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CATEGORICITY IN $\aleph_1$ OF SENTENCES IN $L_{\omega_1,\omega}(Q)$

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ABSTRACT
We investigate the categoricity and number of non-isomorphic models in $\aleph_1$ of sentences in $L_{\omega_1,\omega}(Q)$. Assuming $V = L$ we prove that no sentence in $L_{\omega_1,\omega}(Q)$ has exactly one uncountable model. Thus partially answering problem 24 of a problem list by Friedman.

1. Introduction

After the solution of the problem of the categoricity-spectrum of first-order theories by Morley [9] (for countable theories) and Shelah [14] it is natural to look at categoricity of sentences in wider logics. Keisler [5] deals with categoricity of $\psi \in L_{\omega_1,\omega}$ and, assuming the existence of appropriate $\aleph_1$-homogeneous models, gets full results. Unfortunately this is not the general case. Marcus [8] proved the existence of a minimal countable model which contains an infinite set of elements indiscernible in a strong sense, and the author observed this implies there is $\psi \in L_{\omega_1,\omega}$ categorical in every λ, but no model of which is $(L_{\omega_1,\omega}, \aleph_1)$-homogeneous.

Several years ago the author investigated $\psi \in L_{\omega_1,\omega}$ categorical in $\aleph_1$, (which should be the easiest case) and got a picture quite similar to the one for first-order theories (the most significant result is mentioned in [8]). Unfortunately the existence of prime models over appropriate sets was not proven. Hence the categoricity was not proven. Also the amalgamation property was not proven. Later and independently Knight [7] obtained also some of those results.

A common device is that when your methods do not answer your questions, change your question. The following question (due to Baldwin) appeared in Friedman [3] (question 24):

Can a sentence $\psi \in L(Q)$ have exactly one uncountable model?

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We answer negatively, assuming $V = L$, even for sentences in $L_{\omega_1,\omega}(Q)$, by proving that if such ψ has $< 2^{\aleph_1}$, but at least one, models of cardinality $\aleph_1$, then it has a model of cardinality $\aleph_2$.

The following example is interesting. Let $\psi^R \in L(Q)$ be the sentence saying: $<$ is a dense linear order with no first nor last element, each interval is uncountable, but $\{x: P(x)\}$ is a dense countable subset. By Baumgartner [1] it is consistent with ZFC $+ 2^{\aleph_0} = \aleph_2$ that ψ^R is categorical in $\aleph_1$, but it is not even $(\aleph_0, 1)$-stable (see Def. 3.5)

We can replace the quantifier (Qx) by some stronger quantifiers without changing much. Let $M \models (Q^{st} P)\varphi(P)$ (P varies over one-place predicates) mean that the family $\{P \subseteq |M|: M \models \varphi[P]\}$ does not contain a subfamily **P**, of consistent with ZFC $+ 2^{\aleph_0} = \aleph_2$ that ψ^R is categorical in $\aleph_1$, but it is not even bounded (i.e. $(\forall P)(\exists P_1)(P \subseteq |M| \wedge |P| \leqq \aleph_0 \to P \subseteq P_1 \in \mathbf{P})$]. Notice $((Qz)\varphi(z) \equiv \neg(Q^{st}P)(\forall z)(\varphi(z) \to P(z))$. By Shelah [16] th. 2.14, $L(Q^{st})$ is very similar to $L(Q)$ for models of power $\aleph_1$, and in fact also $L_{\omega_1,\omega}(Q^{st})$ is very similar to $L_{\omega_1,\omega}(Q)$. The results of Secs. 2, 3 and 4 generalize easily to $L_{\omega_1,\omega}(Q^{st})$, moreover by [16] clearly if $\psi \in L_{\omega_1,\omega}(Q^{st})$, $I(\aleph_1, \psi) < 2^{\aleph_1}$, $M \models \psi, \|M\| = \aleph_1$ then e.g. for no $\bar{a} \in |M|$ and $\varphi \in L_{\omega_1,\omega}(Q^{st})$ does $M \models (Q^{st}P)\varphi(P, \bar{a}) \wedge (Q^{st}P) \neg\varphi(P, \bar{a})$.

But Sec. 5 does not generalize, as shown by the following $\psi \in L(Q^{st})$ which has exactly one (uncountable) model: ψ states that $<$ is a dense order, with no first element, each initial segment is countable, but the model is not, and $\neg(Q^{st}P)(\neg P$ does not have a first element). The model of φ is just $\langle n \cdot \omega_1, < \rangle$.

Notation. L will be a countable first-order language, $L(Q)$ is L when we add to it the quantifier (Qx) meaning: "there exist uncountably many x's such that..." $L_{\omega_1,\omega}$ is L when we allow $\bigwedge_{n<\omega}\varphi_n$, provided that $\bigwedge_{n<\omega}\varphi_n$ has only finitely many free variables. $L_{\omega_1,\omega}(Q)$ is defined similarly. A fragment of $L_{\omega_1,\omega}(Q)$ (or $L_{\omega_1,\omega}$) is a *countable* subset, closed under: taking subformulas, changing names of free variables and applying the finite connectives, and the quantifiers $(\exists x), (\forall x)$. Let φ, θ, be formulas, ψ a sentence, R, P predicates.

If $L \subseteq L^1$, $\psi \in L^1_{\omega_1,\omega}(Q)$ then $PC(\psi, L)$ is the class of L-reducts of models of ψ, and $I(\lambda, \psi, L)$ is the number of non-isomorphic models in $PC(\psi, L)$ of cardinality λ. If $L = L^1$ we write $I(\lambda, \psi)$ for $I(\lambda, \psi, L)$.

By $\varphi = \varphi(x_1 \cdots x_m) = \varphi(\bar{x})$ we mean every free variable of φ appears in $\bar{x}$. For $L^* \subseteq L_{\omega_1,\omega}(Q)$ the L^*-type $\bar{a}$ realizes in M (a model) over $A \subseteq |M|$ (= the universe of M) is

$$tp(\bar{a}, A, L^*, M) = \{\varphi(\bar{x}, \bar{b}) \colon \varphi \in L^*,\ \bar{b} \in A,\ M \models \varphi\,[\bar{a}, \bar{b}]\}$$

$$(\bar{a} = \langle a_1 \cdots a_m \rangle \in A \text{ means } a_1 \cdots a_m \in A).$$

If the length of $\bar{a}$, $l(\bar{a})$, is m, it is a L^*-m-type. If not said otherwise, $A = \phi$.

2. Pseudo-elementary classes

LEMMA 2.1. *Let $L \subseteq L^1$, $\psi \in L^1_{\omega_1,\omega}(Q)$, and L^* a fragment of $L_{\omega_1,\omega}(Q)$. Then:*

(A) *If in some model M of ψ of cardinality $\geqq \aleph_1$, uncountably many L^*-types are realized* then $I(\aleph_1, \psi, L) = 2^{\aleph_1}$

(B) *If for some model M of ψ, of cardinality $\geqq \aleph_1$, there is a countable $A \subseteq |M|$, such that in M over A uncountably many L^*-types are realized* then $I(\aleph_1, \psi, L) = 2^{\aleph_1}$ *provided that* $2^{\aleph_1} > 2^{\aleph_0}$.

PROOF.

(1) This is theorem 5.1 of [6].

(2) This follows easily from (1).

LEMMA 2.2. *Let $L \subseteq L^1$, $\psi \in L^1_{\omega_1,\omega}(Q)$, L^* a fragment of $L_{\omega_1,\omega}(Q)$. Assume $\{p\colon p$ is an L^*-type and there is an uncountable model of ψ in which p is realized$\}$ is uncountable. Then $I(\aleph_1, \psi, L) \geqq 2^{\aleph_0}$.*

PROOF. By Keisler [6], just as in Morley [10], it follows that the set of L^*-types realized in uncountable models of ψ, is analytic and its cardinality is $\leqq \aleph_0$ or is $2^{\aleph_0}$. So by the hypothesis the cardinality is $2^{\aleph_0}$. By the downward Löwenheim-Skolem theorem (for $L^1_{\omega_1,\omega}(Q)$) each such type is realized in a model (of ψ) of cardinality $\aleph_1$. So if $I(\aleph_1, \psi, L) < 2^{\aleph_0}$, then in some model of ψ of cardinality $\aleph_1$, at least $\aleph_1$ types are realized, and we get a contradiction by 2.1(A).

THEOREM 2.3. *Let $L \subseteq L^1$, $\psi \in L^1_{\omega_1,\omega}(Q)$, $M \models \psi$, $\|M\| = \aleph_1$.*

(A) *If for every fragment L^*, in M only countably many L^*-types are realized,* then *ψ has a model N, $\|N\| = \aleph_1$ in which only $\aleph_0$ $L_{\omega_1,\omega}(Q)$-types are realized.*

(B) *If for every fragment L^*, over every countable $A \subseteq |M|$ in M only countably many L^*-types are realized* then *ψ has a model N, $\|N\| = \aleph_1$, in which only $\aleph_0$ $L_{\omega_1,\omega}(Q)$-types are realized over any countable $A \subseteq |M|$.*

PROOF.

(A) Define by induction on $\alpha < \omega_1$ the fragment L^*_α of $L_{\omega_1,\omega}(Q)$:

$$L_0^* = L(Q),$$

$$L_\alpha^* = \bigcup_{\beta<\alpha} L_\beta^* \text{ for limit } \alpha$$

and $L_{\alpha+1}^*$ is the minimal fragment closed under (Qx) which contains

$$L_\alpha^* \cup \{\wedge tp(\bar{a}, \phi, L_\alpha^*, M): \bar{a} \in |M|\}.$$

We can prove inductively that L_α^* is indeed countable: for $\alpha = 0$, α limit it is immediate, and for α a successor it follows by the hypothesis.

Now w.l.o.g. we can assume that $|M|$, the universe of M, is ω_1. Expand M to the model

$$M' = (M, <, E_0, \cdots, E_n, \cdots, F_0, \cdots, F_n, \cdots)_{n<\omega}$$

where:

(1) $<$ is the usual order of the ordinals,

(2) $E_n = \{\langle\alpha\rangle \hat{}\, \bar{a} \,\hat{}\, \bar{b}: l(\bar{a}) = l(\bar{b}) = n;\ \bar{a}, \bar{b} \in |M|;$

$$tp(\bar{a}, \phi, L_\alpha^*, M) = tp(\bar{b}, \phi, L_\alpha^*, M)\}$$

(3) F_n is an $n+1$-place function, and $F_n(\alpha, \bar{a}) \in \{m : m < \omega\}$ and $F_n(\alpha, \bar{a}) = F_n(\alpha, \bar{b}) \Leftrightarrow E_n(\alpha, \bar{a}, \bar{b})$.

