

HOMOGENEOUS FAMILIES AND THEIR AUTOMORPHISM GROUPS

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ABSTRACT

A homogeneous family of subsets over a given set is one with a very 'rich' automorphism group. We prove the existence of bi-universal element in the class of homogeneous families over a given infinite set and give an explicit construction of $2^{2^{\aleph_0}}$ isomorphism types of homogeneous families over a countable set.

0. Introduction

Homogeneous objects are often defined in terms of their automorphism groups. Rado's graph Γ , also known as the countable random graph, has the property that for any isomorphism f between two finite induced subgraphs of Γ there is an automorphism of Γ extending f . This property is the homogeneity of Rado's graph; and any graph whose automorphism group satisfies this condition is called homogeneous.

The automorphism group of Rado's graph was studied by Truss in [9], and shown to be simple. Truss studied also the group $\text{AAut}(\Gamma)$ of *almost automorphisms* of Rado's graph (see [11] and also [6]). This is a *highly transitive* group extending $\text{Aut}(\Gamma)$ (where 'highly transitive' stands for ' n -transitive for all n '; the group $\text{Aut}(\Gamma)$ is not highly transitive).

In this paper we shall study *homogeneous families of sets* over infinite sets. Our definition of homogeneity of a family of sets implies that its automorphism group satisfies, among other conditions, that it is highly transitive. However, while all homogeneous graphs over a countable set are classified (see [5]), this is not the case with homogeneous families over a countable set.

We shall show that there are $2^{2^{\aleph_0}}$ isomorphism types of homogeneous families over a countable set. This is done in Section 4. From the proof we shall get $2^{2^{\aleph_0}}$ permutation groups, each acting homogeneously on some family over ω , and each being isomorphic to the free group on 2^{\aleph_0} generators, but such that no two are conjugate in $\text{Sym}(\omega)$.

In Section 3 we prove the existence of a bi-universal homogeneous family over any given infinite set. The definitions of bi-embedding and bi-universality are generalizations of definitions made by Truss in his study of universal permutation groups [10]. A short survey of results concerning the existence of universal objects can be found in the introduction to [2]. Results concerning abelian groups are in [3], and results on stable unsuperstable first order theories are in [4].

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Homogeneous families were studied in [1] (where they were treated as bipartite graphs). There it was shown that the number of isomorphism types of homogeneous families over ω of size \aleph_1 is independent of ZFC and may be 1 as well as 2^{\aleph_1} in different models of set theory.

Model theorists will recognize that uncountable homogeneous families over a countable set are examples of two-cardinal models which are ω -homogeneous as well. Set theorists may be interested in the following.

0.1 PROBLEM. Is it consistent that 2^{\aleph_0} is large and that in some uncountable $\lambda < 2^{\aleph_0}$ there is a maximal homogeneous family (with respect to inclusion)?

Note added in proof. The authors have recently proved the existence of maximal homogeneous families over ω from CH; see M. Kojman and S. Shelah, 'There is a maximal homogeneous family over ω ', *Proc. Amer. Math. Soc.*, to appear.

We wish to remark finally that the existence of $2^{2^{\aleph_0}}$ isomorphism types of homogeneous families over ω follows from a general theorem about non-standard logics [7, VIII, §1] (for more details see also [8]). The virtue of the proof here (besides being elementary) is its explicitness and the information it gives about the embeddability of an arbitrary family in a homogeneous one.

NOTATION. We denote disjoint unions by $\dot{\cup}$ and $\dot{\bigcup}$. A natural number n is the set $\{0, 1, \dots, n-1\}$ of all smaller natural numbers.

1. Getting started

Let $\mathcal{F} \subseteq \mathcal{P}(A)$ be a family of subsets of a given infinite set A . An *automorphism* of \mathcal{F} is a permutation $\sigma \in \text{Sym}(A)$ which satisfies the condition that $X \in \mathcal{F} \Leftrightarrow \sigma[X] \in \mathcal{F}$ for every $X \subseteq A$. (By $\sigma[X]$ we denote $\{\sigma(x) : x \in X\}$ for $X \subseteq A$.) The group $\text{Aut}(\mathcal{F}) \subseteq \text{Sym}(A)$ is the group of all automorphisms of \mathcal{F} .

One way of defining when a family $\mathcal{F} \subseteq \mathcal{P}(A)$ is homogeneous is to demand that the bipartite graph $\langle A, \mathcal{F}, \epsilon \rangle$ is homogeneous, namely that every finite partial automorphism of this graph which respects the sides extends to a total automorphism. We shall write a more complicated (though equivalent) definition. This will be needed in what follows.

1.1 DEFINITION. Suppose that $\mathcal{F} \subseteq \mathcal{P}(A)$ is a given family of subsets of a set A . A *demand* on \mathcal{F} is a pair $d = (h^d, f^d)$ such that h^d is a finite one-to-one function from A to A , f^d is a finite 1-1 function from \mathcal{F} to \mathcal{F} and $x \in X \Leftrightarrow h^d(x) \in f^d(X)$ for every $x \in \text{dom } h^d$, $X \in \text{dom } f^d$. We denote by $D = D(A, \mathcal{F})$ the set of all demands on \mathcal{F} . Let $\text{FG}(D)$ be the free group over the set $D(A, \mathcal{F})$. We say that an automorphism $g \in \text{Aut}(\mathcal{F})$ satisfies a demand d if $g(x) = h^d(x)$ for $x \in \text{dom } h^d$ and $g[X] = f^d(X)$ for $X \in \text{dom } f^d$. We call a partial homomorphism $\phi : \text{FG}(D) \rightarrow \text{Aut}(\mathcal{F})$ a *satisfying* homomorphism if $\phi(d)$ satisfies d for $d \in \text{dom } \phi$. (By 'partial' we mean that ϕ need not be defined on all generators of $\text{FG}(D)$.)

1.2 DEFINITION. A family \mathcal{F} is *homogeneous* if and only if every $d \in D$ is satisfiable if and only if there is a (total) satisfying homomorphism $\phi : \text{FG}(D) \rightarrow \text{Aut}(\mathcal{F})$. A group $G \subseteq \text{Aut } \mathcal{F}$ acts *homogeneously* on \mathcal{F} if and only if G contains the image

of a total satisfying homomorphism, or, equivalently if and only if every demand is satisfied by some element in G .

When ϕ is a homomorphism as above, we say that ϕ *testifies* the homogeneity of \mathcal{F} .

When the set A is clear from the context, we write $D(\mathcal{F})$ instead of $D(A, \mathcal{F})$.

1.3 EXAMPLES. (1) The family $\mathcal{F} = \{\{x\}: x \in A\}$ of all singletons is homogeneous. The group $\text{Aut}(\mathcal{F})$ is the group $\text{Sym}(A)$ of all symmetries of A .

(2) The family $\text{Fin}(A)$ of all finite subsets of A is not homogeneous, although $\text{Aut}(\text{Fin}(A)) = \text{Sym}(A)$, because a demand $d = (\emptyset, \{(X, Y)\})$ cannot be satisfied when X and Y are finite sets of different cardinalities.

(3) A countable family of random subsets of ω is homogeneous in probability 1. The membership of a point in a random set is determined by flipping a coin.

In [1] the following was proved.

1.4 THEOREM. *Every homogeneous family of subsets of an infinite set A satisfies exactly one of the conditions below:*

- (1) $\mathcal{F} = \{\emptyset\}$;
- (2) $\mathcal{F} = \{A\}$;
- (3) \mathcal{F} is the family of all singletons of A ;
- (4) \mathcal{F} is the family of all co-singletons of A ;
- (5) \mathcal{F} is an independent family, namely for every finite function $\tau: \mathcal{F} \rightarrow \{+, -\}$, the set

$$B_\tau = \bigcap_{X \in \tau^{-1}(+)} X \cap \bigcap_{Y \in \tau^{-1}(-)} A \setminus Y$$

is infinite, and \mathcal{F} is dually independent, namely for every function $\tau: A \rightarrow \{+, -\}$ there are infinitely many members of \mathcal{F} containing $\tau^{-1}(+)$ and avoiding $\tau^{-1}(-)$. Equivalently, the first order theory of $\langle \mathcal{A}, \mathcal{F}, \in \rangle$ is the first order theory of the random countable bipartite graph.

