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Cofinality of normal ideals on $[\lambda]^{<\kappa}$ I

Pierre Matet[1] · **Cédric Péan**[2] ·
Saharon Shelah[3,4]

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Abstract An ideal J on $[\lambda]^{<\kappa}$ is said to be $[\delta]^{<\theta}$-normal, where δ is an ordinal less than or equal to λ, and θ a cardinal less than or equal to κ, if given $B_e \in J$ for $e \in [\delta]^{<\theta}$, the set of all $a \in [\lambda]^{<\kappa}$ such that $a \in B_e$ for some $e \in [a \cap \delta]^{<|a\cap\theta|}$ lies in J. We give necessary and sufficient conditions for the existence of such ideals and describe the smallest one, denoted by $NS_{\kappa,\lambda}^{[\delta]^{<\theta}}$. We compute the cofinality of $NS_{\kappa,\lambda}^{[\delta]^{<\theta}}$.

Keywords $[\lambda]^{<\kappa}$ · Normal ideal

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✉ Pierre Matet
pierre.matet@unicaen.fr

Saharon Shelah
shelah@math.huji.ac.il

1 Laboratoire de Mathématiques, CNRS, Université de Caen, BP 5186, 14032 Caen Cedex, France

2 Nexways, 131 Boulevard de Sébastopol, 75002 Paris, France

3 Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Edmond J. Safra Campus, Givat Ram, 91904 Jerusalem, Israel

4 Department of Mathematics, Hill Center for the Mathematical Sciences, Rutgers University, 110 Frelinghuysen Rd, Piscataway, NJ 08854-8019, USA

1 Introduction

Let κ be a regular uncountable cardinal, and λ be a cardinal greater than or equal to κ. An ideal on $[\lambda]^{<\kappa}$ is said to be normal if it is closed under diagonal unions of λ many of its members. Building on work of Jech [10] and Menas [19], Carr [2] described the smallest (in terms of inclusion) such ideal. Called the nonstationary ideal on $[\lambda]^{<\kappa}$, it is usually denoted by $NS_{\kappa,\lambda}$. Numerous variations on the original notion of normality have been considered over the years. We are interested in two of these variants. First there is the notion of strong normality that has been rather extensively studied (see e.g. [3,4,8,13]). The definition involves diagonal unions of length $\lambda^{<\kappa}$. Carr et al. [3] give necessary and sufficient conditions for the existence of strongly normal ideals on $[\lambda]^{<\kappa}$ and describes the least such ideal when there is one. As the terminology implies, any strongly normal ideal is normal. The other notion is that of δ-normality for an ordinal $\delta \leq \lambda$. An ideal on $[\lambda]^{<\kappa}$ is called δ-normal if it is closed under diagonal unions of length δ. Thus λ-normality is the same as normality. This notion of δ-normality has been studied by Abe [1] who gave a description of the smallest δ-normal ideal on $[\lambda]^{<\kappa}$.

We introduce a more general concept, that of $[\delta]^{<\theta}$-normality, where δ is, as above, an ordinal less than or equal to λ, and θ a cardinal less than or equal to κ. The definition is similar to that of strong normality, with this difference that diagonal unions are now indexed by $[\delta]^{<\theta}$. So $[\lambda]^{<\kappa}$-normality is identical with strong normality, whereas $[\delta]^{<2}$-normality is the same as δ-normality.

We give necessary and sufficient conditions for the existence of $[\delta]^{<\theta}$-normal ideals on $[\lambda]^{<\kappa}$ and describe the least such ideal, which we denote by $NS_{\kappa,\lambda}^{[\delta]^{<\theta}}$.

The notion of $[\lambda]^{<\theta}$-normality (with θ a regular infinite cardinal less than κ) has been independently studied by Džamonja [6]. In particular, Claims 2.9 and Corollary 2.13 of [6] provide alternative descriptions of $NS_{\kappa,\lambda}^{[\lambda]^{<\theta}}$.

By the cofinality of an ideal J, we mean the least number of generators for J, that is the least size of any subcollection X of J such that each member of the ideal is included in some element of X. We determine the cofinality of $NS_{\kappa,\lambda}^{[\delta]^{<\theta}}$. Its computation involves a multidimensional version of the dominating number $\mathfrak{d}_\kappa$, which is no surprise, as Landver (Lemma 1.16 in [12]) proved that the cofinality of the minimal normal ideal on κ is $\mathfrak{d}_\kappa$.

Part of the paper is concerned with the problem of comparing the various ideals that are considered. Given two pairs (δ, θ) and (δ', θ'), we investigate whether $NS_{\kappa,\lambda}^{[\delta]^{<\theta}}$ and $NS_{\kappa,\lambda}^{[\delta']^{<\theta'}}$ are equal, and, more generally, whether one of the two ideals is a restriction of the other (there is more about this in [18]). For instance, it is shown that $NS_{\kappa,\lambda}^{[\delta]^{<\theta}} = NS_{\kappa,\lambda}^{[|\delta|]^{<\theta}} | A$ for some A.

The paper is organized as follows. Section 2 collects basic definitions and facts concerning ideals on $[\lambda]^{<\kappa}$. This is standard material except for Proposition 2.6. In Sect. 3 we introduce the property of $[\delta]^{<\theta}$-normality and state necessary and sufficient conditions for the existence of a $[\delta]^{<\theta}$-normal ideal on $[\lambda]^{<\kappa}$. The discussion is very much like the one regarding the existence of a strongly normal ideal, and arguments are routine. We briefly consider various weaker properties (compare e.g. Proposition 3.4 ((iii) and (iv)) and Corollary 3.8 (ii) with Proposition 3.6 (ii)) and characterize the ideals that satisfy them. In Sects. 4 and 5 we show that we could without loss of generality assume that θ is an infinite cardinal, and δ either a cardinal less than κ, or an ordinal multiple of κ. We describe the smallest $[\delta]^{<\theta}$-normal ideal on $[\lambda]^{<\kappa}$, which we denote by $NS_{\kappa,\lambda}^{[\delta]^{<\theta}}$. Section 6 is concerned with the case when θ is a limit cardinal. It is proved among other things that if $\delta \geq \kappa$ and θ is a singular strong limit cardinal, then any $[\delta]^{<\theta}$-normal ideal on $[\lambda]^{<\kappa}$ is $[\delta]^{<\theta^+}$-normal. Sections 7 and 8 deal with the question of the existence of an ordered pair $(\delta', \theta') \neq (\delta, \theta)$ such that $\delta' \leq \delta$, $\theta' \leq \theta$ and $NS_{\kappa,\lambda}^{[\delta]^{<\theta}} = NS_{\kappa,\lambda}^{[\delta']^{<\theta'}} | A$ for some A. In Sect. 9 we show that for any cardinal λ' with $\kappa \leq \lambda' < \lambda$, $NS_{\kappa,\lambda'}^{[\min(\delta,\lambda')]^{<\theta}}$ can be obtained as a projection of $NS_{\kappa,\lambda}^{[\delta]^{<\theta}}$. This generalizes a well-known result of Menas [19].

In Sect. 10 we introduce a three-cardinal version, denoted by $\mathfrak{d}_{\kappa,\lambda}^{\mu}$, of the dominating number $\mathfrak{d}_\kappa$. There are many identities involving the $\mathfrak{d}_{\kappa,\lambda}^{\mu}$'s and we present some of them. Finally, the cofinality of $NS_{\kappa,\lambda}^{[\delta]^{<\theta}}$ is computed in Sect. 11.

2 Ideals

Definition *For a set A and a cardinal τ, we set $[A]^{<\tau} = \{a \subseteq A : |a| < \tau\}$.*

Throughout the section ρ will denote an infinite cardinal, and μ a cardinal greater than or equal to ρ.

The section presents some basic material concerning ideals on $[\mu]^{<\rho}$. We start by recalling a few definitions.

Definition *For $a \in [\mu]^{<\rho}$, we set $\widehat{a} = \{b \in [\mu]^{<\rho} : a \subseteq b\}$.*

Definition *$I_{\rho,\mu}$ denotes the collection of all $A \subseteq [\mu]^{<\rho}$ such that $B \cap \widehat{a} = \emptyset$ for some $a \in [\mu]^{<\rho}$.*

Definition *By an ideal on $[\mu]^{<\rho}$ we mean a collection J of subsets of $[\mu]^{<\rho}$ such that*

- $[\mu]^{<\rho} \notin J$.
- $I_{\rho,\mu} \subseteq J$.
- $P(A) \subseteq J$ *for all* $A \in J$.
- $\bigcup Y \in J$ *whenever* $Y \in [J]^{<\mathrm{cf}(\rho)}$.

The following is readily checked.

Fact 2.1 (folklore) *$I_{\rho,\mu}$ is an ideal on $[\mu]^{<\rho}$.*

Definition *Given a partially ordered set $(P, \leq)$, we let cof$(P, \leq)$ denote the least cardinality of any subset D of P such that for any $p \in P$, there is $d \in D$ with $p \leq d$.*

Definition *Let J be an ideal on $[\mu]^{<\rho}$. We set $J^+ = \{A \subseteq [\mu]^{<\rho} : A \notin J\}$, $J^* = \{A \subseteq [\mu]^{<\rho} : [\mu]^{<\rho} \setminus A \in J\}$, and $J|A = \{B \subseteq [\mu]^{<\rho} : A \cap B \in J\}$ for every $A \in J^+$.*

We let non(J) *denote the least cardinality of any* $A \subseteq [\mu]^{<\rho}$ *with* $A \in J^+$.
We set $\mathrm{cof}(J) = cof(J, \subseteq)$.

Fact 2.2 (folklore) *Let J be an ideal on $[\mu]^{<\rho}$, and $A \in J^+$. Then $J|A$ is an ideal on $[\mu]^{<\rho}$ extending J. Moreover,* $\mathrm{cof}(J|A) \leq \mathrm{cof}(J)$.

Proof Use the fact that for any $B \subseteq [\mu]^{<\rho}$, $B \in J|A$ if and only if $B \subseteq E \cup ([\mu]^{<\rho} \setminus A)$ for some $E \in J$. □

We will also use the following observation.

Fact 2.3 (folklore) *Let I, J, K be three ideals on $[\mu]^{<\rho}$ such that $I \subseteq J \subseteq K$. Suppose that there is $A \in I^+$ such that $K = I|A$. Then $J|A = I|A$.*

Proof Since $A \in K^*$, we have $K|A = K = I|A$. Being sandwiched between $I|A$ and $K|A$, the ideal $J|A$ must be equal to both of them. □

Fact 2.4 (folklore) *Let J be an ideal on $[\mu]^{<\rho}$. Then* $\mathrm{non}(J) \leq \mathrm{cof}(J)$.

Proof Let $S \subseteq J$ be such that $J = \bigcup_{B \in S} P(B)$. Pick $a_B \in [\mu]^{<\rho} \setminus B$ for $B \in S$. Then $\{a_B : B \in S\} \in J^+$. □

Definition *We put* $u(\rho, \mu) = \mathrm{non}(I_{\rho,\mu})$.

Proposition 2.5 (i) $\mu \leq u(\rho, \mu)$.
(ii) $\mathrm{cf}(\rho) \leq \mathrm{cf}(u(\rho, \mu))$.
(iii) $u(\rho, \mu) = cof([\mu]^{<\rho}, \subseteq) = \mathrm{cof}(I_{\rho,\mu})$.

Proof (i) Given $A \in I^+_{\rho,\mu}$, we have $\mu = \bigcup A$ and therefore $\mu \leq \max(\rho, |A|)$. This proves the desired inequality in case $\mu > \rho$. Given $B \subseteq [\rho]^{<\rho}$ with $|B| < \rho$, pick $\alpha_b \in \rho \setminus b$ for $b \in B$. Then $\{\alpha_d : d \in B\} \setminus b \neq \emptyset$ for all $b \in B$, and consequently $B \in I_{\rho,\rho}$. Hence $u(\rho, \rho) \geq \rho$.
(ii) Use the fact that $[\mu]^{<\rho}$ is closed under unions of less than $\mathrm{cf}(\rho)$ many of its members.
(iii) By Fact 2.4, $u(\rho, \mu) \leq \mathrm{cof}(I_{\rho,\mu})$. If $A \subseteq [\mu]^{<\rho}$ is such that $[\mu]^{<\rho} = \bigcup_{a \in A} P(a)$, then clearly, $I_{\rho,\mu} = \{B \subseteq [\mu]^{<\rho} : \exists a \in A \ (B \cap \widehat{a} = \emptyset)\}$. It follows that $\mathrm{cof}(I_{\rho,\mu}) \leq cof([\mu]^{<\rho}, \subseteq)$. Finally, $cof([\mu]^{<\rho}, \subseteq) \leq u(\rho, \mu)$ because $[\mu]^{<\rho} = \bigcup_{a \in H} P(a)$ for any $H \in I^+_{\rho,\mu}$. □

The following will be used in Sect. 10.

Proposition 2.6 *Let K be an ideal on $[\mu]^{<\rho}$. Set $\chi = \min(\{|C| : C \in K^*\})$. Suppose that* $\mathrm{cof}(K) \leq \chi$. *Then χ = the largest cardinal τ such that there exists a partition of $[\mu]^{<\rho}$ into τ sets in K^+.*

Proof Select $D \in K^*$ with $|D| = \chi$. First there is no partition Π of $[\mu]^{<\rho}$ into more than χ sets in K^+ because D has to meet every set in Π. Let us next show that there is a partition $\Pi = \{P_\gamma : \gamma < \chi\}$ of $[\mu]^{<\rho}$ into χ sets in K^+. Fix a family $F = \{B_\alpha : \alpha < \chi\}$ cofinal in K. For $\alpha < \chi$, set $C_\alpha = [\mu]^{<\rho} \setminus B_\alpha$ and let

$\langle c_{\alpha,j} : j < \chi \rangle$ be a one-to-one enumeration of $D \cap C_\alpha$. Let $\langle (\alpha_i, \beta_i) : i < \chi \rangle$ be a one-to-one enumeration of $\chi \times \chi$. By induction define $j_i < \chi$ for $i < \chi$ by $j_i =$ the least $j < \chi$ such that $c_{a_i,j} \notin \{c_{\alpha_l, j_l} : l < i\}$. Now given $\gamma < \chi$, put $A_\gamma = \{c_{\alpha_i, j_i} : i < \chi$ and $\gamma = \beta_i\}$. Let H_γ be the set of all $a \in (\bigcup_{\alpha<\chi} C_\alpha) \setminus (\bigcup_{\xi<\chi} A_\xi)$ with the property that $\gamma =$ the least $\alpha < \chi$ such that $a \in C_\alpha$. Finally put $P_\gamma = A_\gamma \cup H_\gamma$ if $\gamma \neq 0$, and $P_0 = A_0 \cup H_0 \cup ([\mu]^{<\rho} \setminus (\bigcup_{\alpha<\chi} C_\alpha))$. Note that for each $\gamma < \chi$, $|P_\gamma| \geq \chi$, and moreover, $P_\gamma \subseteq C_\gamma \cup ([\mu]^{<\rho} \setminus (\bigcup_{\alpha<\chi} C_\alpha))$. □

Corollary 2.7 *There is a partition $\Pi = \{P_e : e \in [\mu]^{<\rho}\}$ of $[\mu]^{<\rho}$ such that for any $e \in [\mu]^{<\rho}$, $P_e \in I^+_{\rho,\mu} \cap P(\widehat{e})$ and $|P_e| = \mu^{<\rho}$.*

Proof We have that

$$\mathrm{cof}(I_{\rho,\mu}) = u(\rho, \mu) \leq \mu^{<\rho} = \min(\{|C| : C \in I^*_{\rho,\mu}\}.$$

Let $\langle e_\alpha : \alpha < \mu^{<\rho} \rangle$ be a one-to-one enumeration of $[\mu]^{<\rho}$. Then by the proof of Proposition 2.6, there is a partition $\Pi = \{P_\alpha : \alpha < \mu^{<\rho}\}$ of $[\mu]^{<\rho}$ such that for each $\alpha < \mu^{<\rho}$, $|P_\alpha| = \mu^{<\rho}$ and $P_\alpha \subseteq \widehat{e_\alpha}$. □

3 $[\delta]^{<\theta}$-normality

Throughout the remainder of the paper κ denotes a regular infinite cardinal, λ a cardinal greater than or equal to κ, θ a cardinal with $2 \leq \theta \leq \kappa$, and δ an ordinal with $1 \leq \delta \leq \lambda$.