(We can define F_n because the number of L_α^*-types realized in M is countable). It is easy to note that

(i) $E_n(\alpha, \bar{x}, \bar{y})$ is an equivalence relation (in M); it refines $E_n(\beta, \bar{x}, \bar{y})$ for $\beta < \alpha$; and it has $\leqq \aleph_0$ equivalence classes; and $<$ is an order with first element, 0, and $E_n(0, \bar{a}, \bar{b})$ iff the L_0^*-types of $\bar{a}$ and $\bar{b}$ are equal.

(ii) If $N \models E_n(\alpha+1, \bar{a}, \bar{b})$ then for every $c_1 \in N$ there is $c_2 \in N$ such that $N \models E_{n+1}(\alpha, \bar{a} \hat{}\langle c_1\rangle, \bar{b} \hat{}\langle c_2\rangle)$. Moreover if for $\aleph_1$ c's $N \models E_{n+1}(\alpha, \bar{a} \hat{}\langle c\rangle, \bar{a} \hat{}\langle c_1\rangle)$, then for $\aleph_1$, c's $N \models E_{n+1}(\alpha, \bar{b} \hat{}\langle c\rangle, \bar{b} \hat{}\langle c_2\rangle)$.

Clearly (i) and (ii) can be "expressed" by sentences ψ_1, ψ_2 of $L_{\omega_1,\omega}(Q)$ respectively (for (i) we need the F_n's).

By [5] there is a model N', such that: $\|N'\| = \aleph_1$, N' is a model of $\psi \wedge \psi_1 \wedge \psi_2$, $<^{N'}$ is not a well-ordering.

Clearly $N \models \psi$, $\|N\| = \aleph_1$, where N is the L^1-reduct of N'. So let $d_n \in |N'|$ $(n < \omega)$ be such that $N' \models d_{n+1} < d_n$. Let us define E_n^+: for sequences $\bar{a}, \bar{b}$, from $|N'|$ of length n, $\bar{a} E_n^+ \bar{b}$ holds *iff* for some m $N' \models E_n(d_m, \bar{a}, \bar{b})$.

As $N' \models \psi_1 \wedge \psi_2$ it is easy to check that the analogs of (i) and (ii) holds for N'. So it is easy to prove that for every $\varphi(\bar{x}) \in L_{\omega_1,\omega}(Q)$, $\bar{a} E_n^+ \bar{b} \Rightarrow N' \models \varphi[\bar{a}] \equiv \varphi[\bar{b}]$ (by induction on φ). As

$$N' \models E_n(d_0, \bar{a}, \bar{b}) \Rightarrow \bar{a} E_n^+ \bar{b} \Rightarrow tp(\bar{a}, \phi, L_{\omega_1,\omega}(Q), N) = tp(\bar{b}, \phi, L_{\omega_1,\omega}(Q), N)$$

and $E_n(d_0, \bar{x}, \bar{y})$ has $\leqq \aleph_0$ equivalence classes (in N') clearly $\{tp(\bar{a}, \phi, L_{\omega_1,\omega}(Q), N): \bar{a} \in N\}$ is countable, so N is the model we want.

(B) Essentially the same proof.

Lemma 2.4. *If $I(\aleph_1, \psi, L) \leqq \aleph_0, M \models \psi$ then in M only countably many $L_{\omega_1,\omega}(Q)$-types are realized.*

Proof. Let $\{M_i: i < \alpha\}$ be a maximal set of models of ψ of cardinality $\aleph_1$, realizing only countably many $L_{\omega_1,\omega}(Q)$-types, and with pairwise non-isomorphic L-reducts. By the hypothesis $I(\aleph_1, \psi, L) \leqq \aleph_0$, so clearly $\alpha < \omega_1$. Suppose that in M uncountably many $L_{\omega_1,\omega}(Q)$-types are realized and we shall get a contadiction.

Let L_i^* be a (countable) fragment of $L_{\omega_1,\omega}(Q)$ such that if $\bar{a}, \bar{b} \in |M_i|$ then

$$tp(\bar{a}, \phi, L_i^*, M_i) = tp(\bar{b}, \phi, L_i^*, M_i) \Leftrightarrow tp(\bar{a}, \phi, L_{\omega_1,\omega}(Q), M_i) = tp(\bar{b}, \phi, L_{\omega_1,\omega}(Q), M_i)$$

(exists by the choice of the M_i's).

Let L^* be a fragment of $L_{\omega_1,\omega}(Q)$ such that $L_i^* \subseteq L^*$ for $i < \alpha$ (exists as $\alpha < \omega_1$). As $I(\aleph_1, \psi, L) \leqq \aleph_0$, by 2.1(A) in M only countably many L^*-types are realized. As uncountably many $L_{\omega_1,\omega}(Q)$-types are realized, there are $\bar{a}, \bar{b} \in |M|$, which realized the same L^*-types, but for some $\varphi(\bar{x}) \in L_{\omega_1,\omega}(Q)$ $M \models \varphi[\bar{a}] \equiv \neg\varphi(\bar{b})$. Let

$$\psi_1 = (\exists \bar{x})(\exists \bar{y})(\varphi(\bar{x}) \equiv \neg\varphi(\bar{y}) \wedge \bigwedge_{\theta \in L^*} \theta(\bar{x}) \equiv \theta(\bar{y})).$$

So clearly $M_i \models \neg\psi_1$, $M \models \psi_1$, by the hypothesis on M and 2.3 there is a model N, $\|N\| = \aleph_1$, $N \models \psi \wedge \psi_1$ and in N only countably many $L_{\omega_1,\omega}(Q)$-types are realized. Clearly N contradicts the maximality of $\{M_i: i < \alpha\}$.

Definition 2.1. M is $(L^*, \aleph_0)$-homogeneous if when $tp(\bar{a}, \phi, L^*, M) = tp(\bar{b}, \phi, L^*, M)$, then for every $\bar{c} \in |M|$ there is $\bar{d} \in |M|$ such that

$$tp(\bar{a} \frown \bar{c}, \phi, L^*, M) = tp(\bar{b} \frown \bar{d}, \phi, L^*, M).$$

LEMMA 2.5. *Let $L \subseteq L^1$, M an L^1-model, and in M only countably many $L_{\omega_1,\omega}(Q)$-types are realized. Then (A) For some fragment L^* of $L_{\omega_1,\omega}(Q)$, M is $(L^*, \aleph_0)$-homogeneous.*

(B) *Moreover we can choose L^* so that for every $\bar{a} \in |M|$ there is $\varphi(\bar{x}) \in L^*$, such that $M \models \varphi[\bar{a}]$, and $\varphi(\bar{x})$ is $L_{\omega_1,\omega}(Q)$-complete, i.e., $\varphi(\bar{x}) \vdash tp(\bar{a}, \phi, L_{\omega_1,\omega}(Q), M))$.*

(C) *The sentence $\psi_1 = \wedge\{\psi : \psi \in L^*, M \models \psi\}$ is $L_{\omega_1,\omega}(Q)$-complete.*

PROOF. Easy.

3. Nice sentences and the amalgamation property

Here always $\psi \in L_{\omega_1,\omega}(Q)$, M and N are L-models.

DEFINITION 3.1. The sentence $\psi \in L_{\omega_1,\omega}(Q)$ is L^*-almost-nice (L^* a fragment of $L_{\omega_1,\omega}(Q)$) if

(1) $\psi \vdash (Qx)x = x$, ψ has a model and is $L_{\omega_1,\omega}(Q)$-complete

(2) every model of ψ is $(L^*, \aleph_0)$-homogeneous

(3) moreover if $M \models \psi$, $\bar{a} \in |M|$ then for some $\varphi(\bar{x}) \in L^*$, $M \models \varphi[\bar{a}]$ and $\varphi(\bar{x})$ is $L_{\omega_1,\omega}(Q)$-complete.

DEFINITION 3.2.

(A) The sentence ψ is almost nice if it is L^*-almost-nice for some L^*.

(B) The sentence ψ is nice if it is L-almost-nice and in (3) of Def. 3.1 the formula φ is atomic;

(C) $M \models$ "ψ" if M is a (first-order) atomic model of $T(\psi) = \{\psi_1 : \psi_1 \in L, M \models \psi \Rightarrow M \models \psi_1\}$. M is a non-standard model of ψ if $M \models \neg\psi$, $M \models$ "ψ".

(D) $M \models$ "$\varphi[\bar{a}]$" ($\varphi \in L_{\omega_1,\omega}(Q)$) if $\psi \vdash (\forall \bar{x})(\varphi(\bar{x}) \equiv R(\bar{x}))$, $R \in L$, $M \models R[\bar{a}]$, $M \models$ "ψ" and ψ is nice.

REMARK. Notice that $T(\psi)$ is a set of first order sentences. If ψ is nice $\psi = \psi^* \wedge Qx(x = x)$ for some ψ^* a Scott-sentence of a (first-order) prime model in which each type is isolated by a predicate.

LEMMA 3.1.

(A) *For every almost-nice ψ there is $L' \supseteq L$ and a nice $\psi' \in L'_{\omega_1,\omega}(Q)$ such that*

(1) *for every λ $I(\lambda, \psi) = I(\lambda, \psi')$*

(2) *the L-reduct of any model of ψ' is a model of ψ, and every model of ψ can be uniquely expanded to a model of ψ'.*

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(B) *If ψ is nice, there is exactly one model M (up to isomorphism) such that $M \models$"ψ", $\|M\| \leqq \aleph_0$ (this model is the prime model of $T(\psi)$).*

(C) *In Lemma 2.5(C) ψ_1 is almost nice.*

(D) *If M is a model of $T(\psi)$, where ψ is nice then:*

(α) *Assume $N < M$. Then $N \models$"ψ" iff every $\bar{a} \in |N|$ realizes an L-isolated type, i.e. there is $\varphi \in L$, such that $M \models \varphi[\bar{a}]$; $T(\psi)$, $\varphi(\bar{x}) \vdash tp(\bar{a}, \phi, L, M)$*

(β) *If $A \subseteq |M|$, $|A| \leqq \aleph_0$, and every $\bar{a} \in A$ realizes an isolated L-type, then there are N_1, N_2 such that N_2 is a model of $T(\psi)$, $A \subseteq |N_1|$, $N_1 < N_2$ $M < N_2$ and $N_1 \models$"ψ". If M is $\aleph_1$-saturated we can choose $N_2 = M$.*

Proof. Easy.

