A note on κ-freeness of abelian groups

Introduction: Lately Foreman proved that the assertion $(*)_n$ below follows from some axioms (speaking on the $\aleph_n$, seemingly of consistency strength like the determinancy axioms), and $(*)_2$ consistent if some large cardinal axioms ($\approx$ there is a huge cardinal) are consistent.

$(*)_n$ every $\aleph_n$ free abelian group of power $\aleph_n$ is the union $\aleph_1$ free subgroups.

Let in this note a group mean an abelian group.

We consider mainly some variants (which his proofs easily gives); give some sufficient conditions in ZFC, and find the consistency strength for $n=2$ which is Mahlo, and prove the consistency of $(*)_n$ using super compact cardinals.

1. Definition : 1) $P(\lambda,\kappa) \overset{def}{=}$ if G is λ-free of power λ, $G = \bigcup_{i<\lambda} G_i, |G_i| < \lambda, G_i$ increasing continuous *then* $\{i: G/G_i \text{ not } \kappa\text{-free}\}$ is not stationary.

2) Let $P^+(\lambda,\kappa)$ mean that every λ-free group of power λ is κ-freely represented (see 2(4)).

2. Definition : 1) A group G is (μ,κ)-coverable if we can find $H_\alpha(\alpha < \mu)$, free pure subgroups of G, such that: for every $A \subseteq G$ of power $< \kappa$, for some α, $A \subseteq H_\alpha$.

2) We define "weakly (μ,κ)-coverable" similarly if omitting the "purity".

3) G is (μ,κ)-freely represented if it has a (μ,κ) free representation i.e.

$\langle G_i : i < i(*) \rangle$, G_i increasing continuous, $G_0 = \{0\}$, $G_{i(*)} = G$, and G_{i+1}/G_i is κ-free of power $\leq \mu$. (κ-free means: every subgroup of rank $< \kappa$ is free, so κ may be finite).

3. Lemma : If G_i is increasing continuous, $G_{i(*)} = G$, $G_0 = \{0\}$, $\mu \leq \lambda$ *then* G is (μ,κ)-coverable provided that:

(*) there are sequences $\langle H_{i,\xi} : \xi < \mu \rangle$ of pure subgroups of G_{i+1} such that

(a) $H_{i,\xi} + G_i / G_i$ is a free and a pure subgroup of G_{i+1}/G_i.

(b) if $A \subseteq G, |A| < \kappa$ then for some set $S \subseteq i(*)$, and ξ such that:

(α) $A \subseteq \sum_{i \in S} H_{i,\xi}$ and

(β) $(\forall i < j)([i \in S \wedge j \in S \rightarrow H_{j,\xi} \cap G_{i+1} \subseteq H_{i,\xi} + G_i]$ and

(γ) $(\forall j \in S)$ $(\forall i < j)$ $[H_{j,\xi} \cap (G_{i+1} - G_i) \neq \phi \rightarrow i \in S]$.

Proof: Define $K_{i,\xi}$ by induction on $i < i(*)$:
$K_{0,\xi} = \{0\}$, $K_{\delta,\xi} = \bigcup_{i<\delta} K_{i,\xi}$,
$K_{i+1,\xi}$ is $K_{i,\xi} + H_{i,\xi}$ if $H_{i,\xi} \cap G_i \subseteq K_{i,\xi}$, and $K_{i,\xi}$ otherwise.

Easily by (a) of (*) $K_{i(*),\xi}$ is a pure subgroup of G. Now we should prove: for every $A \subseteq G$, if $|A| < \kappa$, then $(\exists \xi < \mu) A \subseteq K_{\lambda,\xi}$. Let S,ξ be as in (b) of (*) (for the set A). We prove by induction on $i \in S$ that:

(i) $H_{i,\xi} \cap G_i \subseteq K_{i,\xi}$

(ii) $H_{i,\xi} \subseteq K_{i+1,\xi}$

For $i = 0$, everything is trivial as $G_i = \{0\}$; when we arrive to i, if (i), fail, choose $j \leq i$ minimal such that $H_{i,\xi} \cap G_j \not\subseteq K_{j,\xi}$, necessarily j is successor, so $H_{i,\xi} \cap (G_j - G_{j-1}) \neq \phi$ so by (*) (b) (γ) $(j-1) \in S$. By the minimality of j, $H_{i,\xi} \cap G_{j-1} \subseteq K_{j-1,\xi}$ and as by (β) of (b) $H_{i,\xi} \cap G_{(j-1)+1} \subseteq H_{j-1,\xi} + G_{j-1}$, by the choice of $K_{j,\xi} = K_{(j-1)1,\xi}$, $H_{i,\xi} \cap G_j \subseteq K_{j,\xi}$, contradicting the choice of j. Now we can prove (ii).

