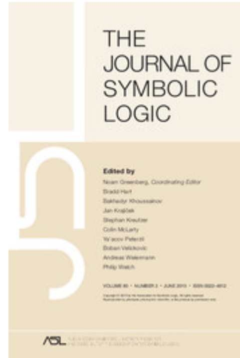
The Journal of Symbolic Logic

http://journals.cambridge.org/JSL



The cofinality spectrum of the infinite symmetric group

Saharon Shelah and Simon Thomas

The Journal of Symbolic Logic / Volume 62 / Issue 03 / September 1997, pp 902 - 916
DOI: 10.2307/2275578, Published online: 12 March 2014

THE JOURNAL OF SYMBOLIC LOGIC
Volume 62, Number 3, Sept. 1997

THE COFINALITY SPECTRUM OF THE INFINITE SYMMETRIC GROUP

SAHARON SHELAH AND SIMON THOMAS

Abstract. Let S be the group of all permutations of the set of natural numbers. The cofinality spectrum $CF(S)$ of S is the set of all regular cardinals λ such that S can be expressed as the union of a chain of λ proper subgroups. This paper investigates which sets C of regular uncountable cardinals can be the cofinality spectrum of S. The following theorem is the main result of this paper.

THEOREM. *Suppose that $V \models GCH$. Let C be a set of regular uncountable cardinals which satisfies the following conditions.*

(a) *C contains a maximum element.*

(b) *If μ is an inaccessible cardinal such that $\mu = \sup(C \cap \mu)$, then $\mu \in C$.*

(c) *If μ is a singular cardinal such that $\mu = \sup(C \cap \mu)$, then $\mu^+ \in C$.*

Then there exists a c.c.c. notion of forcing $\mathbb{P}$ such that $V^{\mathbb{P}} \models CF(S) = C$.

We shall also investigate the connections between the cofinality spectrum and *pcf* theory; and show that $CF(S)$ cannot be an arbitrarily prescribed set of regular uncountable cardinals.

§1. Introduction. Suppose that G is a group that is not finitely generated. Then G can be written as the union of a chain of proper subgroups. The *cofinality spectrum* of G, written $CF(S)$, is the set of regular cardinals λ such that G can be expressed as the union of a chain of λ proper subgroups. The *cofinality* of G, written $c(G)$, is the least element of $CF(G)$.

Throughout this paper, S will denote the group $\mathrm{Sym}(\omega)$ of all permutations of the set of natural numbers. In [5], Macpherson and Neumann proved that $c(S) > \aleph_0$. In [6] and [7], the possibilities for the value of $c(S)$ were studied. In particular, it was shown that it is consistent that $c(S)$ and $2^{\aleph_0}$ can be any two prescribed regular uncountable cardinals, subject only to the obvious requirement that $c(S) \leq 2^{\aleph_0}$. In this paper, we shall begin the study of the possibilities for the set $CF(S)$.

There is one obvious constraint on the set $CF(S)$, arising from the fact that S can be expressed as the union of a chain of $2^{\aleph_0}$ proper subgroups; namely, that $cf(2^{\aleph_0}) \in CF(S)$. Initially it is difficult to think of any other constraints on $CF(S)$. And we shall show that it is consistent that $CF(S)$ is quite a bizarre set of cardinals. For example, the following result is a special case of our main theorem.

THEOREM 1.1. *Let T be any subset of $\omega \setminus \{0\}$. Then it is consistent that $\aleph_n \in CF(S)$ if and only if $n \in T$.*

Received June 23, 1995; revised November 15, 1995.
Research partially supported by the BSF. Publication 524 of the first author.
Research partially supported by NSF Grants.

0022-4812/97/6203-0012/$2.50

After seeing this result, the reader might suspect that it is consistent that $CF(S)$ is an arbitrarily prescribed set of regular uncountable cardinals, subject only to the above mentioned constraint. However, this is not the case.

THEOREM 1.2. *If $\aleph_n \in CF(S)$ for all $n \in \omega \smallsetminus \{0\}$, then $\aleph_{\omega+1} \in CF(S)$.*

(Of course, this result is only interesting when $2^{\aleph_0} > \aleph_{\omega+1}$.) In Section 2, we shall use *pcf* theory to prove Theorem 1.2, together with some further results which restrict the possibilities for $CF(S)$. In Section 3, we shall prove the following result.

THEOREM 1.3. *Suppose that $V \models GCH$. Let C be a set of regular uncountable cardinals which satisfies the following conditions.*

(1.4)

(a) *C contains a maximum element.*

(b) *If μ is an inaccessible cardinal such that $\mu = \sup(C \cap \mu)$, then $\mu \in C$.*

(c) *If μ is a singular cardinal such that $\mu = \sup(C \cap \mu)$, then $\mu^+ \in C$.*

Then there exists a c.c.c. notion of forcing $\mathbb{P}$ such that $V^{\mathbb{P}} \models CF(S) = C$.

This is not the best possible result. In particular, clause (1.4)(c) can be improved so that we gain a little more control over what occurs at successors of singular cardinals. This matter will be discussed more fully at the end of Section 2. Also clause (1.4)(a) is not a necessary condition. For example, let $V \models GCH$ and let $C = \{\aleph_{\alpha+1} \mid \alpha < \omega_1\}$. At the end of Section 3, we shall show that if κ is any singular cardinal such that $cf(\kappa) \in C$, then there exists a *c.c.c.* notion of forcing $\mathbb{P}$ such that $V^{\mathbb{P}} \models CF(S) = C$ and $2^{\aleph_0} = \kappa$. In particular, $2^{\aleph_0}$ cannot be bounded in terms of the set $CF(S)$.

In this paper, we have made no attempt to control what occurs at inaccessible cardinals μ such that $\mu = \sup(C \cap \mu)$. We intend to deal with this matter in a second paper, which is in preparation. In this second paper, we also hope to give a complete characterisation of those sets C for which there exists a *c.c.c.* notion of forcing $\mathbb{P}$ such that $V^{\mathbb{P}} \models CF(S) = C$.

Our notation mainly follows that of Kunen [4]. Thus if $\mathbb{P}$ is a notion of forcing and $p, q \in \mathbb{P}$, then $q \leq p$ means that q is a strengthening of p. If V is the ground model, then we often denote the generic extension by $V^{\mathbb{P}}$ if we do not wish to specify a particular generic filter $G \subseteq \mathbb{P}$. If we want to emphasize that the term t is to be interpreted in the model M of ZFC, then we write t^M; for example, $\mathrm{Sym}(\omega)^M$. If $A \subseteq \omega$, then $S_{(A)}$ denotes the pointwise stabilizer of A. $\mathrm{Fin}(\omega)$ denotes the subgroup of elements $\pi \in S$ such that the set $\{n < \omega \mid \pi(n) \neq n\}$ is finite. If $\phi, \psi \in S$, then we define $\phi =^* \psi$ if and only if $\phi\psi^{-1} \in \mathrm{Fin}(\omega)$.

§2. Some applications of *pcf* theory. Let $\langle \lambda_i \mid i \in I \rangle$ be an indexed set of regular cardinals. Then $\prod_{i \in I} \lambda_i$ denotes the set of all functions f such that $\mathrm{dom}\, f = I$ and $f(i) \in \lambda_i$ for all $i \in I$. If $\mathscr{F}$ is a filter on I and $\mathscr{I}$ is the dual ideal, then we write either $\prod_{i \in I} \lambda_i / \mathscr{F}$ or $\prod_{i \in I} \lambda_i / \mathscr{I}$ for the corresponding reduced product. We shall usually prefer to work with functions $f \in \prod_{i \in I} \lambda_i$ rather than with the corresponding equivalence classes in $\prod_{i \in I} \lambda_i / \mathscr{I}$. For $f, g, \in \prod_{i \in I} \lambda_i$, we define

$$f \leq_{\mathscr{I}} g \text{ iff } \{i \in I \mid f(i) > g(i)\} \in \mathscr{I}$$

$$f <_{\mathscr{I}} g \text{ iff } \{i \in I \mid f(i) \geq g(i)\} \in \mathscr{I}.$$

We shall sometimes write $f \leq_{\mathscr{J}} g, f <_{\mathscr{J}} g$ instead of $f \leq_{\mathscr{I}} g, f <_{\mathscr{I}} g$ respectively. If $\mathscr{I} = \{\phi\}$, then we shall write $f \leq g, f < g$. Suppose that there exists a regular cardinal λ and a sequence $\langle f_\alpha \mid \alpha < \lambda\rangle$ of elements of $\prod_{i\in I} \lambda_i$ such that

(a) if $\alpha < \beta < \lambda$, then $f_\alpha <_{\mathscr{I}} f_\beta$; and
(b) for all $h \in \prod_{i\in I} \lambda_i$, there exists $\alpha < \lambda$ such that $h <_{\mathscr{I}} f_\alpha$.