Lemma 3.2. *If $I(\aleph_1, \psi) \leqq \aleph_0$, then there are almost-nice sentences ψ_n $n \leqq \alpha \leqq \omega$ such that $\vdash [\psi \wedge (Q)x)(x = x)] \equiv \bigvee_{n<\alpha}\psi_n$.*

Proof. Let M_n $n < \alpha \leqq \omega$ be the models of ψ of cardinality $\aleph_1$. By Lemma 2.4 each M_n realizes only countably many $L_{\omega_1,\omega}(Q)$-types. Hence by 2.5 and 3.1(C) there is an almost nice sentence ψ_n^1 such that $M_n \models \psi_n^1$. Then $\psi_n = \psi \wedge \psi_n^1$ satisfies our requirements.

Definition 3.3. Let ψ be nice, $M \models$"ψ", $N \models$"ψ".

(A) $M < N$ *if* M is an elementary submodel of N.

(B) $M <^* N$ *if* $M < N$ and if $R(x, \bar{y}) \in L$, $\bar{a} \in |M|$, and $M \models$ "$\neg(Qx)R(x, \bar{a})$" then for no $c \in |N| - |M|$ does $N \models R[c, \bar{a}]$.

(C) $M <^{**} N$ *if* $M <^* N$ and if $R(x, \bar{y}) \in L$, $\bar{a} \in |M|$ and $M \models$"$(Qx)R(x\bar{a})$" then for some $c \in |N| - |M|$, $N \models R[c, \bar{a}]$.

Remark. Notice that if $M <^{**} N$ then $M \neq N$ (if there is a nice ψ such that $M \models$"ψ").

Lemma 3.3.

(A) *If ψ is nice, $M_i \models$"ψ" for $i < \omega_1$, $M_i <^* M_{i+1}$ for $i < j$, $M_\delta = \bigcup_{i<\delta} M_i$ for limit δ, and $\{i: M_i <^{**} M_{i+1}\}$ has cardinality $\aleph_1$* then $\bigcup_{i<\omega_1} M_i \models \psi$

(B) *If ψ is nice, $M \models$"ψ", $\|M\| = \aleph_0$ then for some N, $M <^{**} N \models$"ψ"*

(C) *The relations $<, <^*, <^{**}$ are transitive, and if $M_0 <^* M_1 <^{**} M_2$ or $M_0 <^{**} M_1 <^* M_2$ then $M_0 <^{**} M_2$.*

Proof. Immediate.

Definition 3.4. A nice sentence ψ has the λ-amalgamation property when: *if* $N_l \models$"ψ" for $l = 0, 1, 2$, $N_0 <^* N_l$, $\|N_l\| \leqq \lambda$ *then* there are M, f_1, f_2 such that $N_0 <^* M$, $M \models$"ψ", f_l is an embedding of N_l into M, $f_l \restriction |N_0| =$ the identity and $M \restriction \text{Range}(f_l) <^* M$ (for $l = 1, 2$).

LEMMA 3.4. *Suppose $V = L$ or even $\diamondsuit_{\aleph_1}$.*

If ψ is nice but does not have the $\aleph_0$-amalgamation property then $I(\aleph_1, \psi) = 2^{\aleph_1}$.

PROOF. Trivially $I(\aleph_1, \psi) \leqq 2^{\aleph_1}$. Let $\{S_i : i < \omega_1\}$ be a partition of ω_1 to $\aleph_1$ pairwise disjoint stationary sets (see e.g. [17]), by Jensen's diamond [4] there are for $\alpha < \omega_1$, a function $f_\alpha : \alpha \to \alpha$, and L-models M^0_α, M^1_α with universe $\omega(1+\alpha)$ such that for every function $g : \omega_1 \to \omega_1$, and L-models M_0, M_1 with universe ω_1; $\{\alpha : \alpha \in S_i, g \restriction \alpha = f_\alpha, M_l \restriction \omega(1+\alpha) = M^{l,i}_\alpha$ for $l = 0, 1\}$ is stationary for every $i < \omega_1$. Let N_0, N_1, N_2 contradict the $\aleph_0$-amalgamation property and w.l.o.g. $N_0 <^{**} N_1$, $N_0 <^{**} N_2$. Now for any set $S \subseteq \omega_1$ we define M^S_α $(\alpha < \omega_1)$ by induction on α, such that $|M^S_\alpha| = \omega(1+\alpha)$, $M^S_\alpha \models$"ψ", $\beta < \alpha \Rightarrow M^S_\beta <^* M^S_\alpha$. For $\alpha = 0$, or α a limit ordinal there is no problem. If M^S_α is defined let g be an isomorphism from N_0 onto M^0_α. If $M^S_\alpha = M^l_\alpha$, $\alpha \in S_i$, and $i \in S \Leftrightarrow l = 0$ choose $M^S_{\alpha+1}$ so that g (if $l = 0$) or $f_\alpha g$ (if $l = 1$) cannot be extended to an isomorphism from N_l onto $M^S_{\alpha+1}$. In any case choose $M^S_{\alpha+1}$ so that $|M^S_{\alpha+1}| = \omega(1+\alpha+1)$, $M^S_\alpha <^{**} M^S_{\alpha+1}$.

Let $M^S = \bigcup_{\alpha<\omega_1} M^S_\alpha$, so clearly $M^S \models \psi$, $\|M^S\| = \aleph_1$. It is easy to see that $M^{S(1)} \cong M^{S(2)}$ implies that $\cup\{S_i : i \in S(1)\}$, $\cup\{S_i : i \in S(2)\}$ are equal modulo the filter of closed unbounded subsets of ω_1, hence $S(1) = S(2)$.

DEFINITION 3.5.

(A) A nice ψ is $(\lambda, 1)$-stable *if* $M \models$"ψ", $A \subseteq |M|$, $|A| \leqq \lambda$, *implies* $|\{tp(\bar{a}, A, L, M) : \bar{a} \in |M|\}| \leqq \lambda$

(B) A nice ψ is λ-stable if $M \models$"ψ", $A \subseteq |M|$, $|A| \leqq \lambda$ implies

$$|\{tp(\bar{a}, A, L, N) : \bar{a} \in N, N \models \text{“}\psi\text{”}, M <^* N\}| \leqq \lambda.$$

LEMMA 3.5. *Assume ψ is nice and has the $\aleph_0$-amalgamation property,*

(A) *ψ is $\aleph_0$-stable iff ψ is $(\aleph_0, 1)$-stable.*

(B) *Assume $2^{\aleph_0} = \aleph_1$; then ψ has an $\aleph_1$-model-homogeneous M of power $\aleph_1$ (i.e.* if $N_1 <^* M$, $N_2 <^* M$, $\|N_1\| = \aleph_0$, *f an isomorphism from N_1 onto N_2,* then f *can be extended to an automorphism of M).*

PROOF.

(A) The direction $\Rightarrow$ is always true, and the direction $\Leftarrow$ follows by the $\aleph_0$-amalgamation property.

(B) Easy.

4. Rank

Let $\psi \in L_{\omega_1,\omega}(Q)$ be nice.

DEFINITION 4.1. Suppose ψ is nice, $M \models$"ψ". For every L-type p with m variable over a finite subset of $|M|$ we define its rank $R^m(p) = R^m(p, M)$ as an ordinal, -1, or ∞, as follows: We define by induction when $R(p) \geqq \alpha$, and then

$$R(p) = -1 \Leftrightarrow R(p) \not\geqq 0,$$

$$R(p) = \alpha \Leftrightarrow R(p) \geqq \alpha \wedge R(p) \not\geqq \alpha + 1,$$

$$R(p) = \infty \Leftrightarrow (\forall \alpha) R(p) \geqq \alpha.$$

(A) $R(p) \geqq 0$ if p is realized in M.

(B) $R(p) \geqq \delta$ (for a limit ordinal δ) if for every $\alpha < \delta\, R(p) \geqq \alpha$.

(C) $R(p) \geqq \alpha + 1$ if the following conditions are satisfied

(α) there are $\varphi \in L$ and $\bar{a} \in |M|$ such that $R^m(p \cup \{\varphi(\bar{x}, \bar{a})\}) \geqq \alpha$, $R^m(p \cup \{\neg\varphi(\bar{x}, \bar{a})\}) \geqq \alpha$

(β) for every $\bar{a} \in |M|$ there is $P(\bar{x}, \bar{a})$ and $\bar{c} \in |M|$ $(l(\bar{x}) = l(\bar{c}) = m)$ such that $P(\bar{x}, \bar{a}) \vdash tp(\bar{c}, \bar{a}, L, M)$ (so $P(\bar{x}, \bar{a},)$ is complete), $R^m(p \cup \{P(\bar{x}, \bar{a},)\}) \geqq \alpha$

(γ) If $M \models$"$\neg(Qy)P(y, \bar{a})$" and $p \vdash (\exists y)[\psi(y, \bar{x}, \bar{c}) \wedge P(y, \bar{a})]$ *then* for some $d \in |M|$, $M \models P[d, \bar{a}]$ and $R^m(p \cup \{\psi(d, \bar{x}, \bar{c})\}) \geqq \alpha$.

REMARK. A natural ordering is defined among the possible ranks by stipulating $-1 < \alpha < \infty$ for any ordinal α.

DEFINITION 4.2. For any not necessarily finite p,

$$R^m(p) = \min\{R^m(q) \colon q \subseteq p, |q| < \aleph_0\}$$

LEMMA 4.1.

(A) *$R^m(\varphi(\bar{x}, \bar{a}), M)$ depends only on $tp(\bar{a}, \phi, L, M)$.*

(B) *$p \vdash q$ implies $R^m(p) \leqq R^m(q)$.*

(C) *$R^m(p) \geqq \omega_1$ implies $R^m(p) = \infty$.*

(D) If $M < {}^*N$, $N \models$"ψ", $M \models$"ψ", $\bar{b} \in |M|$, $\bar{a} \in N$, $\models \varphi[\bar{a}, \bar{b}]$, $R^m(tp(\bar{a}, |M|, L, N) = R^m(\{\varphi(\bar{x}, \bar{b}\})$, $A \subseteq |N|$, $\bar{b} \in A$ then *there is a unique complete L-type p_A over A realized in some N', $N < {}^*N' \models$"ψ", which contains $\varphi(\bar{x}, \bar{b})$ and has the same rank. So $A \subseteq B \Rightarrow p_A \subseteq p_B$ and p_A does not split over* $\bar{b}$, i.e. *if*

$$\bar{c}_1, \bar{c}_2 \in A, tp(\bar{c}_1, \bar{a}, L, N) = tp(\bar{c}_2, \bar{a}, L, N)$$

and $\psi \in L$ then $\psi(\bar{x}, \bar{c}_1, \bar{a}) \in p_A \Leftrightarrow \psi(\bar{x}, \bar{c}_2, \bar{a}) \in p_A$.