We let $\bar{\theta}$ denote the supremum of all the cardinals that are both less than κ and less than or equal to θ.

Thus $\bar{\theta} = \theta$ if $\theta < \kappa$, or $\theta = \kappa$ and κ is a limit cardinal, and $\bar{\theta} = \nu$ if $\theta = \kappa = \nu^+$.

Throughout the remainder of the paper J denotes a fixed ideal on $[\lambda]^{<\kappa}$.

In this section we introduce the notion of $[\delta]^{<\theta}$-normal ideal on $[\lambda]^{<\kappa}$ and describe necessary and sufficient conditions for the existence of such ideals. We start with a few definitions.

Recall that J is normal if it is closed under diagonal unions indexed by λ, i.e. if $\nabla_{\alpha<\lambda} B_\alpha \in J$ whenever $\{B_\alpha : \alpha < \lambda\} \subseteq J$, where $\nabla_{\alpha<\lambda} B_\alpha = \{a \in [\lambda]^{<\kappa} : \exists \alpha \in a\ (a \in B_\alpha)\}$. We could choose to work with $\{\alpha\}$ instead of α, which would lead us to replace in the definition of the diagonal union "there is an element of a" by "there is a subset of a of size 1". The diagonal unions indexed by $[\delta]^{<\theta}$ that we will now introduce are defined in this spirit. This time we consider subsets of a (or rather, of $a \cap \delta$) that are small in the sense that they have size less than $|a \cap \theta|$.

Definition *Given $X_e \subseteq [\lambda]^{<\kappa}$ for $e \in [\delta]^{<\theta}$, we let*

$$\nabla_{e\in[\delta]^{<\theta}} X_e = \{a \in [\lambda]^{<\kappa} : \exists e \in [a \cap \delta]^{<|a\cap\theta|}\ (a \in X_e)\}.$$

and

$$\Delta_{e\in[\delta]^{<\theta}} X_e = \{a \in [\lambda]^{<\kappa} : \forall e \in [a \cap \delta]^{<|a\cap\theta|}\ (a \in X_e)\},$$

Notice that the set $\{a \in [\lambda]^{<\kappa} : a \cap \theta = \emptyset\}$ is included in $\Delta_{e \in [\delta]^{<\theta}} X_e$ and disjoint from $\nabla_{e \in [\delta]^{<\theta}} X_e$.

The following is readily checked.

Lemma 3.1 (i) *Let $X_e \subseteq [\lambda]^{<\kappa}$ for $e \in [\delta]^{<\theta}$. Then*

$$\Delta_{e \in [\delta]^{<\theta}} X_e = [\lambda]^{<\kappa} \setminus (\nabla_{e \in [\delta]^{<\theta}} ([\lambda]^{<\kappa} \setminus X_e)).$$

(ii) *Let $A \subseteq [\lambda]^{<\kappa}$, and $X_e \subseteq [\lambda]^{<\kappa}$ for $e \in [\delta]^{<\theta}$. Then*

$$\Delta_{e \in [\delta]^{<\theta}} (X_e \cap A) = \{a \in [\lambda]^{<\kappa} : a \cap \theta = \emptyset\} \cup ((\Delta_{e \in [\delta]^{<\theta}} X_e) \cap A).$$

(iii) *Let $\rho > 0$ be a cardinal, and $X_e^\alpha \subseteq [\lambda]^{<\kappa}$ for $e \in [\delta]^{<\theta}$ and $\alpha < \rho$. Then*

$$\bigcup_{\alpha < \rho} (\nabla_{e \in [\delta]^{<\theta}} X_e^\alpha) = \nabla_{e \in [\delta]^{<\theta}} \left(\bigcup_{\alpha < \rho} X_e^\alpha \right).$$

Definition *We let $\nabla^{[\delta]^{<\theta}} J$ denote the collection of all $B \subseteq [\lambda]^{<\kappa}$ for which one may find $B_e \in J$ for $e \in [\delta]^{<\theta}$ such that*

$$B \subseteq \{a \in [\lambda]^{<\kappa} : a \cap \theta = \emptyset\} \cup (\nabla_{e \in [\delta]^{<\theta}} B_e).$$

Lemma 3.2 (i) *$J \subseteq \nabla^{[\delta]^{<\theta}} J$.*

(ii) *$\bigcup Y \in \nabla^{[\delta]^{<\theta}} J$ for all $Y \in [\nabla^{[\delta]^{<\theta}}]^{<\kappa}$.*

(iii) *Suppose that δ' is an ordinal with $\delta \leq \delta' \leq \lambda$, θ' is a cardinal with $\theta \leq \theta' \leq \kappa$, and J' is an ideal on $[\lambda]^{<\kappa}$ with $J \subseteq J'$. Then $\nabla^{[\delta]^{<\theta}} J \subseteq \nabla^{[\delta']^{<\theta'}} J'$.*

Proof (i) It suffices to observe that for any $B \in J$,

$$B \subseteq \{a \in [\lambda]^{<\kappa} : a \cap \theta = \emptyset\} \cup (\nabla_{e \in [\delta]^{<\theta}} B).$$

(ii) Use Lemma 3.1 (iii).

(iii) Use (i) and (ii) and the following observation. Let $B_e \in J$ for $e \in [\delta]^{<\theta}$. For $d \in [\delta']^{<\theta'}$, define X_d by: $X_d = B_d$ if $d \in [\delta]^{<\theta}$, and $X_d = \emptyset$ otherwise. Then $\nabla_{e \in [\delta]^{<\theta}} B_e \subseteq \nabla_{d \in [\delta']^{<\theta'}} X_d$, and consequently $\nabla_{e \in [\delta]^{<\theta}} B_e \in \nabla^{[\delta']^{<\theta'}} J'$. □

Proposition 3.3 (i) *$\nabla^{[\delta]^{<\theta}} J = \nabla^{[\delta]^{<\bar{\theta}}} J$.*

(ii) *If $|[\delta]^{<\theta}| < \kappa$, then $J = \nabla^{[\delta]^{<\theta}} J$.*

Proof (i) Suppose that $\theta = \kappa = \nu^+$. Then clearly, $P(\widehat{\nu}) \cap \nabla^{[\delta]^{<\kappa}} J = P(\widehat{\nu}) \cap \nabla^{[\delta]^{<\nu}} J$. Hence by Lemma 3.2 ((i) and (ii)), $\nabla^{[\delta]^{<\kappa}} J = \nabla^{[\delta]^{<\nu}} J$.

(ii) Use Lemma 3.2 (i). □

Definition *Given $A \subseteq [\lambda]^{<\kappa}$, $f : A \to [\delta]^{<\theta}$ is $[\delta]^{<\theta}$-regressive if $f(a) \in [a \cap \delta]^{<|a \cap \theta|}$ for all $a \in A$ with $a \cap \theta \neq \emptyset$.*

Proposition 3.4 *The following are equivalent:*

(i) $[\lambda]^{<\kappa} \notin \nabla^{[\delta]^{<\theta}} J$.
(ii) $\nabla^{[\delta]^{<\theta}} J$ *is an ideal on* $[\lambda]^{<\kappa}$.
(iii) $\Delta_{e\in[\delta]^{<\theta}} C_e \in J^+$ *whenever* $C_e \in J^*$ *for* $e \in [\delta]^{<\theta}$.
(iv) $\Delta_{e\in[\delta]^{<\theta}} C_e \in I^+_{\kappa,\lambda}$ *whenever* $C_e \in J^*$ *for* $e \in [\delta]^{<\theta}$.
(v) *For any* $[\delta]^{<\theta}$*-regressive* $f : [\lambda]^{<\kappa} \to [\delta]^{<\theta}$*, there is* $D \in J^+$ *such that* f *is constant on* D.

Proof (i) → (ii) : By Lemma 3.2 ((i) and (ii)).
(ii) → (iii) : Use Lemmas 3.1 (i) and 3.2 (i).
(iii) → (iv) : Trivial.
(iv) → (v) : Use the fact that for any $[\delta]^{<\theta}$-regressive $f : [\lambda]^{<\kappa} \to [\delta]^{<\theta}$,

$$\Delta_{e\in[\delta]^{<\theta}}([\lambda]^{<\kappa} \setminus f^{-1}(\{e\})) = \{a \in [\lambda]^{<\kappa} : a \cap \theta = \emptyset\}.$$

(v) → (i) : Suppose that we may find $B_e \in J$ for $e \in [\delta]^{<\theta}$ such that $\{a \in [\lambda]^{<\kappa} : a \cap \theta \neq \emptyset\} \subseteq \nabla_{e\in[\delta]^{<\theta}} B_e$. Then there is a $[\delta]^{<\theta}$-regressive $f : [\lambda]^{<\kappa} \to [\delta]^{<\theta}$ with the property that $a \in B_{f(a)}$ for all $a \in [\lambda]^{<\kappa}$ with $a \cap \theta \neq \emptyset$. Clearly, $f^{-1}(\{e\}) \in J$ for every $e \in [\delta]^{<\theta}$. □

Definition J *is* $[\delta]^{<\theta}$*-normal if* $J = \nabla^{[\delta]^{<\theta}} J$.

Proposition 3.5 *Let* δ' *be an ordinal with* $1 \leq \delta' \leq \delta$*, and* θ' *be a cardinal with* $2 \leq \theta' \leq \theta$*. Then every* $[\delta]^{<\theta}$*-normal ideal on* $[\lambda]^{<\kappa}$ *is* $[\delta']^{<\theta'}$*-normal.*

Proof By Lemma 3.2 ((i) and (iii)). □

Proposition 3.6 *The following are equivalent:*

(i) J *is* $[\delta]^{<\theta}$*-normal.*
(ii) $\Delta_{e\in[\delta]^{<\theta}} C_e \in J^*$ *whenever* $C_e \in J^*$ *for* $e \in [\delta]^{<\theta}$.
(iii) $[\lambda]^{<\kappa} \notin \nabla^{[\delta]^{<\theta}}(J|A)$ *for all* $A \in J^+$.
(iv) *Given* $A \in J^+$ *and a* $[\delta]^{<\theta}$*-regressive* $f : A \to [\delta]^{<\theta}$*, there is* $D \in J^+ \cap P(A)$ *such that* f *is constant on* D.

Proof (i) ↔ (ii) : Use Lemmas 3.1 (i) and 3.2 (i).
(ii) ↔ (iii) : By Lemma 3.1 (ii) and Proposition 3.4 ((i) ↔ (iii)).
(iii) ↔ (iv) : By Proposition 3.4 ((i) ↔ (v)). □

Proposition 3.6 ((i) ↔ (iii)) shows that the $[\delta]^{<\theta}$-normality of J can be seen as the global version of the local property "$[\lambda]^{<\kappa} \notin \nabla^{[\delta]^{<\theta}} J$". Let us next briefly consider another, weaker local property. The corresponding global property will be dealt with in Corollary 3.8.

Definition *Two ideals* I, K *on* $[\lambda]^{<\kappa}$ *cohere if* $I \cup K \subseteq H$ *for some ideal* H *on* $[\lambda]^{<\kappa}$.

Proposition 3.7 *Let* K *be an ideal on* $[\lambda]^{<\kappa}$ *such that* $K \subseteq J$ *and* $[\lambda]^{<\kappa} \notin \nabla^{[\delta]^{<\theta}} K$. *Then the following are equivalent:*

(i) *J and $\nabla^{[\delta]^{<\theta}} K$ cohere.*
(ii) $\Delta_{e\in[\delta]^{<\theta}} C_e \in J^+$ *whenever* $C_e \in K^*$ *for* $e \in [\delta]^{<\theta}$.
(iii) *Given $A \in J^*$ and a $[\delta]^{<\theta}$-regressive $f : A \to [\delta]^{<\theta}$, there is $D \in K^+ \cap P(A)$ such that f is constant on D.*

Proof (i) $\to$ (ii) : Straightforward.

(ii) $\to$ (iii) : Suppose that $A \in J^*$ and $f : A \to [\delta]^{<\theta}$ are such that $f^{-1}(\{e\}) \in K$ for every $e \in [\delta]^{<\theta}$. Then $f(a) \notin [a \cap \delta]^{<|a\cap\theta|}$ for all $a \in A \cap \Delta_{e\in[\delta]^{<\theta}}([\lambda]^{<\kappa} \setminus f^{-1}(\{e\}))$.