2. Direct limits and homogeneity

In this section we exhibit a method of constructing homogeneous families as direct limits. This method will be used in the following sections.

Homogeneity is not, in general, preserved under the usual direct limits of families. For example, an increasing union of homogeneous families need not be homogeneous itself. We therefore consider here a stronger relation of embeddability, called here 'multi-embeddability', which, roughly speaking, preserves the satisfaction of previously satisfied demands. Direct limits of *this* relation can be made homogeneous, as we shall presently see.

2.1 DEFINITION. Let $T_i = \langle A_i, \mathcal{F}_i, D_i, G_i, \phi_i \rangle$, (for $i = 0, 1$), be respectively, a set A_i , a family of subsets $\mathcal{F}_i \subseteq \mathcal{P}(A_i)$, the collection of demands $D_i = D(\mathcal{F}_i)$, an automorphism group $G_i \subseteq \text{Aut}(\mathcal{F}_i)$ and a partial satisfying homomorphism $\phi_i: \text{FG}(D_i) \rightarrow G_i$. Let $\bar{T}_i = A_i \cup \mathcal{F}_i \cup D_i \cup G_i$. We call a function $\Phi: \bar{T}_0 \rightarrow \bar{T}_1$ a *multi-embedding of T_0 into T_1* (and write $\Phi: T_0 \rightarrow T_1$) if:

- (1) $\Phi \upharpoonright A_0$ is a one-to-one function into A_1 ,
- (2) $\Phi \upharpoonright \mathcal{F}_0$ is a one-to-one function into \mathcal{F}_1 ,
- (3) $\Phi \upharpoonright D_0$ is a one-to-one function into D_1 ,
- (4) $\Phi \upharpoonright G_0$ is a group monomorphism into G_1 , and the following rules hold for $x \in A_0$, $X \in \mathcal{F}_0$, $d \in D_0$ and $g \in G_0$:

- (a) $x \in X \Leftrightarrow \Phi(x) \in \Phi(X)$,
- (b) $\Phi[\text{dom } h^d] = \text{dom } h^{\Phi(d)}$, $\Phi[\text{dom } f^d] = \text{dom } f^{\Phi(d)}$, $\Phi((h^d(x))) = h^{\Phi(d)}(\Phi(x))$ and $\Phi((f^d(X))) = f^{\Phi(d)}(\Phi(X))$,
- (c) $\Phi(g(x)) = \Phi(g)(\Phi(x))$ and $\Phi(g[X]) = \Phi(g)[\Phi(X)]$,
- (d) $\Phi(d) \in \text{dom } \phi_1$ and $\Phi(\phi_0(d)) = \phi_1(\Phi(d))$ for every $d \in \text{dom } \phi_0$.

We say that a multi-embedding Φ is *successful* if in addition to the conditions above the following also holds:

- (e) $\Phi(d) \in \text{dom } \phi_1$ for every $d \in D_0$.

2.2 DEFINITION. Suppose that I is a directed set and $T_i = \langle A_i, \mathcal{F}_i, D_i, G_i, \phi_i \rangle$ is as in Definition 2.1 above for $i \in I$. Suppose that $\Phi_i^j: T_i \rightarrow T_j$ is a multi-embedding for $i \leq j$, and

- (i) $\Phi_i^i = \text{id}$,
- (ii) $\Phi_j^k \Phi_i^j = \Phi_i^k$ for $i \leq j \leq k$.

Then we call $\mathbf{T} = \langle T_i: (i \in I); \Phi_i^j \rangle$ a *direct system of multi-embeddings*. We call \mathbf{T} *successful* if in addition to (i) and (ii) the following condition holds:

- (iii) for every $i \in I$ there is $j \geq i$ such that Φ_i^j is successful.

2.3 THEOREM. *Suppose that $\langle T_i: (i \in I); \Phi_i^j \rangle$ is a successful direct system of embeddings. Let $T^* = \langle A^*, \mathcal{F}^*, D^*, G^*, \phi^* \rangle := \lim_{-1} T_i$. Then \mathcal{F}^* is homogeneous, with ϕ^* testifying homogeneity, and the canonical mapping $\Phi_i: T_i \rightarrow T^*$ is a successful multi-embedding.*

Proof. We first recall the definition of a direct limit.

An equivalence relation \sim is defined over $\bigcup_{i \in I} \bar{T}_i$ as follows:

$$a \sim b \Leftrightarrow (\exists i \leq j)(\Phi_i^j(a) = b \vee \Phi_i^j(b) = a).$$

Conditions (i) to (ii) above imply that \sim is indeed an equivalence relation. We define the canonical map $\Phi_i(a) = [a]_{\sim}$. Next we set $A^* = \bigcup_{i \in I} A_i / \sim$ and observe the following.

2.4 FACT. *For every infinite cardinal κ , if I and every A_i are of cardinality at most κ , then $|A^*| \leq \kappa$.*

We let $\mathcal{F}^* = \bigcup_{i \in I} \mathcal{F}_i / \sim$, $G^* = \bigcup_{i \in I} G_i / \sim$ and $D^* = \bigcup_{i \in I} D_i / \sim$.

For $x^*, y^* \in A^*$, $X^* \in \mathcal{F}^*$, $d^* \in D^*$ and $g^* \in G^*$ we note that

- (1) $x^* \in X^*$ if and only if there is some $i \in I$ and $x \in A_i$, $X \in \mathcal{F}_i$ such that $x \in X$ and $\Phi_i(x) = x^*$, $\Phi_i(X) = X^*$;
- (2) $g^*(x^*) = y^*$ if and only if $g(x) = y$ for some $i \in I$ such that $x \in A_i$, $g \in G_i$ and $\Phi_i(x) = x^*$, $\Phi_i(y) = y^*$ and $\Phi_i(g) = g^*$;
- (3) $\phi^*(d^*) = g^*$ if and only if there is $i \in I$ such that $\phi_i(d) = g$ and $\Phi_i(d) = d^*$, $\Phi_i(g) = g^*$.

We leave verification of this to the reader and that the following hold:

- (a) $\mathcal{F}^* \subseteq \mathcal{P}(A^*)$,
- (b) $G^* \subseteq \text{Aut}(\mathcal{F}^*)$,

- (c) $D^* = D(\mathcal{F}^*)$,
 (d) $\phi^*: D^* \rightarrow G^*$ is a (total) satisfying homomorphism,
 (e) $\Phi_j \Phi_i^j = \Phi_i$ for $i \leq j$ in I .

We conclude that $\Phi_i: T_i \rightarrow T^*$ is a successful embedding for every $i \in I$. Homogeneity of \mathcal{F}^* follows readily from (c) and (d) above.

3. Bi-universal homogeneous families

The result proved in this section is the existence of a bi-universal member in the class of homogeneous families over a given infinite set.

Let us make the following definition.

3.1 DEFINITION. We call an embedding of structures $\Phi: M \rightarrow N$ a *bi-embedding* if there is a group monomorphism $\Theta: \text{Aut } M \rightarrow \text{Aut } N$ such that $\Phi g \Phi^{-1} \Theta(g) \upharpoonright \text{ran } \Phi$ for all $g \in \text{Aut } M$.

3.2 DEFINITION. A structure M^* in a class of structures K is *bi-universal* if for every structure $M \in K$ there is a bi-embedding $\Phi: M \rightarrow M^*$.

3.3 REMARKS. (1) The definition of embedding of permutation groups (see [10]) is obtained from 3.2 by adding the condition that Φ is onto.

(2) Example 1.3(1) above indicates that if a bi-universal family \mathcal{F}^* over a set A^* exists, then for some $A \subseteq A^*$ of cardinality $|A^*|$ the restrictions of automorphisms of \mathcal{F}^* to A include the full symmetric group $\text{Sym}(A)$.

