda}, J, \kappa)$ is a pcf-case (may omit J in the case $J = [\kappa]^{<\kappa}$) when:

(a) $\bar{\lambda} = \langle \lambda_i : i < \kappa \rangle$ is a sequence of regular cardinals $> \kappa$,

(b) J is an ideal on κ,

(c) $\lambda = \operatorname{tcf}\Big(\prod\limits_{i<\kappa} \lambda_i, <_J \Big)$.

2) We say $\bar{f}$ witnesses a pcf-case $(\lambda, \bar{\lambda}, J, \kappa)$ or is a witness for it when $\bar{f}$ is $<_J$-increasing and $<_J$-cofinal in $\Big(\prod\limits_{i<\kappa} \lambda_i, <_J \Big)$.

3) We say $\bar{f}$ obeys $(\lambda, \bar{\lambda}, J, \bar{\mathscr{P}}, \kappa)$ when for some $\bar{g}$ the sequence $\bar{f}$ obeys $(\lambda, \bar{\lambda}, J, \kappa, \bar{\mathscr{P}})$ as witnessed by $\bar{g}$, see part (4) below and $\bar{f}$ witnesses the pcf-case $(\lambda, \bar{\lambda}, J, \kappa)$. Not mentioning $\bar{g}$ means for some $\bar{g}$.

4) We say that $\bar{f}$ obeys $(\lambda, \bar{\mu}, J, \kappa, \bar{\mathscr{P}})$ as witnessed by $\bar{g}$ when:

(a) $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$,

(b) J is an ideal on κ and $\bar{\mu} = \langle \mu_i : i < \kappa \rangle$,

(c) $f_\alpha \in {}^\kappa\mathrm{Ord}$,

(d) $\bar{f}$ is $<_J$-increasing,

(e) $\bar{\mathscr{P}} = \langle \mathscr{P}_\alpha : \alpha < \lambda \rangle$ is a $(\lambda, \lambda, \leqq 2^\lambda)$-system (normally a $(\lambda, \lambda, < \lambda)$-system) so without loss of generality $\subseteq$-increasing,

(f) $\bar{g} = \Big\langle g_a : a \in \bigcup\limits_\alpha \mathscr{P}_\alpha \Big\rangle$,

(g) $g_a \in {}^\kappa\mathrm{Ord}$,

(h) $g_a(i) < g_b(i)$ when $a \triangleleft b$ are from $\mathscr{P}_{<\lambda}$ and $|b| < \mu_i$ where $\mathscr{P}_{<\alpha} := \cup\{\mathscr{P}_\beta : \beta < \alpha\}$,

(i) if $a \in \mathscr{P}_\alpha$ then $g_a <_J f_\alpha$,

(j) if $\beta \in a \in \mathscr{P}_\alpha$, $i < \kappa$ and $|a| < \mu_i$ then $f_\beta(i) < g_a(i)$.

Convention 1.7. We may allow $\bar f = \langle f_\alpha : \alpha \in S\rangle$ where $S \subsetneqq \lambda = \sup(S)$, that is, say $\bar f$ obeys $(\lambda, \bar\mu, J, \kappa, \bar{\mathscr{P}})$ as witnessed by some $\bar g$ when $\langle f'_\alpha : \alpha < \lambda\rangle$ satisfies the demands there where $\alpha \in S \Rightarrow f'_{\mathrm{otp}(S\cap\alpha)} = f_\alpha$.

Claim 1.8. *Assume $(\lambda, \bar\lambda, J, \kappa)$ is a pcf-case, $\mu = \liminf_J(\bar\lambda)$ and $\bar{\mathscr{P}}$ is a $(\lambda, \mu, < \lambda)$-system.*

1) *There is $\bar f$ obeying $(\lambda, \bar\lambda, J, \kappa, \bar{\mathscr{P}})$.*

2) *For every $\bar f$ witnessing $(\lambda, \bar\lambda, J, \kappa)$, for some unbounded $S \subseteqq \lambda, \bar f \restriction S$ obeys $(\lambda, \bar\lambda, J, \kappa, \bar{\mathscr{P}})$.*

3) *If $\bar f$ obeys $(\lambda, \bar\lambda, J, \kappa, \bar{\mathscr{P}})$ and $\theta = \mathrm{cf}(\theta) < \liminf_J(\bar\lambda)$ then $S_{\mathrm{gd}}[\bar f] \supseteqq \mathrm{good}''_\theta(\bar{\mathscr{P}})$.*

Remark 1.9. The proof is like the ones in [4, Ch. I], [14].

Proof. 1) Follows by (2).

2) Let $\bar f = \langle f_\alpha : \alpha < \lambda\rangle$ witness the pcf-case $(\lambda, \bar\lambda, J, \kappa)$, trivially exists. By induction on $\beta < \lambda$ we choose $\langle g_a : a \in \mathscr{P}_\beta\rangle$ and $\alpha(\beta)$ such that

$\boxplus$ (a) $g_a \in \Pi\bar\lambda$,

(b) if $i < \kappa, b \triangleleft a$ and $\{a, b\} \subseteqq \mathscr{P}_{<\beta}$ and $|a| < \lambda_i$ then $g_b(i) < g_a(i)$,

(c) $\alpha(\beta) < \lambda$ and $\beta_1 < \beta \Rightarrow \alpha(\beta_1) < \alpha(\beta)$,

(d) if $i < \kappa, \beta_1 \in a \in \mathscr{P}_\beta$ and $|a| < \lambda_i$ then $f_{\alpha(\beta_1)}(i) < g_a(i)$,

(e) if $a \in \mathscr{P}_{\leqq\beta}$ then $g_a <_J f_{\alpha(\beta)}$.

In stage β we first choose g_a for $a \in \mathscr{P}_\beta\backslash\mathscr{P}_{<\beta}$, note that this means that for every $i < \kappa$, we have to choose $g_a(i)$ as an ordinal $< \lambda_i$, which is a regular cardinal and if $|a| < \lambda_i$ it should be bigger than $\leqq |a|$ ordinals $< \lambda_i$, so this is easy.