So $\{K_{\lambda,\xi}:\xi < \mu\}$ exemplify G is (μ,κ)-coverable.

4. Lemma: If G is (μ,κ)-freely representable, $\kappa > \aleph_0$, and $\mathcal{P}_{<\kappa}(\mu)$ has a stationary subset S of power μ^1 (see below) *then* G is (μ^1,κ)-coverable.

Proof: We use Lemma 3.

Let $\langle G_i : i \le i(*)\rangle$ be a (μ,κ)-free representation of G, and let $L_i \subset G_{i+1}$ be such that $G_{i+1} = G_i + L_i$, (L_i a pure subgroup of G_{i+1}) and $|L_i| = \mu$ (but maybe $G_i \cap L_i \ne \{0\}$). Let g_i be a one-to-one mapping from μ onto L_i.

Let $S = \{s_\xi : \xi < \mu^1\}$, be an enumeration of S in increasing order, and $H_{i,\xi}$ be the subgroup of G_{i+1} (of L_i in fact) generate by $\{g_i(x) : x \in s_\xi\}$. We can finish as:

$\otimes \langle H_{i,\xi} : \xi < \mu\rangle$ $(i<i(*))$ satisfies $(*)$

(apply second sentence of 5(2)). Remember:

5. Definition : 1) $\mathcal{P}_{<\kappa}(A) = \{s : s \subseteq A, |s| < \kappa\}$.

2) $S \subseteq \mathcal{P}_{<\kappa}(A)$ is stationary, if for every $< \kappa$ (finitary) functions from A to A, some $s \in S$ is closed under all of them.

Note: if $B \supseteq A$, $\gamma < \kappa$, f_i is an n_i -place function from B to B for $i < \gamma$, and from each $\alpha \in B, g_\alpha$ is a one-to-one map from A onto some $B_\alpha \subseteq B$, then for some $s \in \mathcal{P}_{<\kappa}(B)$ closed under the f_i's, $s \cap A \in S$ and for every $\alpha \in s$, $s \cap B_\alpha = \{g_\alpha(x) : x \in (s \cap A\})$.

6. Fact: 1) If κ is regular, $\mu = \kappa^{+n}$ then $\mathcal{P}_{<\kappa}(\mu)$ has a stationary subset of power μ.

2) $\mathcal{P}_{<\kappa}(\mu)$ has a stationary set of power $\mu^{<\kappa}$.

7. Lemma : If $\kappa \le \mu$, $\mathcal{P}_{<\kappa}(\mu)$ has no stationary subset of power μ *then* $0^\#$ exist, there is an inner model with a measurable cardinal B, etc.

Proof: By [Sh 3] Ch. XIII.

8. Lemma: Suppose $\|G\| = \lambda, G$ has a (μ,κ)-free representation, $\kappa = \aleph_0$,

$2^{\mu} \geq \lambda$. Then G is (μ,κ)-coverable.

Proof : Let $\langle G_i : i \leq i(*) \rangle$ be a (μ,κ)-representation of G, $L_i \subseteq G_{i+1}, |L_i| = \mu$, $G_{i+1} = G_i + L_i$, L_i a pure subgroup of G_{i+1}. Let $\langle H^0_{i,\xi} : \xi < \mu \rangle$ be a list of all pure subgroups of L_i of finite rank.

Let $g_i : \mu \to \mu$ $(i < i(*)$ be functions such that for every distinct $i_1, \ldots, i_m (m < \omega)$ and $\xi_1, \ldots, \xi_m < \mu$ for some $\alpha < \mu$ $g_{i_\ell}(\alpha) = \xi_\ell$ for $\ell = 1,m$ (exists by Engelking and Karlowiz [EK] as w.l.o.g. $|i(*)| \leq \|G\| \leq 2^{\mu}$.)