(3) If, for simplicity, M is bi-embedded in N via the identity map on M , then the restriction map $g \mapsto g \upharpoonright M$ is a group homomorphism from the set-wise stabilizer of M in $\text{Aut}(N)$ onto $\text{Aut}(M)$ that *splits*, namely has a left inverse.

3.4 LEMMA. For every infinite $T = \langle A, \mathcal{F}, D, G, \phi \rangle$ there is a set B such that $|A| = |B|$ and a successful multi-embedding

$$\Phi: T \longrightarrow \langle A \dot{\cup} B, \mathcal{P}(A \dot{\cup} B), D(A \dot{\cup} B, \mathcal{P}(A \dot{\cup} B)), \text{Sym}(A \dot{\cup} B), \phi' \rangle.$$

Proof. We specify the points of B . A point in B is a finite function from the power set of a finite subset of A to $\{0, 1\}$, namely $f \in B \Leftrightarrow f: \mathcal{P}(D_f) \rightarrow \{0, 1\}$ and $D_f \subseteq A$ is finite. We let $\Phi \upharpoonright A = \text{id}$. For $X \in \mathcal{F}$ we define $\Phi(X)$ as follows:

$$\Phi(X) = X \cup \{f \in B: f(X \cap D_f) = 1\}.$$

We let $\Phi(\sigma) \upharpoonright A = \sigma$ and let $\Phi(\sigma)(f) = g \Leftrightarrow \sigma[D_f] = D_g \wedge f(X) = g(\sigma[X])$ for all $X \subseteq D_f$. It is straightforward to verify that $\Phi \upharpoonright \text{Sym}(A)$ is a group monomorphism. We verify condition (c) in the definition of successful embedding (Definition 2.1 above). Suppose that $X \subseteq A$ and $\sigma \in \text{Sym}(A)$ are given. Then

$$\begin{aligned} \Phi(\sigma)[\Phi(X)] &= \sigma[X] \dot{\cup} \Phi(\sigma)[\{g \in B: g(X \cap D_g) = 1\}] \\ &= \sigma[X] \dot{\cup} \{\Phi(\sigma)(g): g \in B \wedge g(X \cap D_g) = 1\} \\ &= \sigma[X] \dot{\cup} \{f \in B: f(\sigma[X] \cap D_f) = 1\} = \Phi(\sigma[X]). \end{aligned}$$

The definition of $\Phi \upharpoonright D(\mathcal{P}(A))$ is determined uniquely by condition (b) in 2.1 above. We need to specify ϕ' and prove that (d) holds. For this we notice the following.

3.5 CLAIM. The family $\mathcal{F} = \{\Phi(X): X \subseteq A\}$ satisfies the condition that for every finite function $\tau: \mathcal{F} \rightarrow \{+, -\}$ the set $B_\tau = \bigcap_{X \in \tau^{-1}(+)} \Phi(X) \cap \bigcap_{Y \in \tau^{-1}(-)} (A \dot{\cup} B) \setminus \Phi(Y)$ has the same cardinality as $A \dot{\cup} B$.

Proof. The proof of this is well known.

3.6 COROLLARY. *For every demand d on \mathcal{F} there is a permutation $\sigma \in \text{Sym}(A \dot{\cup} B)$ such that $\sigma(x) = h^d(x)$ and $\Phi(\sigma[X]) = \Phi[f^d(X)]$ for every $x \in \text{dom } h^d$ and $X \in \text{dom } f^d$.*

Proof. For every $\tau: \text{dom } f^d \rightarrow \{+, -\}$ we have that $|B_\tau| = |A \dot{\cup} B| = |B'_\tau|$, where

$$B'_\tau = \bigcap_{x \in \tau^{-1}(+)} \Phi(f^d(X)) \cap \bigcap_{x \in \tau^{-1}(-)} \Phi(A \setminus f^d(X)).$$

(This means, informally, that every ‘cell’ in the Venn diagram of $\text{dom } f^{\Phi(d)}$ and every ‘cell’ of the Venn diagram of $\text{ran } f^{\Phi(d)}$ is of cardinality $|A \dot{\cup} B|$.) Therefore it is trivial to extend h^d to a permutation that carries B_τ onto B'_τ for every τ .

Now let us define $\phi'(\Phi(d)) = \Phi(\phi(d))$ for every $d \in \text{dom } \phi$, and for all $d \in D \setminus \text{dom } \phi$ let us pick, using 3.5 and 3.6 above, a permutation $\phi'(\Phi(d))$ that extends $\Phi(d)$. This completes the proof of Lemma 3.4.

3.7 THEOREM. *Suppose that A_0 is a given infinite set. There is a successful direct system of embeddings $\mathbf{T} = \langle T_n: (n \in \omega); (\Phi_n^n, \phi_n^n) \rangle$ such that:*

- (1) A_n is of cardinality $|A_0|$,
- (2) $\mathcal{F}_n = \mathcal{P}(A_n)$,
- (3) $G_n = \text{Sym}(A_n)$.

Proof. Let $T_0 = \langle A_0, \mathcal{P}(A_0), D(\mathcal{F}_0), \text{Sym}(A_0), \phi_0: \{e\} \rightarrow \{\text{id}_{A_0}\} \rangle$. Now use Lemma 3.4 inductively.

3.8 THEOREM. *For every infinite set A^* there is a homogeneous family $\mathcal{F}^* \subseteq \mathcal{P}(A^*)$, an infinite subset $A \subseteq A^*$ of cardinality $|A^*|$ and a group $G \subseteq \text{Aut}(\mathcal{F})^*$ such that $\mathcal{P}(A) = \{X \cap A: X \in \mathcal{F}^*\}$ and $\text{Sym}(A) = \{g \upharpoonright A: g \in G\}$. Therefore any injection $f: A^* \rightarrow A$ is a bi-embedding of every family $\mathcal{F} \subseteq \mathcal{P}(A^*)$ (not necessarily homogeneous) into \mathcal{F}^* . In particular, \mathcal{F}^* is bi-universal in the class of all homogeneous families over A^* .*

Proof. By Theorem 3.7 there is a successful direct system of embeddings $\mathbf{T} = \langle T_n: (n \in \omega); (\Phi_n^n, \phi_n^n) \rangle$ such that

- (1) $|A_n| = |A^*|$,
- (2) $\mathcal{F}_n = \mathcal{P}(A_n)$,
- (3) $G_n = \text{Sym}(A_n)$.

By Theorem 2.3 and the side remark 2.4 it follows that the family \mathcal{F}^* obtained by the direct limit is a homogeneous family of subsets of a set A^{**} of size $|A^*|$, and we may assume that $A^{**} = A^*$. The canonical map Φ_0 is a successful multi-embedding, and therefore in particular a bi-embedding. Let A be the image of A_0 under Φ_0 and let G be $\Phi_0[\text{Sym}(A_0)]$. The theorem is now obvious.

3.9 REMARK. It is not hard to verify that both the family \mathcal{F}^* and the automorphism group G^* given by the proof of Theorem 3.8 are Borel, in fact F_σ , in the usual (product) topology on $\mathcal{P}(A)$. We thank Alain Louveau for pointing this out to us. A homogeneous family is always dense by 1.4(5), and therefore is never closed. We pose the following problem.

3.10 PROBLEM. Suppose that $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is Borel, and is homogeneous. Is there always a Borel group of automorphisms that acts homogeneously on \mathcal{F} ?

4. The number of isomorphism types of homogeneous families over ω

In this section we make a second use of the method of direct limits, as introduced in Section 2, to determine the number of isomorphism types of homogeneous families over a countable set. It was conjectured in [1] that this number is the maximum possible, namely $2^{2^{\aleph_0}}$. An isomorphism between two families $\mathcal{F}_0 \subseteq \mathcal{P}(A_0)$ and $\mathcal{F}_1 \subseteq \mathcal{P}(A_1)$ is, of course, a one-to-one onto function $f: A_0 \rightarrow A_1$ which satisfies $X \in \mathcal{F}_0 \Leftrightarrow f[X] \in \mathcal{F}_1$.