Then we say that λ is the *true cofinality* of $\prod_{i\in I} \lambda_i/\mathscr{I}$, and write *tcf* $(\prod_{i\in I} \lambda_i/\mathscr{I}) = \lambda$. Furthermore, we say that $\langle f_\alpha \mid \alpha < \lambda\rangle$ *witnesses* that *tcf* $(\prod_{i\in I} \lambda_i/\mathscr{I}) = \lambda$. For example, if $\mathscr{D}$ is an ultrafilter on I, then $\prod_{i\in I} \lambda_i/\mathscr{D}$ is a linearly ordered set and hence has a true cofinality. A cardinal λ is a *possible cofinality* of $\prod_{i\in I} \lambda_i$ if there exists an ultrafilter $\mathscr{D}$ on I such that *tcf* $(\prod_{i\in I} \lambda_i/\mathscr{D}) = \lambda$. The set of all possible cofinalities of $\prod_{i\in I} \lambda_i$ is *pcf* $(\prod_{i\in I} \lambda_i)$.

In recent years, Shelah has developed a deep and beautiful theory of the structure of *pcf* $(\prod_{i\in I} \lambda_i)$ when $|I| < \min\{\lambda_i \mid i \in I\}$. A thorough development of *pcf* theory and an account of many of its applications can be found in [13]. [1] is a self-contained survey of the basic elements of *pcf* theory. In this section of the paper, we shall see that *pcf* theory imposes a number of constraints on the possible structure of $CF(S)$. (Whenever it is possible, we shall give references to both [13] and [1] for the results in *pcf* theory that we use.)

THEOREM 2.1. *Suppose that $\langle \lambda_n \mid n < \omega\rangle$ is a strictly increasing sequence of cardinals such that $\lambda_n \in CF(S)$ for all $n < \omega$. Let $\mathscr{D}$ be a nonprincipal ultrafilter on ω, and let tcf $(\prod_{n<\omega} \lambda_n/\mathscr{D}) = \lambda$. Then $\lambda \in CF(S)$.*

PROOF. For each $n < \omega$, express $S = \bigcup_{i<\lambda_n} G_i^n$ as the union of a chain of λ_n proper subgroups. Let $\langle f_\alpha \mid \alpha < \lambda\rangle$ be a sequence in $\prod_{n<\omega} \lambda_n$ which witnesses that *tcf* $(\prod_{n<\omega} \lambda_n/\mathscr{D}) = \lambda$. For each $\alpha < \lambda$, let H_α be the *set* of all $g \in S$ such that $\{n < \omega \mid g \in G^n_{f_\alpha(n)}\} \in \mathscr{D}$. Then it is easily checked that H_α is a subgroup of S, and that $H_\alpha \subseteq H_\beta$ for all $\alpha < \beta < \lambda$. Suppose that $g \in S$ is an arbitrary element. Define $f \in \prod_{n<\omega} \lambda_n$ by $f(n) = \min\{i \mid g \in G_i^n\}$. Then there exists $\alpha < \lambda$ such that $f <_{\mathscr{D}} f_\alpha$. Hence $g \in H_\alpha$. Thus $S = \bigcup_{\alpha<\lambda} H_\alpha$.

So it suffices to prove that H_α is a proper subgroup of S for each $\alpha < \lambda$. Fix some $\alpha < \lambda$. Lemma 2.4 [5] implies that for each $n < \omega$, $i < \lambda_n$ and $X \in [\omega]^\omega$, the setwise stabilizer of X in G_i^n does not induce $\mathrm{Sym}(X)$ on X. Express $\omega = \bigcup_{n<\omega} X_n$ as the disjoint union of countably many infinite subsets X_n. For each $n < \omega$, choose $\pi_n \in \mathrm{Sym}(X_n)$ such that $g \restriction X_n \neq \pi_n$ for all $g \in G^n_{f_\alpha(n)}$. Then $\pi = \bigcup_{n<\omega} \pi_n \in S \setminus H_\alpha$. ⊣

PROOF OF THEOREM 1.2. By [13, II 1.5] (or see [1, 2.1]), there exists an ultrafilter $\mathscr{D}$ on ω such that *tcf* $(\prod_{n<\omega} \aleph_n/\mathscr{D}) = \aleph_{\omega+1}$. ⊣

If we assume MA_κ, then we can obtain the analogous result for cardinals κ such that $\aleph_0 < \kappa < 2^{\aleph_0}$. (In Section 3, we shall prove that the following result cannot be proved in *ZFC*.)

THEOREM 2.2 (MA_κ). *Suppose that $\langle \lambda_\alpha \mid \alpha < \kappa\rangle$ is a strictly increasing sequence of cardinals such that $\lambda_\alpha \in CF(S)$ for all $\alpha < \kappa$. Let $\mathscr{D}$ be a nonprincipal ultrafilter on κ, and let tcf $(\prod_{\alpha<\kappa} \lambda_\alpha/\mathscr{D}) = \lambda$. Then $\lambda \in CF(S)$.*

PROOF. For each $\alpha < \kappa$, express $S = \bigcup_{i<\lambda_\alpha} G_i^\alpha$ as the union of a chain of λ_α proper subgroups. Let $\langle f_\beta \mid \beta < \lambda\rangle$ be a sequence in $\prod_{\alpha<\kappa} \lambda_\alpha$ which witnesses that

$tcf\left(\prod_{\alpha<\kappa}\lambda_\alpha/\mathscr{D}\right) = \lambda$. For each $\beta < \lambda$, let H_β be the set of all $g \in S$ such that $\{\alpha < \kappa \mid g \in G^\alpha_{f_\beta(\alpha)}\} \in \mathscr{D}$. Arguing as in the proof of Theorem 2.1, it is easily checked that $\langle H_\beta \mid \beta < \lambda\rangle$ is a chain of subgroups such that $S = \bigcup_{\beta<\lambda} H_\beta$.

Thus it suffices to prove that H_β is a proper subgroup of S for each $\beta < \lambda$. Fix some $\beta < \lambda$. Suppose that we can find an element $g \in S \setminus \bigcup_{\alpha<\kappa} G^\alpha_{f_\beta(\alpha)}$.

Then clearly $g \notin H_\beta$. But the existence of such an element g is an immediate consequence of the following theorem. ⊣

THEOREM 2.3 (MA_κ). *Suppose that for each $\alpha < \kappa$, $S = \bigcup_{i<\theta_\alpha} H^\alpha_i$ is the union of the chain of proper subgroups H^α_i. Then for each $f \in \prod_{\alpha<\kappa}\theta_\alpha$, $S \neq \bigcup_{\alpha<\kappa} H^\alpha_{f(\alpha)}$.*

REMARK 2.4. In [6], it was shown that MA_κ implies that $c(S) > \kappa$. This result is an easy consequence of Theorem 2.3.

REMARK 2.5. In [5], Macpherson and Neumann proved that if $\{H_n \mid n < \omega\}$ is an *arbitrary* set of proper subgroups of S, then $S \neq \bigcup_{n<\omega} H_n$. It is an open question whether MA_κ implies the analogous statement for cardinals κ such that $\aleph_0 < \kappa < 2^{\aleph_0}$. Regard S as a Polish space in the usual way. Then the proof of Theorem 2.3 shows that the following result holds.

THEOREM 2.6 (MA_κ). *Suppose that for each $\alpha < \kappa$, H_α is a nonmeagre proper subgroup of S. Then $S \neq \bigcup_{\alpha<\kappa} H_\alpha$.*

Unfortunately there exist maximal subgroups H of S such that H is meagre. For example, let $\omega = \Omega_1 \cup \Omega_2$ be a partition of ω into two infinite pieces. Let

$$H = \{g \in S \,\big|\, |g[\Omega_1]\triangle\Omega_i| < \aleph_0 \text{ for some } i \in \{1,2\}\}.$$

(Here $\triangle$ denotes the symmetric difference.) Then H is a maximal subgroup of S; and it is easily checked that H is meagre.

PROOF OF THEOREM 2.3 (MA_κ). We shall make use of the technique of generic sequences of elements of S, as developed in [3]. (The slight differences in notation between this paper and [3] arise from the fact that permutations act on the left in this paper.)

DEFINITION 2.7. A finite sequence $\langle g_1, \dots, g_n\rangle \in S^n$ is *generic* if the following two conditions hold.

(1) For all $A \in [\omega]^{<\omega}$, there exists $A \subseteq B \in [\omega]^{<\omega}$ such that $g_i[B] = B$ for all $1 \leq i \leq n$.
(2) Suppose that $A \in [\omega]^{<\omega}$ and that $g_i[A] = A$ for all $1 \leq i \leq n$. Suppose further that $A \subseteq B \in [\omega]^{<\omega}$ and that $h_i \in \mathrm{Sym}(B)$ extends $g_i \restriction A$ for all $1 \leq i \leq n$. Then there exists $\pi \in S_{(A)}$ such that $\pi g_i \pi^{-1}$ extends h_i for all $1 \leq i \leq n$.

CLAIM 2.8. *If $\langle g_1, \dots, g_n\rangle, \langle h_1, \dots, h_n\rangle \in S^n$ are generic, then there exists $f \in S$ such that $f g_i f^{-1} = h_i$ for all $1 \leq i \leq n$.*

PROOF OF CLAIM 2.8. This follows from [3, Proposition 2.3]. ⊣

From now on, regard S as a Polish space in the usual way.