As for $\alpha(\beta)$ for each $a \in \mathscr{P}_{\leqq\beta}$, as $\bar f$ is cofinal in $(\Pi\bar\lambda, <_J)$ there is $\gamma_{\bar a} < \lambda$ such that $g_a <_J f_{\gamma_a}$. So $\alpha(\beta)$ should be an ordinal $< \lambda$ and $> \sup\big\{\alpha(\beta_1); \beta_1 < \beta\big\}$ which is an ordinal $< \lambda$, as λ is regular and it also should be $> \sup\{\gamma_a : a \in \mathscr{P}_{\leqq\beta}\}$ which is $< \lambda$ as λ is regular $> |\mathscr{P}_\alpha|$.

3) Straight. $\square$

Definition 1.10. Let J be an ideal on κ, we may omit it below when $J = J^{\mathrm{bd}}_\kappa$.

1) A set $\mathscr{F} \subseteqq {}^\kappa\mathrm{Ord}$ is J-free when there is a sequence $\langle a_f : f \in \mathscr{F}\rangle$ of members of J such that $f_1 \neq f_2 \wedge \{f_1, f_2\} \subseteqq \mathscr{F} \wedge i \in \kappa\backslash a_{f_1}\backslash a_{f_2} \Rightarrow f_1(i) \neq f_2(i)$.

2) A set $\mathscr{F} \subseteq {}^\kappa\mathrm{Ord}$ is (θ, J)-free when $\mathscr{F}'$ is J-free whenever $\mathscr{F}' \subseteq \mathscr{F}$ has cardinality $< \theta$.

3) A sequence $\langle f_\alpha : \alpha < \alpha_* \rangle$ of members of ${}^\kappa\mathrm{Ord}$ is a (θ, J)-free sequence when, for every $u \in [\alpha_*]^{<\theta}$ there is a sequence $\langle a_\alpha : \alpha \in u \rangle$ of members of J such that: if $\alpha < \beta$ are from u then $i \in \kappa \backslash a_\alpha \backslash a_\beta \Rightarrow f_\alpha(i) < f_\beta(i)$.

4) A set $\mathscr{F} \subseteq {}^\kappa\mathrm{Ord}$ (we may use a sequence listing it) is called $(\theta_2, \theta_1, J)^*$-free[1] when for every $\mathscr{F}' \subseteq \mathscr{F}$ of cardinality $< \theta_2$, we can find a partition $\langle \mathscr{F}'_\varepsilon : \varepsilon < \varepsilon(*) \rangle$ of $\mathscr{F}'$ such that:

- each $\mathscr{F}'_\varepsilon$ has cardinality $< \theta_1$,
- we can find a sequence $\langle s_f : f \in \mathscr{F}' \rangle$ of members of J such that $f_1 \in \mathscr{F}'_{\varepsilon_1} \wedge f_2 \in \mathscr{F}'_{\varepsilon_2} \wedge \varepsilon_1 \neq \varepsilon_2 \wedge i \in \kappa \backslash s_{f_1} \backslash s_{f_2} \Rightarrow f_1(i) \neq f_2(i)$.

4A) A set $\mathscr{F} \subseteq {}^\kappa\mathrm{Ord}$ is called $\langle \theta_2, \theta_1, J \rangle$-free when for every $\mathscr{F}' \subseteq \mathscr{F}$ of cardinality θ_2, there is a J-free $\mathscr{F}'' \subseteq \mathscr{F}'$ of cardinality θ_1.

4B) Similarly to 4), 4A) for a sequence $\langle f_\alpha : \alpha < \alpha_* \rangle$ of members of ${}^\kappa\mathrm{Ord}$ means that it is with no repetitions and $\{f_\alpha : \alpha < \alpha_*\}$ satisfies the requirement.

5) A set $\mathscr{F} \subseteq {}^\kappa\mathrm{Ord}$ is called $\langle \theta_2, \theta_1, J \rangle$-stable when for every $u \subseteq \mathrm{Ord}$ of cardinality $< \theta_1$ the set $\big\{ f \in \mathscr{F} : i$ the set $\{ i < \kappa : f(i) \in u \}$ is not in $J \big\}$ has cardinality $< \theta_2$.

5A) A set $\mathscr{F} \subseteq {}^\kappa\mathrm{Ord}$ is (θ, J)-stable when it is (θ, θ, J)-stable.

5B) A set $\mathscr{F} \subseteq {}^\kappa\mathrm{Ord}$ is (θ_2, θ_1, J)-stable when for every $\theta \in [\theta_2, \theta_1)$ is (θ, J)-stable.

Toward proving Theorem 0.4 we prove

CLAIM 1.11. *If* (A) *then* (B) *where*:

(A) (a) $(\lambda, \bar{\lambda}, J, \kappa)$ *is a pcf-case*,

(b) $M_* \prec \big(\mathscr{H}(\lambda^+), \theta, <^*_{\lambda^+} \big)$ *has cardinality* $< \lambda$, $M_* \cap \lambda \in \lambda$ *and* $(\lambda, \bar{\lambda}, J, \kappa) \in M_*$; (*clearly exists and by* 1.1 *and* 1.8 *there are* $\bar{\mathscr{P}}$, E, $\bar{f}$, *as required below*),

(c) $\bar{\mathscr{P}}$, E *are as in* 1.1 *for our* λ, M_*,

(c) $\bar{f}^1$ *obeys* $(\lambda, \bar{\lambda}, J, \kappa, \bar{\mathscr{P}})$,

(d) μ *is a limit uncountable cardinal*,

(e) $\mu = \liminf_J(\bar{\lambda})$, *i.e.* $\mu = \min \big\{ \chi$: *the set* $\{i < \kappa : \lambda_i < \chi\}$ *is not from* $J \big\}$,

(f) $\partial = \mathrm{cf}(\partial) < \kappa$, J *is* ∂^+*-complete*,

(g) $S \subseteq S^\lambda_\partial$ *is stationary such that* $\delta \in S \Rightarrow (\mu^2$ *divide* $\delta)$,

(h) $\bar{\alpha} = \langle \alpha_{\delta,i} : \delta \in S,\ i < \partial \rangle$ *where* $\bar{\alpha}_\delta = \langle \alpha_{\delta,i} : i < \partial \rangle$ *is increasing continuous with limit* δ *such that* $\alpha_{\delta,i}$ *is divisible by* μ,

(i) $\bar{f} = \bar{f}^2 = \big\langle f^2_\delta : \delta \in S \big\rangle$ *is where* $f^2_\delta : \partial \times \kappa \to \delta$ *is defined by* $f^2_\delta(i,j) = \alpha_{\delta,i} + f^1_\delta(j)$,

[1] In Definition [18, 1.2(1)], a variant is (θ_2, θ_1)-free is defined, when $\theta_1 = \mathrm{cf}(\theta_1) > \kappa = \big|\mathrm{Dom}(J)\big|$ the two versions are equivalent.