(iii) $\to$ (i) : Assume that (iii) holds. Given $B_e \in K$ for $e \in [\delta]^{<\theta}$, define $f : \nabla^{[\delta]^{<\theta}} B_e \to [\delta]^{<\theta}$ so that for any $a \in \nabla^{[\delta]^{<\theta}} B_e$, $f(a) \in [a \cap \delta]^{<|a\cap\theta|}$ and $a \in B_{f(a)}$. Then f is $[\delta]^{<\theta}$-regressive. Moreover, $f^{-1}(\{e\}) \in K$ for every $e \in [\delta]^{<\theta}$. It follows that $\nabla^{[\delta]^{<\theta}} B_e \notin J^*$. Hence

$$H = \{B \cup E : B \in J \text{ and } E \in \nabla^{[\delta]^{<\theta}} K\}$$

is an ideal on $[\lambda]^{<\kappa}$ that extends both J and $\nabla^{[\delta]^{<\theta}} K$. □

Corollary 3.8 *Let K be an ideal on $[\lambda]^{<\kappa}$ such that $K \subseteq J$ and $[\lambda]^{<\kappa} \notin \nabla^{[\delta]^{<\theta}} K$. Then the following are equivalent:*

(i) *$J|A$ and $\nabla^{[\delta]^{<\theta}} K$ cohere for every $A \in J^+$.*
(ii) $\Delta_{e\in[\delta]^{<\theta}} C_e \in J^*$ *whenever* $C_e \in K^*$ *for* $e \in [\delta]^{<\theta}$.
(iii) *Given $A \in J^+$ and a $[\delta]^{<\theta}$-regressive $f : A \to [\delta]^{<\theta}$, there is $D \in K^+ \cap P(A)$ such that f is constant on D.*
(iv) $\nabla^{[\delta]^{<\theta}} K \subseteq J$.

We will now show that δ-normality, which was studied by Abe [1], is the same as $[\delta]^{<2}$-normality.

Definition *Given $X_\alpha \subseteq [\lambda]^{<\kappa}$ for $\alpha < \delta$, we let*

$$\Delta_{\alpha<\delta} X_\alpha = \bigcap_{\alpha<\delta} (X_\alpha \cup ([\lambda]^{<\kappa} \setminus \widehat{\{\alpha\}}))$$

and

$$\nabla_{\alpha<\delta} X_\alpha = \bigcup_{\alpha<\delta} (X_\alpha \cap \widehat{\{\alpha\}}).$$

Definition *Given $K \subseteq P([\lambda]^{<\kappa})$, we let $\nabla_{\alpha<\delta} K$ denote the collection of all $B \subseteq [\lambda]^{<\kappa}$ for which one may find $B_\alpha \in K$ for $\alpha < \delta$ such that*

$$B \subseteq ([\lambda]^{<\kappa} \setminus \widehat{\{0\}}) \cup \nabla_{\alpha<\delta} B_\alpha.$$

Definition *J is δ-normal if $J = \nabla_{\alpha<\delta} J$.*

Proposition 3.9 $\nabla_{\alpha<\delta} J = \nabla^{[\delta]^{<2}} J$.

Proof The result easily follows from the following two remarks:

(1) Let $X_\alpha \subseteq [\lambda]^{<\kappa}$ for $\alpha < \delta$. Define Y_e for $e \in [\delta]^{<2}$ by: $Y_{\{\alpha\}} = X_\alpha$ for $\alpha \in \delta$, and $Y_\emptyset = \emptyset$. Then

$$([\lambda]^{<\kappa} \backslash \widehat{2}) \cup \nabla_{\alpha<\delta} X_\alpha = ([\lambda]^{<\kappa} \backslash \widehat{2}) \cup \nabla_{e\in[\delta]^{<2}} Y_e.$$

(2) Let $X_e \subseteq [\lambda]^{<\kappa}$ for $e \in [\delta]^{<2}$. Define Y_α for $\alpha < \delta$ by $Y_\alpha = X_{\{\alpha\}}$. Then

$$([\lambda]^{<\kappa} \backslash \widehat{2}) \cup X_\emptyset \cup \nabla_{\alpha<\delta} Y_\alpha = ([\lambda]^{<\kappa} \backslash \widehat{2}) \cup \nabla_{e\in[\delta]^{<2}} X_e.$$

□

Corollary 3.10 *J is δ-normal if and only if it is $[\delta]^{<2}$-normal.*

We finally turn to the question of the existence of $[\delta]^{<\theta}$-normal ideals. Let us first deal with the degenerate case $\kappa = \omega$.

Proposition 3.11 *Assume $\kappa = \omega$. Then there exists a $[\delta]^{<\theta}$-normal ideal on $[\lambda]^{<\kappa}$ if and only if $\delta < \omega$.*

Proof The right-to-left implication is immediate from Proposition 3.3 (ii). For the reverse implication, observe that $[\lambda]^{<\omega} = ([\lambda]^{<\omega} \backslash \widehat{2}) \cup \nabla_{e\in[\omega]^{<2}} B_e$, where $B_\emptyset = \emptyset$ and

$$B_{\{n\}} = \{a \in [\lambda]^{<\omega} : \max(a \cap \omega) = n\}$$

for $n \in \omega$. Hence by Lemma 3.2 ((i) and (ii)), $[\lambda]^{<\omega} \in \nabla^{[\omega]^{<2}} I_{\omega,\lambda}$. If $\delta \geq \omega$, then $[\lambda]^{<\omega} \in \nabla^{[\delta]^{<\theta}} J$ by Lemma 3.2 (iii), and therefore J is not $[\delta]^{<\theta}$-normal. □

We will now look for sufficient conditions for the existence of $[\delta]^{<\theta}$-normal ideals on $[\lambda]^{<\kappa}$ in the case $\kappa > \omega$. The following is a key lemma.

Lemma 3.12 (i) *Suppose that* $\max(\omega, \theta) < \kappa$, *and* $|[\mu]^{<\theta}| < \kappa$ *for every cardinal* $\mu < \kappa$. *Then* $[\lambda]^{<\kappa} \notin \nabla^{[\lambda]^{<\theta}} I_{\kappa,\lambda}$.

(ii) ([13]) *Suppose that κ is Mahlo. Then* $[\lambda]^{<\kappa} \notin \nabla^{[\lambda]^{<\kappa}} I_{\kappa,\lambda}$.

Proof (i) Let $b_e \in [\lambda]^{<\kappa}$ for $e \in [\lambda]^{<\theta}$, and $a \in [\lambda]^{<\kappa}$. Set $\rho = \max(\omega, \theta)$ if $\max(\omega, \theta)$ is regular, and $\rho = (\max(\omega, \theta))^+$ otherwise. Note that $\rho < \kappa$. Now define $x_\alpha \in [\lambda]^{<\kappa}$ for $\alpha < \rho$ so that

- $x_0 = a \cup \theta$.
- If $\alpha > 0$, then $\bigcup_{\beta<\alpha} x_\beta \subseteq x_\alpha$, and moreover $x_\alpha \in \bigcap\{\widehat{b_e} : e \in [\bigcup_{\beta<\alpha} x_\beta]^{<\theta}\}$.

Set $x = \bigcup_{\alpha<\rho} x_\alpha$. Given $e \in [x]^{<|x\cap\theta|}$, there must be $\beta < \rho$ such that $e \in [x_\beta]^{<\theta}$. Then $b_e \subseteq x_{\beta+1} \subseteq x$. Thus $\widehat{a} \cap \Delta_{e\in[\lambda]^{<\theta}} \widehat{b_e} \neq \emptyset$. By Proposition 3.4 ((iv) $\rightarrow$ (i)), it follows that $[\lambda]^{<\kappa} \notin \nabla^{[\lambda]^{<\theta}} I_{\kappa,\lambda}$.

(ii) Let $b_e \in [\lambda]^{<\kappa}$ for $e \in [\lambda]^{<\kappa}$, and $a \in [\lambda]^{<\kappa}$. Define $x_\alpha \in [\lambda]^{<\kappa}$ for $\alpha < \kappa$ so that
- $x_0 = a$.
- $x_\alpha \cup ((\sup(x_\alpha \cap \kappa)) + 1) \cup (\bigcup_{e \subseteq x_\alpha} b_e) \subseteq x_{\alpha+1}$.
- $x_\alpha = \bigcup_{\beta<\alpha} x_\beta$ in case α is an infinite limit ordinal.

There must be a regular infinite cardinal $\tau < \kappa$ such that $x_\tau \cap \kappa = \tau$. Then $x_\tau \in \widehat{a} \cap \Delta_{e\in[\lambda]^{<\kappa}} \widehat{b_e}$. Hence by Proposition 3.4 ((iv) $\to$ (i)), $[\lambda]^{<\kappa} \notin \nabla^{[\lambda]^{<\kappa}} I_{\kappa,\lambda}$. □

Menas [19] proved that $NS_{\kappa,\lambda}$ (the smallest normal ideal on $[\lambda]^{<\kappa}$) is generated by sets of the form

$$[\lambda]^{<\kappa} \backslash \{a \in [\lambda]^{<\kappa} \backslash \{\emptyset\} : \forall \alpha, \beta \in a\ (f(\alpha, \beta) \subseteq a)\},$$

where f is a function from $\lambda \times \lambda$ to $[\lambda]^{<\kappa}$. We are looking for an analogous result concerning the smallest $[\delta]^{<\theta}$-normal ideal on $[\lambda]^{<\kappa}$. This motivates the following definition.

Definition *For $f : [\delta]^{<\theta} \to [\lambda]^{<\kappa}$, $C_f^{\kappa,\lambda}$ denotes the set of all $a \in [\lambda]^{<\kappa}$ such that $a \cap \theta \neq \emptyset$, and $f(e) \subseteq a$ for every $e \in [a \cap \delta]^{<|a\cap\theta|}$.*

The following is straightforward.

Lemma 3.13 *Given $B \subseteq [\lambda]^{<\kappa}$, $B \in \nabla^{[\delta]^{<\theta}} I_{\kappa,\lambda}$ if and only if $B \cap C_f^{\kappa,\lambda} = \emptyset$ for some $f : [\delta]^{<\theta} \to [\lambda]^{<\kappa}$.*

Lemma 3.14 *Assume that $\delta \geq \kappa$ and either $\theta = \kappa$ and κ is Mahlo, or $3 \leq \theta$, $\max(\omega, \theta) < \kappa$ and $|[\mu]^{<\theta}| < \kappa$ for every cardinal $\mu < \kappa$. Then $\nabla^{[\delta]^{<\theta}} I_{\kappa,\lambda}$ is a $[\delta]^{<\theta}$-normal ideal on $[\lambda]^{<\kappa}$.*

Proof By Lemmas 3.12 and 3.2 (iii) and Proposition 3.4 ((i) $\to$ (ii)), $\nabla^{[\delta]^{<\theta}} I_{\kappa,\lambda}$ is an ideal on $[\lambda]^{<\kappa}$. Let us first suppose that $\theta \geq \omega$. Given $g_b : [\delta]^{<\theta} \to [\lambda]^{<\kappa}$ for $b \in [\delta]^{<\theta}$, define $f : [\delta]^{<\theta} \to [\lambda]^{<\kappa}$ by $f(e) = \bigcup_{b,c\subseteq e} g_b(c)$. Then $\widehat{\omega} \cap C_f^{\kappa,\lambda} \subseteq \Delta_{b\in[\delta]^{<\theta}} C_{g_b}^{\kappa,\lambda}$. Hence by Proposition 3.6 ((ii) $\to$ (i)) and Lemma 3.13, $\nabla^{[\delta]^{<\theta}} I_{\kappa,\lambda}$ is $[\delta]^{<\theta}$-normal.

Suppose next that $3 \leq \theta < \omega$. Select a bijection $j : [\delta]^{<\theta} \to [\delta]^{<2}$. Given $g_b : [\delta]^{<\theta} \to [\lambda]^{<\kappa}$ for $b \in [\delta]^{<\theta}$, define $f : [\delta]^{<\theta} \to [\lambda]^{<\kappa}$ by

$$f(e) = \bigcup \{g_b(c) : b, c \in [\delta]^{<\theta} \text{ and } j(b) \cup j(c) \subseteq e\}.$$

Then $\widehat{\theta} \cap C_j^{\kappa,\lambda} \cap C_f^{\kappa,\lambda} \subseteq \Delta_{b\in[\delta]^{<\theta}} C_{g_b}^{\kappa,\lambda}$. Hence by Proposition 3.6 ((ii) $\to$ (i)) and Lemma 3.13, $\nabla^{[\delta]^{<\theta}} I_{\kappa,\lambda}$ is $[\delta]^{<\theta}$-normal. □

Lemma 3.15 *Assume that J is $[\delta]^{<\max(3,\bar{\theta})}$-normal. Then J is $[\delta]^{<\theta}$-normal.*

Proof If $\bar{\theta} \geq 3$, then by Proposition 3.3 (i) $J = \nabla^{[\delta]^{<\max(3,\bar{\theta})}} J = \nabla^{[\delta]^{<\theta}} J$. If $\bar{\theta} < 3$, then by Lemma 3.2 ((i) and (iii)), $J \subseteq \nabla^{[\delta]^{<\theta}} J \subseteq \nabla^{[\delta]^{<3}} J \subseteq J$. □

It remains to show that our sufficient conditions are also necessary ones.

Lemma 3.16 *Assume that* $[\lambda]^{<\kappa} \notin \nabla^{[\delta]^{<\theta}} I_{\kappa,\lambda}$*, and let* μ *and* τ *be two cardinals such that* $\mu < \min(\kappa, \delta + 1)$ *and* $0 < \tau < \min(\theta^+, \kappa)$*. Then* $|[\mu]^{<\tau}| < \kappa$*.*

Proof Suppose otherwise, and pick a one-to-one $j : \kappa \to [\mu]^{<\tau}$. Define $f : \widehat{\mu \cup \tau} \to [\mu]^{<\tau}$ by $f(a) = j(\sup(a \cap \kappa))$. Then f is $[\delta]^{<\theta}$-regressive, which contradicts Proposition 3.4 ((i) $\to$ (v)). □

Lemma 3.17 (i) *Suppose that* $\delta \geq \kappa > \omega$ *and* δ *is a limit ordinal. Then the set of all* $a \in [\lambda]^{<\kappa}$ *such that* $\sup(a \cap \delta)$ *is a limit ordinal that does not belong to* a *lies in* $(\nabla^{[\delta]^{<2}} I_{\kappa,\lambda})^*$*.*

(ii) *Suppose* $\delta \geq \kappa > \omega$*. Then the set of all* $a \in [\lambda]^{<\kappa}$ *such that* $\mathrm{cf}(\sup(a \cap \eta)) < |a \cap \theta|$ *for some limit ordinal* η *with* $\kappa \leq \eta \leq \delta$ *and* $\mathrm{cf}(\eta) \geq \bar{\theta}$ *lies in* $\nabla^{[\delta]^{<\theta}} I_{\kappa,\lambda}$*.*

(iii) *Suppose* $\kappa > \omega$*, and let* C *be a closed unbounded subset of* κ*. Then*

$$\{a \in [\lambda]^{<\kappa} : a \cap \kappa \in C\} \in (\nabla^{[\kappa]^{<2}} I_{\kappa,\lambda})^*.$$

Proof Use Lemmas 3.2 (iii) and 3.12 (i) and Propositions 3.3 (i) and 3.4 ((i) $\to$ (v)). □

Lemma 3.18 *Assume that* κ *is an uncountable limit cardinal and* $[\lambda]^{<\kappa} \notin \nabla^{[\kappa]^{<\kappa}} I_{\kappa,\lambda}$*. Then* κ *is Mahlo.*

Proof κ is inaccessible by Lemma 3.16, and weakly Mahlo by Lemmas 3.2 (iii) and 3.17. □

Our study of the case $\kappa > \omega$ culminates in the following.