CLAIM 2.9. *The set* $\{\langle g_1, \ldots, g_n \rangle \in S^n \mid \langle g_1, \ldots, g_n \rangle$ *is generic*$\}$ *is comeagre in* S^n *in the product topology.*

PROOF OF CLAIM 2.9. This follows from [3, Theorem 2.9]. $\dashv$

CLAIM 2.10. *If* $\langle g_1, \ldots, g_{n+1} \rangle \in S^{n+1}$ *is generic, then for each* $A \in [\omega]^{<\omega}, m \in \omega \setminus A$ *and* $1 \leq \ell \leq n+1$, *the following condition holds.*

$(2.11)_{A,m,\ell}$ *Let* $\Omega = \{i \mid 1 \leq i \leq n+1, i \neq \ell\}$. *If* $g_i[A] = A$ *for all* $i \in \Omega$, *then there exists* $B \in [\omega \setminus A]^{<\omega}$ *such that*

(a) $m \in B$;

(b) $g_i[B] = B$ *for all* $i \in \Omega$;

(c) $g_\ell[A \cup B] = A \cup B$;

(d) *for all* $\pi \in Sym(\Omega)$, *there exists* $\phi \in Sym(B)$ *such that* $\phi(g_i \upharpoonright B)\phi^{-1} = g_{\pi(i)} \upharpoonright B$ *for all* $i \in \Omega$.

PROOF OF CLAIM 2.10. For each $A \in [\omega]^{<\omega}, m \in \omega \setminus A$ and $1 \leq \ell \leq n+1$, let $C^{n+1}(A, m, \ell)$ consist of the sequences $\langle g_1, \ldots, g_{n+1} \rangle \in S^{n+1}$ which satisfy $(2.11)_{A,m,\ell}$. Then it is easily checked that $C^{n+1}(A, m, \ell)$ is open and dense in S^{n+1}. Hence $C^{n+1} = \bigcap_{A,m,\ell} C^{n+1}(A, m, \ell)$ is comeagre in S^{n+1}. Claim 2.9 implies that there exists a generic sequence $\langle g_1, \ldots, g_{n+1} \rangle \in C^{n+1}$. So the result follows easily from Claim 2.8. $\dashv$

DEFINITION 2.12. If σ is an infinite ordinal, then the sequence $\langle g_i \mid i < \sigma \rangle$ of elements of S is *generic* if for every finite subsequence $i_1 < \ldots < i_n < \sigma$, $\langle g_{i_1}, \ldots, g_{i_n} \rangle$ is generic.

We have now developed enough of the theory of generic sequences to allow us to begin the proof of Theorem 2.3. Consider the chains of proper subgroups, $S = \bigcup_{i<\theta_\alpha} H_i^\alpha$ for $\alpha < \kappa$. We can assume that $\mathrm{Fin}(\omega) \leq H_o^\alpha$ for all $\alpha < \kappa$. Let $f \in \prod_{\alpha<\kappa} \theta_\alpha$. We must find an element $\pi \in S \setminus \bigcup_{\alpha<\kappa} H_{f(\alpha)}^\alpha$. We shall begin by inductively constructing a generic sequence of elements of S

$$\langle g_o^o, g_o^1, \ldots, g_\alpha^o, g_\alpha^1, \ldots \rangle_{\alpha<\kappa}$$

such that for all $\alpha < \kappa$, there exist $f(\alpha) \leq \gamma_\alpha < \theta_\alpha$ such that $g_\alpha^o \in H_{\gamma_\alpha}^\alpha$ and $g_\alpha^1 \notin H_{\gamma_\alpha}^\alpha$. Then we shall find an element $\pi \in S$ such that $\pi g_\alpha^o \pi^{-1} =^* g_\alpha^1$ for all $\alpha < \kappa$. This implies that $\pi \notin \bigcup_{\alpha<\kappa} H_{\gamma_\alpha}^\alpha \supseteq \bigcup_{\alpha<\kappa} H_{f(\alpha)}^\alpha$.

Suppose that we have constructed g_β^o, g_β^1 for $\beta < \alpha$. For each finite subsequence $\bar{g}$ of $\langle g_\beta^o, g_\beta^1 \mid \beta < \alpha \rangle$, the set $\{h \in S \mid \bar{g}^\frown h$ is generic$\}$ is comeagre in S. (See [3, p. 216].) Since MA_κ implies that the union of κ meagre subsets of a Polish space is meagre, the set

$$\{h \in S \mid \langle g_\beta^o, g_\beta^1 \mid \beta < \alpha \rangle^\frown h \text{ is generic}\}$$

is also comeagre in S. So we can choose a suitable g_α^o and $f(\alpha) \leq \gamma_\alpha < \theta_\alpha$ with $g_\alpha^o \in H_{\gamma_\alpha}^\alpha$. The set

$$C = \{h \in S \mid \langle g_\beta^o, g_\beta^1 \mid \beta < \alpha \rangle^\frown g_\alpha^{o\frown} h \text{ is generic}\}$$

is also comeagre in S. Since $H_{\gamma_\alpha}^\alpha$ is a proper subgroup of S, we have that $C \setminus H_{\gamma_\alpha}^\alpha \neq \emptyset$. (If not, then $H_{\gamma_\alpha}^\alpha$ is comeagre and hence so are each of its cosets in S. As any two comeagre subsets of S intersect, this is impossible.) Hence we can choose a suitable $g_\alpha^1 \in C \setminus H_{\gamma_\alpha}^\alpha$. Thus the desired generic sequence can be constructed.

LEMMA 2.13. *Let $\langle g_\alpha^o, g_\alpha^1 \mid \alpha < \kappa \rangle$ be a generic sequence of elements of S. Then there exists a σ-centred notion of forcing $\mathbb{P}$ such that*

$$\Vdash_{\mathbb{P}} \text{There exists } \pi \in \mathrm{Sym}(\omega) \text{ such that } \pi g_\alpha^o \pi^{-1} =^* g_\alpha^1 \text{ for all } \alpha < \kappa.$$

PROOF OF LEMMA 2.13. Let $\mathbb{P}$ consist of the conditions $p = \langle h, F \rangle$ such that
(a) there exists $A \in [\omega]^{<\omega}$ such that $h \in \mathrm{Sym}(A)$;
(b) $F \in [\kappa]^{<\omega}$;
(c) for each $\alpha \in F$ and $\tau \in \{0, 1\}$, $g_\alpha^\tau[A] = A$.
We define $\langle h_2, F_2 \rangle \leq \langle h_1, F_1 \rangle$ iff the following two conditions hold.
(1) $h_1 \subseteq h_2$ and $F_1 \subseteq F_2$.
(2) Let $B = \mathrm{dom}\, h_2 \setminus \mathrm{dom}\, h_1$ and let $\phi = h_2 \restriction B$. Then $\phi(g_\alpha^o \restriction B)\phi^{-1} = g_\alpha^1 \restriction B$ for each $\alpha \in F_1$.

Clearly $\mathbb{P}$ is σ−centered. Claim 2.10 implies that each of the sets

$$D_m = \{\langle h, F \rangle \mid m \in \mathrm{dom}\, h\}, \quad m < \omega$$

and

$$E_\alpha = \{\langle h, F \rangle \mid \alpha \in F\}, \quad \alpha < \kappa,$$

are dense in $\mathbb{P}$. The result follows. ⊣

This completes the proof of Theorem 2.3. ⊣

The following theorem goes some way towards explaining why we have assumed that C satisfies condition (1.4)(c) in the statement of Theorem 1.3. (We will discuss this matter fully after we have proved Theorem 2.15.)

DEFINITION 2.14. If δ is a limit ordinal, then J_δ^{bd} is the ideal on δ defined by

$$J_\delta^{bd} = \{B \mid \text{There exists } i < \delta \text{ such that } B \subseteq i\}.$$

THEOREM 2.15. *Let κ be a regular cardinal, and suppose that $\langle \lambda_\alpha \mid \alpha < \kappa \rangle$ is a strictly increasing sequence of cardinals such that $\lambda_\alpha \in CF(S)$ for all $\alpha < \kappa$. Suppose further that $tcf\left(\prod_{\alpha<\kappa} \lambda_\alpha / J_\kappa^{bd}\right) = \lambda$. Then either $\kappa \in CF(S)$ or $\lambda \in CF(S)$.*

PROOF. Suppose that $\kappa \notin CF(S)$. For each $\alpha < \kappa$, express $S = \bigcup_{i<\lambda_\alpha} G_i^\alpha$ as the union of a chain of λ_α proper subgroups. Let $\langle f_\beta \mid \beta < \lambda \rangle$ be a sequence in $\prod_{\alpha<\kappa} \lambda_\alpha$ which witnesses that $tcf\left(\prod_{\alpha<\kappa} \lambda_\alpha / J_\kappa^{bd}\right) = \lambda$. For each $\beta < \lambda$, let G_β^* be the set of all $g \in S$ such that $\kappa \setminus \{\alpha < \kappa \mid g \in G_{f_\beta(\alpha)}^\alpha\} \in J_\kappa^{bd}$. Arguing as before, it is easily checked that $\langle G_\beta^* \mid \beta < \lambda \rangle$ is a chain of subgroups such that $S = \bigcup_{\beta<\lambda} G_\beta^*$.