Let $H^0_{i,\alpha} = H^0_{i,g_i(\alpha)}$.

Now apply 3 to $\langle \langle H_{i,\alpha} : \alpha < \mu \rangle : i < i(*) \rangle$.

9. Lemma : Suppose $\|G\| = \lambda \leq 2^{\mu}$, G has a (μ,κ)-free representation, $\kappa < \aleph_0$. Then G is (μ,κ)-coverable.

Proof : Like 8 but $G_{i+1} = L_i$ and we restrict ourselves to $H = H^0_{i,\xi}$ disjoint to G_i (more exactly, $H^0_{i,\xi} \cap G_i = \{0\}$).

We prove by induction on i, that for $A \subseteq G_i, |A| < \kappa$, the (*) (b) of Lemma (3) holds. For $i = 0,i$ limit - no problem. For $i+1$: let $A = \{a_\ell : \ell < |A|\}$, w.l.o.g. a_ℓ belong to the pure closure of $\langle G_i, a_0, \ldots, a_{\ell-1} \rangle$ iff $\ell \geq m$. We first define by induction on $\ell < m$, $b_\ell \in G_{i+1}$, $c_\ell \in G_i$.

(i) $\{b_0 + G_i, \ldots, b_\ell + G_i\}$ is independent, and generates a pure subgroup of G_{i+1}/G_i (of course $b_\ell + G_i$ is not torsion).

(ii) $a_\ell \in \langle b_0, \ldots, b_{\ell-1}, b_\ell, c_\ell \rangle_G$ (= the subgroup generated by then).

As $m \leq |A| < k$ in the ℓ-stage, $G_{i+1}/\langle G_i, a_0, \ldots, a_{\ell-1} \rangle$ is $(\kappa - \ell)$-free, so there is a maximal integer n_ℓ dividing $a_\ell + \langle G_i, b_0, \ldots, b_{\ell-1} \rangle$, and let b_ℓ be such that $n_\ell b_\ell - a_\ell \in \langle G_i b_0, \ldots, b_{\ell-1} \rangle$. So for some $n_{\ell,0}, \ldots, n_{\ell,\ell-1}$; $a_\ell - n_\ell b_\ell + n_{\ell,0} b_0 + \cdots + n_{\ell,\ell-1} b_{\ell-1} \in G_i$, and call it c_ℓ.

Now we define for $m \leq \ell < |A|, c_\ell$ such that

(iii) $a_\ell \in \langle c_0, \ldots, c_\ell \rangle$

Arriving to ℓ, for some $m_\ell \neq 0$ $m_\ell a_\ell \in \langle G_i, a_0, \ldots, a_{\ell-1} \rangle$, hence $m_\ell a_\ell + G_i \in \langle a_0+G_i, \ldots, a_{\ell-1}+G_i \rangle \subseteq \langle b_0+G_i, \ldots, b_{m-1}+G_i \rangle$, but the latter is pure so or some $n_{\ell,0}, \ldots, n_{\ell,m-1}$, $a_\ell+G_i \in \langle b_0+G_i, \ldots, b_{m-1}+G_i \rangle$, so for some $n_{\ell,0}, \ldots, n_{\ell,m-1}$ the following equation holds $a_\ell - n_{\ell,0} b_0 - \ldots - n_{\ell,m-1} b_m \stackrel{def}{=} c_i \in G_i$.

Now use then induction hypothesis on $\{c_0, c_1, \cdots\}$.

10. Fact: If $P(\lambda,\kappa)$, $\lambda = \mu^+$ *then* every λ-free group of power λ is (μ,κ)-represented.

The following is a (strong) converse to 4,8,9 (so under suitable condition (μ,κ)-coverable $\equiv$ weakly (μ,κ)-coverable.)

11. Lemma : 1) Suppose $\lambda = \mu^+$, $|G| = \lambda$ and G is (μ,κ)-coverable *then* G is (μ,κ)-freely represented.

2) Then $\kappa > \aleph_0$ G weakly (μ,κ)-coverable is enough.

Proof: 1) Let $|G| = \lambda$ (i.e., the universe = the set of elements of G, is λ), $G = \bigcup_{\xi<\mu} H_\xi$, each H_ξ is a free pure subgroup of G, and $(\forall A \subseteq G)[|A| < \kappa \rightarrow (\exists \xi) A \subseteq H_\xi]$.