PROOF.

(A) Prove by induction on α that the truth of $R^m(\varphi(\bar{x},\bar{a}),M)\geqq\alpha$ depends only on $tp(\bar{a},\phi,L,M)$.

(B) Easy.

(C) By (A) the number of possible ranks is countable, hence necessarily for some $\alpha_0<\omega_1$ for no p $R^m(p,M)=\alpha_0$. Now prove by induction on $\alpha\geqq\alpha_0$ that $R^m(p,M)\geqq\alpha_0$ implies $R^m(p,M)\geqq\alpha+1$ (for α_0 this is the definition of α_0, for α limit — immediate and $\alpha=\beta+1$ use the definition of rank and the induction hypothesis).

(D) Easy.

LEMMA 4.2. *The following conditions on* ψ *satisfy* (B) $\Rightarrow$ (A) $\Leftrightarrow$ (C) $\Rightarrow$ (D)

(A) ψ *is* $\aleph_0$*-stable.*

(B) ψ *is* $(\aleph_0,1)$*-stable and has the* $\aleph_0$*-amalgamation property.*

(C) *For every finite* p *over* M, $M\models$"ψ", $R^m(p,M)<\infty$.

(D) (α) ψ *is* $(\aleph_0,1)$*-stable, and*

(β) *if* $N,M\models$"ψ", $N<^*M$, $\bar{a}\in|M|$, *then* $tp(\bar{a},|N|,L,M)$, *is definable over a finite set* $\subseteq|N|$, *where*

DEFINITION 4.3. Let $A\subseteq B\subseteq M\models$"$\psi$", $\bar{a}\in|M|$, then $tp(\bar{a},B,L,M)$ is definable over A, if for every $P_1(\bar{x},\bar{y})$ there is $P(\bar{y},\bar{b})$, $\bar{b}\in A$ such that for every $\bar{c}\in|B|$, $M\models P_1(\bar{a},\bar{c})\Leftrightarrow M\models P(\bar{c},\bar{b})$.

REMARK. Not necessarily all the conditions are equivalent.

PROOF.

(B) $\Rightarrow$ (A): This holds by 3.5(A).

(A) $\Rightarrow$ (C): Let M be an $\aleph_1$-saturated model of $T(\psi)$ and $N<M$, $\|N\|=\aleph_0$, $N=$"ψ". Then we prove by standard techniques (see e.g. Keisler [6]).

CLAIM 4.3. Let M be an $\aleph_1$-saturated model of $T(\psi)$, $A\subseteq|M|$, $|A|\leqq\aleph_0$. Then there is a model N, such that

(i) $N<M$, $A\subseteq|N|$, $\|N\|=\aleph_0$

(ii) let $\bar{a}\in A$, $M\models$"$\neg(Qx)\varphi(x,\bar{a})$" $(\varphi\in L)$ then for some $c\in|N|-A$, $M\models\varphi[c,\bar{a}]$ *iff* there are $\theta\in L$, $\bar{b}\in A$,

$$M\models(\exists y)\theta(y,\bar{b})\wedge(\forall y)(\theta(y,\bar{b})\rightarrow\varphi(y,\bar{a}))$$

but for no $c\in A$, $M\models\theta(c,\bar{b})$. Then it is easy to prove that if $R^m(p)=\infty$, for some p, then there are in M $\bar{a}_i$, $i<2^{\aleph_0}$, satisfying the conditions of 4.3, and realizing in M over $|N|$ distinct L-types such that by 4.3 there are N_i,

$|N| \cup \bar{a}_i \subseteq |N_i|$, $N <^* N_i$, $N_i < M$ (remember $R^m(p) \geqq \omega_1 \Rightarrow R^m(p) > \omega_1$, and notice that the definition of rank is tailored for this proof.

(C) $\Rightarrow$ (D), (A): Let $N \models$"ψ", $\|N\| = \aleph_0$, $N <^* M \models$"ψ", and $\bar{a} \in |M|$. Then by (C) and 4.1 there is $P(\bar{x}, \bar{b}) \in p_{\bar{a}} = tp(\bar{a}, |N|, L, M)$ with minimal rank, which is $\alpha < \infty$. Clearly by the definition of rank and the choice of $P(\bar{x}, \bar{b})$, $R^m(\{P(\bar{x}, \bar{b})\}) \not\geqq \alpha + 1$ implies that for no $P_1(\bar{x}, \bar{b}_1)(\bar{b}_1 \in |N|)$ do

$$R^m(\{P(\bar{x}, \bar{b}), P_1(\bar{x}, \bar{b}_1)\}) \geqq \alpha$$

$$R^m(\{P(\bar{x}, \bar{b}), \neg P_1(\bar{x}, \bar{b}_1)\}) \geqq \alpha,$$

both hold; so exactly one holds, the one contained in $p_{\bar{a}}$. This proves that $p_{\bar{a}}$ is definable over a finite subset of $N(=\bar{b})$ so (D) (β) holds. As the number of such definitions is $\leqq \|N\| + \aleph_0$ also (D) (α) (A) holds.

Lemma 4.4. *Suppose ψ is nice and $\aleph_0$-stable, $M <^* N$, $\|N\| = \aleph_0$ $M \models$"ψ", $N \models$"ψ", $\bar{a} \in |N|$. Then there is a prime model M' over $|M| \cup \bar{a}$, i.e. $M <^* M' < N$, and if $M <^* N'$, $\bar{a}' \in N'$, $tp(\bar{a}, |M|, L, N) = tp(\bar{a}', |M|, L, N')$, then there is an elementary imbedding f of M' into N', which is the identity over $|M|$, and $f(\bar{a}) = \bar{a}'$, and $N' \restriction$ Range $f < N'$.*

M' is, in fact, the prime model of the first-order theory of $(N, c)_{c \in |M| \cup \bar{a}}$.

Question. Can we demand $M' <^* N$, $N' \restriction$ Range $f <^* N'$?

Remark. (Until then this lemma is interesting mainly for $\psi \in L_{\omega_1,\omega}$.)

Proof. Clearly it suffices to prove:

(*) *If $N \models (\exists y) \varphi(y, \bar{a}, \bar{b})$ ($\varphi \in L$) where $\bar{b} \in |M|$, then there is $\varphi_1(y, \bar{a}, \bar{b}_1)$ ($\bar{b}_1 \in |M|$, $\varphi_1 \in L$) such that $N \models (\forall y)(\varphi_1(y, \bar{a}, \bar{b}_1) \rightarrow \varphi(y, \bar{a}, \bar{b}))$ and $\varphi_1(y, \bar{a}, \bar{b}_1)$ isolates a complete L-type of y over $|M| \cup \bar{a}$, and $N \models (\exists y) \varphi_1(y, \bar{a}, \bar{b}_1)$.*

Proof of (*). Choose $\theta(y, \bar{x}, \bar{c})(\bar{c} \in |M|, \theta \in L)$ such that

(i) $N \models (\exists y)(\theta(y, \bar{a}, \bar{c}) \wedge \varphi(y, \bar{a}, \bar{b}))$

(ii) $R^{m+1}(tp(\bar{a}, |M|)) \cup \{\theta(y, \bar{x}, \bar{b})\})$ $(m = l(\bar{a}))$ is minimal assuming (i) holds.

It is easy to see that $\theta(y, \bar{a}, \bar{c}) \wedge \varphi(y, \bar{a}, \bar{b})$ isolates a complete L-type over $|M| \cup \bar{a}$, so we finish.

5: The order property

Let ψ be nice and $\aleph_0$-stable.

Definition 5.1. We say that ψ has the order property if there is a model M of ψ and $\bar{a}_\alpha \in |M|$ $(\alpha < \omega_1)$ and formula $\varphi(\bar{x}, \bar{y}) \in L$ such that $M \models \varphi[\bar{a}_\alpha, \bar{a}_\beta] \Leftrightarrow \alpha \leqq \beta$.

DEFINITION 5.2.

(A) We say that ψ has the symmetry property *if* for $M <^* N$, $N \models$ "ψ", $M \models$ "ψ"; $\bar{a}, \bar{b} \in |N|$

$$R(tp(\bar{a}, |M| \cup \bar{b}, L, N) = R(tp(\bar{a}, |M| L, N))$$

iff

$$R(tp(\bar{b}, |M| \cup \bar{a}, L, N)) = R(tp(\bar{b}, |M|, L, M)).$$

(B) We say that ψ has the asymmetry property if there are $M, N, \bar{a}, \bar{b}$ as above such that

(i) $R(tp(\bar{a}, |M| \cup \bar{b}, L, N)) = R(tp(\bar{a}, |M|, L, N))$

(ii) for some $E = E(\bar{x}_1, \bar{x}_2, \bar{z}) \in L$, $E(\bar{x}_1, \bar{x}_2, \bar{a})$ is an equivalence relation with $\aleph_0$ equivalence classes(in any model $N'; N <^* N' \models \psi$) and $\bar{b}$ is not $E(\bar{x}_1, \bar{x}_2, \bar{a})$ equivalent to any sequence from $|M|$.

THEOREM 5.1. *The following properties of ψ are equivalent (for nice $\aleph_0$-stable ψ)*

(A) *ψ has the order property.*

(B) *ψ does not have the symmetry property.*

(C) *ψ has the asymmetry property.*

PROOF.

(B) $\Rightarrow$ (A).

Let $M, N, \bar{a}, \bar{b}$ be a counter example to the symmetry property, and let $\varphi(\bar{x}, \bar{y}, \bar{c})$ ($\bar{c} \in |M|, \varphi \in L$) be such that:

(i) $N \models \varphi[\bar{a}, \bar{b}, \bar{c}]$

(ii) $R(\{\varphi(\bar{x}, \bar{b}, \bar{c})\}) < R(tp(\bar{a}, |M|, L, M))$

(by the symmetry between $\bar{a}$ and $\bar{b}$ we can assume this). We can also assume w.l.o.g. that $\|N\| = \aleph_0$.