Thus it suffices to prove that G_β^* is a proper subgroup of S for each $\beta < \lambda$. So suppose that $G_\beta^* = S$ for some $\beta < \lambda$. For each $\alpha < \kappa$, define $H_\alpha = \bigcap\{G_{f_\beta(\gamma)}^\gamma \mid \alpha \leq \gamma < \kappa\}$. Then $\langle H_\alpha \mid \alpha < \kappa \rangle$ is a chain of subgroups such that $S = \bigcup_{\alpha<\kappa} H_\alpha$. If $\alpha < \kappa$, then $H_\alpha \leq G_{f_\beta(\alpha)}^\alpha$ and so H_α is a proper subgroup of S. But this contradicts the assumption that $\kappa \notin CF(S)$. ⊣

Suppose that $V \models GCH$, and that μ is a singular cardinal. Let $\langle \theta_i \mid i < \eta \rangle$ be the strictly increasing enumeration of all regular uncountable cardinals θ such that $\theta < \mu$. Let $\mathscr{F} = \prod_{i<\eta} \theta_i$. Then $|\mathscr{F}| = \mu^+$. Now let $\mathbb{P}$ be any c.c.c. notion of forcing. From now on, we shall work in $V^{\mathbb{P}}$. Since $\mathbb{P}$ is c.c.c., for each $g \in \prod_{i<\eta} \theta_i$, there exists $f \in \mathscr{F}$ such that $g \leq f$. Suppose now that $\langle \lambda_\alpha \mid \alpha < \delta \rangle$ is an increasing subsequence of $\langle \theta_i \mid i < \eta \rangle$ such that $|\delta| < \lambda_o$ and $\sup_{\alpha<\delta} \lambda_\alpha = \mu$. Let

$$\mathscr{F}^* = \{f \in \prod_{\alpha<\delta} \lambda_\alpha \,|\, \text{There exists } h \in \mathscr{F} \text{ such that } f \subseteq h\}.$$

Then for all $g \in \prod_{\alpha<\delta} \lambda_\alpha$, there exists $f \in \mathscr{F}^*$ such that $g \leq f$. This implies that $\max(pcf(\prod_{\alpha<\delta} \lambda_\alpha)) = \mu^+$. By [13, I] (or see [1, 4.3]), we obtain that $tcf(\prod_{\alpha<\delta} \lambda_\alpha / J^{bd}_\delta) = \mu^+$. In summary, we have shown that the following statement is true in $V^{\mathbb{P}}$.

THE STRONG HYPOTHESIS (2.16). *Let δ be a limit ordinal, and let $\langle \lambda_\alpha \mid \alpha < \delta \rangle$ be a strictly increasing sequence of regular cardinals such that $|\delta| < \lambda_o$. Then* $tcf\left(\prod_{\alpha<\delta} \lambda_\alpha / J^{bd}_\delta\right) = (\sup_{\alpha<\delta} \lambda_\alpha)^+$.

In particular, using Theorem 2.15 and the Strong Hypothesis, we see that the following statement is true in $V^{\mathbb{P}}$.

$(*)$ If μ is a singular cardinal such that $\mu = \sup(CF(S) \cap \mu)$, then either $cf(\mu) \in CF(S)$ or $\mu^+ \in CF(S)$.

This suggests that we might try to replace condition (1.4)(c) of Theorem 1.3 by the following condition.

$(1.4)(c)'$ If μ is a singular cardinal such that $\mu = \sup(C \cap \mu)$, then either $cf(\mu) \in C$ or $\mu^+ \in C$.

However, Theorem 2.19 shows that this cannot be done. For example, Theorem 2.19 implies that if

$$C = \{\aleph_1\} \cup \{\aleph_{\delta+1} | \delta < \omega_2, cf(\delta) = \omega\} \cup \{\aleph_{\omega_2+1}\},$$

then there does not exist a c.c.c. notion of forcing $\mathbb{P}$ such that $V^{\mathbb{P}} \models CF(S) = C$.

REMARK 2.17. The Strong Hypothesis is usually taken to be the following apparently weaker statement.

(2.18) For all singular cardinals μ, $pp(\mu) = \mu^+$.

(For the definition of $pp(\mu)$, see [11].) However, Shelah [12, 6.3 (1)] has shown that (2.16) and (2.18) are equivalent.

THEOREM 2.19 (The Strong Hypothesis). *Let κ be a regular uncountable cardinal, and suppose that $\langle \lambda_\alpha \mid \alpha < \kappa \rangle$ is a strictly increasing sequence of cardinals such that $\lambda_\alpha \in CF(S)$ for all $\alpha < \kappa$. Suppose further that*

(a) $\kappa < \lambda_o$;

(b) $E = \{\delta < \kappa \,|\, \lim \delta, (\sup_{\alpha<\delta} \lambda_\alpha)^+ \notin CF(S)\}$ *is a stationary subset of κ.*

Then $\kappa \in CF(S)$.

PROOF. For each $\alpha < \kappa$, express $S = \bigcup_{i<\lambda_\alpha} G_i^\alpha$ as the union of a chain of λ_α proper subgroups. For each $\delta \in E$, let $\mu_\delta = \sup_{\alpha<\delta} \lambda_\alpha$. By the Strong Hypothesis,

$tcf\left(\prod_{\alpha<\delta}\lambda_\alpha/_{J^{bd}_\delta}\right) = \mu^+_\delta$. Let $\langle f^\delta_\xi \mid \xi < \mu^+_\delta\rangle$ be a sequence in $\prod_{\alpha<\delta}\lambda_\alpha$ which witnesses that $tcf\left(\prod_{\alpha<\delta}\lambda_\alpha/_{J^{bd}_\delta}\right) = \mu^+_\delta$. For each $\xi < \mu^+_\delta$, let H^δ_ξ be the set of all $g \in S$ such that $\delta \setminus \{\alpha < \delta \mid g \in G^\alpha_{f^\delta_\xi(\alpha)}\} \in J^{bd}_\delta$. Once again, it is easily checked that $\langle H^\delta_\xi \mid \xi < \mu^+_\delta\rangle$ is a chain of subgroups such that $S = \bigcup_{\xi<\mu^+_\delta} H^\delta_\xi$. Since $\mu^+_\delta \notin CF(S)$, there exists $\pi(\delta) < \mu^+_\delta$ such that $H^\delta_{\pi(\delta)} = S$.

Since $\kappa < \lambda_o$, there exists $f \in \prod_{\alpha<\kappa}\lambda_\alpha$ such that $f(\alpha) > \sup\{f^\delta_{\pi(\delta)}(\alpha) \mid \alpha < \delta \in E\}$ for all $\alpha < \kappa$. Let $g \in S$. Then for each $\delta \in E, g \in H^\delta_{\pi(\delta)}$; and so there exists $\gamma(g,\delta) < \delta$ such that $g \in G^\alpha_{f^\delta_{\pi(\delta)}(\alpha)} \subseteq G^\alpha_{f(\alpha)}$ for all $\gamma(g,\delta) \leq \alpha < \delta$. By Fodor's Theorem, there exists an ordinal $\gamma(g) < \kappa$ and a stationary subset D of E such that $\gamma(g,\delta) = \gamma(g)$ for all $\delta \in D$. This means that $g \in \bigcap\{G^\alpha_{f(\alpha)} \mid \gamma(g) \leq \alpha < \kappa\}$.

For each $\gamma < \kappa$, let $\Gamma_\gamma = \bigcap\{G^\alpha_{f(\alpha)} \mid \gamma \leq \alpha < \kappa\}$. Then $\langle \Gamma_\gamma \mid \gamma < \kappa\rangle$ is a chain of subgroups such that $S = \bigcup_{\gamma<\kappa}\Gamma_\gamma$. Finally note that $\Gamma_\gamma \subseteq G^\gamma_{f(\gamma)}$, and so Γ_γ is a proper subgroup of S for all $\gamma < \kappa$. Thus $\kappa \in CF(S)$. ⊣

§3. The main theorem. In this section, we shall prove Theorem 1.3. Our notation generally follows that of Kunen [4]. We shall only be using finite support iterations. An iteration of length α will be written as $\langle \mathbb{P}_\beta, \tilde{\mathbb{Q}}_\gamma \mid \beta \leq \alpha, \gamma < \alpha\rangle$, where $\mathbb{P}_\beta$ is the result of the first β stages of the iteration, and for each $\beta < \alpha$ there is some $\mathbb{P}_\beta$-name $\tilde{\mathbb{Q}}_\beta$ such that

$$\Vdash_{\mathbb{P}_\beta} \tilde{\mathbb{Q}}_\beta \text{ is a partial ordering}$$

and $\mathbb{P}_{\beta+1}$ is isomorphic to $\mathbb{P}_\beta * \tilde{\mathbb{Q}}_\beta$. If $p \in \mathbb{P}_\alpha$, then $\mathrm{supt}(p)$ denotes the support of p.