Let $G = \bigcup G_i$, G_i increasing continuous, $\|G_i\| < \lambda$ and let $S = \{i < \lambda$: G/G_i is not κ-free$\}$, we assume S is stationary and will arrive at contradiction thus finishing. For $i \in S$, let L_i be a pure subgroup of G of rank $< \kappa$, such that $L_i + G_i / L_i$ not free. Let $A_i \subseteq L_i$ be such that $|A_i| < \kappa$, and $L_i + G_i$ is the pure closure of $\langle G_i \cup A_i \rangle$.

So for every $i \in S$ for some $\xi(i) < \mu, A_i \subseteq H_{\xi(i)}$. So for some $\xi, T = \{i \in S: \xi(i) = \xi\}$ is stationary. Let N be an elementary submodel of an appropriate expansion of G, with universal $|G_i| = i \in T$. We shall prove that: (the pure closure of $G_i \cup A_i$ in G)$/G_i \cong$ pure closure of $(H_\xi \cap G_i) \cup A_i$ in $H_\xi / H_\xi \cap G_i$.

This suffices. So it suffices to show.

(*) if $a_1, \ldots, a_n \in A_i$, $0 < k < \omega$, $b \in G_i$, $b + \sum_{i=1}^{n} m^\ell a_\ell$ is divisible by k (in G), then we can find such $b \in H_\xi \cap G_i$ divisible by k even in H_ξ.

Proof of (*): As N is an elementary submodel, $b \in N$ as $b \in G_i = i$ we can find $a'_\ell \in H_\xi \cap N = H_\xi \cap G_i$ such that $b + \sum m^\ell a'_\ell$ is divisible by k (in G and even in G_i). Now let $b' = 0 - \sum m^\ell a'_\ell \in H_\xi \cap G_i$, and divisibility is in H_ξ using: $H_\xi \subseteq G$ purely.

2) Just take care that $A_i = L_i$, $L_i, G_i + L_i$ and $G \cap L_i$ will be pure subgroups of G.

We now restrict ourselves for a while to $\lambda = \aleph_2$ $\mu = \aleph_1$.

12. Lemma : The following are equivalent.

A) $P(\aleph_2, \aleph_1)$

B) every $\aleph_2$-free group of power $\aleph_2$ is $(\aleph_1, \aleph_1)$-coverable.

C) every $\aleph_2$-free group of power $\aleph_2$ is $(\aleph_1, 2)$-coverable.

D) If $S \subseteq \{\delta : \delta < \aleph_2, cf\ \delta = \aleph_0\}$ is stationary. $A_\delta \subseteq \delta$ (is countable for $\delta \in S$ *then* there is a stationary $T \subseteq \aleph_1$ and $f : T \to S$ one-to-one such that $\{\xi \in T : A_{f(\xi)} \subseteq \bigcup_{\zeta < \xi} A_{f(\zeta)}\}$ is stationary.

Proof : (A) $\Longrightarrow$ (B) by 10, 4+6(1).

(B) $\Longrightarrow$ (C) trivial

(C) $\Longrightarrow$ (D). We prove $\neg$(D) $\Longrightarrow$ $\neg$(C).

Let $\{A_\delta : \delta \in S\}$ be a counterexample to (D). Let $A_\delta = \{a_{\delta,n} : n < \omega\}$. Let G be freely generated by $x_\eta (\eta \in {}^{\omega>}\aleph_2)$, $y_{\delta,n} (n < \omega, \delta \in S)$ except the relations (letting $\eta_\delta = \langle a_{\delta,0}, a_{\delta,1}, \cdots \rangle$)

$$p \ \ y_{\delta,n+1} = y_{\delta,n} - x_{\eta_\delta \restriction n}$$

(p a fixed prime but you can make it a natural number ≥ 1 depending on δ, n) Easily G is not $(\aleph_1, 2)$-freely represented and by 1) we get a contradiction.

(D) $\Longrightarrow$ (A): See [Sh 2] for much more

13 Theorem : (D) is equi consistent with Mahlo..

Proof : See Harrington Shelah [H Sh].

14. Lemma: We can move the cardinals in 11, e.g. let $\mu = \mu^{<\kappa}$, $\kappa > \aleph_0$ then the following are equivalent.