Proposition 3.19 (i) *Suppose that* $\kappa > \omega$*. Suppose further that either* $\delta < \kappa$*, or* $\theta < \kappa$*, or* κ *is not a limit cardinal. Then there exists a* $[\delta]^{<\theta}$*-normal ideal on* $[\lambda]^{<\kappa}$ *if and only if* $|[\mu]^{<\bar{\theta}}| < \kappa$ *for every cardinal* $\mu < \min(\kappa, \delta + 1)$*.*

(ii) *Suppose that* $\delta \geq \kappa = \theta > \omega$ *and* κ *is a limit cardinal. Then there exists a* $[\delta]^{<\theta}$*-normal ideal on* $[\lambda]^{<\kappa}$ *if and only if* κ *is Mahlo.*

Proof (i) Let us first assume that there exists a $[\delta]^{<\theta}$-normal ideal on $[\lambda]^{<\kappa}$. Then by Lemma 3.2 (iii), $[\lambda]^{<\kappa} \notin \nabla^{[\delta]^{<\theta}} I_{\kappa,\lambda}$. Observe that if $\delta < \kappa = \theta$ and κ is a limit cardinal, then setting $\tau = |\delta|^+$, we have that $0 < \tau < \min(\theta^+, \kappa)$ and $[|\delta|]^{<\bar{\theta}} = [|\delta|]^{<\tau}$. Hence by Lemma 3.16, $|[\mu]^{<\bar{\theta}}| < \kappa$ for every cardinal $\mu < \min(\kappa, \delta + 1)$.

Conversely, assume that $|[\mu]^{<\bar{\theta}}| < \kappa$ for any cardinal $\mu < \min(\kappa, \delta + 1)$. If $\delta < \kappa$, then $|[\delta]^{<\max(3,\bar{\theta})}| < \kappa$, and therefore by Proposition 3.3 (ii), $I_{\kappa,\lambda}$ is $[\delta]^{<\max(3,\bar{\theta})}$-normal. If $\delta \geq \kappa$, then $\bar{\theta} < \kappa$, and consequently by Lemma 3.14, $\nabla^{[\delta]^{<\max(3,\bar{\theta})}} I_{\kappa,\lambda}$ is a $[\delta]^{<\max(3,\bar{\theta})}$-normal ideal on $[\lambda]^{<\kappa}$. Thus by Lemma 3.15, there exists a $[\delta]^{<\theta}$-normal ideal on $[\lambda]^{<\kappa}$.

(ii) If κ is Mahlo, then by Lemma 3.14, $\nabla^{[\delta]^{<\theta}} I_{\kappa,\lambda}$ is a $[\delta]^{<\theta}$-normal ideal on $[\lambda]^{<\kappa}$. Conversely, if there exists a $[\delta]^{<\theta}$-normal ideal on $[\lambda]^{<\kappa}$, then by Lemmas 3.2 (iii) and 3.18, κ is Mahlo.

$\square$

Corollary 3.20 *There exists a $[\delta]^{<\theta}$-normal ideal on $[\lambda]^{<\kappa}$ if and only if there exists a $[\min(\delta, \kappa)]^{<\min(\theta, |\delta|^+)}$-normal ideal on $[\kappa]^{<\kappa}$.*

Proof By Propositions 3.11 and 3.19. $\square$

Corollary 3.21 *Assume that $\delta < \kappa$ and there exists a $[\delta]^{<\theta}$-normal ideal on $[\lambda]^{<\kappa}$. Then every ideal on $[\lambda]^{<\kappa}$ is $[\delta]^{<\theta}$-normal.*

Proof By Propositions 3.3 and 3.19 (i). $\square$

The following (see e.g. Theorem 7.12 in [7]) is due independently to Hajnal and Shelah.

Fact 3.22 *Let μ be an infinite cardinal. Then μ^ρ assumes only finitely many values for ρ with $2^\rho < \mu$.*

Lemma 3.23 *Let μ, χ be two infinite cardinals such that $2^{<\chi} \leq \mu$. Then $(\mu^{<\chi})^{<\chi} = \mu^{<\chi}$.*

Proof If there exists a cardinal $\tau < \chi$ such that $2^\tau = \mu$, then $\mu^{<\chi} = (2^\tau)^{<\chi} = 2^{<\chi} = \mu$. Otherwise, there exists by Fact 3.22 a cardinal $\rho < \chi$ such that $\mu^{<\chi} = \mu^\rho$. Then $(\mu^{<\chi})^{<\chi} = (\mu^\rho)^{<\chi} = \mu^{<\chi}$. $\square$

Proposition 3.24 *Assume that there exists a $[\kappa]^{<\theta}$-normal ideal on $[\lambda]^{<\kappa}$. Then the following hold :*

(i) $\kappa^{<\bar{\theta}} = \kappa$.
(ii) $(\mu^{<\bar{\theta}})^{<\bar{\theta}} = \mu^{<\bar{\theta}}$ *for every cardinal $\mu > \kappa$.*

Proof A proof of (i) can be found in [15]. As for (ii), it follows from Lemma 3.23, since by Proposition 3.19, $2^{<\bar{\theta}} \leq \kappa$. $\square$

4 $NS_{\kappa,\lambda}^{[\delta]^{<\theta}}$

In this section we describe the smallest $[\delta]^{<\theta}$-normal ideal on $[\lambda]^{<\kappa}$. We start with the following that shows that we could without loss of generality assume θ to be an infinite cardinal.

Proposition 4.1 *Assume that J is $[\delta]^{<\theta}$-normal. Then J is $[\delta]^{<\max(\omega,\theta)}$-normal.*

Proof We can assume that $\theta < \omega$ since otherwise the result is trivial. The desired conclusion is immediate from Proposition 3.3 (ii) in case $\delta < \omega$. Now assume $\delta \geq \omega$. We have $\kappa > \omega$ by Proposition 3.11. Fix $A \in J^+$ and a $[\delta]^{<\omega}$-regressive $f : A \to [\delta]^{<\omega}$. We define a $[\delta]^{<\theta}$-regressive $g : A \cap \widehat{\omega} \to [\delta]^{<\theta}$ by $g(a) = \{|f(a)|\}$. By

Proposition 3.6 ((i) → (iv)), we may find $C \in J^+ \cap P(A \cap \widehat{\omega})$ and $n \in \omega$ such that g is identically n on C. If $n = 0$, then f is clearly constant on C. Otherwise select a bijection $j_n : n \to f(a)$ for each $a \in C$. Using Proposition 3.6 ((i) → (iv)), define $C_k \in J^+$ for $k \leq n$, and $h_i : C_i \to [\delta]^{<\theta}$ for $i < n$ so that

- $C_0 = C$.
- $C_{i+1} \subseteq C_i$.
- $h_i(a) = \{j_a(i)\}$.
- h_i is constant on C_{i+1}.

Then f is constant on C_n. Hence by Proposition 3.6 ((iv) → (i)), J is $[\delta]^{<\omega}$-normal. □

Proposition 4.2 *If there exists a $[\delta]^{<\theta}$-normal ideal on $[\lambda]^{<\kappa}$, then the smallest such ideal is $\nabla^{[\delta]^{<\max(3,\bar{\theta})}} I_{\kappa,\lambda}$.*

Proof Assume that there exists a $[\delta]^{<\theta}$-normal ideal on $[\lambda]^{<\kappa}$. Then by Lemma 3.2 (iii) and Proposition 4.1, $\nabla^{[\delta]^{<\max(3,\bar{\theta})}} I_{\kappa,\lambda} \subseteq K$ for every $[\delta]^{<\theta}$-normal ideal K on $[\lambda]^{<\kappa}$. Morever by the proofs of Propositions 3.11 and 3.19, $\nabla^{[\delta]^{<\max(3,\bar{\theta})}} I_{\kappa,\lambda}$ is itself a $[\delta]^{<\theta}$-normal ideal on $[\lambda]^{<\kappa}$. □

Definition *Assuming the existence of a $[\delta]^{<\theta}$-normal ideal on $[\lambda]^{<\kappa}$, we set $NS_{\kappa,\lambda}^{[\delta]^{<\theta}} = \nabla^{[\delta]^{<\max(3,\bar{\theta})}} I_{\kappa,\lambda}$.*

Proposition 4.3 *Let δ' be an ordinal with $1 \leq \delta' \leq \delta$, and θ' be a cardinal with $2 \leq \theta' \leq \theta$. Then $NS_{\kappa,\lambda}^{[\delta']^{<\theta'}} \subseteq NS_{\kappa,\lambda}^{[\delta]^{<\theta}}$.*

Proof By Proposition 3.5. □

Proposition 4.4 $NS_{\kappa,\lambda}^{[\delta]^{<\theta}} = NS_{\kappa,\lambda}^{[\delta]^{<\max(\omega,\theta)}} = NS_{\kappa,\lambda}^{[\delta]^{<\bar{\theta}}}$.

Proof By Propositions 3.3 (i), 4.1, 4.2 and 4.3,

$$NS_{\kappa,\lambda}^{[\delta]^{<\max(\omega,\theta)}} \subseteq NS_{\kappa,\lambda}^{[\delta]^{<\theta}} \subseteq NS_{\kappa,\lambda}^{[\delta]^{<\bar{\theta}}} \subseteq NS_{\kappa,\lambda}^{[\delta]^{<\max(\omega,\theta)}}.$$

□

Proposition 4.5 *If $\delta < \kappa$, then $NS_{\kappa,\lambda}^{[\delta]^{<\theta}} = I_{\kappa,\lambda}$.*

Proof By Corollary 3.21. □

Definition *We put $NS_{\kappa,\lambda}^{\delta} = NS_{\kappa,\lambda}^{[\delta]^{<2}}$.*

It follows from Corollary 3.10 and Proposition 4.2 that $NS_{\kappa,\lambda}^{\delta}$ is the smallest δ-normal ideal on $[\lambda]^{<\kappa}$. We will conform to usage and denote $NS_{\kappa,\lambda}^{\lambda}$ by $NS_{\kappa,\lambda}$.

The following is due to Abe [1].

Proposition 4.6 *Assume $\kappa \leq \delta < \kappa^+$. Then $NS_{\kappa,\lambda}^{\delta} = \nabla^{[\delta]^{<2}} I_{\kappa,\lambda}$.*

Proof Let us first prove the assertion for $\delta = \kappa$. Given $f_b : [\kappa]^{<2} \to [\lambda]^{<\kappa}$ for $b \in [\kappa]^{<2}$, define $f : [\kappa]^{<2} \to [\lambda]^{<\kappa}$ by $f(e) = \bigcup_{b,c \in [(\cup e)+1]^{<2}} f_b(c)$. Then $C_f^{\kappa,\lambda} \subseteq \Delta_{b \in [\kappa]^{<2}} C_{f_b}^{\kappa,\lambda}$. Hence by Lemma 3.13 and Proposition 3.6 ((iii) $\to$ (i)), $\nabla^{[\kappa]^{<2}} I_{\kappa,\lambda}$ is $[\kappa]^{<2}$-normal. By Proposition 4.2 and Lemma 3.2 (iii), it follows that $NS_{\kappa,\lambda}^{\kappa} = \nabla^{[\kappa]^{<2}} I_{\kappa,\lambda}$.

Now assume $\kappa < \delta < \kappa^+$. By Propositions 3.19 (i) and 5.4 (below), there is $A \in (\nabla^{[\delta]^{<2}} I_{\kappa,\lambda})^*$ such that $NS_{\kappa,\lambda}^{[\delta]^{<2}} = NS_{\kappa,\lambda}^{[\kappa]^{<2}} | A$. Then by Lemma 3.2 (iii),

$$\nabla^{[\delta]^{<2}} I_{\kappa,\lambda} \subseteq NS_{\kappa,\lambda}^{[\delta]^{<2}} = (\nabla^{[\kappa]^{<2}} I_{\kappa,\lambda}) | A \subseteq \nabla^{[\delta]^{<2}} I_{\kappa,\lambda}.$$

$\square$

Abe [1] also showed that for $\delta \geq \kappa^+$, $NS_{\kappa,\lambda}^{\delta} \setminus \nabla^{[\delta]^{<2}} I_{\kappa,\lambda} \neq \emptyset$. In fact,

$$(\nabla^{[\kappa]^{<2}} (\nabla^{[\kappa^+]^{<2}} I_{\kappa,\lambda})) \setminus \nabla^{[\kappa]^{<2}} I_{\kappa,\lambda} \neq \emptyset.$$

By Lemma 3.13, $NS_{\kappa,\lambda}^{[\delta]^{<\theta}}$ is the set of all $B \subseteq [\lambda]^{<\kappa}$ such that $B \cap C_f^{\kappa,\lambda} = \emptyset$ for some $f : [\delta]^{<\max(3,\bar\theta)} \to [\lambda]^{<\kappa}$. The following generalizes a well-known (see e.g. Lemma 1.13 in [19] and Proposition 1.4 in [14]) characterization of $NS_{\kappa,\lambda}$.

Proposition 4.7 *Assume $\delta \geq \kappa$. Then given $B \subseteq [\lambda]^{<\kappa}$, $B \in NS_{\kappa,\lambda}^{[\delta]^{<\theta}}$ if and only if $B \cap \{a \in C_g^{\kappa,\lambda} : a \cap \kappa \in \kappa\} = \emptyset$ for some $g : [\lambda]^{<\max(3,\bar\theta)} \to [\lambda]^{<3}$.*

Proof Set $\tau = 2$ if $\bar\theta < \omega$ and $\delta < \kappa^+$, $\tau = 3$ if $\bar\theta < \omega$ and $\delta \geq \kappa^+$, and $\tau = \bar\theta$ if $\bar\theta \geq \omega$. Then by Lemma 3.13 and Propositions 4.4 and 4.6, it suffices to show that for any $f : [\delta]^{<\tau} \to [\lambda]^{<\kappa}$, there exists $g : [\delta]^{<\max(3,\bar\theta)} \to [\lambda]^{<3}$ with the property that $\{a \in C_g^{\kappa,\lambda} : a \cap \kappa \in \kappa\} \subseteq C_f^{\kappa,\lambda}$. Thus fix $f : [\delta]^{<\tau} \to [\lambda]^{<\kappa}$. Pick a bijection $j_e : |f(e)| \to f(e)$ for each $e \in [\delta]^{<\tau}$.