There is one important difference between our notation and that of Kunen. Unlike Kunen, we shall *not* use $V^{\mathbb{P}}$ to denote the class of $\mathbb{P}$-names for a notion of forcing $\mathbb{P}$. Instead we are using $V^{\mathbb{P}}$ to denote the generic extension, when we do not wish to specify a particular generic filter $G \subseteq \mathbb{P}$. Normally it would be harmless to use $V^{\mathbb{P}}$ in both of the above senses, but there is a point in this section where this notational ambiguity could be genuinely confusing. Suppose that $\mathbb{Q}$ is an arbitrary suborder of $\mathbb{P}$. Then the class of $\mathbb{Q}$-names is always a subclass of the class of $\mathbb{P}$-names. (Of course, a $\mathbb{Q}$-name τ might have very different properties when regarded as a $\mathbb{P}$-name. For example, it is possible that $\Vdash_{\mathbb{Q}} \tau$ is a function, whilst $\nVdash_{\mathbb{P}} \tau$ is a function.) However, we will not always have that $V^{\mathbb{Q}} \subseteq V^{\mathbb{P}}$; where this means that $V[G \cap \mathbb{Q}] \subseteq V[G]$ for some unspecified generic filter $G \subseteq \mathbb{P}$.

Definition 3.1. Let $\mathbb{Q}$ be a suborder of $\mathbb{P}$. $\mathbb{Q}$ is a *complete* suborder of $\mathbb{P}$, written $\mathbb{Q} \lessdot \mathbb{P}$, if the following two conditions hold.

1. If $q_1, q_2 \in \mathbb{Q}$ and there exists $p \in \mathbb{P}$ such that $p \leq q_1, q_2$, then there exists $r \in \mathbb{Q}$ such that $r \leq q_1, q_2$.
2. For all $p \in \mathbb{P}$, there exists $q \in \mathbb{Q}$ such that whenever $q' \in \mathbb{Q}$ satisfies $q' \leq q$, then q' and p are compatible in $\mathbb{P}$. (We say that q is a *reduction* of p to $\mathbb{Q}$.)

It is wellknown that if $\mathbb{Q} \lessdot \mathbb{P}$, then $V^{\mathbb{Q}} \subseteq V^{\mathbb{P}}$; and we shall only write $V^{\mathbb{Q}} \subseteq V^{\mathbb{P}}$ when $\mathbb{Q} \lessdot \mathbb{P}$.

We are now ready to explain the idea behind the proof of Theorem 1.3. Let $V \models GCH$, and let C be a set of regular uncountable cardinals which contains a maximum element κ. We seek a c.c.c. $\mathbb{P}$ such that $V^{\mathbb{P}} \models 2^{\omega} = \kappa \wedge CF(S) = C$. The easiest part of our task is to ensure that $V^{\mathbb{P}} \models C \subseteq CF(S)$. We shall accomplish this by constructing $\mathbb{P}$ so that the following property holds for each $\lambda \in C$.

$(3.2)_\lambda$ There exists a sequence $\langle \mathbb{P}^{\lambda}_{\xi} \mid \xi < \lambda \rangle \in V$ of suborders of $\mathbb{P}$ such that

(a) if $\xi < \eta < \lambda$, then $\mathbb{P}^{\lambda}_{\xi} \lessdot \mathbb{P}^{\lambda}_{\eta} \lessdot \mathbb{P}$;

(b) for each $\pi \in \mathrm{Sym}(\omega)^{V^{\mathbb{P}}}$, there exists $\xi < \lambda$ such that $\pi \in \mathrm{Sym}(\omega)^{V^{\mathbb{P}^{\lambda}_{\xi}}}$;

(c) for each $\xi < \lambda$, there exists $\pi \in \mathrm{Sym}(\omega)^{V^{\mathbb{P}}} \smallsetminus \mathrm{Sym}(\omega)^{V^{\mathbb{P}^{\lambda}_{\xi}}}$.

The harder part is to ensure that $V^{\mathbb{P}} \models CF(S) \subseteq C$. This includes the requirement that $(3.2)_\lambda$ fails for every $\lambda \notin C$. So, roughly speaking, we are seeking a c.c.c. $\mathbb{P}$ which can be regarded as a "kind of iteration" of length λ precisely when $\lambda \in C$. We shall use the technique of [10, Section 3] to construct such a notion of forcing $\mathbb{P}$.

Definition 3.3. Let $\langle a_i \mid i < \alpha \rangle$ be a sequence of subsets of α. We say that $b \subseteq \alpha$ is *closed* for $\langle a_i \mid i < \alpha \rangle$ if $a_i \subseteq b$ for all $i \in b$.

Definition 3.4. Let $\mathscr{C}$ be the class of all sequences

$$\bar{Q} = \langle \mathbb{P}_i, \tilde{\mathbb{Q}}_j, a_j \mid i \leq \alpha, j < \alpha \rangle$$

for some α which satisfy the following conditions. (We say that $\bar{Q}$ has length α and write $\alpha = \lg(\bar{Q})$.)

(a) $a_i \subseteq i$.

(b) a_i is closed for $\langle a_j \mid j < i \rangle$.

(c) $\mathbb{P}_i$ is a notion of forcing and $\tilde{\mathbb{Q}}_j$ is a $\mathbb{P}_j$-name such that $\Vdash_{\mathbb{P}_j} \tilde{\mathbb{Q}}_j$ is a c.c.c. partial order.

(d) $\langle \mathbb{P}_i, \tilde{\mathbb{Q}}_j \mid i \leq \alpha, j < \alpha \rangle$ is a finite support iteration.

(e) For each $j < \alpha$, define the suborder $\mathbb{P}^*_{a_j}$ of $\mathbb{P}_j$ inductively by

$$\mathbb{P}^*_{a_j} = \{p \in \mathbb{P}_j \mid \mathrm{supt}(p) \subseteq a_j \text{ and } p(k) \text{ is a } \mathbb{P}^*_{a_k} - \text{name for all } k \in \mathrm{supt}(p)\}.$$

Then $\tilde{\mathbb{Q}}_j$ is a $\mathbb{P}^*_{a_j}$-name. (At this stage, we do not know whether $\mathbb{P}^*_{a_j}$ is a complete suborder of $\mathbb{P}_j$. It is for this reason that we are being careful with our notation. However, we shall soon see that $\mathbb{P}^*_{a_j} \lessdot \mathbb{P}_j$, and then we can relax again.)

Definition 3.5. Let $\bar{Q} \in \mathscr{C}$ be as above, so that $\alpha = \lg(\bar{Q})$.

(a) We say that $b \subseteq \alpha$ is closed for $\bar{Q}$ if b is closed for $\langle a_j \mid j < \alpha \rangle$.

(b) If $b \subseteq \alpha$ is closed for $\bar{Q}$, then we define $\mathbb{P}^*_b = \{p \in \mathbb{P}_\alpha \mid \mathrm{supt}(p) \subseteq b$ and $p(k)$ is a $\mathbb{P}^*_{a_k}$-name for all $k \in \mathrm{supt}(p)\}$.

If $\beta < \alpha$, then we identify $\mathbb{P}_\beta$ with the corresponding complete suborder of $\mathbb{P}_\alpha$ in the usual way. If $b \subseteq \alpha$, then $p \restriction b$ denotes the α-sequence defined by

$$\begin{aligned}(p \restriction b)(\xi) &= p(\xi) \text{ if } \xi \in b \\ &= \mathbf{1}_{\tilde{\mathbb{Q}}_\xi} \text{ otherwise.}\end{aligned}$$

LEMMA 3.6. *Let $\bar{Q} \in \mathscr{C}$ and let $\alpha = lg(\bar{Q})$. Suppose that $b \subseteq c \subseteq \beta \leq \alpha$, and that b and c are closed for $\bar{Q}$.*

(1) *β is closed for $\bar{Q}$, and $\mathbb{P}_\beta = \mathbb{P}^*_\beta$.*

(2) *If $p \in \mathbb{P}_\beta$ and $i \in supt(p)$, then $p \restriction a_i \Vdash p(i) \in \tilde{\mathbb{Q}}_i$.*

(3) *Suppose that $p, q \in \mathbb{P}_\beta$ and $p \leq q$. If $i \in supt(q)$, then $p \restriction a_i \Vdash p(i) \leq q(i)$.*

(4) *If $p \in \mathbb{P}^*_c$, then $p \restriction b \in \mathbb{P}^*_b$.*

(5) *Suppose that $p \in \mathbb{P}^*_c, q \in \mathbb{P}^*_b$ and $p \leq q$. Then $p \restriction b \leq q$.*

(6) *Suppose that $p \in \mathbb{P}^*_c, q \in \mathbb{P}_\beta$ and $p \leq q \restriction c$. Define the α-sequence r by*

$$\begin{aligned} r(\xi) &= p(\xi) \text{ if } \xi \in c \\ &= q(\xi) \text{ otherwise.} \end{aligned}$$

Then $r \in \mathbb{P}_\beta$ and $r \leq p, q$.

(7) *$\mathbb{P}^*_c \lessdot \mathbb{P}_\beta$.*

PROOF. This is left as a straightforward exercise for the reader. ⊣

LEMMA 3.7. *Let $\bar{Q} \in \mathscr{C}$ and let $\alpha = lg(\bar{Q})$. Suppose that $b \subset \alpha$ is closed for $\bar{Q}$ and that $i \in \alpha \setminus b$.*

(1) *$c = b \cup i$ and $c \cup \{i\}$ are closed for $\bar{Q}$.*

(2) *$\mathbb{P}^*_b \lessdot \mathbb{P}^*_c \lessdot \mathbb{P}^*_{c\cup\{i\}} \lessdot \mathbb{P}_\alpha$.*

(3) *$\mathbb{P}^*_{c\cup\{i\}}$ is isomorphic to $\mathbb{P}^*_c * \tilde{\mathbb{Q}}_i$.*

PROOF. Once again left to the reader. ⊣

Now we are ready to begin the proof of Theorem 1.3. Suppose that $V \models GCH$, and let C be a set of regular uncountable cardinals which satisfies the following conditions.