To obtain $2^{2^{\aleph_0}}$ non isomorphic homogeneous families over a countable set, it is enough to obtain $2^{2^{\aleph_0}}$ different such families; for then dividing by isomorphism, the size of each class is 2^{\aleph_0} , and therefore there are 2^{\aleph_0} classes (see below).

The technique used to achieve this is by embedding a family $\mathcal{F} \subseteq \mathcal{P}(A)$ in a homogeneous family $\mathcal{F}' \subseteq \mathcal{P}(A^*)$ for $A^* \supseteq A$ in such a way that $\{X \cap A: X \in \mathcal{F}'\} = \mathcal{F}$. In other words, we will 'homogenize' a family \mathcal{F} 'without adding sets' to \mathcal{F} . Thus, starting with distinct families \mathcal{F} we obtain distinct homogeneous families \mathcal{F}' .

4.1 LEMMA. *There is a pair of countable sets $A_0 \subseteq A^*$ (in fact, for every pair $A_0 \subseteq A^*$ of countable sets satisfying $A^* \setminus A_0$ infinite) such that for every family $\mathcal{F} \subseteq \mathcal{P}(A_0)$ satisfying $\text{Fin}(A_0) \subseteq \mathcal{F}$ there is a homogeneous family $\mathcal{F}' \subseteq \mathcal{P}(A^*)$ satisfying $\{X \cap A_0: X \in \mathcal{F}'\} = \mathcal{F}$.*

This lemma determines the number of isomorphism types of homogeneous families over a countable set.

4.2 COROLLARY. *There are $2^{2^{\aleph_0}}$ isomorphism types of homogeneous families over a countable set.*

Proof. There are $2^{2^{\aleph_0}}$ different families $\{\mathcal{F}_\alpha: \alpha < 2^{2^{\aleph_0}}\}$, such that $\text{Fin}(A_0) \subseteq \mathcal{F}_\alpha \subseteq \mathcal{P}(A_0)$. For each \mathcal{F}_α there is, by 4.1, a homogeneous family $\mathcal{F}'_\alpha \subseteq \mathcal{P}(A^*)$ that satisfies $\{X \cap A_0: X \in \mathcal{F}'_\alpha\} = \mathcal{F}_\alpha$. Therefore, $\alpha \neq \beta$ implies that $\mathcal{F}'_\alpha \neq \mathcal{F}'_\beta$. Let us define an equivalence relation over $2^{2^{\aleph_0}}$ by $\alpha \sim \beta$ if and only if there is an isomorphism between \mathcal{F}'_α and \mathcal{F}'_β . There are at most 2^{\aleph_0} members in an equivalence class $[\alpha]_\sim$, as there are 2^{\aleph_0} permutations of A^* , and therefore at most 2^{\aleph_0} different isomorphic images of \mathcal{F}'_α . As $2^{\aleph_0} \times 2^{\aleph_0} = 2^{\aleph_0}$, while $2^{2^{\aleph_0}} > 2^{\aleph_0}$, there must be $2^{2^{\aleph_0}}$ equivalence classes over \sim , and therefore $2^{2^{\aleph_0}}$ isomorphism types of homogeneous families over A^* .

We prepare for the proof Lemma 4.1. Before plunging into the formalism, let us state the idea behind the proof. We use the set of demands over a family and the free group associated with this set to construct a successful extension in which the automorphisms act freely. Thus, we can control sets in the orbit of an 'old' set so that their intersections with the 'old' set are either finite or 'old'.

We need some notation. Let $\text{FG}(D)$ be the free group over the set $D = D(\mathcal{F})$ for some family \mathcal{F} . If \mathcal{F} is countable, this group is also countable. We view $\text{FG}(D)$ as the collection of all reduced words in the alphabet $C = D \cup \{d^{-1}: d \in D\}$ (a word is reduced if there is no occurrence of dd^{-1} or $d^{-1}d$ in it) and the group operation, denoted by \circ , is juxtaposition and cancellation (so $w_1 \circ w_2$ is a reduced word, and its length

may be strictly smaller than $\lg w_1 + \lg w_2$). We let c range over the alphabet C , and let c^{-1} denote d^{-1} if $c = d$ or d if $c = d^{-1}$. We denote by e the unit of the free group, which is the empty sequence $\langle \rangle$. For convenient discussion we also adopt the notation h^c and f^c , by which we mean h^d and f^d if $c = d$ and the respective inverses $(h^d)^{-1}$ and $(f^d)^{-1}$ otherwise. Now we can make the following definitions.

4.3 DEFINITION. Suppose that $\mathbf{T} = \langle T_i : (i \in I); \Phi_i' \rangle$ is a successful direct system of multi-embeddings. For every $j \in I$ we have the following.

(1) A homomorphism $\xi_j: \text{FG}(D_j) \rightarrow G_j$ is defined by $\xi_j(d) := \phi_j(\Phi(d))$.

(2) We call a word $w = c_0 \dots c_k \in \text{FG}(D_j)$ *new* if c_i is not in the range of Φ_i' for all $l \leq k$ and all $i < j$. A word $w \in \text{FG}(D_j)$ is *old* if it is in the range of Φ_i' for some $i < j$.

(3) For a word $w \in \text{FG}(D_j)$ and $X \in \mathcal{F}_i$ we define what $f^w(X)$ is. Let $w = w_0 w_1 \dots w_l$ where for each $k \leq l$ the word w_k is either new or old. For a new word $w_k = c_k^0 \dots c_k^{l(k)}$ we denote by f^{w_k} the composition $f^{c_k^{l(k)}} \dots f^{c_k^0}$. If this composition is empty, we say that f^{w_k} is not defined. If w_k is old, then $\xi_j(w) \in \text{Aut}(\mathcal{F}_j)$ and induces a one-to-one function $f^{w_k}: \mathcal{F}_j \rightarrow \mathcal{F}_j$. Let f^w be the composition $f^{w_l} \dots f^{w_0}$. If this composition is empty, we say that f^w is not defined.

(4) Analogously to the definition in (3), we define h^w .

To prove Lemma 4.1 we need an expansion of the technique of direct limits by some more structure. This is needed to enable us to handle uncountably many demands by adding just countably many points. We first define (a particular case of) inverse systems. Then we form direct limits of inverse systems to obtain a pair of sets as required by the lemma.

4.4 DEFINITION. A sequence $\mathbf{T} = \langle T^m : m < \omega \rangle$, where

$$T^m = \langle A^m, \mathcal{F}^m, D^m, G^m, \phi^m \rangle,$$

is called an *inverse system* if:

(1) $\mathcal{F}^m \subseteq \mathcal{P}(A^m)$, $D = D(A^m, \mathcal{F}^m)$, $G^m \subseteq \text{Aut}(\mathcal{F}^m)$ and $\phi^m: \text{FG}(D^m) \rightarrow G^m$ is a partial satisfying homomorphism,

(2) A^m and \mathcal{F}^m are **countable**,

and for $m \leq m'$ the following hold:

(3) $A^m \subseteq A^{m'}$,

(4) $\mathcal{F}^m \subseteq \{X \cap A^m : X \in \mathcal{F}^{m'}\}$,

(5) $G^m \subseteq \{g \upharpoonright A^m : g \in G^{m'}, g \upharpoonright A^m \in \text{Sym}(A^m)\}$.

For a demand $d \in D^{m'}$ we define $d \upharpoonright A^m$ if and only if $\text{dom } h^d \cup \text{ran } h^d \subseteq A^m$ and for every distinct $X, Y \in \text{dom } f^d \cup \text{ran } f^d$ the sets $X \cap A^m$ and $Y \cap A^m$ are distinct. When $d \upharpoonright A^m$ is defined, $h^{d \upharpoonright A^m} = h^d$ and $f^{d \upharpoonright A^m}$ is obtained from f^d by replacing every $X \in \text{dom } f^d \cup \text{ran } f^d$ by $X \cap A^m$. Clearly, when $d \upharpoonright A^m$ is defined, it belongs to D^m , and every $d \in D^m$ equals $d' \upharpoonright A^m$ for some $d' \in D^{m'}$ by (3) and (4).