Now define by induction on $\alpha < \omega_1$ models N_α; and sequences $\bar{a}_\alpha$, $\bar{b}_\alpha$ for limit α only such that:

(1) $\|N_\alpha\| = \aleph_0$

(2) for limit $\alpha, N_\alpha = \bigcup_{\beta<\alpha} N_\beta$ and $N_0 = N$

(3) $N_\alpha <^* N_{\alpha+1}$, $N_{\alpha+2} <^{**} N_{\alpha+3}$.

(4) for limit α, $\bar{a}_\alpha \in N_{\alpha+1}$ and $tp(\bar{a}_\alpha, |N_\alpha|, L, N_{\alpha+1})$ extends and has the same rank, as $tp(\bar{a}, |M|, L, N)$.

(5) for limit α, $\bar{b}_\alpha \in |N_{\alpha+2}|$ and $tp(\bar{b}_\alpha, |N_{\alpha+1}|, L, N_{\alpha+2})$ extends, and has the same rank, as $tp(\bar{b}, |M|, L, N)$.

This is easy to do. Clearly by (4) and (2) and Lemma 4.1A

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$N_{\alpha+1} \models \neg\varphi[\bar{a}_\alpha, \bar{b}, \bar{c}]$. As $tp(\bar{b}_\beta, |M|, L, N_{\beta+2}) = tp(\bar{b}, |M|, L, N_{\beta+2})$ and as by 4.1D $tp(\bar{a}_\alpha, |N_\alpha|, L, N_{\alpha+1})$ does not split over $|M|$, necessarily $\beta < \alpha \Rightarrow N_{\alpha+1} \models \neg\varphi[\bar{a}_\alpha, \bar{b}_\beta, \bar{c}]$.

Similarly we can prove that for $\alpha \leqq \beta$,

$$tp(\bar{a}_\alpha{}^\wedge \bar{b}_\beta, |M|, L, N_{\beta+2}) = tp(\bar{a}{}^\wedge \bar{b}, |M|, L, N_{\beta+2})$$

hence $N_{\beta+2} \models \varphi[\bar{a}_\alpha, \bar{b}_\beta, \bar{c}]$. As $N^* = \bigcup_{\alpha<\omega_1} N_\alpha$ is a model of ψ (by 3.3(A)) letting $\bar{c}_\alpha = \bar{a}_\alpha{}^\wedge \bar{b}_\alpha{}^\wedge \bar{c}$ and $\theta(\bar{x}_1, \bar{y}_1, \bar{z}_1; \bar{x}_2, \bar{y}_2, \bar{z}_2) = \varphi(\bar{x}_1, \bar{y}_2, z_2)$ we find that $N \models \psi$ and $N \models \theta[\bar{c}_\alpha, \bar{c}_\beta] \Leftrightarrow \alpha \leqq \beta$. So we finish.

(C) $\Rightarrow$ (B).

Let $M, N, \bar{a}, \bar{b}, E$ be as in Definition 5.2(B). Clearly it suffices to prove $p_1 = tp(\bar{b}, |M| \cup \bar{a}, L, N)$ has rank smaller than that of $p_2 = tp(\bar{b}, |M|, L, N)$. Suppose not, and let $\varphi(\bar{x}, \bar{c}) \in p_2$ has the same rank as p_2, so that (using 4.1B) $R(tp(\bar{a}, \bar{c}, L, N)) = R(tp(\bar{a}, M, L, N))$. Choose $\bar{b}' \in |M|$, $tp(\bar{b}', \bar{c}, L, M) = tp(\bar{b}, \bar{c}, L, M)$, and define models $N_\alpha (\alpha < \omega_1)$ so that $N_\alpha <^{**} N_{\alpha+1}$, $N_\delta = \bigcup_{\alpha<\delta} N_\alpha \models$ "ψ", $\|N_\alpha\| = \aleph_0$, and $\bar{b}_\alpha \in N_{\alpha+1}$, $N_\alpha \models$ "$\varphi(\bar{b}_\alpha, \bar{c})$ and $R(tp(\bar{b}_\alpha, N_\alpha, L, N_{\alpha+1})) = R(\varphi(\bar{x}, \bar{c}))$. As $E(\bar{x}_1, \bar{x}_2, \bar{a})$ has in $\bigcup_{\alpha<\omega_1} N_\alpha$ only $\aleph_0$ equivalence classes, for some $\beta < \alpha < \omega_1$, $E(\bar{b}_\alpha, \bar{b}_\beta, \bar{a})$. We can assume not (B), so $R(tp(\bar{b}', \bar{c}{}^\wedge \bar{a}, L, N)) = R(\{\varphi(\bar{x}, \bar{c})\})$, so by 5.2B (below) $E(\bar{b}', \bar{b}, \bar{a})$, contradicting the definition 5.2(B).

(A) $\Rightarrow$ (C)

During this proof we shall prove several claims. Of course we can assume $\|N\| = \aleph_1$.

CLAIM 5.2. Suppose $N \models \psi$, and I^* is a set of $\aleph_1$ sequences from N and $A \subseteq |N|$ is countable, and $\|N\| = \aleph_1$.

(A) We can find an $N_\alpha <^* N$, $A \subseteq |N_0|$, $N_\alpha <^{**} N_{\alpha+1}$, $N_\delta = \bigcup_{\alpha<\delta} N_\alpha$, $N = \bigcup_{\alpha<\omega_1} N_\alpha$ and $\bar{a}_\alpha \in |N_{\alpha+1}|, \bar{a}_\alpha \not\in |N_\alpha|, \bar{a}_\alpha \in I^*$ and $\bar{c} \in |N_0|$ and $\varphi \in L$ such that $N \models \varphi[\bar{a}_\alpha, \bar{c}]$, and $R(tp(\bar{a}_\alpha, |N_\alpha|, L, N)) = R(\{\varphi(\bar{x}, \bar{c})\})$.

(B) The conditions of (A) or even $R(tp(\bar{a}_\alpha, \bigcup_{\beta<\alpha} \bar{a}_\beta \cup A, L, N)) = R(\{\varphi(\bar{x}, \bar{c})\})$ and $N \models \varphi(\bar{a}_\alpha, \bar{c})$ implies $\{\bar{a}_\alpha : \alpha < \omega_1\}$ is an indiscernible sequence over A, i.e. if

$$\alpha(l,1) < l(l,2) \cdots < \alpha(l,n) < \omega_1 (l = 1,2, n < \omega)$$

then

$$tp(\bar{a}_{\alpha(1,1)}{}^\wedge \bar{a}_{\alpha(1,2)}{}^\wedge \cdots {}^\wedge \bar{a}_{\alpha(1,n)}, A, L, N) = tp(\bar{a}_{\alpha(2,1)}{}^\wedge \bar{a}_{\alpha(2,2)}{}^\wedge \cdots {}^\wedge \bar{a}_{\alpha(2,n)}, A, L, N)$$

(in any case we assume $\varphi(\bar{x}, \bar{c})$ is as in (A)).

(C) If ψ does not have the order property, in (B) we get that $\{\bar a_\alpha : \alpha < \omega_1\}$ is an indiscernible set over A (i.e. we demand only that $\{\alpha(l, i): i = 1, n\}$ are distinct.

PROOF.

(A) We can easily find appropriate N_α's. Now for $\alpha < \omega_1$, choose inductively $\bar a^1_\alpha \in I$, $\bar a^1_\alpha \notin |N_\alpha|$, $\bar a^1_\alpha \notin \{\bar a^1_\beta : \beta < \alpha\}$, and choose $\varphi_\alpha \in L$, $\bar b_\alpha \in |N_\alpha|$ so that $R(tp(\bar a^1_\alpha, |N_\alpha|, L, N) = R(\varphi_\alpha(\bar x, \bar b_\alpha))$ and $N \models \varphi_\alpha(\bar a^1_\alpha, b_\alpha)$.

By a theorem of Fodour [2] it follows that there is $S \subseteq \omega_1$, $|S| = \aleph_1$ such that $\alpha \in S \Rightarrow \varphi_\alpha = \varphi$, $\bar b_\alpha = \bar b$. By renaming we get our conclusion.

(B) and (C). The proof essentially is as in Morley [9], Shelah [13].

DEFINITION 5.2. Let $M \models$"ψ", J an ordered set, and $\bar a_t \in |M|$ for $t \in J$. Then the indexed set $\{\bar a_t : t \in J\}$ is called nice in M if for every $\bar b \in |M|$ there is a finite set $S \subseteq J$ such that if $t(1) \approx t(2) \bmod S$ [i.e. $(\forall t \in S)$ $(t < t(1) \equiv t < t(2) \wedge t = t(1) \equiv t = t(2))$] then $tp(\bar a_{t(1)} {}^\wedge \bar b, \phi, L, M) = tp(\bar a_{t(2)} {}^\wedge \bar b, \phi, L, M)$.

CLAIM 5.3.

(A) The indexed set $\{\bar a_\alpha : \alpha < \omega_1\}$ from 5.2A is nice in N

(B) If $\{a_t : t \in J\}$ is nice in M, $M <^* N \models$"ψ" then it is nice in N.

PROOF.

(A) Let $\bar b \in N$, so for some α $\bar b \in |N_{\alpha+1}|$, $\bar b \notin N_\alpha$ or $\bar b \in |N_0|$. If $\bar b \in |N_0|$ clearly $S = \phi$ will do. We prove the existence of $S = S(b)$ by induction on α. So by 4.1C for some $\bar c \in |N_\alpha|$ $tp(\bar b, |N_\alpha|, L, N)$ does not split over $\bar c$. Choose $S(\bar b) = \{\alpha\} \cup S(\bar c)$, and clearly this will do.

(B) For every $\bar b \in N$ choose $\bar c \in |M|$ so that $tp(\bar b, |M|, L, N)$ does not split over $\bar c$. Clearly if $t(1), t(2) \in J$, $t(1) \approx t(2) \bmod S(\bar c)$ ($S(\bar c)$ — the S we can choose for $\bar c$ by Definition 5.3) then $tp(\bar b {}^\wedge \bar a_{t(1)}, \phi, L, N) = tp(\bar b {}^\wedge \bar a_{t(2)}, \phi, L, N)$. So we finish.