Let us first assume that $\bar\theta \geq \omega$. Define $h : [\delta]^{<\tau} \to \kappa$ by

$$h(e) = \max(\omega, ((\sup(e \cap \kappa)) + 1) + |f(e)|).$$

We define $k : [\delta]^{<\tau} \to \lambda$ as follows. Given $e \in [\delta]^{<\tau}$, set $\alpha = \sup(e \cap \kappa)$. We put $k(e) = 0$ if $\alpha \notin e$. Assuming now that $\alpha \in e$, put $c = e \setminus \{\alpha\}$ and $\xi = \sup(c \cap \kappa)$, and let β denote the unique ordinal ζ such that $\alpha = (\xi + 1) + \zeta$. We put $k(e) = j_c(\beta)$ if $\beta \in |f(c)|$, and $k(e) = 0$ otherwise. Finally define $g : [\delta]^{<\tau} \to [\lambda]^{<3}$ by $g(e) = \{h(e), k(e)\}$. Now fix $a \in C_g^{\kappa,\lambda}$ with $a \cap \kappa \in \kappa$, and $c \in [a \cap \delta]^{<|a \cap \tau|}$. Put $\xi = \sup(c \cap \kappa)$. Given $\beta \in |f(c)|$, set $e = c \cup \{(\xi + 1) + \beta\}$. Since $h(c) \subseteq a$, we have $\omega \subseteq a$ and $(\xi + 1) + \beta \in a$, and therefore $e \in [a \cap \delta]^{<|a \cap \tau|}$. Hence $j_c(\beta) \in a$, since clearly $k(e) = j_e(\beta)$. Thus $f(c) \subseteq a$.

Let us next assume that $\bar\theta < \omega$ and $\delta \geq \kappa^+$. Select a bijection $h : [\delta]^{<3} \to \delta \setminus \kappa$. Define $k : [\delta]^{<3} \to \lambda$ so that $k(\emptyset) = 2$, and given $e \in [\delta]^{<3}$, $k(\{h(e)\}) = |f(e)|$ and for each $\beta \in |f(e)|$, $k(\{\beta, h(e)\}) = j_e(\beta)$. Then define $g : [\delta]^{<\max(3,\bar\theta)} \to [\lambda]^{<3}$ so that $g(e) = \{h(e), k(e)\}$ for all $e \in [\delta]^{<3}$. It is readily checked that g is as desired.

Finally assume that $\bar{\theta} < \omega$ and $\delta < \kappa^+$. Define $h : [\delta]^{<2} \to \kappa$ by:

- $h(\emptyset) = 2 + |f(\emptyset)|$.
- $h(\{\alpha\}) = (\alpha + 1) + |f(\{\alpha\})|$ for $\alpha \in \kappa$.
- $h(\{\alpha\}) = |f(\{\alpha\})|$ for $\alpha \in \delta \backslash \kappa$.

Then define $k : [\delta]^{<3} \to \lambda$ so that

- $k(\{\beta\}) = j_{\emptyset}(\beta)$ whenever $\beta \in |f(\emptyset)|$.
- $k(\{\alpha, (\alpha + 1) + \beta\}) = j_{\{\alpha\}}(\beta)$ whenever $\alpha \in \kappa$ and $\beta \in |f(\{\alpha\})|$.
- $k(\{\alpha, \beta\}) = j_{\{\alpha\}}(\beta)$ whenever $\alpha \in \delta\backslash\kappa$ and $\beta \in |f(\{\alpha\})|$.

Finally define $g : [\delta]^{<\max(3,\bar{\theta})} \to [\lambda]^{<3}$ so that $g(e) = \{h(e), k(e)\}$ if $e \in [\delta]^{<2}$, and $g(e) = \{k(e)\}$ if $e \in [\delta]^{<3} \backslash [\delta]^{<2}$. Then g is as desired. □

5 Variations of δ

This section is concerned with the case when δ is not a cardinal.

Throughout the section it is assumed that $\delta \geq \kappa$.

Our first remark is that we do not lose generality by assuming that δ is the ordinal product $\kappa\alpha$ for some $\alpha > 0$. Lemma 5.1 and Proposition 5.2 generalize results of Abe [1].

Lemma 5.1 *Assume that $\delta = \kappa\alpha$ for some ordinal $\alpha > 0$, and J is $[\delta]^{<\theta}$-normal. Then J is $[\delta + \xi]^{<\theta}$-normal for every $\xi < \kappa$.*

Proof Fix $\xi < \kappa$. Since $\xi + \delta = \delta$, we can define $j : \delta + \xi \to \delta$ by: $j(\beta) = \xi + \beta$ for $\beta < \delta$, and $j(\delta + \gamma) = \gamma$ for $\gamma < \xi$. Set

$$C = \widehat{\xi} \cap \{a \in [\lambda]^{<\kappa} : \forall \beta \in a \cap \delta\ (j(\beta) \in a)\}.$$

Then clearly $C \in (NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^*$. Now given $A \in J^+$ and a $[\delta + \xi]^{<\theta}$-regressive $f : A \to [\delta + \xi]^{<\theta}$, define $g : A \cap C \to [\delta]^{<\theta}$ by $g(a) = j``(f(a))$. Since $A \cap C \in J^+$ by Proposition 4.2, and g is $[\delta]^{<\theta}$-regressive, we have by Proposition 3.6 ((i) $\to$ (iv)) that g is constant on some $D \in J^+$. Then f is constant on D. Hence by Proposition 3.6 ((iv) $\to$ (i)), J is $[\delta + \xi]^{<\theta}$-normal. □

Proposition 5.2 *Assume that $\delta = \kappa\alpha$ for some ordinal $\alpha > 0$. Then the following hold:*

(i) $NS_{\kappa,\lambda}^{[\delta]^{<\theta}} = NS_{\kappa,\lambda}^{[\delta+\xi]^{<\theta}}$ *for every $\xi < \kappa$.*

(ii) $NS_{\kappa,\lambda}^{[\delta+\kappa]^{<2}} \backslash NS_{\kappa,\lambda}^{[\delta]^{<\theta}} \neq \emptyset$.

Proof (i) By Lemma 5.1 and Propositions 4.2 and 4.3.

(ii) Select $f : [\delta + \kappa]^{<2} \to [\lambda]^{<\kappa}$ so that $f(\{\beta\}) = \{\beta + 1\}$ for every $\beta \in \delta + \kappa$. Given $g : [\delta]^{<\max(3,\bar{\theta})} \to [\lambda]^{<\kappa}$, pick $a \in C_g^{\kappa,\lambda}$ and $\gamma \in (\delta + \kappa) \backslash \delta$ with $\gamma \geq \sup(a \cap (\delta + \kappa))$. Then $a \cup \{\gamma\} \in C_g^{\kappa,\lambda} \backslash C_f^{\kappa,\lambda}$. Hence by Lemma 3.13, $[\lambda]^{<\kappa} \backslash C_f^{\kappa,\lambda} \in NS_{\kappa,\lambda}^{[\delta+\kappa]^{<2}} \backslash NS_{\kappa,\lambda}^{[\delta]^{<\theta}}$.

□

Lemma 5.3 *The following are equivalent:*

(i) J *is* $[\delta]^{<\theta}$*-normal.*
(ii) $\nabla^\delta I_{\kappa,\lambda} \subseteq J$ *and* J *is* $[|\delta|]^{<\theta}$*-normal.*

Proof (i) $\rightarrow$ (ii) : By Lemma 3.2 (iii).

(ii) $\rightarrow$ (i) : Select a bijection $j : \delta \rightarrow |\delta|$ and set $D = \Delta_{\alpha<\delta}\widehat{\{j(\alpha)\}}$. Then D lies in $(\nabla^\delta I_{\kappa,\lambda})^*$ and hence in J^*. Now fix $A \in J^+$ and a $[\delta]^{<\max(3,\bar{\theta})}$-regressive $f : A \rightarrow [\delta]^{<\max(3,\bar{\theta})}$. Define $g : A \cap D \rightarrow [|\delta|]^{<\max(3,\bar{\theta})}$ by $g(a) = j``(f(a))$. Since g is $||\delta|]^{<\max(3,\bar{\theta})}$-regressive, we may find $C \in J^+ \cap P(A \cap D)$ and $u \in [|\delta|]^{<\max(3,\bar{\theta})}$ so that $g(a) = u$ for all $a \in C$. Then f takes the constant value $j^{-1}(u)$ on C. □

Let us remark in passing that Lemma 5.3 can be combined with a result of [16] to show that J is $[\delta]^{<\theta}$-normal if and only if it is δ-normal and $(\mu, |\delta|)$-distributive for every infinite cardinal $\mu < \bar{\theta}$.

Proposition 5.4 $NS_{\kappa,\lambda}^{[\delta]^{<\theta}} = NS_{\kappa,\lambda}^{[|\delta|]^{<\theta}} | D$ *for some* $D \in (\nabla^\delta I_{\kappa,\lambda})^*$.

Proof By the proof of Lemma 5.3. □

Using Cantor's normal form for the base $|\delta|$, one easily obtains the following.

Proposition 5.5 *Assume that* $\gamma < \delta \leq \gamma^\gamma$*, where* $\gamma = |\delta|$*. Then* $NS_{\kappa,\lambda}^{\delta} = NS_{\kappa,\lambda}^{\gamma}|A$*, where* A *is the set of all* $a \in [\lambda]^{<\kappa}$ *with the following property : Suppose that* $1 \leq \alpha < \delta$ *and* $\alpha = \gamma^{\eta_1}\xi_1 + \cdots + \gamma^{\eta_p}\xi_p$*, where* $1 \leq p < \omega$*,* $\gamma > \eta_1 > \cdots > \eta_p$*, and* $\gamma > \xi_i \geq 1$ *for* $1 \leq i \leq p$*. Then* $\alpha \in a$ *if and only if* $\{\eta_1, \xi_1, \ldots, \eta_p, \xi_p\} \subseteq a$*.*

Thus for example $NS_{\kappa,\lambda}^{\kappa+\kappa} = NS_{\kappa,\lambda}^{\kappa}|A$, where A is the set of all $a \in [\lambda]^{<\kappa}$ such that $a\backslash\kappa = \{\kappa + \alpha : \alpha \in a \cap \kappa\}$, and $NS_{\kappa,\lambda}^{\kappa^2} = NS_{\kappa,\lambda}^{\kappa}|B$, where B is the set of all $a \in [\lambda]^{<\kappa}$ such that $a\backslash\kappa = \{\kappa\beta + \alpha : \alpha, \beta \in a \cap \kappa$ and $\beta \geq 1\}$.

6 Variations of θ

Proposition 6.1 *Assume that* $\delta \geq \kappa$ *and* $\max(\omega, \bar{\theta})$ *is a regular cardinal, and let* θ' *be a cardinal such that* $\theta' \leq \kappa$ *and* $\max(\omega, \bar{\theta}) < \bar{\theta}'$*. Then* $NS_{\kappa,\lambda}^{[\kappa]^{<\theta'}} \backslash NS_{\kappa,\lambda}^{[\delta]^{<\theta}} \neq \emptyset$ *(and therefore* $NS_{\kappa,\lambda}^{[\delta]^{<\theta}} \neq NS_{\kappa,\lambda}^{[\delta]^{<\theta'}}$*).*

Proof Let us assume that there exists a $[\kappa]^{<\theta'}$-normal ideal on $[\lambda]^{<\kappa}$. Given $f : [\delta]^{<\max(3,\bar{\theta})} \rightarrow [\lambda]^{<\kappa}$, we use Proposition 3.19 (i) to define $a_\alpha \in [\lambda]^{<\kappa}$ for $\alpha < \max(\omega, \bar{\theta})$ as follows:

- $a_0 = \max(3, \bar{\theta})$.
- $a_{\alpha+1} = a_\alpha \cup ((\sup(a_\alpha \cap \kappa)) + 1) \cup (\bigcup f``([a_\alpha \cap \delta]^{<\max(3,\bar{\theta})})$.
- $a_\alpha = \bigcup_{\beta<\alpha} a_\beta$ in case α is an infinite limit ordinal.

Put $a = \bigcup_{\alpha<\max(\omega,\bar{\theta})} a_\alpha$. Then $a \in C_f^{\kappa,\lambda}$, and moreover $\mathrm{cf}(\sup(a \cap \kappa)) = \max(\omega, \bar{\theta})$. Hence by Lemma 3.13,

$$\{a \in [\lambda]^{<\kappa} : \mathrm{cf}(\sup(a \cap \kappa)) = \max(\omega, \bar{\theta})\} \in (NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^+.$$

It remains to observe that by Lemma 3.17 (ii),

$$\{a \in [\lambda]^{<\kappa} : \mathrm{cf}(\sup(a \cap \kappa)) > \max(\omega, \bar{\theta})\} \in (\nabla^{[\kappa]^{<\bar{\theta}'}} I_{\kappa,\lambda})^*.$$

□

We will see that the conclusion of Proposition 6.1 may fail if $\bar{\theta}$ is a singular cardinal. The remainder of the section is concerned with the case when θ is a limit cardinal.

The following is immediate from Proposition 3.19 (i).

Proposition 6.2 *Suppose that θ is a limit cardinal less than κ. Then the following are equivalent:*

(i) *There exists a $[\delta]^{<\theta}$-normal ideal on $[\lambda]^{<\kappa}$.*
(ii) *For each cardinal ρ with $2 \le \rho < \theta$, there exists a $[\delta]^{<\rho}$-normal ideal on $[\lambda]^{<\kappa}$.*

Notice that if $\theta = \kappa$ and κ is an inaccessible cardinal that is not Mahlo, then by Proposition 3.19, (ii) holds but (i) does not.

Proposition 6.3 *Assume that $\delta \ge \kappa$ and θ is a limit cardinal . Then the following are equivalent:*

(i) *J is $[\delta]^{<\theta}$-normal.*
(ii) *J is $[\delta]^{<\rho}$-normal for every cardinal ρ with $2 \le \rho < \theta$.*

Proof (i) $\to$ (ii) : By Lemma 3.2 (iii).

(ii) $\to$ (i) : By Proposition 3.6, it suffices to show that if $A \in J^+$ and $f : A \to [\delta]^{<\theta}$ is $[\delta]^{<\theta}$-regressive, then $f|D$ is $[\delta]^{<\rho}$-regressive for some $D \in J^+ \cap P(A)$ and some cardinal ρ with $2 \le \rho < \theta$. This is clear if $\theta < \kappa$. Assuming $\theta = \kappa$, put $B = \{a \in A : a \cap \kappa \in \kappa\}$. Then $|f(a)| \in a \cap \kappa$ for every $a \in B$ with $a \cap \kappa \ne \emptyset$. It remains to observe that by Lemmas 3.2 (iii) and 3.17 (iii), J is $[\kappa]^{<2}$-normal and $B \cap \widehat{2} \in J^+$. □

We have the following corresponding characterization of $NS_{\kappa,\lambda}^{[\delta]^{<\theta}}$.