If $w = c_0 \dots c_k \in \text{FG}(D^{m'})$ and $c_i \upharpoonright A^m$ is defined for every $i \leq k$, we define $w \upharpoonright A^m$ as $c_0 \upharpoonright A^m \dots c_k \upharpoonright A^m$ (it is obvious what $c \upharpoonright A^m$ is). The restriction \upharpoonright is a partial homomorphism from $\text{FG}(D^{m'})$ onto $\text{FG}(D^m)$. The last condition is

(6) If $d \in \text{dom } \phi^{m'}$ and $d \upharpoonright A^m$ is defined, then $d \upharpoonright A^m \in \text{dom } \phi^m$ and $\phi^m(d \upharpoonright A^m) = \phi^{m'}(d) \upharpoonright A^m$ (the operation of $\phi^{m'}(d)$ on A^m depends only on $d \upharpoonright A^m$ when $d \upharpoonright A^m$ is defined).

4.5 DEFINITION. Given an inverse system $\mathbf{T} = \langle T^m : m < \omega \rangle$ we define the inverse limit $\lim \mathbf{T} = T^* = \langle A^*, \mathcal{F}^*, D^*, G^*, \phi^* \rangle$ as follows.

(a) $A^* = \bigcup_m A^m$. For every $x \in A^*$ let $m(x)$ be the least m such that $x \in A^m$.

(b) $\mathcal{F}^* = \{X \subseteq A^* : (X \cap A^m \in \mathcal{F}^m) \text{ for all but finitely many } m\}$. For $X \in \mathcal{F}^*$ we let $m(X)$ be the least such that $X \cap A^m \in \mathcal{F}^m$ for every $m \geq m(X)$. We call $X \in \mathcal{F}^*$ bounded if $X \subseteq A^m$ for some m .

(c) $G^* = \{g \in \text{Sym}(A^*) : (g \upharpoonright A^m \in G^m) \text{ for all but finitely many } m\}$. Let $m(g)$ be the least such that $g \upharpoonright A^m \in G^m$ for every $m \geq m_g$. It is easy to verify that $G^* \subseteq \text{Aut}(\mathcal{F}^*)$.

(d) $D^* = D(\mathcal{F}^*)$. It is easy to verify that for every $d^* \in D^*$ there is some $m(d^*)$ such that for all $m \geq m(d^*)$ it is true that $d^* \upharpoonright A^m$ is defined, and $d^* \upharpoonright A^m \in D^m$.

(e) Finally, $\phi^*(d^*) = \bigcup_{m \geq m_d} \phi^m(d^* \upharpoonright A^m)$ and is defined if and only if $d^* \upharpoonright A^m \in \text{dom } \phi^m$ for all $m \geq m_d$.

4.6 DEFINITION. Suppose that $\mathbf{T}_0 = \langle T_0^m : m < \omega \rangle$ and $\mathbf{T}_1 = \langle T_1^m : m < \omega \rangle$ are inverse systems, and let

$$\lim \mathbf{T}_0 = T_0 = \langle A_0, \mathcal{F}_0, D_0, G_0, \phi_0 \rangle \quad \text{and} \quad \lim \mathbf{T}_1 = T_1 = \langle A_1, \mathcal{F}_1, D_1, G_1, \phi_1 \rangle$$

be their respective inverse limits. We call a sequence $\langle \overset{m}{\Phi} : T_0^m \rightarrow T_1^m : m < \omega \rangle$ of multi-embeddings an *inverse system of multi-embeddings* if for $m \leq m'$ we have:

$$(1) \quad \overset{m'}{\Phi} \upharpoonright A_0^m = \overset{m}{\Phi} \upharpoonright A_0^m,$$

$$(2) \quad \overset{m'}{\Phi}(X) \upharpoonright A_1^m = \overset{m}{\Phi}(X \cap A_0^m) \text{ for every } X \in \mathcal{F}_0^{m'} \text{ for which } X \cap A_0^m \in \mathcal{F}_0^m,$$

$$(3) \quad \overset{m'}{\Phi}(g) \upharpoonright A_1^m = \overset{m}{\Phi}(g \upharpoonright A_0^m) \text{ for every } g \in G_0^m \text{ for which } g \upharpoonright A_0^m \in G^m.$$

When $\langle \overset{m}{\Phi} : T_0^m \rightarrow T_1^m : m < \omega \rangle$ is an inverse system of multi-embeddings we define a multi-embedding $\Phi = \lim \overset{m}{\Phi} : T_0 \rightarrow T_1$ as follows:

$$\Phi \upharpoonright A_0^* = \bigcup \overset{m}{\Phi} \upharpoonright A_0^m,$$

$$\Phi(X) = \bigcup_{m \geq m(X)} \overset{m}{\Phi}(X \cap A_0^m) \text{ for } X \in \mathcal{F}_0^*,$$

$$\Phi(g) = \bigcup_{m \geq m_g} \overset{m}{\Phi}(g \upharpoonright A_0^m) \text{ for } g \in G_0^*.$$

Call $\Phi = \lim \overset{m}{\Phi}$ a *multi-embedding of inverse systems*.

4.7 CLAIM. If $\mathbf{T}_0 = \langle T_0^m : m < \omega \rangle$ and $\mathbf{T}_1 = \langle T_1^m : m < \omega \rangle$ are inverse system and $\langle \overset{m}{\Phi} : T_0^m \rightarrow T_1^m : m < \omega \rangle$ is an inverse system of multi-embeddings such that every $\overset{m}{\Phi}$ is successful, then $\Phi = \lim \overset{m}{\Phi}$ is also successful.

Proof. Suppose that $d \in D_0$; we shall show that $\Phi(d) \in \text{dom } \phi_1$. There is some m_d such that for all $m \geq m_d$ the restriction $d \upharpoonright A_m$ is defined. As $\overset{m}{\Phi}$ is successful, $\overset{m}{\Phi}(d \upharpoonright A_m)$ belongs to $\text{dom } \phi_1^m$ for $m \geq m_d$. Therefore $\phi_1(\bigcup_{m \geq m_d} \overset{m}{\Phi}(d \upharpoonright A_m)) = \phi_1(\Phi(d))$ exists and belongs to G_1 .

We shall construct a two dimensional system $\mathbf{T} = \langle T_n^m : n, m < \omega \rangle$ and successful multi-embeddings $\Phi_n^{m+1} : T_n^m \rightarrow T_{n+1}^m$ such that for every n ,

- (1) $\mathbf{T}_n = \langle T_n^m : m < \omega \rangle$ is an inverse system,
- (2) $\langle \Phi_n^{m+1} : m < \omega \rangle$ is an inverse system of successful multi-embeddings.

Then a direct system will result: $T_n = \varinjlim \mathbf{T}_n$ and $\Phi_n^{n+1} = \varinjlim \Phi_n^{n+1}$.

Let $T_0^m = \langle m+1, \mathcal{P}(m+1), D(\mathcal{P}(m+1)), \{\text{id}\}, \{(e, \text{id})\} \rangle$. Clearly, $T_0 = \varinjlim \mathbf{T}_0 = \langle \omega, \mathcal{P}(\omega), D(\mathcal{P}(\omega)), \{\text{id}\}, \{(e, \text{id})\} \rangle$.

Suppose now that $T_n = \varinjlim T_n^m$ is defined, where $T_n^m = \langle A_n^m, \mathcal{F}_n^m, D_n^m, G_n^m, \phi_n^m \rangle$, and that $\Phi_{n-1}^n = \varinjlim \Phi_{n-1}^n$ is also defined (when $n > 0$).