(1.4)

(a) C contains a maximum element, say κ.

(b) If μ is an inaccessible cardinal such that $\mu = \sup(C \cap \mu)$, then $\mu \in C$.

(c) If μ is a singular cardinal such that $\mu = \sup(C \cap \mu)$, then $\mu^+ \in C$.

DEFINITION 3.8. (a) ΠC denotes the set of all functions f such that dom $f = C$ and $f(\lambda) \in \lambda$ for all $\lambda \in C$.

(b) $\mathscr{F}_C$ is the set of all functions $f \in \Pi C$ which satisfy the following condition.

($*$) If μ is an inaccessible cardinal such that $\mu = \sup(C \cap \mu)$, then there exists $\lambda < \mu$ such that $f(\theta) = 0$ for all $\lambda \leq \theta \in C \cap \mu$.

DEFINITION 3.9. In V, we define a sequence

$$\langle \mathbb{P}_i, \tilde{\mathbb{Q}}_j, f_j \mid i \leq \kappa, j < \kappa \rangle$$

such that the following conditions are satisfied.

(a) $f_i \in \mathscr{F}_C$.

(b) Let $a_i = \{j < i \mid f_j \leq f_i\}$. Then $\bar{Q} = \langle \mathbb{P}_i, \tilde{\mathbb{Q}}_j, a_j \mid i \leq \kappa, j < \kappa \rangle \in \mathscr{C}$.

(c) For each $f \in \mathscr{F}_C$, there exists a cofinal set of ordinals $j < \kappa$ such that $f_j = f$.

(d) Suppose that $i < \kappa$ and that $\tilde{\mathbb{Q}}$ is a $\mathbb{P}^*_{a_i}$-name with $|\tilde{\mathbb{Q}}| < \kappa$. Then there exists $i < j < \kappa$ such that

(1) $f_j = f_i$, and so $a_i \subseteq a_j$;
(2) if $\Vdash_{\mathbb{P}_j} \tilde{\mathbb{Q}}$ is c.c.c., then $\tilde{\mathbb{Q}}_j = \tilde{\mathbb{Q}}$.

We shall prove that $V^{\mathbb{P}_\kappa} \models CF(S) = C$. From now on, we shall work inside $V^{\mathbb{P}_\kappa}$.

DEFINITION 3.10. If $b \subseteq \kappa$ is closed for $\bar{Q}$, then $S^b = \mathrm{Sym}(\omega)^{V^{\mathbb{P}^*_b}}$.

First we shall show that $C \subseteq CF(S)$. Fix some $\mu \in C$. For each $\xi < \mu$, let $b_\xi = \{i < \kappa \mid f_i(\mu) \leq \xi\}$. Clearly b_ξ is closed for $\bar{Q}$; and if $\xi < \eta < \kappa$, then $b_\xi \subseteq b_\eta$. Thus $\langle S^{b_\xi} \mid \xi < \mu\rangle$ is a chain of subgroups of S.

LEMMA 3.11. *For each $\xi < \mu$, S^{b_ξ} is a proper subgroup of S.*

PROOF. Let $\xi < \mu$ and let $i < \kappa$ satisfy $f_i(\mu) > \xi$. Let $\mathbb{Q}$ be the partial order of finite injective functions $q : \omega \to \omega$, and let $\tilde{\mathbb{Q}}$ be the canonical $\mathbb{P}^*_{a_i}$-name for $\mathbb{Q}$. Then there exists $i < j < \kappa$ such that $f_j = f_i$ and $\tilde{\mathbb{Q}}_j = \tilde{\mathbb{Q}}$. Clearly $j \notin b_\xi$. Let $c = b_\xi \cup j$. By Lemma 3.7, $\tilde{\mathbb{Q}}_j$ adjoins a permutation π of ω such that $\pi \notin V^{\mathbb{P}^*_c}$. It follows that $\pi \notin S^{b_\xi}$. ⊣

LEMMA 3.12. $S = \bigcup_{\xi<\mu} S^{b_\xi}$.

PROOF. Let $\pi \in S$. Let $\tilde{g}$ be a nice $\mathbb{P}^*_\kappa$-name for π. (Remember that $\mathbb{P}_\kappa = \mathbb{P}^*_\kappa$.) Thus there exist antichains $A_{\ell,m}$ of $\mathbb{P}^*_\kappa$ for each $\langle \ell, m\rangle \in \omega \times \omega$ such that $\tilde{g} = \bigcup_{\ell,m}\{\langle \ell, m\rangle\} \times A_{\ell,m}$. Let $\bigcup\{\mathrm{supt}(p) \mid p \in \bigcup_{\ell,m} A_{\ell,m}\} = \{\alpha_k \mid k < \omega\}$. Let $\xi = \sup\{f_{\alpha_k}(\mu) \mid k < \omega\}$. Then $p \in \mathbb{P}^*_{b_\xi}$ for each $p \in \bigcup_{\ell,m} A_{\ell,m}$, and so $\tilde{g}$ is a nice $\mathbb{P}^*_{b_\xi}$-name. Hence $\pi \in S^{b_\xi}$. ⊣

This completes the proof of the following result.

LEMMA 3.13. *If $\mu \in C$, then $\mu \in CF(S)$.*

To complete the proof of Theorem 1.3, we must show that if $\mu \notin C$, then $\mu \notin CF(S)$. We shall make use of the following easy observation.

LEMMA 3.14. *Let $M \models ZFC$, and let $\langle g_\beta \mid \beta < \alpha\rangle \subseteq M$ be a generic sequence of elements of* Sym(ω). *Let $\mathbb{Q}$ be the partial order of finite injective functions $q : \omega \to \omega$, and let $\pi \in M^{\mathbb{Q}}$ be the $\mathbb{Q}$-generic permutation. Then for all $h \in Sym(\omega)^M$, $\langle g_\beta \mid \beta < \alpha\rangle^\frown h\pi$ is generic.*

PROOF. For each finite subsequence $\beta_1 < \cdots < \beta_n < \alpha$, the set $C(\alpha_1, \dots, \alpha_n) = \{\phi \in \mathrm{Sym}(\omega) \mid \langle g_{\alpha_1}, \dots, g_{\alpha_n}\rangle^\frown\phi$ is generic$\}$ is comeagre in $\mathrm{Sym}(\omega)$. Hence $h^{-1}C(\alpha_1, \dots, \alpha_n)$ is also comeagre for each $h \in \mathrm{Sym}(\omega)$. So for each $h \in \mathrm{Sym}(\omega)^M, \pi \in h^{-1}C(\alpha_1, \dots, \alpha_n)$. The result follows. ⊣

LEMMA 3.15. *Suppose that $\alpha < \kappa$ and that $\langle g_\beta \mid \beta < \alpha\rangle$ is a generic sequence of elements of Sym(ω). If H is any proper subgroups of Sym(ω), then there exists a permutation $\phi \notin H$ such that $\langle g_\beta \mid \beta < \alpha\rangle^\frown\phi$ is generic.*

PROOF. Let $h \in \mathrm{Sym}(\omega) \setminus H$. Then there exists $i < \kappa$ such that $h, \langle g_\beta \mid \beta < \alpha\rangle \in V^{\mathbb{P}_i}$. There exists $i < j < \kappa$ such that $\tilde{\mathbb{Q}}_j$ is the canonical $\mathbb{P}^*_{a_j}$-name for the partial order $\mathbb{Q}$ of finite injective functions $q : \omega \to \omega$. By Lemma 3.14, there exists a permutation $\pi \in V^{\mathbb{P}_{j+1}}$ such that both $\langle g_\beta \mid \beta < \alpha \rangle^\frown \pi$ and $\langle g_\beta \mid \beta < \alpha \rangle^\frown h\pi$ are generic. Clearly either $\pi \notin H$ or $h\pi \notin H$. ⊣

Now fix some $\mu \notin C$, and suppose that $\mu \in CF(S)$. It is easily checked that $2^{\aleph_0} = \kappa$, and so we can suppose that μ is a regular uncountable cardinal such that $\mu < \kappa$. Express $S = \bigcup_{\alpha<\mu} G_\alpha$ as the union of a chain of μ proper subgroups. We can suppose that $\mathrm{Fin}(\omega) \le G_o$. Using Lemma 3.15, we can inductively construct a generic sequence of elements of S

$$\langle g_o^o, g_o^1, \dots, g_\alpha^o, g_\alpha^1, \dots \rangle_{\alpha<\mu}$$

such that for each $\alpha < \mu$, there exists $\alpha \le \gamma_\alpha < \mu$ such that $g_\alpha^o \in G_{\gamma_\alpha}$ and $g_\alpha^1 \notin G_{\gamma_\alpha}$.