(j) $J_* = J^{\rm bd}_\partial \times J = \{u \subseteq \partial \times \kappa$: *for every* $i < \partial$ *large enough*, $\{j < \kappa : (i,j) \in u\} \in J\}$; *of course, we can translate* J_* *to an ideal on* κ, *that is* $\{v \subseteq \kappa : \{(i,j) \in \partial \times \kappa : \partial \cdot j + i \in v\} \in J_*\}$.

(B) (a) (α) *if* $\theta \in [\kappa, \mu)$ *then the sequence* $\bar f^2$ *is* $(\theta^{+\,\mathrm{comp}\,(J)+1}, \theta^{+4}, J_*)$*-free recalling* $\partial < \mathrm{comp}\,(J) \leqq \kappa$, *see* 1.13 *and* 0.8(5),

(β) $\bar f^2$ *is* $(\,\mathrm{comp}\,(J), J_*)$*-free*,

(γ) *if* $\theta \in [\kappa, \mu)$ *is a limit cardinal and* $\mathrm{cf}\,(\theta) \notin [\,\mathrm{comp}\,(J), \kappa^+)$ *then* $\bar f^2$ *is* $(\theta^{+\,\mathrm{comp}\,(J)+1}, \theta^+, J_*)$*-free*,

(b) *if* σ *is regular and* $\delta \in S^\lambda_\sigma$ *and* $\sigma < \mu$ *then, see Definition* 0.18:

(α) $\kappa^{+4} \leqq \sigma \Rightarrow \delta \notin S^{\rm ch}_J[\bar f]$,

(β) $\kappa^+ < \sigma < \kappa^{+\,\mathrm{comp}\,(J)+1} \Rightarrow \delta \notin S^{\rm bd}_J[\bar f]$,

(γ) $\kappa \leqq \theta \wedge \theta^{+4} \leqq \sigma < \theta^{+\,\mathrm{comp}\,(J)+1} \Rightarrow \delta \notin S^{\rm bd}_J[\bar f]$.

REMARK 1.12. This continues [9] and [11]; note that here $\partial < \kappa$. This helps; there are relatives with $\sigma \geqq \kappa$ but not needed at present.

PROOF. Note that

$\boxplus_0$ if $\theta = \mathrm{cf}\,(\theta) \in \mu \backslash \kappa^+$ then $S_{\rm gd}[\bar f] \cap S^\lambda_\theta \supseteq \mathrm{good}''_\theta[\bar{\mathscr{P}}]$.

[By 1.8(3).]

$\boxplus_1$ $\begin{cases} \text{if } \theta,\ \sigma \text{ are regular cardinals from } (\kappa,\mu) \text{ and } \theta^{+2} < \sigma \text{ then } S_{\rm gd}[\bar f] \\ \cap\, S^\lambda_\theta \text{ reflect in every } \delta \in S^\lambda_\sigma. \end{cases}$

[Let $\Upsilon = \theta^{+2}$, hence by 0.17(3) there is a $(\Upsilon, \theta, <\Upsilon)$-system such that $S^\Upsilon_\theta \notin I^{\rm cg}_\theta[\Upsilon]$, see Definition 0.13(1) hence by 1.1, that is the choice of $\bar{\mathscr{P}}$, the set $\mathrm{good}''_\theta(\bar{\mathscr{P}}) \subseteq S^\lambda_\theta$ reflect in every $\delta \in S^\lambda_\sigma$, and so by $\boxplus_0$ we are done.]

$\boxplus_2$ $\begin{cases} \text{if } \theta = \mathrm{cf}\,(\theta) \in [\kappa^{+4}, \lambda) \text{ then } \mathrm{refl}\,\mathrm{good}''_\theta[\bar{\mathscr{P}}] \text{ includes } S^\lambda_\theta \text{ hence} \\ S^{\rm ch}_\theta[\bar f, J] \text{ is non-stationary.} \end{cases}$

[As in the proof of $\boxplus_1$, only simpler.]

$\boxplus_3$ $\begin{cases} S^{\rm gd}_J[\bar f] \text{ include } \{\delta < \lambda : \theta^{+4} \leqq \mathrm{cf}\,(\delta) < \theta^{+\,\mathrm{comp}\,(J)+1}\} \text{ when} \\ \theta \in [\kappa, \mu). \end{cases}$

[By $\boxplus_1$, 0.19(2), 0.20.]

So we have proved (b) of (B); concerning (B)(b)(γ) recall that

• if $\delta \in S^{\rm ch}_I[\bar f]$ then for some club e of δ we have $\alpha \in e \wedge \mathrm{cf}\,(\alpha) > \kappa \Rightarrow \alpha \in S^{\rm ch}_J[\bar f]$, (similarly for $S^{\rm gd}_J[\bar f]$)

$\boxplus_4$ $\begin{cases} \bar f^2 \text{ is } (\kappa^{+\,\mathrm{comp}\,(J)+1}, \kappa^{+4}, J)\text{-free, see Definition 1.10(4), that is as} \\ \text{a set.} \end{cases}$

[By $\boxplus_6$ proved below using $\boxplus_3$.]

$\boxplus_5$ if $\theta \in [\kappa, \mu)$ then $\bar{f}^2$ is $\big(\theta^{+\,\mathrm{comp}\,(J)+1}, \theta^{+4}, J\big)$-free.