CONTINUATION OF THE PROOF OF 5.1, (A) $\Rightarrow$ (C)

So let $N, N_\alpha, \bar a_\alpha, \varphi(\bar x, \bar c)$ $(\alpha < \omega_1)$ be as in 5.2A. We can assume $|N| = \omega_1$, $|N_\alpha| = \omega\alpha$.

Now it is known (see e.g. [5]) that if $\theta \in L_{\omega_1, \omega}(Q)$ has a model of order type ω_1, then it has a model which is countable and has an order type which contains a copy of the rationals.

Hence, using extra-predicates, there is an ordered set J, models $N_t (t \in J)$ and elements $\bar a_t (t \in J)$ such that

(1) J, N_t are countable, and $N_{t(0)} = N_0$ where $t(0)$ is the first element of J, and J contains a copy of the rationals.

(2) $N_t \models$ "ψ"

(3) $t(1) < t(2) \in J \Rightarrow N_{t(1)} <^{**} N_{t(2)}$, and let $N^* = \bigcup_{t \in J} N_t$

(4) for each $\bar{a} \in \bigcup_{t\in J} |N_t| - |N_{t(0)}|$ there is $t = t_{\bar{a}} \in J$ such that $a \in |M_{t+1}|$, $\bar{a} \notin |M_t|$ ($t+1$ – the successor of t)

(5) $\bar{c} \in |N_{t(0)}|$, $N_{t+1} \models \varphi[\bar{a}_t, \bar{c}]$ and $tp(\bar{a}_t, |N_t|, L, N_{t+1})$ has the same rank as $\varphi(\bar{x}, \bar{c})$

(6) for each $\bar{a} \in N^*$ there is a finite $S(\bar{a}) \subseteq J$ such that $t(1), t(2) \in J$, $t(1) \sim t(2) \bmod S(\bar{a})$ implies $tp(\bar{a} \,^\wedge \bar{a}_{t(1)}, \phi, L, N^*) = tp(\bar{a} \,^\wedge \bar{a}_{t(2)}, \phi, L, N^*)$

(7) for each $\bar{b} \in |N_{t+1}| - |N_t|$ there are $n, t(1) < \cdots < t(n) = t$ and $\bar{b}_0 \in |N_o|$, and $\bar{b}_l \in |N_{t(l)+1}|$, $\bar{b}_l \notin |N_{t(l)}$, such that, for $0 \leqq k \leqq l \leqq n$, $tp(\bar{b}_l, N_{t(k)}, L, N^*)$, $tp(\bar{b}_l, \bar{b}_k, L, N^*)$ have the same rank.

Remark. For the original N_α's, (7) follows immediately.

As J contains a copy of the rational order, it has a Dedekind cut (J_1, J_2) (J_1 — the lower part) with no last element in J_1 nor first element in J_2, (and $J_1 \neq \emptyset$, $J_2 \neq \emptyset$).

By (6) there is an $\aleph_1$-saturated model M of $T(\psi)$, $N^* < M$, and $\bar{a}^* \in |M|$ so that for $\bar{b} \in N^*$, $\varphi \in L$.

$M \models \varphi(\bar{a}^*, \bar{b}) \Leftrightarrow$ there are $t(1) \in I_1$, $t(2) \in I_2$ so that $t(1) < t < t(2)$ implies $N^* \models \varphi[\bar{a}_t, \bar{b}]$.

Clearly for every $\bar{c} \in |N^*| \cup \bar{a}^*$, $tp(\bar{c}, \phi, L, M)$ is isolated. If there is a model M', $N^* <^* M' < M$, $\bar{a}^* \in M'$, $M' \models$ "ψ", then $tp(\bar{a}^*, |N^*|, L, M')$ split over every finite set $\subseteq |N^*|$, contradiction. By 4.3 there are $\bar{c}_1 \in |N^*|$, $\theta_1, \theta_2 \in L$ such that

(α) $N^* \models$ "$\neg(Qx)\,\theta_1(x, \bar{c}_1)$"

(β) $M \models (\exists\, y)\,\theta_2(y, \bar{a}^*, \bar{c}_1)$

(γ) $M \models (\forall\, y)(\forall\, \bar{x})(\forall\, \bar{z})[\theta_2(y, \bar{x}, \bar{z}) \rightarrow \theta_1(y, \bar{z})]$

(δ) for no $d \in |N^*|$, $N^* \models \theta_1(d, \bar{c}_1)$ and $M \models \theta_2[d, \bar{a}^*, \bar{c}_1]$.

By (7) we can find $t(1) \in I_1$, $t(2) \in I_2$ and $\bar{c}_2 \in N_{t(1)}$ such that $tp(\bar{c}_1, |N_{t(2)}|, L, N^*)$, $tp(\bar{c}_1, \bar{c}_2, L, N^*)$ have the same rank. By notational changes we can assume $t(1) = t(0)$, $\bar{c}_2 = \bar{c}$, $\bar{c}_1 \in N_{t(2)+1}$. Let

$$E(\bar{x}_1, \bar{x}_2; \bar{z}) = (\forall\, y)\,[\theta_2(y, \bar{x}_1, \bar{z}) \equiv \theta_2(y, \bar{x}_2, \bar{z})].$$

Clearly $E(\bar{x}_1, \bar{x}_2; \bar{z})$ is an equivalence relation, and if $N^* \alpha^*\, M_1 \models$ "ψ", $\bar{c}' \in M_1$. $M_1 \models$ "$\neg(Qy)\theta_1(y, \bar{c}')$" then in M_1 $E(\bar{x}_1, \bar{x}_2; \bar{c}')$ has $\leqq \aleph_0$ equivalence classes (by the $\aleph_0$-stability of ψ). Hence if $\bar{c}^1 \in |M_1|$, $M_1 <^* M_2 \models$ "ψ", $M_1 \models$ "$\neg(Qy)\theta_1(y, \bar{c}^1)$" then there is in M_2 no new $E(\bar{x}_1, \bar{x}_2; \bar{c}^1)$-equivalence class.

So $E(\bar{x}_1, \bar{x}_2; \bar{c}_1)$ has $\aleph_0$ equivalence classes: it has $\leq \aleph_0$ by the previous argument, and $t(3) < t(4) < t(2)$ implies $N^* \models \neg E(\bar{a}_{t(3)}, \bar{a}_{t(4)}; \bar{c}_1)$. The last formula implies of course that $a_{t(0)}$ is not $E(\bar{x}_1, \bar{x}_2; \bar{c}_1)$-equivalent to any sequence from $N_{t(0)}$. So clearly (C) holds with $N_0, N^*, \bar{c}_1, \bar{a}_{t(0)}$ for $M, N, \bar{a}, \bar{b}$ respectively.

THEOREM 5.4. *If* ψ *(is nice,* $\aleph_0$*-stable and) has the asymmetry property then* $I(\aleph_1, \psi) = 2^{\aleph_1}$.

PROOF. Let $M, N, \bar{a}, \bar{b}, E$ be as in Definition 5.2(B). $\|N\| = \aleph_0$ w.l.o.g. Now we define by induction on $\alpha < \omega_1$ models N_α such that:

(1) $N_0 = N$

(2) $N_\alpha \models$ "ψ", $\|N_\alpha\| = \aleph_0$

(3) $N_\alpha <^* N_{\alpha+1}$ and $N_{\alpha+1} <^{**} N_{\alpha+2}$. Moreover every L-type over $N_{\alpha+1}$ realized in some N', $N_{\alpha+1} <^* N'$, is realized in $N_{\alpha+2}$.

(4) $N_\delta = \bigcup_{i<\delta} N_i$ for limit δ

(5) $N_{\delta+1}$ is prime over $|N_\delta| \cup \bar{a}_\delta$ (see Lemma 4.4) where $tp(\bar{a}_\delta, |N_\delta|, L, N_{\delta+1})$ extend and has the same rank as $tp(\bar{a}, |M|, L, N)$; for limit δ.

(6) $\bar{b}_{\beta+1} \in |N_{\beta+2}|$

Where $tp(b_{\beta+1}, |N_{\beta+1}|, L, N_{\beta+2})$ extend and has the same rank as $tp(\bar{b}, |M|, L, N)$

So clearly $N^* = \bigcup_{\alpha<\omega_1} N_\alpha \models \psi$. Note that if $\delta < \omega_1$ (is a limit ordinal and $\bar{c} \in |N_\delta|$ then for every $\alpha < \delta$, $\bar{c} \in |M_\alpha|$ and for all $\beta, \alpha < \beta < \delta$ the types $\mathrm{tp}(\bar{c} {}^\wedge \bar{b}_{\beta+1} {}^\wedge \bar{a}_\delta, \phi, L, N^*)$ are equal. (i.e., the type does not depend on β nor on δ).

Notice that all the $E(\bar{x}, \bar{y}; \bar{a}_\delta)$ equivalence classes are representable in $N_{\delta+1}$ (otherwise we can get a contradiction to the choice of E by (3)). Now for no $\bar{b}' \in N^*$ is $tp(\bar{a}_\delta {}^\wedge \bar{b}', |N_\delta|, L, N^*) = tp(\bar{a}_{\delta+\omega}, \bar{b}_{\delta+1}, |N_\delta|, L, N^*)$. Otherwise choose $\bar{b}'' \in N_{\delta+1}$ such that $N^* \models E[\bar{b}', \bar{b}'', \bar{a}_\delta]$, so by the conditions in Definition 5.2 (B), $N^* \models \neg E[b'', \bar{b}_\alpha, \bar{a}_\delta]$ for any $\alpha < \delta$. By 4.4 we can choose $\bar{c} \in |N_\delta|$ and φ so that $N^* \models \varphi[\bar{b}'', \bar{a}_\delta, \bar{c}]$ and $\varphi(\bar{x}, \bar{a}_\delta, \bar{c}) \vdash tp(\bar{b}'', \bar{a}_\delta \cup |N_\delta|, L, N^*)$ and let $\bar{c} \in |N_\alpha|$, $\alpha < \delta$ and $\alpha < \beta < \delta$. Then $\varphi(\bar{x}, \bar{a}_\delta, \bar{c}) \vdash \neg E(\bar{x}, \bar{b}_\beta, \bar{c})$ hence

$$\varphi_1(y_1, \bar{a}_\delta, \bar{c}) \overset{df}{=} (\exists\, y)(E(\bar{x}, \bar{y}, \bar{a}_\delta) \wedge \varphi(\bar{y}, \bar{a}_\delta, \bar{c})) \vdash \neg E(\bar{x}, \bar{b}_\beta, \bar{c})$$

but $N^* \models \varphi_1[\bar{b}_\beta, \bar{a}_\delta, \bar{c}]$ so $N^* \models \neg E(\bar{b}_\beta, \bar{b}_\beta, \bar{a}_\delta)$, a contradiction.