Proposition 6.4 *Assume that $\delta \ge \kappa$ and θ is a limit cardinal. Then $NS_{\kappa,\lambda}^{[\delta]^{<\theta}} = \nabla^{\mathrm{cf}(\theta)}(\bigcup_{2\le\rho<\theta} NS_{\kappa,\lambda}^{[\delta]^{<\rho}})$.*

Proof By Lemma 3.2 (iii) and Proposition 3.9,

$$\nabla^{\mathrm{cf}(\theta)}(\bigcup_{2\le\rho<\theta} NS_{\kappa,\lambda}^{[\delta]^{<\rho}}) \subseteq \nabla^{\mathrm{cf}(\theta)} NS_{\kappa,\lambda}^{[\delta]^{<\theta}} \subseteq NS_{\kappa,\lambda}^{[\delta]^{<\theta}}.$$

For the reverse inclusion, select an increasing, continuous sequence $\langle \rho_i : i < \mathrm{cf}(\theta)\rangle$ of cardinals greater than or equal to 2 with supremum θ. Define D by: $D = \widehat{\theta}$ if $\theta < \kappa$, and

$$D = \{a \in [\lambda]^{<\kappa} : a \cap \kappa \text{ is an infinite limit ordinal}\}$$

otherwise. Note that $D \in (NS_{\kappa,\lambda}^{[\kappa]^{<2}})^*$ by Lemma 3.17 (iii). Set

$$H = \{a \in [\lambda]^{<\kappa} : \forall i \in a \cap \mathrm{cf}(\theta)\ (\rho_i \in a)\}.$$

Note that $H \in (NS_{\kappa,\lambda}^{[\kappa]^{<2}})^*$. Moreover, $H \subseteq \{a \in [\lambda]^{<\kappa} : a \cap \kappa = \rho_{a\cap\kappa}\}$ in case $\theta = \kappa$. Now fix $B \in NS_{\kappa,\lambda}^{[\delta]^{<\theta}}$. Then by Lemma 3.13, there is $f : [\lambda]^{<\theta} \to [\lambda]^{<\kappa}$ such that $B \cap C_f^{\kappa,\lambda} = \emptyset$. For $i < \mathrm{cf}(\theta)$, set $f_i = f|[\delta]^{<\rho_i}$. It is simple to see that $D \cap H \cap \Delta_{i<\mathrm{cf}(\theta)} C_{f_i}^{\kappa,\lambda} \subseteq C_f^{\kappa,\lambda}$. Hence, $B \subseteq ([\lambda]^{<\kappa} \setminus \widehat{\{0\}}) \cup \nabla_{i<\mathrm{cf}(\theta)} B_i$, where B_i equals $[\lambda]^{<\kappa} \setminus (D \cap H \cap C_{f_0}^{\kappa,\lambda})$ if $i = 0$, and $[\lambda]^{<\kappa} \setminus C_{f_i}^{\kappa,\lambda}$ otherwise, and consequently $B \in \nabla^{\mathrm{cf}(\theta)}(\bigcup_{2\le\rho<\theta} NS_{\kappa,\lambda}^{[\delta]^{<\rho}})$. □

Let us now concentrate on the case when θ is a singular cardinal.

Proposition 6.5 *Suppose that there exists a $[\delta]^{<\theta}$-normal ideal on $[\lambda]^{<\kappa}$, θ is a singular cardinal, and either $\delta \ge 2^{<\theta}$, or $\delta \ge \theta$ and $\mathrm{cf}(\theta^{<\theta}) \ne \mathrm{cf}(\theta)$. Then there exists a $[\delta]^{<\theta^+}$-normal ideal on $[\lambda]^{<\kappa}$.*

Proof Note that by Proposition 3.19 (i), $2^{<\theta} \le \theta^{<\theta} < \kappa$. First suppose that $\theta \le \delta < 2^{<\theta}$ and $\mathrm{cf}(\theta^{<\theta}) \ne \mathrm{cf}(\theta)$. Then there is a cardinal $\tau < \theta$ such that $\theta^{<\theta} = \theta^\tau$. We get

$$|\delta|^\theta \le (2^\theta)^\theta = \theta^\theta = (\theta^{<\theta})^{\mathrm{cf}(\theta)} = \theta^{\max(\tau,\mathrm{cf}(\theta))} = \theta^{<\theta},$$

so the desired conclusion follows from Proposition 3.19 (i). Now suppose $\delta \ge 2^{<\theta}$. Let μ be a cardinal with $2^{<\theta} \le \mu < \min(\kappa, \delta+1)$. Then by Lemma 3.23 and Proposition 3.19 (i), $\mu^\theta = (\mu^{<\theta})^{<\theta} = \mu^{<\theta} < \kappa$. From this together with Proposition 3.19 (i), we get the desired conclusion. □

Observe that if θ is a singular cardinal with $\mathrm{cf}(\theta^{<\theta}) = \mathrm{cf}(\theta)$, then for $\delta = \theta$ and $\kappa = (\theta^{<\theta})^+$, (a) there is a $[\delta]^{<\theta}$-normal ideal on $[\lambda]^{<\kappa}$, but (b) there is no $[\delta]^{<\theta^+}$-normal ideal on $[\lambda]^{<\kappa}$ (since $\theta^\theta = (\theta^{<\theta})^{\mathrm{cf}(\theta)} \ge \kappa$).

Corollary 6.6 *Assume that there exists a $[\delta]^{<\theta}$-normal ideal on $[\lambda]^{<\kappa}$, θ is a singular cardinal and $\delta \ge \kappa$. Then there exists a $[\delta]^{<\theta^+}$-normal ideal on $[\lambda]^{<\kappa}$.*

Proof This is immediate from Proposition 6.5, since by Proposition 3.19 (i), $2^{<\theta} \le \theta^{<\theta} < \kappa \le \delta$. □

It is then natural to ask whether, under the assumptions of Corollary 6.6, the notions of $[\delta]^{<\theta}$-normality and $[\delta]^{<\theta^+}$-normality coincide. We will see that they do in a number of cases. We start by recalling a few facts concerning covering numbers.

Definition *Given four cardinals $\rho_1, \rho_2, \rho_3, \rho_4$ greater than or equal to 2, we let* $\mathrm{cov}(\rho_1, \rho_2, \rho_3, \rho_4)$ = *the least cardinality of any X in $\mathfrak{X}_{\rho_1\rho_2\rho_3\rho_4}$ if $\mathfrak{X}_{\rho_1\rho_2\rho_3\rho_4} \neq \emptyset$, and* $\mathrm{cov}(\rho_1, \rho_2, \rho_3, \rho_4) = 0$ *otherwise, where $\mathfrak{X}_{\rho_1\rho_2\rho_3\rho_4}$ is the collection of all $X \subseteq [\rho_1]^{<\rho_2}$ such that for any $a \in [\rho_1]^{<\rho_3}$, there is $Q \in [X]^{<\rho_4}$ with $a \subseteq \bigcup Q$.*

It is simple to see that $\mathrm{cov}(\rho_1, \rho_2, \rho_3, \rho_4) = 1$ if $\rho_2 > \rho_1$. Note that if $\omega \leq \rho_3 = \rho_2 \leq \rho_1$ and $\rho_4 = 2$, then $\mathrm{cov}(\rho_1, \rho_2, \rho_3, \rho_4) = u(\rho_2, \rho_1)$. We are interested in situations when $\rho_2 = \rho_3$ and $\mathrm{cov}(\rho_1, \rho_2, \rho_3, \rho_4) = \rho_1$.

Fact 6.7 ([17], [20, pp. 85–86]) *Let ρ_1, ρ_2, ρ_3 and ρ_4 be four cardinals such that $\rho_1 \geq \rho_2 \geq \rho_3 \geq \omega$ and $\rho_3 \geq \rho_4 \geq 2$. Then the following hold:*

(i) *If $\rho_1 = \rho_2$ and either $\mathrm{cf}(\rho_1) < \rho_4$ or $\mathrm{cf}(\rho_1) \geq \rho_3$, then $\mathrm{cov}(\rho_1, \rho_2, \rho_3, \rho_4) = \mathrm{cf}(\rho_1)$.*
(ii) *If either $\rho_1 > \rho_2$, or $\rho_1 = \rho_2$ and $\rho_4 \leq \mathrm{cf}(\rho_1) < \rho_3$, then $\mathrm{cov}(\rho_1, \rho_2, \rho_3, \rho_4) \geq \rho_1$.*
(iii) $\mathrm{cov}(\rho_1, \rho_2, \rho_3, \rho_4) = \mathrm{cov}(\rho_1, \rho_2, \rho_3, \max(\omega, \rho_4))$.
(iv) $\mathrm{cov}(\rho_1^+, \rho_2, \rho_3, \rho_4) = \max(\rho_1^+, \mathrm{cov}(\rho_1, \rho_2, \rho_3, \rho_4))$.
(v) *If $\rho_1 > \rho_2$ and $\mathrm{cf}(\rho_1) < \rho_4 = \mathrm{cf}(\rho_4)$, then*

$$\mathrm{cov}(\rho_1, \rho_2, \rho_3, \rho_4) = \sup(\{\mathrm{cov}(\rho, \rho_2, \rho_3, \rho_4) : \rho_2 \leq \rho < \rho_1\}).$$

(vi) *If ρ_1 is a limit cardinal such that $\rho_1 > \rho_2$ and $\mathrm{cf}(\rho_1) \geq \rho_3$, then*

$$\mathrm{cov}(\rho_1, \rho_2, \rho_3, \rho_4) = \sup(\{\mathrm{cov}(\rho, \rho_2, \rho_3, \rho_4) : \rho_2 \leq \rho < \rho_1\}).$$

(vii) *If $\rho_3 > \rho_4 \geq \omega$, then*

$$\mathrm{cov}(\rho_1, \rho_2, \rho_3, \rho_4) = \sup(\{\mathrm{cov}(\rho_1, \rho_2, \rho^+, \rho_4) : \rho_4 \leq \rho < \rho_3\}).$$

(viii) *If $\rho_3 \leq \mathrm{cf}(\rho_2) = \rho_2$, $\omega \leq \mathrm{cf}(\rho_4) = \rho_4$ and $\rho_1 < \rho_2^{+\rho_4}$, then $\mathrm{cov}(\rho_1, \rho_2, \rho_3, \rho_4) = \rho_1$.*
(ix) *If $\rho_3 = \mathrm{cf}(\rho_3)$, then either $\mathrm{cf}(\mathrm{cov}(\rho_1, \rho_2, \rho_3, \rho_4)) < \rho_4$, or $\mathrm{cf}(\mathrm{cov}(\rho_1, \rho_2, \rho_3, \rho_4)) > \rho_1$.*

Fact 6.8 ([21]) *Let π, τ and σ be three infinite cardinals such that $\pi > \tau > \sigma$ and $\mathrm{cf}(\sigma) > \mathrm{cf}(\tau)$. Then $\mathrm{cov}(\pi, \tau, \tau, \sigma) = \mathrm{cov}(\pi, \tau^+, \tau^+, \sigma)$.*

We omit the definition of the pp function, which can be found on p. 41 of [20]. Shelah's Strong Hypothesis (SSH) asserts that $\mathrm{pp}(\chi) = \chi^+$ for any singular cardinal χ. Its failure (the exact consistency strength of which is not known) entails the existence of inner models with large large cardinals.

Lemma 6.9 *Let π, τ and σ be three infinite cardinals such that $\pi > \tau > \sigma = \mathrm{cf}(\sigma) > \mathrm{cf}(\tau)$, and either $\mathrm{cf}(\pi) < \sigma$, or $\mathrm{cf}(\pi) > \tau$. Suppose that $\mathrm{pp}(\chi) = \chi^+$ for any cardinal χ such that $\mathrm{cf}(\chi) = \sigma < \chi < \pi$. Then $\mathrm{cov}(\pi, \tau^+, \tau^+, \sigma) = \pi$.*

Proof By Proposition 3.1 in [17], $\mathrm{cov}(\mu, \tau^+, \tau^+, \sigma) \leq \mu^+$ for any cardinal μ with $\tau < \mu < \pi$. The desired conclusion now follows from Fact 6.7. □

Let us finally recall the statement of Shelah's Revised GCH Theorem.

Fact 6.10 ([21]) *Let ρ be a singular strong limit cardinal, and $\pi > \rho$ be a cardinal. Then there is a regular cardinal $\sigma < \rho$ such that $\mathrm{cov}(\pi, \tau, \tau, \sigma) = \pi$ for every cardinal τ with $\sigma < \tau \leq \rho$.*

Proposition 6.11 *Assume that θ is a singular cardinal, $\delta \geq \kappa$ and there is a cardinal σ such that $2 \leq \sigma < \theta$ and $\mathrm{cov}(|\delta|, \theta, \theta, \sigma) = |\delta|$. Then every $[\delta]^{<\theta}$-normal ideal on $[\lambda]^{<\kappa}$ is $[\delta]^{<\theta^+}$-normal.*

Proof Suppose that J is $[\delta]^{<\theta}$-normal. Since by Proposition 3.19 (i) $2^{<\theta} < \delta$, we may find $x_\xi \in [\delta]^{<\theta}$ for $\xi \in \delta$, and $f : [\delta]^{<\theta} \to [\delta]^{<\sigma}$ such that $c = \bigcup_{\xi \in f(c)} x_\xi$ for every $c \in [\delta]^{<\theta}$. Now fix $A_e \in J^*$ for $e \in [\delta]^{<\theta^+}$. Put $B_d = A_{\bigcup_{\xi \in d} x_\xi}$ for $d \in [\delta]^{<\theta}$. Set $C = \Delta_{c \in [\delta]^{<\theta}} \widehat{f(c)}$, $D = \Delta_{d \in [\delta]^{<\theta}} B_d$ and $E = C \cap D \cap \widehat{\theta}$. Then by Proposition 3.6, $E \in J^*$. Let $a \in E$ and $e \in [a \cap \delta]^{<|a \cap \theta^+|}$ be given. Select $c_\zeta \in [\delta]^{<\theta}$ for $\zeta < \mathrm{cf}(\theta)$ so that $e = \bigcup_{\zeta < \mathrm{cf}(\theta)} c_\zeta$. For each $\zeta < \mathrm{cf}(\theta)$, we have $c_\zeta \in [a \cap \delta]^{<|a \cap \theta|}$ and therefore $f(c_\zeta) \subseteq a$. So setting $d = \bigcup_{\zeta < \mathrm{cf}(\theta} f(c_\zeta)$, we have $d \in [a \cap \delta]^{<|a \cap \theta|}$ and consequently $a \in B_d$. Notice that $B_d = A_e$, since

$$\bigcup_{\xi \in d} x_\xi = \bigcup_{\zeta < \mathrm{cf}(\theta)} \bigcup_{\xi \in f(c_\zeta)} x_\xi = \bigcup_{\zeta < \mathrm{cf}(\theta)} c_\zeta = e.$$

Thus $E \subseteq \Delta_{e \in [\delta]^{<\theta^+}} A_e$, and therefore $\Delta_{e \in [\delta]^{<\theta^+}} A_e \in J^*$. Hence by Proposition 3.6, J is $[\delta]^{<\theta^+}$-normal. □

Corollary 6.12 *Suppose that θ is a singular cardinal, $\delta \geq \kappa$ and one of the following holds:*

(i) SSH *holds.*
(ii) $|\delta| < \theta^{+\theta}$.
(iii) *θ is a strong limit cardinal.*

Then any $[\delta]^{<\theta}$-normal ideal on $[\lambda]^{<\kappa}$ is $[\delta]^{<\theta^+}$-normal.