[By $\boxplus_6$ below using $\boxplus_3$.]

$$\boxplus_6 \quad \begin{cases} \text{if } \theta_2 > \theta_1 = \mathrm{cf}\,(\theta_1) > \kappa \text{ and } \delta < \lambda \wedge \theta_1 \leqq \mathrm{cf}\,(\delta) < \theta_2 \Rightarrow \delta \in S^{\mathrm{gd}}_J[\bar{f}] \\ \text{then } \bar{f}^2 \text{ is } (\theta_2, \theta_1, J_*)\text{-free.} \end{cases}$$

Toward this we consider for $\theta \in [\theta_1, \theta_2)$ the statement

$$\oplus_{\bar{f},\theta} \quad \begin{cases} \text{if } u \subseteqq S, \text{ recalling } S \subseteqq S^\lambda_\partial, \ |u| = \theta \text{ then we can find } \bar{s} = \\ \langle s_\alpha : \alpha \in u\rangle \in {}^u(J_*) \text{ such that in the graph } (u, R_{\bar{s}}) \text{ every node} \\ \text{has valency } < \theta_1 \text{ where for } u \subseteqq \lambda \text{ and } \bar{s} \in {}^uJ_* \text{ let } (u, R_{\bar{s}}) \text{ be the} \\ \text{following graph: } \alpha R_{\bar{s}} \beta \text{ iff } \alpha \neq \beta \in u \text{ and for some } (i,j) \in \partial \times \kappa, \\ \text{we have } (i,j) \notin s_\alpha \cup s_\beta \text{ and } f^2_\alpha(i,j) = f^2_\beta(i,j). \end{cases}$$

Why this suffice? As then let $\langle u_t : t \in I\rangle$ list the components of the graph $(u, R_{\bar{s}})$, so necessarily each component has cardinality $< \theta_1$, recalling θ_1 is regular, so $\langle \{f_\alpha : \alpha \in u_t\} : t \in I\rangle$ is a partition as required in Definition 1.10(4). We prove this by induction on $\mathrm{otp}\,(u)$.

Case 1: $\mathrm{otp}\,(u) < \theta_1$. Let $s_\alpha = \emptyset \in J_*$ for $\alpha \in u$, clearly as required.

Case 2: $\mathrm{otp}\,(u) = \zeta + 1$. Let $\alpha = \max\,(u)$, let $\bar{s}^1 \in {}^{u\cap\alpha}(J_*)$ be as promised for $u \cap \alpha$ and let $\bar{s}^2 = \big\langle s^2_\beta : \beta \in \alpha \big\rangle$ be defined by $s^2_\beta = \big\{ (i,j) \in \partial \times \kappa : f_\beta(i,j) = f_\alpha(i,j)\big\}$, so $s^2_\beta \in J_*$.

Lastly, define $\bar{s} \in {}^u(J_*)$ by: s_β is $s^1_\beta \cap s^2_\beta$ if $\beta < \alpha$ and is $\emptyset$ if $\beta = \alpha$, now check.

Case 3: $\delta = \mathrm{otp}\,(u)$ is a limit ordinal of cofinality $< \theta_1$. Let $\sigma := \mathrm{cf}\,(\delta)$ and $\langle \alpha_\varepsilon : \varepsilon < \sigma\rangle$ be increasing continuous with limit $\sup\,(u)$ such that $\alpha_0 = 0$. For $\varepsilon < \sigma$ let $u_\varepsilon = u \cap [\alpha_\varepsilon, \alpha_{\varepsilon+1})$ and let $\bar{s}_\varepsilon = \langle s_\alpha : \alpha \in u_\varepsilon\rangle$ be as required for u_ε, exists as $\mathrm{otp}\,(u_\varepsilon) < \mathrm{otp}\,(u)$. So $\bar{s} = \langle s_\alpha : \alpha \in u\rangle$ is well defined. Now for each $\beta \in u, (i_*, j_*) \in \partial \times \kappa$ and ε the set $w_{\beta,\varepsilon,i_*,j_*} = \big\{ \gamma \in u_\varepsilon : (i_*, j_*) \notin s_\gamma \text{ and } f^2_\gamma(i_*, j_*) = f^2_\beta(i_*, j_*)\big\}$ has cardinality $< \theta_1$ because $\gamma_1, \gamma_2 \in w_{\beta,\varepsilon,i_*,j_*} \Rightarrow \big((i_*, j_*) \in \partial \times \kappa \backslash (s_{\gamma_1} \cup s_{\gamma_2})\big) \wedge f^2_{\gamma_1}(i_*, j_*) = f^2_{\gamma_2}(i_*, j_*)$; hence $w_\beta := \cup\{w_{\beta,\varepsilon,i,j} : \varepsilon < \sigma \text{ and } i < \partial,\ j < \kappa\}$ has cardinality $< \theta_1$ and $\bar{s}$ is as required.

Case 4: $\delta = \mathrm{otp}\,(u)$ has cofinality $\geqq \theta_1$. We choose $\bar{\beta}, \bar{a}^1$ such that:

$(*)_{6.1}$ (a) $\bar{\beta} = \big\langle \beta_\varepsilon : \varepsilon < \mathrm{cf}\,(\delta)\big\rangle$ is increasing continuous,
(b) $\beta_0 = 0$,
(c) $\cup\big\{ \beta_\varepsilon : i < \mathrm{cf}\,(\delta)\big\} = \sup\,(u)$,
(d) $\bar{a}^1 = \big\langle a^1_\varepsilon : \varepsilon < \mathrm{cf}\,(\delta) \text{ non-limit}\big\rangle$,

(e) $a^1_\varepsilon \in J$,
(f) if $\varepsilon > 0$ then $\beta_\varepsilon = \sup(u \cap \beta_\varepsilon)$,
(g) if $\varepsilon < \zeta < \mathrm{cf}(\delta)$ are non-limit and $j \in \kappa \backslash a^1_\varepsilon \backslash a^1_\zeta$ then $f^1_{\beta_\varepsilon}(j) < f^1_{\beta_\zeta}(j)$,
(h) $\beta_\varepsilon \in S^\lambda_\partial$ iff $\mathrm{cf}(\varepsilon) = \partial$.