(A)' $P(\mu^+,\kappa)$.

(B)' every μ^+-free group of power μ^+ is (μ,κ)-coverable.

(C)' every μ^+-free group of power μ^+ is weakly (μ,κ)-coverable.

(D)' for some regular χ, $\vartheta < \kappa+\aleph_1$, there are a stationary $S \subseteq \{\delta < \mu^+ : cf\ \delta = \vartheta\}$, and $A_\delta \subseteq \delta$ of order type $\vartheta\chi$,Sup $A_\delta = \delta$, $A_\delta = \bigcup_{i<\vartheta} A_{\delta,i}, A_{\delta,i} < A_{\delta,j}$ for $i < j$ otop $(A_{\delta,\alpha}) = \chi$, such that for every $i < \mu^+$ we can find pairwise disjoint $B_\delta \subseteq A_\delta$, such that $(\exists^{<\vartheta} i)(\exists^{<\chi} j \in A_{\delta,i}) j \not\in B_\delta$ (if $\kappa = \aleph_1$, (D)' can be replaced by "A_δ of order type ω , $|A_\delta - B_\delta| < \aleph_0$".

The consistency strength, for μ regular is as in 13.

Proof: As in [Sh 2].

However.

15 Observation: Suppose $\lambda_0 \le \lambda$, $(\exists n)\lambda \le \lambda_0^{+n}$ λ is regular, and

(A) for every χ, $\lambda < \chi^+ \le \lambda$, every (χ^+)-free group of power χ^+ is χ-freely represented (i.e $P(\chi^+,\chi)$.

(B) every λ_0-free group of power λ_0 is (μ,κ)-freely represented.

Then every λ-free group of cardinality λ is (μ,κ)- freely represented.

Proof: By induction on λ. For $\lambda = \lambda_0$ this is (B) for λ a successor cardinal use (A).

Remark. We can phrase similar things for $\lambda \ge \lambda_0^{+\omega}$, but then for λ singular every λ-free group of power λ will by free be [Sh 1] so this is not an interesting case.

The consistency strength is much higher by Magidor [Ma].

Now by 14 and 15 and known set theory we can get positive results e.g. (using $\aleph_1$ for simplicity).

16 Theorem: 1) Suppose $2 < n < \omega$ and $P(\aleph_{m+1},\aleph_m)$ holds when $1 \le m < m$. Then every $\aleph_n$-free group of cardinality $\aleph_n$ is $(\aleph_1,\aleph_1)$-freely represented hence in $(\aleph_1,\aleph_1)$-coverable.

2) From the consistency of $(n-1)$ supercompact cardinals we can get the consistency of $\bigwedge_{m=1}^{n-1} P(\aleph_{m+1},\aleph_m)$ and G.C.H. [$\aleph_0 < \kappa_1 < \cdots < \kappa_n$ are supercompact, w.l.o.g. satisfying Laver's conclusion [L], and use Levi collapse to make κ_ℓ to $\aleph_\ell$ ($\ell = 1,n$) and use Baumgartner [B] argument.]

Note

17. Lemma : 1) Let U be an abelian group, and let

$F = \{(A,B): \langle A \cup B \rangle_G / \langle B \rangle_G$ is (μ,κ)-represented$\}$.

Then (in the context of [Sh 1], §1, or [Sh 2] §1 the following axioms holds) with χ there standing for μ here: II, III, IV, VI, VII.

18. Lemma: 1) If G is (μ,κ)-coverable then G is (μ,κ)-represented.

2) If $\kappa > \aleph_0$ weakly (μ,κ)-coverable suffice.

Proof: We can prove this by induction on $\|G\|$. If $\|G\| \le \mu$ this is trivial. For $\|G\| > \mu$ a singular cardinal use the compactness theorem of [Sh 1] (where Lemma 17 shows the assumption holds. For $\|G\| > \mu$ a regular cardinal repeats the proof of 11.

19 Conclusion: Suppose $\kappa > \aleph_0$ and $\mathcal{P}_{<\kappa}(\mu)$ has a stationary subset of cardinality μ.

For any group G, G is (μ,κ)-represented *iff* G is (μ,κ)-coverable if G is weakly (μ,κ)-coverable.

Proof: The first implies the secod by Lemma 4, the second implies the third trivially, the third implies the first by Lemma 18.

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