As in the Proof of 5.1 (A)$\to$(C), using [16], 2.14, for every set $S \subseteq \omega_1$ we can find an order J, and models N_t, $t \in J$, and sequences $\bar{a}_t, \bar{b}_t$, such that

(A) $J = \bigcup_{\alpha<\omega_1} J_\alpha$, $|J_\alpha| = \aleph_0$, $|J| = \aleph_1$, J_α is an initial segment of J; $J - J_\alpha$ has a first element iff $\alpha \in S$; and J is elementarily equivalent to ω_1. Also $\alpha < \beta \Rightarrow J_\alpha \subseteq J_\beta$ and $J_\delta = \bigcup_{\alpha<\delta} J_\alpha$ for limit δ.

(B) The conditions parallel to (1)–(6) above holds. We denote $\bigcup_{\iota\in J} N_\iota$, which is a model of ψ of cardinality $\aleph_1$, by N_S. Let $\bar{c}\in M$, $\varphi_1,\varphi_2\in L$ be such that $N\models\varphi_1[\bar{a},\bar{c}]\wedge\varphi_2[\bar{a}\,{}^\wedge\bar{b},\bar{c}]$ and $\varphi_1(\bar{x},\bar{c}),\varphi_2(\bar{x},\bar{y},\bar{c})$ has the same rank as $tp(\bar{a},|M|,L,N)$, $tp(\bar{a}\,{}^\wedge\bar{b},|M|,L,N)$ resp.

Now clearly

(*) Let $\alpha<\omega_1$, $N^\alpha=\bigcup_{\iota\in J_\alpha} N_\iota$. Then $\alpha\in S$ *iff* there are $\bar{c}'\in N^\alpha$,

$tp(\bar{c},\phi,L,N)=tp(\bar{c}',\phi,L,N^\alpha)$, and $\bar{a}'\in N_S$, $N_S\models\varphi_1[\bar{a}',\bar{c}']$, and $\varphi_1(\bar{x},\bar{c}')$ has the same rank as $tp(\bar{a}',|N^\alpha|,L,N_S)$ *such that* for no $\bar{b}'\in|N_S|$ does $N_S\models\varphi_2[\bar{a}'\,{}^\wedge\bar{b}',\bar{c}']$ and $\varphi_2(\bar{x},\bar{y},\bar{c}')$ has the same rank as $tp(\bar{a}'\,{}^\wedge\bar{b}',|N^\alpha|,L,N_S)$.

(**)
If $N_S=\bigcup N^1_\alpha(\alpha<\omega_1)$, $N^1_\alpha<^*N_S$, $\|N^1_\alpha\|=\aleph_0$, $N^1_\alpha<^*N^1_{\alpha+1}$, $N^1_\delta=\bigcup_{\alpha<\delta}N^1_\alpha$

then $\{\alpha: N^1_\alpha=N^\alpha\}$ is a closed and unbounded subset of ω_1.

We can easily conclude that $N_{S_1}\cong N_{S_2}$ implies that S_1,S_2 are equal modulo the filter on ω_1 generated by the closed unbounded subsets of ω_1. Hence e.g. by Solovay [17], $I(\aleph_1,\psi)=2^{\aleph_1}$.

THE $\aleph_0$-AMALGAMATION LEMMA 5.5.

(A) *Let ψ be nice and $\aleph_0$-stable, $N\models$"ψ", $(l=0,1,2)N_0<^*N_1$, $N_0<^*N_2$. Then there is a model M of $T(\psi)$ and elementary embeddings f_l of N_l into M $f_l\restriction|N_0|=$ the identity, f_l maps N_l onto N'_l $(l=1,2)$, and for $\bar{a}\in|N'_2|$ $tp(\bar{a},N'_1,L,M)$ has the same rank as $tp(\bar{a},|N_0|,L,M)$.*

(B) *Under the conditions of (A), if $\|N_1\|=\|N_2\|=\aleph_0$ there is $M'<M$, $M'\models$"ψ", $N'_l<^*M'$.*

(C) *If ψ has the symmetry property, then in (B) we can have also $N'_2<^*M'$.*

(D) *If ψ has the symmetry property, it has the $\aleph_0$-amalgamation property.*

PROOF.

(A) Immediate.

(B) Follows by claim 4.3.

(C) Immediate by 4.3, as then the conditions in (A) are symmetric for N'_1 and N'_2.

(D) Immediate by (C).

LEMMA 5.6. *Suppose ψ is nice, $\aleph_0$-stable and with the symmetry property.*

(A) *If $N\models\psi$, $\|N\|=\aleph_1$ then there is M, $M\models\psi$, $N<^*M$, $M\neq N$.*

(B) *Moreover there is such an M of cardinality $\aleph_2$.*

PROOF.

(A) Let $N = \bigcup_{\alpha<\omega_1} N_\alpha$, $\|N_\alpha\| = \aleph_0$, $N_\alpha <^{**} N_{\alpha+1}$, $N_\delta = \bigcup_{\alpha<\omega_1} N_\alpha$, and let $N < M$, M an $\aleph_2$-saturated model of $T(\psi)$. We now define by induction on α models M_α and embedding $f_{\beta,\alpha}$ (for $\beta < \alpha$) such that:

(1) $N_\alpha <^* M_\alpha$, $M_0 \neq N_0$

(2) $f_{\beta,\alpha}$ is an elementary embedding of M_β into M_α

(3) $M_\alpha \restriction \text{Range } f_{\beta,\alpha} <^* M_\alpha$

(4) $f_{\beta,\alpha} \restriction N_\beta =$ the identity

(5) if $\gamma < \beta < \alpha$ then $f_{\gamma,\alpha} = f_{\beta\alpha} f_{\alpha,\beta}$

(6) if $\bar{a} \in |M_\beta|$, $\beta < \alpha$, then $tp(\bar{a}, |N_\beta|, L, M_\beta)$ has the same rank as $tp(f_{\beta,\alpha}(a), N_\alpha, L, M_\alpha)$.

We can define $M_0 = N_1$, and then proceed by 5.5 for successor ordinal, and using the limit for limit ordinal. We can assume $M_\beta <^* M_\alpha$ for $\beta < \alpha$.

Clearly $\bigcup_{\alpha<\omega_1} M_\alpha$ is the required model.

(B) By repeating (A) we get $M_\alpha (\alpha < \omega_2)$, $M_\beta <^* M_\alpha \neq M_\beta$ for $\beta < \alpha$, $M_0 = N$. Clearly $\bigcup_{\alpha<\omega_2} M_\alpha$ is as required.

Without any assumptions on ψ let us prove.

MAIN THEOREM 5.7. *($V = L$ or $\diamondsuit_{\aleph_1}$) If $\psi \in L_{\omega_1,\omega}(Q)$, $I(\aleph_1, \psi) < 2^{\aleph_1}$, but ψ has an uncountable model, then ψ has a model of cardinality $\aleph_2$.*

PROOF. Clearly we can replace in the proof ψ by ψ' if $I(\lambda, \psi') \leqq I(\lambda, \psi)$ for $\lambda > \aleph_0$, but $I(\aleph_1, \psi') \geqq 1$.

Let M be an uncountable model of ψ, so by the downward Löwenheim-Skolem theorem we can assume $\|M\| = \aleph_1$.

By 2.1A for every fragment L^* of $L_{\omega_1,\omega}(Q)$, only countably many L^*-types are realized in M. By Theorem 2.3A, ψ has a model M_1 of cardinality $\aleph_1$ in which only countably many $L_{\omega_1,\omega}(Q)$-types are realized. By 2.5A for some fragment L^* of $L_{\omega_1,\omega}(Q)$, M_1 is $(L^*, \aleph_0)$-homogeneous. By 3.1(C), 2.5(C) for some almost nice ψ_1, $M_1 \models \psi_1$, $\psi_1 \vdash \psi$, so we can replace ψ by ψ_1. By 3.1(A) we can replace ψ_1 by a nice ψ_2. By 3.4 ψ_2 has the $\aleph_0$-amalgamation property, and by 2.1(B) it is $(\aleph_0, 1)$-stable. By Theorem 4.2 ψ_2 is $\aleph_0$-stable. By Theorem 5.4 ψ_2 does not have the asymmetry property, hence by 5.1 it has the symmetry property. Hence by 5.7 ψ_2 has a model of cardinality $\aleph_2$.

CONJECTURE. *If $\psi \in L_{\omega_1,\omega}(Q)$ has an uncountable model, then it has at least $2^{\aleph_1}$ non-isomorphic models.*

6. Various results

We give here various additional results, but do not elaborate the proofs or omit them.

LEMMA 6.1. *Suppose $\psi \in L_{\omega_1,\omega}(Q)$ has a model of cardinality $\beth_{\omega_1}$.*

(A) *Then some model of ψ of cardinality $\geqq \beth_{\omega_1}$ satisfies an almost-nice sentence ψ'.*

(B) *So $\lambda > \aleph_0 \Rightarrow I(\lambda, \psi) \geqq I(\lambda, \psi')$ and equality holds if ψ is categorical in some $\mu \leqq \lambda$.*

(C) *If ψ is categorical in $\aleph_1$ then it is $(\aleph_0, 1)$-stable.*

PROOF. Let M be an Ehrenfeucht-Mostowski model of ψ of cardinality $\beth_{\omega_1}$ (see e.g. [5]), with dense skeleton. Then in M only countably many $L_{\omega_1,\omega}(Q)$-types are realized. Hence we finish (A), and (B) is immediate. By the proof of Morley [9] (C) is immediate.