LEMMA 3.16. *There exists a subset $X \in [\mu]^\mu$ and an ordinal $\xi < \kappa$ such that $\langle g_\alpha^o, g_\alpha^1 \mid \alpha \in X \rangle \in V^{\mathbb{P}^*_{\alpha_\xi}}$.*

PROOF. For each $\alpha < \mu$ and $\tau \in \{0,1\}$, let $\tilde{g}_\alpha^\tau$ be a nice $\mathbb{P}^*_\kappa$-name for g_α^τ. Thus there exist antichains $A_{\ell,m}^{\alpha,\tau}$ of $\mathbb{P}^*_\kappa$ for each $\langle \ell, m \rangle \in \omega \times \omega$ such that

$$\tilde{g}_\alpha^\tau = \bigcup_{\ell,m} \{\langle \ell, m \rangle\} \times A_{\ell,m}^{\alpha,\tau}.$$

For each $\alpha < \mu$, let $\bigcup\{\,\mathrm{supt}(p) \mid p \in \bigcup_{\ell,m} A_{\ell,m}^{\alpha,o} \cup \bigcup_{\ell,m} A_{\ell,m}^{\alpha,1}\} = \{\beta_k^\alpha \mid k < \omega\}$. Define $h_\alpha \in \mathscr{F}_C$ by $h_\alpha(\lambda) = \sup\{f_{\beta_k^\alpha}(\lambda) \mid k < \omega\}$ for each $\lambda \in C$.

It is easily checked that there are less than μ possibilities for the restriction $h_\alpha \restriction C \cap \mu$. (This calculation is the only point in the proof of Theorem 1.3 where we make use of the hypothesis that C satisfies conditions (1.4)(b) and (1.4)(c).) Hence there exists $X \in [\mu]^\mu$ such that $h_\alpha \restriction C \cap \mu = h_\beta \restriction C \cap \mu$ for all $\alpha, \beta \in X$. Define the function $f \in \Pi C$ by $f \restriction C \cap \mu = h_\alpha \restriction C \cap \mu$, where $\alpha \in X$, and $f(\lambda) = \sup\{h_\alpha(\lambda) \mid \alpha \in X\}$ for each $\lambda \in C \setminus \mu$. Then it is easily checked that $f \in \mathscr{F}_C$; and clearly $f_{\beta_k^\alpha} \le h_\alpha \le f$ for all $\alpha \in X$ and $k < \omega$. Now choose $\xi > \sup\{\beta_k^\alpha \mid \alpha \in X, k < \omega\}$ such that $f_\xi = f$. If $\alpha \in X$ and $\tau \in \{0,1\}$, then $p \in \mathbb{P}^*_{\alpha_\xi}$ for each $p \in \bigcup_{\ell,m} A_{\ell,m}^{\alpha,\tau}$; and hence $\tilde{g}_\alpha^\tau$ is a nice $\mathbb{P}^*_{\alpha_\xi}$-name. It follows that $\langle g_\alpha^o, g_\alpha^1 \mid \alpha \in X \rangle \in V^{\mathbb{P}^*_{\alpha_\xi}}$. ⊣

By Lemma 2.13, there exists a σ-centred $\mathbb{Q} \in V^{\mathbb{P}^*_{\alpha_\xi}}$ such that

$$\Vdash_{\mathbb{Q}} \text{There exists } \pi \in \mathrm{Sym}(\omega) \text{ such that } \pi g_\alpha^o \pi^{-1} =^* g_\alpha^1 \text{ for all } \alpha \in X.$$

Let $\tilde{\mathbb{Q}}$ be a $\mathbb{P}^*_{\alpha_\xi}$-name for $\mathbb{Q}$. Then there exists $\xi < \eta < \kappa$ such that $f_\eta = f_\xi$ and $\tilde{\mathbb{Q}}_\eta = \tilde{\mathbb{Q}}$. Hence there exists $\pi \in S$ such that $\pi g_\alpha^o \pi^{-1} =^* g_\alpha^1$ for all $\alpha \in X$. But this implies that $\pi \notin \bigcup_{\alpha<\mu} G_\alpha$, which is a contradiction. This completes the proof of Theorem 1.3.

By modifying the choice of the set $\mathscr{F}_C$ of functions, we can obtain some interesting variants of Theorem 1.3. For example, the following theorem shows that Theorem 2.2 cannot be proved in *ZFC*. (Of course, it also shows that (1.4)(c) is not a necessary condition in Theorem 1.3.)

THEOREM 3.17. *Suppose that $V \models GCH$ and that $\kappa > \aleph_{\omega_1+1}$ is regular. Let $C = \{\aleph_{\alpha+1} \mid \alpha < \omega_1\} \cup \{\kappa\}$. Then there exists a c.c.c. notion of forcing $\mathbb{P}$ such that $V^{\mathbb{P}} \models CF(S) = C$.*

PROOF. The proof is almost identical to that of Theorem 1.3. The only change is that we use the set of functions

$$\mathscr{F}_C^* = \{f \in \Pi C \mid \text{There exists } \alpha < \omega_1 \text{ such that } f(\aleph_{\beta+1}) = 0 \text{ for all } \alpha \leq \beta < \omega_1\}$$

in the definition of $\mathbb{P}_\kappa$. This ensures that the counting argument in the analogue of Lemma 3.16 goes through. ⊣

Using some more *pcf* theory, we can prove the following result.

THEOREM 3.18. *Suppose that V satisfies the following statements.*
(a) $2^{\aleph_n} = \aleph_{n+1}$ *for all* $n < \omega$.
(b) $2^{\aleph_\omega} = \aleph_{\xi+1}$ *for some* $\omega < \xi < \omega_1$.
(c) $2^{\aleph_\eta} = \aleph_{\eta+1}$ *for all* $\eta \geq \xi$.
Let $T \in [\omega]^\omega$ and let κ be a regular cardinal such that $\kappa \geq \aleph_{\xi+1}$. Let $C = pcf(\prod_{n\in T} \aleph_n) \cup \{\kappa\}$. Then there exists a c.c.c. notion of forcing $\mathbb{P}$ such that $V^{\mathbb{P}} \models CF(S) = C$.

PROOF. Again we argue as in the proof of Theorem 1.3. This time we use the set of functions, $\mathscr{F}_C^\# = \prod_{n\in T} \aleph_n$, in the definition of $\mathbb{P}_\kappa$. Examining the proof of Lemma 3.16, we see that it is enough to prove that the following statement holds for each regular uncountable $\mu \notin C$.
$(3.19)_\mu$

$$\text{If } \langle h_\alpha \mid \alpha < \mu\rangle \text{ is a sequence in } \prod_{n\in T} \aleph_n, \text{ then there exists } X \in [\mu]^\mu$$
$$\text{and an } f \in \prod_{n\in T} \aleph_n \text{ such that } h_\alpha \leq f \text{ for all } \alpha \in X.$$

This is easy if $\mu < \aleph_\omega$. If $\mu > \aleph_\omega$, then $(3.19)_\mu$ is a consequence of the following result. ⊣

THEOREM 3.20. *Let $\{\lambda_i \mid i \in I\}$ be a set of regular cardinals such that* $\min\{\lambda_i \mid i \in I\} > |I|$. *Let μ be a regular cardinal such that $\mu > 2^{|I|}$ and $\mu \notin pcf(\prod_{i\in I} \lambda_i)$. If $\langle h_\alpha \mid \alpha < \mu\rangle$ is a sequence in $\prod_{i\in I} \lambda_i$, then there exists $X \in [\mu]^\mu$ and $f \in \prod_{i\in I} \lambda_i$ such that $h_\alpha \leq f$ for all $\alpha \in X$.*

PROOF. This is included in the proof of [13, II 3.1]. (More information on this topic is given in [9, Section 5]. Also [8, 6.6D] gives even more information under the hypothesis that $2^{|I|} < \min\{\lambda_i \mid i \in I\}$.) Alternatively, argue as in the proof of [1, 7.11]. ⊣

It is known that, assuming the consistency of a suitable large cardinal hypothesis, for each $\omega < \xi < \omega_1$ there exists a universe which satisfies the hypotheses of Theorem 3.18. (See [2].) Thus the following result shows that Theorem 1.2 cannot be substantially improved in *ZFC*.

COROLLARY 3.21. *Suppose that V satisfies the hypotheses of Theorem* 3.18 *with respect to some $\omega < \xi < \omega_1$. Then for each $\omega \leq \alpha \leq \xi$ and $\kappa \geq \aleph_{\xi+1}$, there exists a set $T \in [\omega]^\omega$ and a c.c.c. notion of forcing $\mathbb{P}$ such that*

$$V^{\mathbb{P}} \models CF(S) = \{\aleph_n \mid n \in T\} \cup \{\aleph_{\alpha+1}\} \cup \{\kappa\}.$$

In particular, if $\omega < \alpha \leq \xi$, then

$$V^{\mathbb{P}} \models \aleph_{\omega+1} \notin CF(S).$$

PROOF. With the above hypotheses, [13, VIII] implies that there exists $T \in [\omega]^\omega$ such that $tcf\left(\prod_{n\in T} \aleph_n/J^{bd}_\omega\right) = \aleph_{\alpha+1}$. It follows that $pcf(\prod_{n\in T} \aleph_n) = \{\aleph_n \mid n \in T\} \cup \{\aleph_{\alpha+1}\}$. So the result is a consequence of Theorem 3.18. ⊣

Finally we shall show that (1.4)(a) is not a necessary condition in Theorem 1.3, and that $2^{\aleph_0}$ cannot be bounded in terms of the set $CF(S)$.