[Why such $\bar{\alpha}$, $\bar{a}$ exist? First, $\sup(u) \in S^{\mathrm{gd}}_J\big[\bar{f}^1\big]$ holds by an assumption of $\boxplus_6$ because $\theta_1 \leqq \mathrm{cf}\big(\sup(u)\big)$ by the case assumption and $\mathrm{cf}\big(\sup(u)\big) < \theta_2$ as $|u| < \theta_2$. Second, use Definition 0.18(1) recalling clause (d) of $(*)_{6.1}$.]

$(*)_{6.2}$ we can find $\bar{a}$ such that:
(a) $\bar{a} = \big\langle a_\varepsilon : \varepsilon < \mathrm{cf}(\delta)\big\rangle$,
(b) $a_\varepsilon = a^1_\varepsilon$ if ε is non-limit,
(c) $a_\varepsilon \in J$,
(d) if $\varepsilon < \zeta < \mathrm{cf}(\delta)$ and $\mathrm{cf}(\zeta) < \mathrm{comp}(J)$ or $\mathrm{cf}(\zeta) > \kappa$ then $j \in \kappa \backslash a_\varepsilon \backslash a_\zeta \Rightarrow f_{\beta_\varepsilon}(j) < f_{\beta_\zeta}(j)$.

[For non-limit $\varepsilon < \mathrm{cf}(\delta)$ let $a_\varepsilon = a^1_\varepsilon$. If $\varepsilon < \mathrm{cf}(\delta)$ and $\aleph_0 \leqq \mathrm{cf}(\varepsilon) < \mathrm{comp}(J)$ then let e_ε be an unbounded subset of ε of order type $\mathrm{cf}(\varepsilon)$ and let $a_\varepsilon = \kappa \backslash \big\{ i < \kappa : i \notin \cup\{a_{\beta_{\zeta+1}} : \zeta \in e_\varepsilon\}$ and $f^1_{\beta_\varepsilon}(i) < f^1_{\beta_{\varepsilon+1}}(i)$ and $\zeta \in e_\varepsilon \Rightarrow f^1_{\beta_{\zeta+1}}(i) < f^1_{\beta_\varepsilon}(i)\big\}$.

As J is $\mathrm{comp}(J)$-complete ideal on κ and $\bar{f}^1$ is $<_J$-increasing clearly $a_\varepsilon \in J$.

If $\varepsilon < \mathrm{cf}(\delta)$ and $\mathrm{cf}(\varepsilon) > \kappa$ then let $a_\varepsilon = \big\{ i < \kappa$: the set $\big\{ \zeta < \varepsilon : i \notin a_{\zeta+1}$ and $f_{\beta_{\zeta+1}}(i) < f_{\beta_\varepsilon}(i) \big\}$ is a bounded subset of $\varepsilon \big\}$.

Toward proving $a_\varepsilon \in J$, first we find $\xi(\varepsilon) < \varepsilon$ such that: if $i < \kappa$ and the set $\big\{ \zeta < \varepsilon : i \in \kappa \backslash a_{\zeta+1}$ and $f^1_{\beta_{\zeta+1}}(i) < f^1_{\beta_\varepsilon}(i) \big\}$ is bounded below ε then it is $\leqq \xi(\varepsilon)$; this is possible as $\mathrm{cf}(\varepsilon) > \kappa$.

So $\kappa \backslash a_\varepsilon \supseteqq \big\{ i < \kappa : f^1_{\beta_{\xi(\varepsilon)+1}} < f^1_{\beta_\varepsilon}(i)$ and $i \notin a_{\xi(\varepsilon)+1} \big\}$ and the latter set is $= \kappa \mod J$ because $\big(a_{\xi(\varepsilon)+1} \in J\big) \wedge (f_{\beta_{\xi(\varepsilon)+1}} <_J f^1_{\beta_\varepsilon})$; it follows that $a_\varepsilon \in J$.

In the remaining cases $\mathrm{cf}(\varepsilon) \in \big[\, \mathrm{comp}(J), \kappa \big]$ let $a_\varepsilon = \kappa \backslash \big\{ i < \kappa : f_{\beta_\varepsilon}(i) < f_{\beta_{\varepsilon+1}}(i)$ and $i \notin a_{\varepsilon+1} \big\}$. Actually only the a_ε for $\varepsilon \in S^{\mathrm{cf}(\delta)}_\partial$ are used later.

Let us check that $\big\langle a_\varepsilon : \varepsilon < \mathrm{cf}(\delta) \big\rangle$ is as required in $(*)_{6.2}$ so assume $\varepsilon < \zeta < \mathrm{cf}(\delta)$ and $i \in \kappa \backslash a_\varepsilon \backslash a_\zeta$. First, without loss of generality ε is a successor ordinal, otherwise we know that $f_{\beta_\varepsilon}(i) < f_{\beta_{\varepsilon+1}}(i)$ and $i \in a_{\varepsilon+1}$ by the choice of a_ε. Second, if ζ is non-limit then $i \in \kappa \backslash a^1_\varepsilon \backslash a^1_\zeta$ hence $f_{\beta_\varepsilon}(i) < f_{\beta_\zeta}(i)$. Third, if $\mathrm{cf}(\zeta) < \mathrm{comp}(J)$ then we can find $\xi \in e_\zeta$ which is $> \varepsilon$, so $i \notin a_{\beta_{\xi+1}}$ as $a_{\beta_{\xi+1}} \subseteqq a_{\beta_\varepsilon}$ hence $f_{\beta_\varepsilon}(i) < f_{\beta_{\xi+1}}(i)$ and by the choice of a_{α_ε} also $f_{\beta_{\xi+1}}(i) < f_{\beta_\zeta}(i)$, together $f_{\beta_\varepsilon}(i) < f_{\beta_\zeta}(i)$. Fourth, if $\mathrm{cf}(\zeta) > \kappa$, let $\xi \in e$ be such that $\varepsilon < \xi$ and $i \notin a_{\xi+1}$ and $f_{\beta_{\xi+1}}(i) < f^1_{\beta_\zeta}(i)$. As $i \notin a_{\beta_{\xi+1}}$ and $i \notin a_{\beta_\varepsilon}$ and $\varepsilon < \xi + 1$ by the "second" we have $f_{\beta_\varepsilon}(i) < f_{\beta_{\xi+1}}(i)$, so recalling the previous sentence $f_{\beta_\varepsilon}(i) < f_{\beta_\zeta}(i)$. So we have proved $(*)_{6.2}$.]