We assume, for simplicity, that $\Phi_{n-1}^n \upharpoonright A_{n-1} = \text{id}$ (if $n > 0$) and, furthermore, identify $\text{FG}(D_{n-1}^m)$ with its image under Φ_{n-1}^n , and write $\text{FG}(D_{n-1}^m) \subseteq \text{FG}(D_n^m)$ as well as $\text{FG}(D_{n-1}) \subseteq \text{FG}(D_n)$. Thus, the new words of $\text{FG}(D_n)$ coincide with $\text{FG}(D_n \setminus D_{n-1})$ and similarly for $\text{FG}(D_n^m)$. Let $\bigcup_m D_n^m$ be the disjoint union of D_n^m . We view A_n as a subset of the set $B_{n+1} = \{xw : x \in A_n, w \in \text{FG}(\bigcup_m D_n^m)\}$. The expression xw is the formal string $xc_0 \dots c_w$ where $w = c_0 \dots c_k$, and x is identified with xe (where e is the empty string).

4.8 FACT. *The set B_{n+1} is countable.*

This fact holds because each D_n^m is countable.

Now define $B_{n+1}^m = \{xw : x \in A_n^m, w \in \text{FG}(\bigcup_{m' \leq m} D_n^{m'})\}$. Clearly, $A_n^m \subseteq B_{n+1}^m$. Next we define an operation $\xi_{n+1}(c) : B_{n+1} \rightarrow B_{n+1}$ for every $c \in D_n$ (there are, of course, uncountably many elements $c!$). We want $\xi_{n+1}(c) \upharpoonright B_{n+1}^m$ to depend only $c \upharpoonright A_n^m$ whenever $c \upharpoonright A_n^m$ is defined. If $x \in \text{dom } h^c$, we let $\xi_{n+1}(c)(x) = h^c(x)$. For all other points in B_{n+1} , we let $\xi_{n+1}(c)(xw) = xw \circ (c \upharpoonright A_0^m)$ if m is the least integer such that $xw \in B_n^m$ and $c \upharpoonright A_0^m (\in C_1^m)$ is defined.

There is a unique extension of ξ_{n+1} to a homomorphism from $\text{FG}(D_n)$ to $\text{Sym}(B_{n+1})$, which we also call ξ_{n+1} .

4.9 CLAIM. *For every $w \in \text{FG}(D_n)$ there is some $m(w)$ such that*

- (1) B_{n+1}^m is invariant under $\xi_{n+1}(w)$ for all $m \geq m_w$,
- (2) if $w \neq e$, then for every $xv \in B_{n+1} \setminus A_{n+1}^{m_w}$ we have $\xi_{n+1}(w)(xv) = xv \circ w \neq xv$.

Proof. (1) This is clear from the definition. For (2) notice that if $c \upharpoonright A_0^m$ is defined then the finitely many points in $\text{dom } h^c$ belong to A_0^m . Then $\xi_{n+1}(c)(xv) = xv \circ (w \upharpoonright A_0^m)$.

From 4.9 (2) it follows readily that ξ_{n+1} is, in fact a *monomorphism*, as for every $w \in \text{FG}(d_0)$ there is some m_w for which $w \upharpoonright A_0^m$ is defined.

Let $\xi_{n+1}^m(c \upharpoonright A_n^m) = \xi_{n+1}(c) \upharpoonright B_{n+1}^m$ for all $c \in D_n$ for which $c \upharpoonright A_n^m$ is defined. Now we can define

$$A_{n+1} = \{\xi_{n+1}(w)(x) : x \in A_n, w \in \text{FG}(D_n)\},$$

$$A_{n+1}^m = A_n \cap B_{n+1}^m = \{\xi_{n+1}^m(w)(x) : x \in A_n^m, w \in \text{FG}(D_n^m)\}.$$

(We remark that $A_{n+1} \neq B_{n+1}$, because when $x \in \text{dom } h^c$, the point $xc \notin A_{n+1}$). Clearly, a_{n+1} is invariant under $\xi_{n+1}(w)$ for every $w \in \text{FG}(D_n)$, and also $A_{n+1}^{m_w}$ is, if $w \upharpoonright A_n^{m_w}$ is

defined. Having defined A_{n+1} we let $\Phi_n^{m+1}: A_n^m \rightarrow A_{n+1}^m$ be the identity. Therefore also $\Phi_n^{n+1} \upharpoonright A_n$ is the identity.

Now let us define $\Phi_n^{m+1} \upharpoonright \mathcal{F}_n^m$. For every $X \in \mathcal{F}_n^m$ and $xw \in A_{n+1}^m$ we determine whether $xw \in \Phi_n^{m+1}(X)$ by induction on the length of w . If $\text{lg } w = 0$ then necessarily $xw = x$, and we let $x \in \Phi_0^m(X) \Leftrightarrow x \in X$ for every $X \in \mathcal{F}_n^m$ and $x \in A_n^m$. Suppose that this is done for all words of length k and that $\text{lg } wc = k + 1$. Distinguish two cases: when c is old and when c is new.

First case: c is old, namely $c \in C_{n-1}^m$ (this case does not exist when $n = 0$).

Here we have that $\xi_n^m(c) = \phi_n^m(c)$ is defined, and is an automorphism of \mathcal{F}_n^m . Let $xwc \in \Phi_0^m(X) \Leftrightarrow xw \in \Phi_n^{n+1}(\xi_{n-1}^m(c^{-1})[X])$.

Second case: c is new. Let $xwc \in \Phi_n^{m+1}(X) \Leftrightarrow xw \in \Phi_n^{m+1}(f^{c^{-1}}(X))$. In the right hand side we mean that $f^{c^{-1}}(X)$ is defined and $xw \in \Phi_n^{m+1}(f^{c^{-1}}(X))$.

Now we can set $\Phi_n^{m+1}(X) = \bigcup_{m \geq m(X)} \Phi_n^{m+1} X \cap A_n^m$.

4.10 FACT. For every old $w \in \text{FG}(D_n)$ and every $X \in \mathcal{F}_n$ we have that $\Phi_n^{n+1}(\phi_n(w)[X]) = \xi_{n+1}(w)[\Phi_n^{n+1}(X)]$ (2.1(c)).

The proof of 4.10 is straightforward using induction on word length.

4.11 CLAIM. For every $w \in \text{FG}(D_n)$ and every $X \in \mathcal{F}_n$ there is $m \geq m_w$ such that

- (1) $f^w(X)$ is defined if and only if $f^{w \upharpoonright A_n^m}(X \cap A_n^m)$ is defined,
- (2) $f^{w^{-1}}(X)$ is defined if and only if $f^{w^{-1} \upharpoonright A_n^m}(X \cap A_n^m)$ is defined,
- (3) for every $x \in A_n$ with $m(x) \geq m$ we have that

$$\xi_{n+1}(w)(x) = x(w \upharpoonright A_n^{m(x)}) \in \Phi_n^{n+1}(X) \Leftrightarrow x \in f^{w^{-1}}(X)$$

(where by $x \in f^{w^{-1}}(X)$ we mean that $f^{w^{-1}}(X)$ is defined and $x \in f^{w^{-1}}(X)$).

Proof. If $f^w(X)$ is defined, then $f^{w \upharpoonright A_n^m}(X \cap A_n^m)$ is defined whenever $w \upharpoonright A_n^m$ is defined and equals $f^w(X) \cap A_n^m$. Conversely, if $f^w(X)$ is not defined, then there is some $m \geq m_w$ such that $X \cap A_n^m \neq Y \cap A_n^m$ for all $Y \in \text{dom } f^w$ (if there is one X for which $f^w(X)$ is not defined, then $\text{dom } f^w$ is necessarily finite) and therefore $f^{w \upharpoonright A_n^m}(X \cap A_n^m)$ is not defined.

From the definition of ξ_{n+1} and the fact that $m(x) \geq m_w$ it follows that $\xi_{n+1}(x) = x(w \upharpoonright A_n^m)$. From the definition of $\Phi_n^{n+1} \upharpoonright \mathcal{F}_n$ it is immediate that $x(w \upharpoonright A_n^m) \in \Phi_n^{n+1}(X) \Leftrightarrow x \in f^{w^{-1}}(X)$.

4.12 FACT. The intersection $\Phi_n^{n+1}(X) \cap A_{n+1}^m$ depends only on $X \cap A_n^m$ whenever $X \cap A_n^m \in \mathcal{F}_n^m$.