Proof For (i), use Fact 6.8, Lemma 6.9 and Proposition 6.11 (with $\sigma = (\mathrm{cf}(\theta))^+$ if $\mathrm{cf}(|\delta|) > \theta$, and $\sigma = \max((\mathrm{cf}(\theta))^+, (\mathrm{cf}(|\delta|))^+)$ otherwise). For (ii), use Fact 6.7 (viii) and Proposition 6.11 (with $\sigma = (\max(\mathrm{cf}(\theta), |\xi|))^+$ if $|\delta| = \theta^{+\xi}$). Finally for (iii), use Fact 6.10 and Proposition 6.11. □

7 The case $\kappa \leq \delta < \kappa^{+\bar{\theta}}$

Definition *We let $E_{\kappa,\lambda}$ denote the set of all $a \in [\lambda]^{<\kappa}$ such that $a \cap \kappa \neq \emptyset$ and $a \cap \kappa = \bigcup (a \cap \kappa)$.*

Fact 7.1 ([15]) *Assuming the existence of a $[\kappa]^{<\theta}$-normal ideal on $[\lambda]^{<\kappa}$, the following are equivalent:*

(i) *J is $[\kappa]^{<\theta}$-normal.*
(ii) *J is κ-normal and $\{a \in E_{\kappa,\lambda} : \mathrm{cf}(a \cap \kappa) \geq \sup(a \cap \bar{\theta})\} \in J^*$.*

We will show that this result can be generalized.

Definition *Let ρ be a cardinal with $\kappa \leq \rho$, and β be an ordinal with $1 \leq \beta < \kappa$. Then $A^{\rho,\beta}_{\kappa,\lambda}$ denotes the set of all $a \in [\lambda]^{<\kappa}$ such that (i) $\alpha+1 \in a$ for every $\alpha \in a \cap (\rho^{+\beta} \backslash \rho)$, and (ii) $\rho^{+\gamma} \in a$ for every $\gamma < \beta$.*

Thus if $a \in A^{\rho,\beta}_{\kappa,\lambda}$ and $\gamma < \beta$, then $\sup(a \cap \rho^{+(\gamma+1)})$ is a limit ordinal that is strictly greater than $\rho^{+\gamma}$ and does not belong to a.

Proposition 7.2 *Assume that $\delta = \rho^{+\beta}$, where ρ is a cardinal with $\kappa \leq \rho$, and β an ordinal with $1 \leq \beta < \bar{\theta}$. Then the following are equivalent:*

(i) *J is $[\delta]^{<\theta}$-normal.*
(ii) *J is $[\delta]^{<|\beta|^+}$-normal and $[\rho]^{<\theta}$-normal, and the set of all $a \in A^{\rho,\beta}_{\kappa,\lambda}$ such that $\mathrm{cf}(\sup(a \cap \rho^{+(\alpha+1)})) \geq \sup(a \cap \bar{\theta})$ for every $\alpha < \beta$ lies in J^*.*

Proof (i) $\to$ (ii) : By Lemma 3.17(ii) and Proposition 3.5.

(ii) $\to$ (i) : By Proposition 6.3 it suffices to prove the result for $\theta < \kappa$. We can also assume that $|\beta|^+ < \theta$ (since otherwise the result is trivial) and (by Proposition 4.1) that θ is an infinite cardinal.

For $\gamma \in \delta \backslash \rho$, select a bijection $\tilde{\gamma} : \gamma \to |\gamma|$. Let B be the set of all $a \in A^{\rho,\beta}_{\kappa,\lambda}$ such that $\theta \subseteq a$, $\mathrm{cf}(\sup(a \cap \rho^{+(\alpha+1)})) \geq \theta$ for all $\alpha < \beta$, and $\tilde{\gamma}(\xi) \in a$ whenever $\gamma \in a \cap (\delta \backslash \rho)$ and $\xi \in a \cap \gamma$. Notice that $B \in J^*$. For $a \in B$ and $\alpha < \beta$, select $z^a_\alpha \subseteq a \cap (\kappa^{+(\alpha+1)} \backslash \kappa^{+\alpha})$ so that $\mathrm{o.t.}(z^a_\alpha) = \mathrm{cf}(\sup(a \cap \kappa^{+(\alpha+1)}))$ and $\sup(z^a_\alpha) = \sup(a \cap \kappa^{+(\alpha+1)})$. Now fix $C \in J^+$ and a $[\delta]^{<\theta}$-regressive $F : C \to [\delta]^{<\theta}$. Set $D = C \cap B$. For $a \in D$ and $1 \leq \eta \leq \beta$, define $k^a_\eta : [a \cap \rho^{+\eta}]^{<\theta} \to [a \cap \rho^{+\eta}]^{<|\eta|^+}$ as follows:

- $k^a_1(e) = \{\gamma\}$, where $\gamma =$ the least $\zeta \in z^a_0$ such that $e \subseteq \zeta$.
- If $e \backslash \rho^{+\eta} \neq \emptyset$, then $k^a_{\eta+1}(e) = \{\gamma\} \cup k^a_\eta(\tilde{\gamma}``e)$, where $\gamma =$ the least $\zeta \in z^a_\eta$ such that $e \subseteq \zeta$. Otherwise, $k^a_{\eta+1}(e) = k^a_\xi(e)$, where $\xi =$ the least $\chi \geq 1$ such that $e \subseteq \rho^{+\chi}$.
- Suppose that η is a limit ordinal. If $\sup(e) = \kappa^{+\eta}$, then $k^a_\eta(e) = \bigcup_{\alpha<\eta} k^a_{\alpha+1}(e \cap \rho^{+(\alpha+1)})$. Otherwise, $k^a_\eta(e) = k^a_\xi(e)$, where $\xi =$ the least $\chi \geq 1$ such that $e \subseteq \rho^{+\chi}$.

Let $a \in D$. For $1 \leq \xi \leq \beta$, let Φ_ξ assert that given $\zeta \in e \in [a \cap \rho^{+\xi}]^{<\theta}$, we may find $n \in \omega$ and $\gamma_0, \ldots, \gamma_n \in k^a_\xi(e)$ such that $\zeta \in \gamma_0$, $(\tilde{\gamma}_j \circ \cdots \circ \tilde{\gamma}_0)(\zeta) \in \gamma_{j+1}$ for $j = 0, \ldots, n-1$, and $(\tilde{\gamma}_n \circ \cdots \circ \tilde{\gamma}_0)(\zeta) \in a \cap \rho$. Let us prove by induction that Φ_ξ holds. For $e \in [a \cap \rho^+]^{<\theta}$, let $k^a_1(e) = \{\gamma\}$. Then $e \subseteq \gamma$, and moreover $\tilde{\gamma}(\zeta) \in a \cap \rho$ for all $\zeta \in e$. Thus Φ_1 holds. Next suppose that $1 < \alpha \leq \beta$ and Φ_ξ holds for $1 \leq \xi < \alpha$. Let $e \in [a \cap \rho^{+\alpha}]^{<\theta}$ be such that $e \backslash \rho^{+\xi} \neq \emptyset$ for every $\xi < \alpha$. Given $\zeta \in e$, define ξ, γ_0 and e' as follows:

- If α is a limit ordinal, then $\xi =$ the least σ such that $\zeta \in \rho^{+(\sigma+1)}$. Otherwise $\xi + 1 = \alpha$.

- $\gamma_0 =$ the least $\gamma \in z^a_\xi$ such that $e \cap \rho^{+(\xi+1)} \subseteq \gamma$.
- $e' = \tilde{\gamma}_0``(e \cap \rho^{+(\xi+1)})$.

Then $\xi < \alpha$ and $\zeta \in \gamma \in z^a_\xi \cap k^a_\alpha(e)$. Moreover $\tilde{\gamma}_0(\zeta) \in e' \in [a \cap \rho^{+\xi}]^{<\theta}$, and $k^a_\xi(e') \subseteq k^a_\alpha(e)$ since

$$\{\gamma_0\} \cup k^a_\xi(e') = k^a_{\xi+1}(e \cap \rho^{+(\xi+1)}) \subseteq k^a_\alpha(e).$$

If $\xi = 0$, then $\tilde{\gamma}_0(\zeta) \in a \cap \rho$. Otherwise, we may find $\gamma_1, \ldots, \gamma_n \in k^a_\xi(e')$, where $1 \leq n < \omega$, such that $\tilde{\gamma}_0(\zeta) \in \gamma_1$, $(\tilde{\gamma}_j \circ \cdots \circ \tilde{\gamma}_1)(\tilde{\gamma}_0(\zeta)) \in \gamma_{j+1}$ for $j = 1, \ldots, n-1$, and $(\tilde{\gamma}_n \circ \cdots \circ \tilde{\gamma}_1)(\tilde{\gamma}_0(\zeta)) \in a \cap \rho$. So Φ_α holds.

Define $G \to [\delta]^{<|\beta|^+}$ by $G(a) = k^a_\beta(F(a))$. Since G is $[\delta]^{<|\beta|^+}$-regressive, there must be $T \in J^+ \cap P(D)$ and $x \in [\delta]^{<|\beta|^+}$ such that G takes the constant value x on T. For $a \in T$ and $\zeta \in F(a)$, we may find $\chi^a_\zeta \in a \cap \rho$, $n \in \omega$ and $\gamma_0, \ldots, \gamma_n \in x$ such that $\zeta \in \gamma_0$, $(\tilde{\gamma}_j \circ \cdots \circ \tilde{\gamma}_0)(\zeta) \in \gamma_{j+1}$ for $j = 0, \ldots, n-1$, and $(\tilde{\gamma}_n \circ \cdots \circ \tilde{\gamma}_0)(\zeta) = \chi^a_\zeta$. Now define $H : T \to [\rho]^{<\theta}$ by $H(a) = \{\chi^a_\zeta : \zeta \in F(a)\}$. Since H is $[\rho]^{<\theta}$-regressive, we may find $W \in J^+ \cap P(T)$ and $y \in [\rho]^{<\theta}$ so that H takes the constant value y on W. Let d be the set of all $\zeta \in \delta$ for which one can find $n \in \omega$ and $\gamma_0, \ldots, \gamma_n \in x$ so that $\zeta \in \gamma_0$, $(\tilde{\gamma}_j \circ \cdots \circ \tilde{\gamma}_0)(\zeta) \in \gamma_{j+1}$ for $j = 0, \ldots, n-1$, and $(\tilde{\gamma}_n \circ \cdots \circ \tilde{\gamma}_0)(\zeta) \in y$. Then $|d| < \theta$ and $F``W \subseteq [d]^{<\theta}$. Since $|[d]^{<\theta}| < \kappa$ by Proposition 3.19 (i), there must be $Z \in J^+ \cap P(W)$ and $v \in [d]^{<\theta}$ such that F takes the constant value v on Z. $\square$

Corollary 7.3 (i) *Suppose that* $|\delta| = \kappa^{+n}$, *where* $n < \omega$. *Then* $NS^{[\delta]^{<\theta}}_{\kappa,\lambda} = NS^{\delta}_{\kappa,\lambda}|C$, *where* C *is the set of all* $a \in [\lambda]^{<\kappa}$ *such that* $\mathrm{cf}(\sup(a \cap \kappa^{+m})) \geq \sup(a \cap \bar{\theta})$ *for all* $m \leq n$.

(ii) *Suppose that* $|\delta| = \kappa^{+\beta}$, *where* $\omega \leq \beta < \bar{\theta}$. *Then* $NS^{[\delta]^{<\theta}}_{\kappa,\lambda} = NS^{[\delta]^{<|\beta|^+}}_{\kappa,\lambda}|C$, *where* C *is the set of all* $a \in [\lambda]^{<\kappa}$ *such that* $\mathrm{cf}(\sup(a \cap \kappa)) \geq \sup(a \cap \bar{\theta})$, *and* $\mathrm{cf}(\sup(a \cap \kappa^{+\alpha+1})) \geq \sup(a \cap \bar{\theta})$ *for all* $\alpha < \beta$.

Proof By Lemma 5.3, Fact 7.1 and Proposition 7.2. $\square$

So for example, if $\kappa > \omega_2$ and $\lambda = \kappa^{+\omega}$, then $NS^{[\lambda]^{<\aleph_2}}_{\kappa,\lambda} = NS^{[\lambda]^{<\aleph_1}}_{\kappa,\lambda}|C$, where C is the set of all $a \in [\lambda]^{<\kappa}$ such that $\mathrm{cf}(\sup(a \cap \kappa^{+n})) \geq \omega_2$ for all $n < \omega$. We will see later (see Proposition 11.6) that if $\lambda^{\aleph_1} = 2^\lambda$, then $NS^{[\lambda]^{<\aleph_2}}_{\kappa,\lambda}|A \neq NS^{[\lambda]^{<\aleph_0}}_{\kappa,\lambda}|A$ for all A.

8 $NS^{[\delta]^{<\theta}}_{\kappa,\lambda}|A$

In this section we continue to investigate whether given $\delta' \geq \delta$ and $\theta' \geq \theta$ with $(\delta', \theta') \neq (\delta, \theta)$, it is possible to find A such that $NS^{[\delta']^{<\theta'}}_{\kappa,\lambda} = NS^{[\delta]^{<\theta}}_{\kappa,\lambda}|A$. The following is obvious.

Proposition 8.1 *Let* δ' *be an ordinal with* $\delta \leq \delta' \leq \lambda$, *and* θ' *be a cardinal with* $\theta \leq \theta' \leq \kappa$. *Then the following are equivalent*:

(i) *There exists* $A \in (NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^+$ *such that* $NS_{\kappa,\lambda}^{[\delta']^{<\theta'}} = NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A$.
(ii) *There is* $f : [\delta']^{<\max(3,\bar{\theta'})} \to [\lambda]^{<\kappa}$ *such that for any* $h : [\delta']^{<\max(3,\bar{\theta'})} \to [\lambda]^{<\kappa}$, *one may find* $k : [\delta]^{<\max(3,\bar{\theta})} \to [\lambda]^{<\kappa}$ *with* $C_f^{\kappa,\lambda} \cap C_k^{\kappa,\lambda} \subseteq C_h^{\kappa,\lambda}$.