THEOREM 3.22. *Suppose that $V \models GCH$ and that $C = \{\aleph_{\alpha+1} \mid \alpha < \omega_1\}$. If κ is any singular cardinal such that $cf(\kappa) \in C$, then there exists a c.c.c notion of forcing $\mathbb{P}$ such that $V^{\mathbb{P}} \models CF(S) = C$ and $2^{\aleph_0} = \kappa$.*

PROOF. Let κ be a singular cardinal such that $cf(\kappa) \in C$. Let $\langle \lambda_\beta \mid \beta < cf(\kappa)\rangle$ be a strictly increasing sequence of regular cardinals such that $\lambda_0 = \aleph_{\omega_1+1}$ and $\sup_{\beta<cf(\kappa)} \lambda_\beta = \kappa$. Let

$$\mathscr{F}^*_C = \{f \in \Pi C \mid \text{There exists } \alpha < \omega_1 \text{ such that } f(\aleph_{\beta+1}) = 0 \text{ for all } \alpha \leq \beta < \omega_1\}.$$

In V, we define a sequence $\langle \mathbb{P}_i, \tilde{\mathbb{Q}}_j, f_j \mid i \leq \kappa, j < \kappa\rangle$ such that the following conditions are satisfied.

(a) $f_i \in \mathscr{F}^*_C$.

(b) Let $a_i = \{j < i \mid f_j \leq f_i\}$. Then $\bar{Q} = \langle \mathbb{P}_i, \tilde{\mathbb{Q}}_j, a_j \mid i \leq \kappa, j < \kappa\rangle \in \mathscr{C}$.

(c) For each $f \in \mathscr{F}^*_C$ and $\beta < cf(\kappa)$, there exists a cofinal set of ordinals $j < \lambda_\beta$ such that $f_j = f$.

(d) Suppose that $\beta < cf(\kappa)$, $i < \lambda_\beta$ and that $\tilde{\mathbb{Q}}$ is a $\mathbb{P}^*_{a_i}$-name with $|\tilde{\mathbb{Q}}| < \lambda_\beta$. Then there exists $i < j < \lambda_\beta$ such that

(1) $f_j = f_i$, and so $a_i \subseteq a_j$;

(2) if $\Vdash_{\mathbb{P}_j} \tilde{\mathbb{Q}}$ is c.c.c., then $\tilde{\mathbb{Q}}_j = \tilde{\mathbb{Q}}$.

Clearly $V^{\mathbb{P}_\kappa} \models 2^{\aleph_0} = \kappa$. Arguing as in the proof of Lemma 3.13, we see that $V^{\mathbb{P}_\kappa} \models C \subseteq CF(S)$. From now on, we shall work inside $V^{\mathbb{P}_\kappa}$. Let μ be a regular cardinal such that $\aleph_{\omega_1+1} \leq \mu < \kappa$. Suppose that we can express $S = \bigcup_{\alpha<\mu} G_\alpha$ as the union of a chain of μ proper subgroups. For each $\alpha < \mu$, choose an element $h_\alpha \in G \setminus G_\alpha$. Then there exists a subset $I \in [\mu]^\mu$ and an ordinal $\beta < cf(\kappa)$ such that $\langle h_\alpha \mid \alpha \in I\rangle \in V^{\mathbb{P}_{\lambda_\beta}}$ and $\mu \leq \lambda_\beta$. In $V^{\mathbb{P}_\kappa}$, we can inductively construct a generic sequence of elements of S

$$\langle g^0_0, g^1_0, \dots, g^0_\alpha, g^1_\alpha, \dots\rangle_{\alpha<\mu}$$

such that for each $\alpha < \mu$

(1) there exists $\alpha \leq \gamma_\alpha < \mu$ such that $g^0_\alpha \in G_{\gamma_\alpha}$ and $g^1_\alpha \notin G_{\gamma_\alpha}$; and

(2) there exists $\lambda_\beta \leq i_\alpha < \lambda_{\beta+1}$ such that $\langle g^0_\delta, g^1_\delta \mid \delta < \alpha\rangle \subseteq V^{\mathbb{P}_{i_\alpha}}$.

For suppose that $\langle g^0_\delta, g^1_\delta \mid \delta < \alpha \rangle$ has been defined. By Lemma 3.14, there exists $i_\alpha < j < \lambda_{\beta+1}$ and $g^0_\alpha \in V^{\mathbb{P}_j}$ such that $\langle g^0_\delta, g^1_\delta \mid \delta < \alpha \rangle \frown g^0_\alpha$ is generic. Choose $\gamma_\alpha \in I$ such that $\alpha \leq \gamma_\alpha < \mu$ and $g^0_\alpha \in G_{\gamma_\alpha}$. By a second application of Lemma 3.14, there exists $j < i_{\alpha+1} < \lambda_{\beta+1}$ and $\pi \in V^{\mathbb{P}_{i_{\alpha+1}}}$ such that both $\langle g^0_\delta, g^1_\delta \mid \delta < \alpha \rangle \frown g^0_\alpha \frown \pi$ and $\langle g^0_\delta, g^1_\delta \mid \delta < \alpha \rangle \frown g^0_\alpha \frown h_{\gamma_\alpha} \pi$ are generic. Clearly either $\pi \notin G_{\gamma_\alpha}$ or $h_{\gamma_\alpha} \pi \notin G_{\gamma_\alpha}$. Hence we can also find a suitable g^1_α.

There exists a subset $J \in [\mu]^\mu$ and an ordinal $\delta < cf(\kappa)$ such that $\langle g^0_\alpha, g^1_\alpha \mid \alpha \in J \rangle \in V^{\mathbb{P}_{\lambda_\delta}}$ and $\mu \leq \lambda_\delta$. Arguing as in the proofs of Theorems 1.3 and 3.17, there exists $\pi \in V^{\mathbb{P}_{\lambda_{\delta+1}}}$ such that $\pi g^0_\alpha \pi^{-1} = {}^* g^1_\alpha$ for all $\alpha \in J$. This is a contradiction. $\dashv$

REFERENCES

[1] M. R. BURKE and M. MAGIDOR, *Shelah's pcf theory and its applications*, ***Annals of Pure and Applied Logic***, vol. 50 (1990), pp. 207–254.

[2] M. GITIK and M. MAGIDOR, *The singular cardinal hypothesis revisited*, ***Set theory of the continuum*** (H. Judah, W. Just, and H. Woodin, editors), Mathematical Sciences Research Institute Publications, vol. 26, Springer-Verlag, 1992, pp. 243–279.

[3] W. HODGES, I. HODKINSON, D. LASCAR, and S. SHELAH, *The small index property for ω-stable ω-categorical structures and for the random graph*, ***Journal of the London Mathematical Society***, vol. 48 (1993), no. 2, pp. 204–218.

[4] K. KUNEN, ***Set theory. an introduction to independence proofs***, North Holland, Amsterdam, 1980.

[5] H. D. MACPHERSON and P. M. NEUMANN, *Subgroups of infinite symmetric groups*, ***Journal of the London Mathematical Society***, vol. 42 (1990), no. 2, pp. 64–84.

[6] J. D. SHARP and SIMON THOMAS, *Uniformisation problems and the cofinality of the infinite symmetric group*, ***Notre Dame Journal of Formal Logic***, vol. 35 (1994), pp. 328–345.

[7] J. D. SHARP and SIMON THOMAS, *Unbounded families and the cofinality of the infinite symmetric group*, ***Arch. Math. Logic***, vol. 34 (1995), pp. 33–45.

[8] S. SHELAH, *Further cardinal arithmetic*, to appear in Israel Journal of Mathematics.

[9] ———, *PCF and infinite free subsets*, submitted to Archive for Mathematical Logic.

[10] ———, *Strong partition relations below the power set: Consistency. Was Sierpinski right? II, in Proceedings of the Conference on Set Theory and its Applications in honor of A. Hajnal and V. T. Sos, Budapest*, ***Sets, Graphs and Numbers***, vol. 60 of Colloquia Mathematica Societatis Janos Bolyai (1991), pp. 637–668.

[11] ———, *Cardinal arithmetic for skeptics*, ***American Mathematical Society Bulletin, New Series***, vol. 26 (1992), pp. 197–210.

[12] ———, *Advances in cardinal arithmetic*, ***Finite and infinite combinatorics in sets and logic*** (N. W. Sauer, R. E. Woodrow, and B. Sands, editors), Kluwer Academic Publishers, 1993, pp. 355–383.

[13] ———, ***Cardinal arithmetic***, Oxford Logic Guides, vol. 29, Oxford University Press, 1994.

MATHEMATICS DEPARTMENT
BILKENT UNIVERSITY
ANKARA, TURKEY

MATHEMATICS DEPARTMENT
THE HEBREW UNIVERSITY
JERUSALEM, ISRAEL
and
MATHEMATICS DEPARTMENT
RUTGERS UNIVERSITY
NEW BRUNSWICK, NEW JERSEY 08903, USA