Now we can define $\mathcal{F}_{n+1}^m = \{\xi_{n+1}(w)[\Phi_n^{n+1}(X)]: X \in \mathcal{F}_n, w \in \text{FG}(D_n)\}$. Let $\mathcal{F}_{n+1}^m = \{\xi_{n+1}(w)(X): X \in \mathcal{F}_{n+1}^m, w \in \text{FG}(D_n^m)\}$.

4.13 CLAIM. With the above notation \mathcal{F}_{n+1}^m is countable for every m .

Proof. The claim follows from the countability of $\text{FG}(D_n^m)$ and 4.12.

We have finished defining T_{n+1} and $\langle \Phi_n^{m+1}: m < \omega \rangle$, and verified that T_{n+1} is an inverse system, that $\langle \Phi_n^{m+1}: m < \omega \rangle$ is an inverse system of successful multi-embedding and that, consequently, $\Phi_n^{n+1}: T_n \rightarrow T_{n+1}$ is a multi-embedding of inverse systems.

Let $T^* = \langle A^*, \mathcal{F}^*, D^*, G^*, \phi^* \rangle = \lim T_n$. We show that the conclusion of Lemma 4.1 holds for the pair of sets A_0 and A^* . Clearly, these sets are countable and $A_0 \subseteq A^*$. So all we need is the following.

4.14 CLAIM. *For every family $\mathcal{F} \subseteq \mathcal{P}(A_0)$ which includes $\text{Fin}(A_0)$ there is a homogeneous family $\mathcal{F}' \subseteq \mathcal{F}^*$ such that $\mathcal{F}' \upharpoonright A_0 = \mathcal{F}$.*

Proof. Suppose that $\mathcal{F} \subseteq \mathcal{P}(A_0)$ is a family which includes $\text{Fin}(A_0)$. We work by induction on n and define $\mathcal{F}'_n \subseteq \mathcal{P}(A_n)$ for every n :

- (1) $\mathcal{F}'_0 = \mathcal{F}$,
- (2) $\mathcal{F}'_{n+1} = \{\xi_n(w)[\Phi_n^{n+1}(X)]: w \in \text{FG}(D(\mathcal{F}'_n)), X \in \mathcal{F}'_n\}$.

Let $\mathcal{F}' = \{\Phi_n(X): X \in \mathcal{F}'_n\}$. We claim that

- (a) $\mathcal{F}' \subseteq \mathcal{F}^*$ and $\phi^* \upharpoonright D(\mathcal{F}')$ testifies that \mathcal{F}' is homogeneous,
- (b) $\{X \cap A_0: X \in \mathcal{F}'\} = \mathcal{F}$.

To prove (a) suppose that $d \in D(\mathcal{F}')$ is a demand. Then there is some n and a demand $d_n \in D(\mathcal{F}'_n)$ such that $\Phi_n(d_n) = d$. As Φ_n^{n+1} is successful, $\xi_n(d) = \phi_{n+1}(\Phi_n^{n+1}(d_n)) =: g$ is defined. Now $\Phi_{n+1}(g) = \phi^*(d)$ satisfies d and is an automorphism of \mathcal{F}^* . Why is it also an automorphism of \mathcal{F}' ? Because of (2) above.

To prove (b) we notice that it is enough to prove by induction that for every n and $X \in \mathcal{F}'_{n+1}$, we have

$$(*)_n \quad X \cap A_n \in \mathcal{F}'_n \text{ or is bounded.}$$

For then it follows by induction that $X \cap A_0 \in \mathcal{F}$ for every n and $X \in \mathcal{F}'_n$: if $X \cap A_n \in \mathcal{F}'_n$ we have that $X \cap A_0 \in \mathcal{F}$ by the induction; if $X \cap A_n$ is bounded, then $X \cap A_0$ is finite and again in \mathcal{F} .

So let us prove $(*)_n$. We have to show that for every $w \in \text{FG}(D_n)$ and every $X \in \mathcal{F}'_n$ the set $\xi_{n+1}[\Phi_n^{n+1}(X)] \cap A_n$ belongs to \mathcal{F}'_n or is bounded. We show something stronger.

$(**)_n$ For every $X \in \mathcal{F}'_n$ and $w \in \text{FG}(D(\mathcal{F}'_n))$, if $f^w(X)$ is defined then $\xi_{n+1}[\Phi_n^{n+1}(X)] = \Phi_n^{n+1}(f^w(X))$ (and therefore $\xi_{n+1}[\Phi_n^{n+1}(X)] \cap A_n = \Phi_n^{n+1}(f^w(X)) \cap A_n = f^w(X) \in \mathcal{F}'_n$). If $f^w(X)$ is not defined, then $\xi_{n+1}[\Phi_n^{n+1}(X)] \cap A_n$ is bounded.

Suppose first that $f^w(X)$ is defined. Then obviously it belongs to \mathcal{F}'_n , because $w \in \text{FG}(D(\mathcal{F}'_n))$. It is easy to check that $\xi_{n+1}(w)(xv) \in \Phi_n^{n+1}(f^w(X)) \Leftrightarrow xv \in \Phi_n^{n+1}(X)$.

So assume that $f^w(X)$ is not defined; we want to prove that $\xi_{n+1}(w)[\Phi_n^{n+1}(X)] \cap A_n$ is bounded. If $f^w(X)$ is not defined, then $X \notin \text{ran } f^{w^{-1}}$. It is sufficient to see that the set

$$\{x \in A_n: \xi_{n+1}(w^{-1})(x) \in \Phi_n^{n+1}(X)\}$$

is bounded, because this set equals $\xi_{n+1}(w)[\Phi_n^{n+1}(X)] \cap A_n$. By 4.11 there is a large enough integer $m > m(w)$ such that for all $x \in A_n$ with $m(x) \geq m$ we have that

$$\xi_{n+1}(w)(x) = \xi_{n+1}^{m(x)}(x) = x(w \upharpoonright A_n^{m(x)}) \in \Phi_n^{n+1}(X \cap A_n^{m(x)}) \Leftrightarrow x \in f^{w^{-1}}(X).$$

But $f^{w^{-1}}(X)$ is not defined, and therefore $\xi_{n+1}(w)(x) \notin \Phi_n^{n+1}(X)$ for all $x \in A_n$ with $m(x) > m$, which is what we wanted. This completes the proof of Lemma 4.1.

We give a corollary of this proof.

4.15 COROLLARY. *There is a collection of $2^{2^{N_0}}$ permutation groups $\langle G_\alpha : \alpha < 2^{2^{N_0}} \rangle$ over ω , such that:*

- (1) every G_α is isomorphic to the free group on 2^{N_0} generators;
- (2) every G_α testifies the homogeneity of some family $\mathcal{F}_\alpha \subseteq \mathcal{P}(\omega)$;
- (3) if $\alpha < \beta < 2^{2^{N_0}}$, then G_α and G_β are not isomorphic as permutation groups.

Proof. We have shown that there are $2^{2^{N_0}}$ many homogeneous sub-families F'_α of F^* for $\alpha < 2^{2^{N_0}}$. The restriction of ϕ^* to $\text{FG}(D(A^*, \mathcal{F}_\alpha))$ is a monomorphism of the free group over a set of cardinality 2^{N_0} into G^* which testifies homogeneity of \mathcal{F}_α . This gives us $2^{2^{N_0}}$ different groups satisfying (1) and (2) in the corollary, since a single group can act homogeneously on at most 2^{N_0} different families, because different families on which the same group acts 1-transitively are disjoint. To obtain (3), divide by the relation 'isomorphic via a permutation of ω ', and pick a member from every equivalence class. As in each class there are 2^{N_0} many members at the most, we get that there are $2^{2^{N_0}}$ classes.

We now wish to show that there is no homogeneous family over ω such that every homogeneous family over ω is isomorphic to one of its subfamilies. This will follow from the next lemma about the number of pairwise incompatible homogeneous families over a countable set. Two families over ω are *incompatible* if for some $X \subseteq \omega$ the set X belongs to one family while the set $\omega \setminus X$ belongs to the other. For every $X \subseteq \omega$ let us denote $X^0 := X$ and $X^1 := \omega \setminus X$.