LEMMA 6.2. *Suppse $\psi \in L_{\omega_1,\omega}(Q)$ is nice and has a model of cardinality $\beth_{\omega_1}$ and is categorical in $\aleph_1$. Then ψ is $\aleph_0$-stable.*

PROOF. Let M^1 be an Ehrenfeucht-Mostowski model of ψ. (M^1 is an L_1-model, $L \subseteq L_1$) which is the closure of the indiscernible sequence $\{y_i: i < \omega_1\}$. Let M^1_α be the closure of $\{y_i: i < \alpha\}$ and $M(M_\alpha)$ the L-reduct of $M^1(M^1_\alpha)$. It is easy to see that $\alpha < \beta \Rightarrow M_\alpha <^* M_\beta$. By [12] in M we cannot find a set of $\aleph_1$ sequence which some $\varphi \in L$ ordered. From this it is not hard to deduce that if $\bar{a} \in |M|$, β limit for some $\alpha < \beta\, tp(\bar{a}, |M_\beta|, L, M)$ does not split over M_α, and there is $\bar{a}' \in |M_\alpha|$ such that $tp(\bar{a}, |M_\beta|, L, M) = tp(a', |M_\beta|, L, M)$. If T is not $\aleph_0$-stable, we can find models N_α ($\alpha < \omega_1$) such that $N_\alpha <^{**} N_{\alpha+1}$ $N_\delta = \bigcup_{\alpha<\delta} N_\alpha$, $\|N_\alpha\| = \aleph_0$, $N_\alpha \models$"ψ" and the condition mentioned above does not hold (i.e. for every δ there is $\bar{a} \in |N_{\delta+1}|$ such that: $tp(\bar{a}, |N_\delta|, L, N_{\delta+1})$ split over every $|N_\alpha|$, ($\alpha < \delta$) *or* for some $\alpha < \delta$, $tp(a, |N_\alpha|, L, N_{\delta+1})$ is not realized in N_δ.)

It is easy to check that $N = \bigcup_{\alpha<\omega_1} N$ is not isomorphic to M, but is a model of ψ of cardinality $\aleph_1$, contradiction.

The following lemma was once used in the proof of 5.6 so we do not prove it.

LEMMA 6.3. *Let ψ be nice, $\aleph_0$-stable, with the symmetry property. Let M be a model of $T(\psi)$, $N_1 < N_2 < M$, $\|N_2\| = \aleph_0$, $\bar{a} \in |M|$, $M_l < M$ is prime over $|N_l| \cup \bar{a}$; and $N_1, N_2, M_1, M_2 \models$"$\psi$". Then there is an elementary embedding f of M_1 into M_2, $f \upharpoonright (|N_1| \cup \bar{a}) =$ the identity and $M_2 \upharpoonright$Range $f <^* M_2$.*

From here we work in $L_{\omega_1,\omega}$.

We could reduce all the previous discussion to $L_{\omega_1,\omega}$. The only noticeable changes are the omitting of (γ) in Definition 4.1 (of rank), and replacing "$\psi \vdash (Qx)x = x$" by "ψ has an uncountable model" in Definition 3.1 (of niceness), and we can drop $<^*$, $<^{**}$ and

LEMMA 6.4. *If ψ is nice and $\aleph_0$-stable,* then *it does not have the order property (and does have the symmetry property.*

PROOF. Follows by the proof of 5.1 (A) $\Rightarrow$ (C) (as we lack the alternative followed there).

DEFINITION 6.1. Let $M \models$"ψ",

(A) the formula $\varphi(\bar{x}, \bar{a})(\bar{a} \in |M|, \varphi \in L)$ is big if there is a model N, $N \models$"ψ", $M <^* N$, and some $\bar{c} \in |N|$, $\bar{c} \not\subseteq |M|$ satisfies $\varphi(\bar{x}, \bar{a})$.

(B) The formula $\varphi(\bar{x}, \bar{a})$ is minimal if it is big but for no $\theta \in L$, $\bar{b} \in |M|$, are both $\varphi(\bar{x}, \bar{a}) \wedge \theta(\bar{x}, \bar{b})$ and $\varphi(\bar{x}, \bar{a}) \wedge \neg\theta(\bar{x}, \bar{b})$ big.

(C) If $\bar{a} \in M$, $A \subseteq M$, $tp(\bar{a}, A, L, M)$ is big (minimal) if some formula in it is.

LEMMA 6.5.

(A) *The properties "$\varphi(\bar{x}, \bar{a})$ is big", "$\varphi(\bar{x}, \bar{a})$ is minimal" depends only on $tp(\bar{a}, \phi, L, M)$*

(B) *If $\varphi(\bar{x}, \bar{a})$ is minimal $\bar{a} \in A \subseteq M \models$"$\psi$",* then *there is a unique complete L-type over A realized in some N, $M <^* N \models$"ψ", which is big and contains $\varphi(\bar{x}, \bar{a})$.*

PROOF. Immediate.

LEMMA 6.6. *Let ψ be nice and $\aleph_0$-stable.*

(A) *If $M \models \psi$ there is a minimal formula $\varphi(\bar{x}, \bar{a})$, $\bar{a} \in A$.*

(B) *If $M \models \psi$, $\bar{a} \in |M|$, $\varphi(\bar{x}, \bar{a})$ is minimal, then the dependence relation among sequences satisfying $\varphi(\bar{x}, \bar{a})$, defined by "$\bar{b}$ depends on $\{\bar{b}_1, \bar{b}_2, \cdots\}$ if $tp(\bar{b}, \bar{a} \bigcup_i \bar{b}_i, L, M)$ is not big" satisfies the axioms for linear dependence* (which enable us to define dimension).

PROOF.

(A) Choose $\varphi(x, \bar{a})$ with minimal rank such that for some N, $M < N$, $N \models \psi$, and $c \in |N| - |M|$, $N \models \varphi[c, \bar{a}]$.

(B) Easy, remembering 6.5.

LEMMA 6.7. *Let ψ be nice and $\aleph_0$-stable. Then ψ is categorical in $\aleph_1$,* iff *for every model N, $\|N\| = \aleph_1$, $N \models \psi$ for every minimal $\varphi(x, \bar{a})$ $(\bar{a} \in N)$ $|\{c \in |N| : N \models \varphi[c, \bar{a}]\}| = \aleph_1$* iff *for every model M, N of ψ, $M < N$, and* *minimal* $\varphi(x,\bar{a})$ $(\bar{a}\in|M|)$ *for some* $c\in|N|-|M|$, $N\models\varphi[c,\bar{a}]$ iff *over every countable* $N\models\psi$, *there is a prime model* M, *of* ψ *i.e.* $N<M\models\psi$, $N\neq M$, *and if* $N<M'\models\psi$, $N\neq M'$, *then there is an elementary embedding of* M *into* M' *which is the identity over* $|N|$.

PROOF. Left to the reader.

This seemed a reasonable characterization of categoricity.

CONCLUSION 6.8. *Let* ψ *be nice,* $\aleph_0$*-stable and categorical in* $\aleph_1$. *Then its model* M *of cardinality* $\aleph_1$ *is* $\aleph_1$*-model-homogeneous, i.e. if* $N_1, N_2<M$, f *an isomorphism from* N_1 *onto* N_2, N_1, N_2 *are countable* then *we can extend* f *to an automorphism of* M.

REMARKS. (1) We can easily generalize Lemma 3.4 (that the lack of the amalgamation property implies $I(\aleph_1,\psi)=2^{\aleph_1}$) to higher cardinals and to pseudo-elementary classes.

(2) If $T\subseteq L(Q)$, and for every finite set of formulas $\Gamma\subseteq L(Q)$ there is a model M of T, $\|T\|=\aleph_1$ such that for every countable $A\subseteq|M|$ $|\{tp(\bar{a},A,\Gamma,M)\colon \bar{a}\in|M|\}|\leqq\aleph_0$ *then* T has a model N, $\|N\|=\aleph_1$, such that the number of $L_{\omega_1,\omega}(Q)$-types realized in N is countable. The proof is analagous to 2.3.

(3) Claim 5.2 generalizes easily to any regular cardinality.

(4) We can strengthen the definition of nice indexed set (Def. 5.2) as in [S6] without changing the conclusions.

(5) We can generalize 6.4–6.8 to $\psi\in L_{\omega_1,\omega}(Q)$.

(6) We can define niceness for all reasonable logics.

Note added October 6, 1974.

(1) A Variant of 2.3 was proved, later and independently by M. Makkai, An addmissible generalization of a theorem on countable Σ_1^1 sets of reals with applications, to appear.

(2) Recently, the author has proven that e.g., if $\psi\in L_{\omega_1,\omega}$ is categorical in $\aleph_n$ for $0<n<\omega$ then ψ is categorical in every $\lambda>\aleph_0$, assuming $V=L$.

REFERENCES

1. Baumgartner, *On $\aleph_1$-dense sets of reals*, Fund. Math.

2. G. Fodor, *Eine Bemerkung zur Theorie der regressiven Funktionen*, Acto. Sci. Math. **17** (1965), 139–142.

3. H. Friedman, *Ninety-four problems in mathematical logic*, to appear in J. Symbolic Logic.

4. R. B. Jensen, *The fine structure of the constructible hierarchy*, Ann. Math. Logic **4** (1972), 224–308.

5. H. J. Keisler, *Model theory for infinitary logic*, North Holland Publ. Co., Amsterdam, 1971.

6. H. J. Keisler, *On the quantifier "there exist uncountably many x"*, Ann. Math. Logic **1** (1970), 1–91.

7. J. Knight, *Minimal sets in infinitary logic*, preprint.

8. L. Marcus, *A prime minimal model with an infinite set of indiscernibles*, Israel J. Math. **11** (1972), 180–183.

9. M. D. Morley, *Categoricity in power*, Trans. Amer. Math. Soc. **114** (1965), 514–538.

10. M. D. Morley, *The number of countable models*, J. Symbolic Logic **35** (1970), 14–18.

11. S. Shelah, *Finite diagrams stable in power*, Ann. Math. Logic **2** (1970), 69–118.

12. S. Shelah, *The number of non-isomorphic models of an unstable first-order theory*, Israel J. Math. **9** (1971), 473–487.

13. S. Shelah, *Stability, the f.c.p. and superstability; model theoretic properties of formulas in first order theory*, Ann. Math. Logic, **3** (1971), 271–362.

14. S. Shelah, *Categoricity of uncountable theories*, Proc. of Symp. in honor of Tarski's Seventieth birthday, Berkeley 1972; Proc. of Symp. in Pure Math., Amer. Math. Soc., Providence, R.I., 1974. pp. 187–204.

15. S. Shelah, *Various results in Math. Logic*, Notices Amer. Math. Soc. **22** (1974).

16. S. Shelah, *Generalized quantifiers and compact logic*, Trans. Amer. Math. Soc., to appear.

17. R. Solovay, *Real-valued measurable cardinals in axiomatic set theory*, Proc. Symp. Pure Math. XIII Part I, Amer. Math. Soc. Providence, R.I., 1971, pp. 397–428.

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