Now for each $\varepsilon < \mathrm{cf}(\delta)$ let $u_\varepsilon = u \cap [\beta_\varepsilon, \beta_{\varepsilon+1})$ hence $\mathrm{otp}(u_\varepsilon) < \mathrm{otp}(u) = \delta$ hence there is a sequence $\langle s^\varepsilon_\alpha : \alpha \in u_\varepsilon \rangle$ of members of J_* as required.

For each $\varepsilon < \mathrm{cf}(\delta)$ and $\beta \in u_\varepsilon \backslash \{\beta_\varepsilon\}$ hence $\beta \in S$, let $\mathbf{i}(\beta) < \partial$ be such that $\big\{\alpha_{\beta,i} : i \in \big[\mathbf{i}(\beta), \sigma\big)\big\} \cap \beta_\varepsilon = \emptyset$ and if $\varepsilon < \mathrm{cf}(\delta)$, $\beta = \beta_\varepsilon \in S$ so $\beta_\varepsilon \in S^\lambda_\partial$ let $\mathbf{i}(\alpha) = 0$.

Lastly, let us define $\bar{s} = \langle s_\beta : \beta \in u\rangle$:

$$(*)_{6.3} \quad \begin{cases} \text{if } \ \beta \in u_\varepsilon \ \text{ then } \ s_\beta := s^\varepsilon_\beta \cup \big\{(i,j) \in \partial \times \kappa : i \leqq \mathbf{i}(\beta)\big\} \cup \big\{(i,j) \\ \in \partial \times \kappa : j \in a_\varepsilon \cup a_{\varepsilon+1}\big\} \cup \big\{(i,j) \in \partial \times \kappa : \neg(f^1_{\beta_\varepsilon}(j) \leqq f^1_\beta(j) \\ < f^1_{\beta_{\varepsilon+1}}(j)\big\}. \end{cases}$$

Let $\beta \in u$ and let $w_\beta = \big\{\gamma \in u$: there is $(i,j) \in \partial \times \kappa \backslash s_\beta \backslash s_\gamma$ satisfying $f^2_\gamma(i,j) = f^2_\beta(i,j)\big\}$ and we have to prove that w_β has cardinality $< \theta_1$. Let $\varepsilon < \mathrm{cf}(\delta)$ be such that $\beta \in u_\varepsilon$ that is $\beta \in [\beta_\varepsilon, \beta_{\varepsilon+1})$, clearly ε exists and is unique. As $s_\beta \supseteqq s^\varepsilon_\beta$ clearly $w_\beta \cap [\beta_\varepsilon, \beta_{\varepsilon+1})$ have cardinality $< \theta_1$. Now if $\gamma \in u \cap \beta_\varepsilon \wedge \beta > \beta_\varepsilon$ then by the choice of s_β we have $s_\beta \supseteqq \mathbf{i}(\beta) \times \kappa$ and by the choice of $\mathbf{i}(\beta)$ we have $\gamma \notin w_\beta$ recalling $\{\alpha_{\gamma,j} : j < \partial\} \subseteqq \beta_\varepsilon$. If $\gamma \in u \cap \beta_\varepsilon \wedge \beta = \beta_\varepsilon$ then necessarily $\beta_\varepsilon \in S^\lambda_\partial$ so $\mathrm{cf}(\beta_\varepsilon) = \partial$ and let $\xi < \mathrm{cf}(\delta)$ be such that $\gamma \in [\beta_\xi, \beta_{\xi+1})$, now if $(i,j) \in \partial \times \kappa \backslash s_\beta \backslash s_\gamma$ then by $(*)_{6.2}$(d) we have $f^1_\gamma(i) < f^1_{\alpha_{\xi+1}}(i) < f^1_{\alpha_\varepsilon}(i)$ so $\gamma \notin w_\beta$. Together $w_\beta \cap \alpha_\varepsilon = \emptyset$.

Next, assume $\gamma \in u \backslash \beta_{\varepsilon+1}$ say $\gamma \in u_\xi$, $\xi > \varepsilon$; if $\mathrm{cf}(\xi) \neq \partial \vee \gamma > \beta_\xi$ we use $\mathbf{i}(\gamma) \times \kappa \subseteqq s_\gamma$ and if $\mathrm{cf}(\xi) = \partial \wedge \gamma = \beta_\xi$ we use the choices of a_ξ, a_ε; hence $w_\beta \backslash \beta_{\varepsilon+1} = \emptyset$.

Together w_β has cardinality $< \theta_1$ as required. So we are done proving Case 4, hence proving $\boxplus_6$.

$$\boxplus_7 \quad \begin{cases} \text{the sequence } \bar{f}^2 \text{ is } \big(\operatorname{comp}(J)^+, J_*\big)\text{-free; this is clause (a)}(\beta) \\ \text{of (B).} \end{cases}$$

[Let $u \subseteqq \lambda$ have cardinality $\leqq \operatorname{comp}(J)$, let $\big\langle \beta_\varepsilon : \varepsilon < |u|\big\rangle$ list u and $a_\varepsilon = \big\{i < \kappa$: for some $\zeta < \varepsilon$ we have $f^1_{\beta_\zeta}(i) = f^1_{\beta_\varepsilon}(i)\big\}$, so as J is $|u|^+$-complete by the assumption clearly $a_\varepsilon \in J$. Let $s_{\beta_\varepsilon} = \partial \times a_\varepsilon$ for $\varepsilon < |u|$, recalls that for each $\zeta < \varepsilon$, $\big\{i < \kappa : f^1_{\beta_\zeta}(i) = f^1_{\beta_\varepsilon}(i)\big\} \in J$ by clause (A)(c) of the assumption and so $\langle s_\beta : \beta \in u\rangle$ is as required.]

$\boxplus_8$ if $\theta \in [\kappa, \mu)$ then $\bar{f}^2$ is $\big(\theta^{+\operatorname{comp}(J)+1}, \theta^{+4}, J_*\big)$-free.