4.16 LEMMA. *There is a collection $\{F_\alpha : \alpha < 2^{2^{N_0}}\}$ of pairwise incompatible homogeneous families over ω .*

4.17 COROLLARY. *There is no homogeneous family over ω such that every homogeneous family over ω is isomorphic to one of its subfamilies.*

Proof of Corollary. Suppose to the contrary that \mathcal{F}^* is a homogeneous family over ω with this property. By Lemma 4.16 pick a collection $\{F_\alpha : \alpha < 2^{2^{N_0}}\}$ of pairwise incompatible homogeneous families over ω . For each $\alpha < 2^{2^{N_0}}$ fix a permutation σ_α which embeds \mathcal{F}_α in \mathcal{F}^* . By the pigeon hole principle there are $\alpha < \beta < 2^{2^{N_0}}$ and a permutation σ such that $\sigma_\alpha = \sigma_\beta = \sigma$. As \mathcal{F}_α and \mathcal{F}_β are incompatible, let us find a set $X \subseteq \omega$ such that $X^0 \in F_\alpha$ and $X^1 \in F_\beta$. Now $\sigma_\alpha(X^0) = \sigma(X^0) \in \mathcal{F}^*$, and $\sigma_\beta(X^1) = \sigma(X^1) \in \mathcal{F}^*$. This means that in \mathcal{F}^* there is a set and its complement. This contradicts Theorem 1.4 which states that every homogeneous family over ω is independent.

Proof of Lemma 4.16. We use the direct system of inverse systems from the proof of Lemma 4.1. The pairwise disjoint families will be over A^* rather than over ω , but as this is a countable set this makes no difference.

Let the variable η range over the set of all functions $\eta: \mathcal{P}(A_0) \rightarrow 2$ which satisfy $\eta(X) + \eta(A_0 \setminus X) = 1$ for all $X \subseteq A_0$. These are functions that select exactly one element

from each pair of a set and its complement (for example, characteristic functions of ultra filters). There are $2^{2^{\aleph_0}}$ such functions.

For every function $\eta: \mathcal{P}(A_0) \rightarrow 2$ as above let $\mathcal{F}_\eta^0 = \{X \subseteq A_0: \eta(X) = 1\}$. The collection $\{\mathcal{F}_\eta: \eta: \mathcal{P}(A_0) \rightarrow 2\}$ is a collection of $2^{2^{\aleph_0}}$ pairwise incompatible families over A_0 . For every $m < \omega$ let $\mathcal{F}_\eta^{0,m}$ be the projection of \mathcal{F}_η on A_0^m .

We know that for every \mathcal{F}_η there is a homogeneous family \mathcal{F}'_η over A^* whose projection on A_0 equals \mathcal{F}_η (modulo finite sets). However, it is NOT true that $\{\mathcal{F}'_\eta: \eta: \mathcal{P}(A_0) \rightarrow 2\}$ is a collection of pairwise incompatible families. In fact, $\Phi_0^1(X^0) \cap \Phi_0^1(X^1)$ is not empty for every $X \subseteq A_0$.

What we shall do now is refine the extension operation in such a way that not only is the projection on A_0 preserved, but also the disjointness of X^0 and X^1 . This will be achieved by removing some of the points of A^* . We define by induction on n a subset $\bar{D}_n \subseteq D_n$ and a subset $E_n \subseteq A_n$. Restricting ourselves to the points of $E = \bigcup_n E_n$ will provide the desired conservation property. Let $E_0 = A_0$. Let $\bar{D}_0 = \bigcup_n D(E_0, \mathcal{F}_\eta^0)$.

4.18 FACT. *If $d \in \bar{D}_0$, then for no $X \subseteq A_0$ is it true that both X^0, X^1 belong to $\text{ran } f^d$.*

We remove, thus, from the collection of demands all demands which mention simultaneously a set and its complement in their range. Let us now define E_1 as follows:

$$E_1 = \{xw: x \in E_0, w = c_0 \dots c_k \in \text{FG}(\bigcup_m \bar{D}_0^m) \text{ and } x \notin \text{dom } f^{c_0}\}.$$

The difference from the proof of 4.1 is that only a proper subset of words is being used. Hence, $E_1 \subseteq A_1$

4.19 CLAIM. *For every $X \subseteq A_0$ we have that $\Phi_0^1(X^0) \cap \Phi_0^1(X^1) \cap E_1 = \emptyset$.*

Proof. By induction on the length of $w \in \text{FG}(\bigcup_m \bar{D}_0^m)$ we shall see that $xw \notin \Phi_0^1(X^0) \cap \Phi_0^1(X^1)$.

If $\text{lg } w = 0$ then $xw = x \in E_0 = A_0$. As $\Phi_0^1(X) \cap A_0 = X$ for all X , it follows that $x \notin \Phi_0^1(X^0) \cap \Phi_0^1(X^1)$.

Now suppose that $\text{lg } wc = k + 1$. By the definition of the \in relation over the set A_1 we know that $xwc \in \Phi_0^1(X^0)$ if and only if there is some Y such that $xw \in \Phi_0^1(Y)$ and $f^c(Y) = X^0$. Similarly, $xwc \in \Phi_0^1(X^1)$ if and only if there is some Z such that $xw \in \Phi_0^1(Z)$ and $f^c(Z) = X^1$. But X^0 and X^1 cannot both appear in $\text{ran } f^c$ because $c \in \bar{D}_0^m$. Therefore xwc is not in the intersection. This completes the proof of 4.19.

Now we notice that E_1 is invariant under $\xi_1(w)$ for all $w \in \text{FG}(\bar{D}_0)$. Also, for every $w \in \text{FG}(\bar{D}_0)$ and every $X \subseteq E_0$ we have that $\xi_1(w)[\Phi_0^1(X^0)] \cap \xi_1(w)[\Phi_0^1(X^1)] \cap E_1 = \emptyset$.

Let $\bar{\mathcal{F}}_1 = \{\xi_1(w)[\Phi_0^1(X)]: X \in \bar{F}_0, w \in \bar{D}_0\}$. We proceed by induction on n , defining \bar{D}_n and E_{n+1} for all $n > 0$.

First, let us view each $\eta: \mathcal{P}(E_0) \rightarrow 2$ as a partial function $\eta: \bar{\mathcal{F}}_1 \rightarrow 2$ by replacing every $X \subseteq E_0$ by $\Phi_0^1(X)$. Next extend each η to contain $\bar{\mathcal{F}}_1$ in its domain, demanding that

$$\eta(\xi_1(w)[\Phi_0^1(X)]) = \eta(X).$$

We refer to the resulting extended function also as η to avoid cumbersome notation. For every η let $\bar{\mathcal{F}}_{\eta,1} = \{X \in \bar{\mathcal{F}}_1: \eta(X) = 1\}$. Now define $\bar{D}_1 = \bigcup_n D(\bar{F}_{\eta,1})$. Define E_{n+1} and $\bar{\mathcal{F}}_{n+1}$ as before. We should check the following claim.

4.20 CLAIM. For all $X \in \mathcal{F}_n$ we have that $\Phi_n^{n+1}(X^0) \cap \Phi_n^{n+1}(X^1) \cap E_{n+1} = \emptyset$.

Proof. By induction on word length. The case which should be added to the proof of 4.19 is the case when c is old, and is easily verified.

Having done the induction, we set $E = \bigcup_n E_n$. For every $\eta: \mathcal{P}(E_0) \rightarrow 2$ let \mathcal{F}'_η be the homogeneous family obtained from \mathcal{F}_η as in the proof of 4.1. The reader will verify that

- (1) for every $X \subseteq E_0$ we have that $\Phi_0(X^0) \cap \Phi_0(X^1) \cap E = \emptyset$,
- (2) for every $\eta: \mathcal{P}(E_0) \rightarrow 2$ the family $\mathcal{F}'_\eta \upharpoonright E$ is homogeneous.

This completes the proof of Lemma 4.16.

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