[By $\boxplus_6$ and (B)(b)(γ) which we have proved in $\boxplus_3$.]

$$\boxplus_9 \quad \begin{cases} \text{if } \theta \in [\kappa, \mu) \text{ is a limit cardinal and } \mathrm{cf}(\theta) \notin \big[\operatorname{comp}(J), \kappa^+\big) \text{ then } \bar{f}^2 \\ \text{is } \big[\theta^{+\operatorname{comp}(J)+1}, \theta, J_*\big)\text{-free. This is clause (B)(a)}(\gamma) \text{ of the desired} \\ \text{conclusion.} \end{cases}$$

[Clearly $\theta \neq \kappa$ hence recalling θ is a limit ordinal $\geqq \kappa$ we have $\theta \geqq \kappa^{+4}$. Again by $\boxplus_6$ it suffices to prove that if $\delta < \lambda$ and $\mathrm{cf}(\delta) \in \big[\theta, \theta^{+\mathrm{comp}(J)+1}\big)$ then $\delta \notin S^{\mathrm{ch}}_J[\bar f]$ and $\delta \notin S^{\mathrm{bd}}_J[f]$.

If $\mathrm{cf}(\delta) \geqq \theta^{+4}$ this holds by $\boxplus_3$, so we can assume $\mathrm{cf}(\delta) \in \{\theta^{+\ell} : \ell \leqq 3\}$. Now $\delta \notin S^{\mathrm{ch}}_J[\bar f]$ as otherwise there is a club e of δ such that $\alpha \in e \wedge \mathrm{cf}(\alpha) > \kappa \Rightarrow \alpha \in S^{\mathrm{ch}}_J[f]$, contradicting $\boxplus_3$ applied to κ^{+4}.

Also $\delta \notin S^{\mathrm{bd}}_J[\bar f]$ as otherwise $\mathrm{cf}(\delta) = \Big(\prod\limits_{i<\kappa} \sigma_i, <_J\Big)$ for some $\sigma_i = \mathrm{cf}(\sigma_i) \in \big(\kappa, \mathrm{cf}(\delta)\big)$ but by $\boxplus_5$ this contradicts the assumption of $\boxplus_9$, e.g. (B)(a)(γ).] $\square$

Proof of 0.4. The proof is by cases.

Case 1: λ is singular. Then there is a μ^+-free $\mathscr{F} \subseteqq {}^\kappa\mu$ of cardinality $2^\mu = \lambda$ by [18, 3.10(3)=1f.28(3)]; more fully by [5, Ch. II, 2.3, p. 53] for every $\chi \in (\mu, \lambda)$ there is a μ^+-free $\mathscr{F}_\chi \subseteqq {}^\kappa\mu$ of cardinality χ; by letting $\bar\chi = \big\langle \chi_\varepsilon : \varepsilon < \mathrm{cf}(\lambda)\big\rangle$ be increasing with limit λ, combining the $\mathscr{F}_{\chi_\varepsilon}$'s and $\mathscr{F}_{\mathrm{cf}(\lambda)}$ we are done. So clause (A) of 0.4 holds and we are done.

Case 2: λ is regular and $|\alpha|^{<\kappa} = \lambda$ for some $\alpha < \lambda$. Then by [18, 3.6 = 1f.21] there is a μ^+-free $\mathscr{F} \subseteqq {}^\kappa\mu$ of cardinality $2^\mu = \lambda$ so again clause (A) of 0.4 holds and we are done.

Case 3: λ is regular and $\alpha < \lambda \Rightarrow |\alpha|^{<\kappa} < \lambda$. Let $E = \{\delta < \lambda : \alpha < \lambda \Rightarrow |\alpha|^{<\kappa} < \delta$ and δ is divisible by $\mu \cdot \mu\}$, clearly a club of λ. Let $S \subseteqq E$ be any stationary subset of S^λ_σ. We choose $\langle \bar\alpha_\delta : \delta \in S\rangle$ such that $\bar\alpha_\delta = \langle \alpha_{\delta,i} : i < \sigma\rangle$ is increasing with limit δ such that each $\alpha_{\delta,i}$ is divisible by μ. By the case assumption we have $S \in \check I_\sigma[\lambda]$, hence without loss of generality $\alpha_{\delta_1,i_1} = \alpha_{\delta_2,i_2} \Rightarrow i_1 = i_2 \wedge (\forall i < i_1)(\alpha_{\delta_1,i} = \alpha_{\delta_2,i})$.

Now as $\mu \in \mathbf{C}_\kappa$, recalling [5, Ch. VIII] there is a sequence $\bar\lambda$ such that $\big(\lambda, \bar\lambda, J^{\mathrm{bd}}_\kappa, \kappa\big)$ is a pcf-case such that $\bar\lambda$ is an increasing sequence of regular cardinals with limit μ. We can choose χ, M_* as in the assumption of 1.1 for λ such that $\mathscr{H}(\mu) \in M_*$ and then choose E, $\bar{\mathscr{P}}$ as in the conclusion of 1.1.

Hence by 1.8(1) we can find $\bar f^1 = \langle f^1_\alpha : \alpha < \lambda\rangle$ obeying $\big(\lambda, \bar\lambda, J^{\mathrm{bd}}_\kappa, \kappa, \bar{\mathscr{P}}\big)$. Let $\mathrm{cd} : {}^{\kappa>}\mu \to \mu$ be one-to-one, we may assume that $(\forall i)\lambda_i > \kappa$ and $\nu \in \prod\limits_{j<\kappa} \lambda_j \wedge i < j < \kappa \Rightarrow \mathrm{cd}(\nu \restriction i) < \mathrm{cd}(\nu \restriction j)$. Define $f^*_\alpha : \kappa \to \mu$ by $f^*_\alpha(i) = \mathrm{cd}\big(f_\alpha \restriction (i+1)\big)$, so f^*_α is increasing.

Lastly, let $\alpha_{\delta,i,j} = \alpha_{\delta,i} + f^*_\delta(j)$ and we should prove that $\langle \alpha_{\delta,i,j} : \delta \in S, i < \sigma, j < \kappa\rangle$ is as required in Definition 0.6, so $\eta_\delta = \big\langle \alpha_{\delta,i,j} : (i,j) \in \sigma \times \kappa\big\rangle$. If we have used f^1_α instead of f^*_α we just have to omit clause (d) of 0.6.

Clauses (a), (c) of 0.6 holds by our choice of η_δ. Clause (b) of 0.6 holds by the choice of S noting that $S \in \check I_\sigma[\lambda]$ as $S \subseteqq E \cap S^\lambda_\sigma$ and the case assumption. Clause (d) of 0.6 holds by the choices of the $\bar\alpha_\delta$'s and of cd, f^*_α recalling

$f^1_\alpha \in {}^\kappa\mu$ and $\alpha_{\delta,i}$ is divisible by μ. Clause (e) holds by 1.11, that is (B)(a) there says $\bar{f} = \bar{f}^2$ is $(\theta^{+\kappa+1}, \theta, J_*)$-free when $\theta \in [\kappa, \mu)$. Also clause (f) of 0.6 that is "$\bar{f}$ is (κ^+, J_*)-free" holds by direct inspection or see clause (B)(a)(β) of 1.11 recalling $J^{\rm bd}_\kappa$ is κ-complete ideal on κ.

Lastly, clause (g)$'$ follows by clause (g) and clause (g) holds by [16]. □

Definition 1.13. Let J be an ideal on κ.

1) We say $\mathscr{F} \subseteq {}^\kappa\mathrm{Ord}$ is strongly semi-$\langle\theta_2, \theta_1, J\rangle$-stable when there are no $f_\varepsilon \in \mathscr{F}$ for $\varepsilon < \theta_2$ and $u \subseteq \mathrm{Ord}$ of cardinality $< \theta_1$ such that for $\varepsilon < \zeta < \theta_2$ the following set $A_{\varepsilon,\zeta} = A_{\kappa,\zeta}\big(u, \langle f_\varepsilon : \varepsilon \in u\rangle\big)$ is $\neq \emptyset \mod J$

$$A_{\varepsilon,\zeta} := \Big\{ i < \kappa : \min\big(u \cup \{\infty\} \backslash f_\varepsilon(i)\big) \neq \min\big(u \cup \{\infty\} \backslash f_\zeta(i)\big) \Big\}.$$

2) For $<_J$-increasing $\bar{f} = \langle f_\alpha : \alpha < \alpha_*\rangle$, $f_\alpha \in {}^\kappa\mathrm{Ord}$ we say $\bar{f}$ is a strongly-semi-$\langle\theta_2, \theta_1, J\rangle$-stable sequence when there are no $v \subseteq \alpha_*$ of cardinality θ_2 and $u \subseteq \mathrm{Ord}$ of cardinality $< \theta_1$ such that: if $\alpha < \beta$ are from v then the following set is $\neq \emptyset \mod J$

$$\Big\{ i < \kappa : \min\big(u \cup \{\infty\} \backslash f_\alpha(i)\big) \nleq \min\big(u \cup \{\infty\} \backslash f_\beta(i)\Big\}.$$

3) In parts (1), (2) above, if $\theta_1 = \theta_2$ we may write (θ, J) instead of (θ_1, θ_2).

4) In parts (1), (2) above writing (θ_2, θ_1, J) instead of $\langle\theta_2, \theta_1, J\rangle$ means: strongly-semi-(θ, J)-stable for every $\theta \in [\theta_1, \theta_2)$.

Claim 1.14. *Assume $\bar{f} = \langle f_\alpha : \alpha < \lambda\rangle$ witness the pcf-case $(\lambda, \bar{\lambda}, J, \kappa)$ and is strongly-semi-(θ_2, θ_1, J)-stable, see 1.13(2), (4) and $\theta_2 < \theta_1^{+\,\mathrm{com}\,(J)}$. Then $S_{\rm gd}[\bar{f}] \supseteq \big\{ \delta < \lambda : \mathrm{cf}\,(\delta) \in [\theta_1, \theta_2) \big\}$.*

Proof. Straightforward. □

Note also

Observation 1.15. *Let J be an ideal on κ.*

1) *If $f_\alpha \in {}^\kappa\mathrm{Ord}$ for $\alpha < \alpha_*$ and the sequence $\langle f_\alpha : \alpha < \alpha_*\rangle$ is (θ, J)-free then the set $\{f_\alpha : \alpha < \alpha_*\}$ is (θ, J)-free and is with no repetitions.*

2) *Similarly for (θ_2, θ_1, J)-free.*

2A) *Similarly for $\langle\theta_2, \theta_1, J\rangle$-free.*

3) *If $\theta_2' \geqq \theta_2 \geqq \theta_1 \geqq \theta_1'$ then,*

(a) *$\mathscr{F}$ is (θ_2, J)-free implies $\mathscr{F}$ is (θ_1, J)-free,*

(b) *similarly for $\bar{f}$,*

(c) *$\mathscr{F}$ is $\langle\theta_2, \theta_1, J\rangle$-stable implies $\mathscr{F}$ is $\langle\theta_2', \theta_1', J\rangle$-stable.*

4) *If $\mathscr{F} \subseteq {}^\kappa\mathrm{Ord}$ is (θ^+, J)-free then it is (θ, J)-stable.*

5) *If $\mathscr{F} \subseteq {}^\kappa\mathrm{Ord}$ is $(\theta_2^+, \theta_1, J)$-free then $\mathscr{F}$ is $\langle\theta_2, \theta_1, J\rangle$-free.*

6) *If $\mathscr{F} \subseteq {}^\kappa\mathrm{Ord}$ is $\langle\theta_2, \theta_1, J\rangle$-free then it is $(\theta_2^+, \theta_1, J)$-stable.*

Remark 1.16. We also have obvious monotonicity in $\mathscr{F}$ and $\bar{f}$ and other obvious implications.

Claim 1.17. 1) *Assume* $\mathscr{F} \subseteq {}^\kappa\mathrm{Ord}$ *is semi-*(θ, J)*-stable or just* J *is* θ_**-complete and* $\varepsilon \leqq \theta$*. Then* $\mathscr{F}$ *is strongly semi-*$(\theta^{+\varepsilon+1}, J)$*-stable.*

2) *Similarly without semi.*

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