We start with a positive result.

Lemma 8.2 *Let* δ' *be an ordinal with* $\delta \leq \delta' \leq \lambda$, *and* θ' *be a cardinal with* $\theta \leq \theta' \leq \kappa$. *Suppose that* $\delta \geq \kappa$ *and* $|\delta|^{<\bar{\theta}} = |\delta'|^{<\bar{\theta}'}$. *Then* $NS_{\kappa,\lambda}^{[\delta']^{<\theta'}} = NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A$ *for some* $A \in (\nabla^{[\delta']^{<\bar{\theta}'}} I_{\kappa,\lambda})^*$.

Proof Select a bijection $j : [\delta']^{<\bar{\theta}'} \to [\delta]^{<\bar{\theta}}$ with $j(\emptyset) = \emptyset$, and let i denote its inverse. Define $f : [\delta']^{<\bar{\theta}'} \to [\lambda]^{<\kappa}$ by: $f(b) = \max(3, \bar{\theta}) \cup j(b)$ if $\bar{\theta} < \kappa$, and $f(b) = |j(b)|^+ \cup j(b)$ otherwise. Then by Lemma 3.13 $C_f^{\kappa,\lambda} \in (\nabla^{[\delta']^{<\bar{\theta}'}} I_{\kappa,\lambda})^*$. Now given $h : [\delta']^{<\max(3,\bar{\theta'})} \to [\lambda]^{<\kappa}$, define $k : [\delta]^{<\max(3,\bar{\theta})} \to [\lambda]^{<\kappa}$ so that

- $k(e) = (h \circ i)(e)$ whenever $e \in [\delta]^{<\bar{\theta}}$.
- If $\bar{\theta}' = 2$, then $k(\{\alpha, \beta\}) = h(i(\{\alpha\}) \cup i(\{\beta\}))$ whenever α and β are two distinct members of δ.

It is readily checked that $C_f^{\kappa,\lambda} \cap C_k^{\kappa,\lambda} \subseteq C_h^{\kappa,\lambda}$. Hence $NS_{\kappa,\lambda}^{[\delta']^{<\theta'}} = NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|C_f^{\kappa,\lambda}$. □

Lemma 8.3 *Assume that there exists a* $[\kappa]^{<\theta}$*-normal ideal on* $[\lambda]^{<\kappa}$. *Let* $\nu > \kappa$ *be a cardinal, and* σ *be the least cardinal* τ *with* $\tau^{<\bar{\theta}} \geq \nu$. *Then the following hold:*

(i) $\sigma > \kappa$.
(ii) $\mu^{<\bar{\theta}} < \sigma$ *for every cardinal* $\mu < \sigma$.
(iii) $\sigma^{<\bar{\theta}} = \nu^{<\bar{\theta}}$.
(iv) $\sigma^{<\bar{\theta}} = \sigma$ *if* $\mathrm{cf}(\sigma) \geq \bar{\theta}$, *and* $\sigma^{<\bar{\theta}} = \sigma^{\mathrm{cf}(\sigma)}$ *otherwise.*

Proof Proposition 3.24 tells us that $\kappa^{<\bar{\theta}} = \kappa$, so $\sigma > \kappa$. Moroever for any cardinal μ with $\kappa < \mu < \sigma$, $\mu^{<\bar{\theta}} < \sigma$ since otherwise by Proposition 3.24 $\mu^{<\bar{\theta}} = (\mu^{<\bar{\theta}})^{<\bar{\theta}} \geq \sigma^{<\bar{\theta}} \geq \nu$, which would contradict the definition of σ. Again by Proposition 3.24, $\sigma^{<\bar{\theta}} = (\sigma^{<\bar{\theta}})^{<\bar{\theta}} \geq \nu^{<\bar{\theta}}$ and hence $\sigma^{<\bar{\theta}} = \nu^{<\bar{\theta}}$. It only remains to prove (iv). We can assume that $\bar{\theta} > \omega$ since otherwise the result is trivial. For any infinite cardinal $\chi < \bar{\theta}$, we have that $\mu^{<\chi} < \sigma$ for every cardinal $\mu < \sigma$, and therefore by Lemma 1.7.3 in [9], σ^{χ} equals σ if $\mathrm{cf}(\sigma) > \chi$, and $\sigma^{\mathrm{cf}(\sigma)}$ otherwise. It immediately follows that $\sigma^{<\bar{\theta}}$ equals σ if $\mathrm{cf}(\sigma) \geq \bar{\theta}$, and $\sigma^{\mathrm{cf}(\sigma)}$ otherwise. □

Proposition 8.4 *Assume* $\delta \geq \kappa$, *and let* σ *be the least cardinal* τ *such that* $\tau^{<\bar{\theta}} \geq |\delta|$. *Then* $NS_{\kappa,\lambda}^{[\delta]^{<\theta}} = NS_{\kappa,\lambda}^{\sigma}|A$ *for some* $A \in (\nabla^{[\delta]^{<\bar{\theta}}} I_{\kappa,\lambda})^*$ *if* $\mathrm{cf}(\sigma) \geq \bar{\theta}$, *and* $NS_{\kappa,\lambda}^{[\delta]^{<\theta}} = NS_{\kappa,\lambda}^{[\sigma]^{<(\mathrm{cf}(\sigma))^+}}|D$ *for some* $D \in (\nabla^{[\delta]^{<\bar{\theta}}} I_{\kappa,\lambda})^*$ *otherwise.*

Proof By Lemmas 8.2 and 8.3. □

Lemma 8.2 has the following generalization.

Proposition 8.5 *Assume $|\delta'|^{<\bar{\theta}'} = |\delta|^{<\bar{\theta}}$, where δ' is an ordinal with $\kappa \leq \delta' \leq \lambda$, and θ' a cardinal with $2 \leq \theta' \leq \kappa$. Then $NS_{\kappa,\lambda}^{[\delta']^{<\theta'}}|C = NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|C$ for some $C \in (\nabla^{[\delta'']^{<\bar{\theta}''}} I_{\kappa,\lambda})^*$, where $\delta'' = max(\delta, \delta')$ and $\theta'' = max(\theta, \theta')$.*

Proof By Lemma 8.2 we may find $A, B \in (\nabla^{[\delta'']^{<\bar{\theta}''}} I_{\kappa,\lambda})^*$ so that $NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A = NS_{\kappa,\lambda}^{[\delta'']^{<\theta''}} = NS_{\kappa,\lambda}^{[\delta']^{<\theta'}}|B$. Then $C = A \cap B$ is as desired. □

We will now describe some situations in which $\delta \leq \delta'$, $\theta \leq \theta'$, $|\delta|^{<\bar{\theta}} < |\delta'|^{<\bar{\theta}'}$ and there is no A such that $NS_{\kappa,\lambda}^{[\delta']^{<\theta'}} = NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A$, thus providing partial converses to Lemma 8.2.

Definition *Assume $\bar{\theta} < \kappa$. Then for $f : [\delta]^{<\max(3,\bar{\theta})} \to [\lambda]^{<\kappa}$ and $X \subseteq \lambda$, we define $\Gamma_f(X)$ as follows. Let $\rho = \max(\omega, \bar{\theta})$ if $\max(\omega, \bar{\theta})$ is a regular cardinal, and $\rho = (\max(\omega, \bar{\theta}))^+$ otherwise. Define $X_\alpha \subseteq \lambda$ for $\alpha < \rho$ by:*

- $X_0 = X$.
- $X_{\alpha+1} = X_\alpha \cup (\bigcup f``([X_\alpha \cap \delta]^{<\max(3,\bar{\theta})}))$.
- $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$ *if α is an infinite limit ordinal.*

Then let $\Gamma_f(X) = \bigcup_{\alpha<\rho} X_\alpha$.

Notice that

$$\Gamma_f(X) = \bigcap\{Y : X \subseteq Y \subseteq \lambda \text{ and } \forall e \in [Y \cap \delta]^{<\max(3,\bar{\theta})}(f(e) \subseteq Y)\}.$$

Definition *Let δ' be an ordinal with $\delta \leq \delta' \leq \lambda$, and θ' be a cardinal with $\theta \leq \theta' \leq \kappa$. Given $f : [\delta']^{<\max(3,\bar{\theta}')} \to [\lambda]^{<\kappa}$ and $k : [\delta]^{<\max(3,\bar{\theta})} \to [\lambda]^{<\kappa}$, we define $\varphi(f, k) : [\delta']^{<\max(3,\bar{\theta}')} \to [\lambda]^{<\kappa}$ by: $(\varphi(f, k))(e) = f(e) \cup k(e)$ if $e \in [\delta]^{<\max(3,\bar{\theta})}$, and $(\varphi(f, k))(e) = f(e)$ otherwise.*

Notice that if $\bar{\theta}' < \kappa$ and there exists a $[\delta']^{<\theta'}$-normal ideal on $[\lambda]^{<\kappa}$, then $\Gamma_{\varphi(f,k)}(a) \in C_f^{\kappa,\lambda} \cap C_k^{\kappa,\lambda}$ for any $a \in [\lambda]^{<\kappa}$ with $\max(3, \bar{\theta}') \subseteq a$.

Proposition 8.6 *Let δ' be an ordinal with $\max(\kappa, \delta) \leq \delta' < \lambda$, and θ' be a cardinal with $\theta \leq \theta' \leq \kappa$. Suppose that $|\delta|^{<\bar{\theta}} < |\delta'|^{<\bar{\theta}'} < \lambda$. Then $NS_{\kappa,\lambda}^{[\delta']^{<\theta'}} \neq NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A$ for all $A \in (NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^+$.*

Proof Fix $f : [\delta']^{<\max(3,\bar{\theta}')} \to [\lambda]^{<\kappa}$. Set $\nu = \max(\kappa, (|\delta|^{<\bar{\theta}})^+)$ and select a one-to-one $i : \nu \to [\delta']^{<\max(3,\bar{\theta}')}$ and a one-to-one $j : \nu \to \lambda \setminus (\nu \cup \delta' \cup (\bigcup ran(f)))$. Define $h : [\delta']^{<\max(3,\bar{\theta}')} \to [\lambda]^{<2}$ so that $h(i(\xi)) = \{j(\xi)\}$ for every $\xi \in \nu$. Now let $k : [\delta]^{<\max(3,\bar{\theta})} \to [\lambda]^{<\kappa}$. Pick $\xi \in \nu$ so that $j(\xi) \notin \bigcup ran(k)$.

First assume $\bar{\theta}' < \kappa$. We set $b = \Gamma_{\varphi(f,k)}(i(\xi) \cup \max(3, \bar{\theta}'))$. Then $b \in C_f^{\kappa,\lambda} \cap C_k^{\kappa,\lambda} \cap \widehat{i(\xi)}$. On the other hand, $b \notin C_h^{\kappa,\lambda}$ since $j(\xi) \notin b$. Next assume $\bar{\theta}' = \kappa$. We define $d_\beta \in [\lambda]^{<\kappa}$ for $\beta < \kappa$ as follows:

- $d_0 = \{0\} \cup i(\xi) \cup |i(\xi)|^+$ if $\bar{\theta} = \kappa$, and $d_0 = \max(3, \bar{\theta}) \cup i(\xi) \cup |i(\xi)|^+$ otherwise.
- $d_{\beta+1} = d_\beta \cup ((\sup(d_\beta \cap \kappa) + 1) \cup (\bigcup\{(\varphi(f,k))(e) : e \in [d_\beta \cap \delta']^{<|d_\beta \cap \kappa|}\})$.
- $d_\beta = \bigcup_{\zeta<\beta} d_\zeta$ if β is an infinite limit ordinal.

Select a regular infinite cardinal $\tau < \kappa$ so that $\sup(d_\tau \cap \kappa) = \tau$, and $\bar{\theta} \leq \tau$ in case $\bar{\theta} < \kappa$. Then $d_\tau \in C_f^{\kappa,\lambda} \cap C_k^{\kappa,\lambda}$. Moreover, $i(\xi) \in [d_\tau \cap \delta']^{<|d_\tau \cap \max(3,\bar{\theta}')|}$ and $j(\xi) \notin d_\tau$, so $d_\tau \notin C_h^{\kappa,\lambda}$. □

Proposition 8.7 *Let μ be a cardinal with $\kappa \leq \mu < \lambda$. Assume that either λ is regular, or $u(\mu^+, \lambda) = \lambda$. Then $NS_{\kappa,\lambda} \neq NS^\mu_{\kappa,\lambda}|A$ for all $A \in (NS^\mu_{\kappa,\lambda})^+$.*

Proof Let us first deal with the case when λ is regular. Fix $f : [\lambda]^{<3} \to [\lambda]^{<\kappa}$. Let C be the set of all $\beta \in \lambda$ such that $f(e) \subseteq \beta$ for every $e \in [\beta]^{<3}$. Notice that C is a closed unbounded set. Define $h : [\lambda]^{<2} \to [\lambda]^{<2}$ so that $h(\{\xi\}) = \{\beta_\xi\}$, where β_ξ is the least element β of C with $\beta > \max(3, \xi)$. Now given $k : [\mu]^{<3} \to [\lambda]^{<\kappa}$, select $\xi \in \lambda$ so that $\bigcup ran(k) \subseteq \xi$. Setting $b = \Gamma_{\varphi(f,k)}(3 \cup \{\xi\})$, we have $b \notin C_h^{\kappa,\lambda}$ since $h(\{\xi\}) \backslash b \neq \emptyset$.

Next suppose that λ is singular. Fix $f : [\lambda]^{<3} \to [\lambda]^{<\kappa}$. Select a one-to-one $j : \lambda \to [\lambda]^{<\mu^+}$ so that $ran(j) \in I_{\mu^+,\lambda}$. Define $h : [\lambda]^{<2} \to [\lambda]^{<2}$ so that $h(\{\xi\}) = \{\beta_\xi\}$, where β_ξ is the least element β of λ with $\beta \notin \Gamma_f(\{\xi\} \cup j(\xi))$. Now given $k : [\mu]^{<3} \to [\lambda]^{<\kappa}$, select $\xi \in \lambda$ so that $3 \cup (\bigcup ran(k)) \subseteq j(\xi)$. Set $b = \Gamma_{\varphi(f,k)}(3 \cup \{\xi\})$. Then $b \subseteq \Gamma_f(\{\xi\} \cup j(\xi))$ and therefore $b \notin C_h^{\kappa,\lambda}$. □

Proposition 8.8 *Let ν and σ be two cardinals such that $\theta = (\mathrm{cf}(\sigma))^+ < \kappa \leq \nu < \sigma \leq \lambda \leq \sigma^{<\theta}$. Suppose that $\mu^{<\theta} < \sigma$ for every cardinal $\mu < \sigma$, and $u(\sigma, \lambda) \leq \lambda^{<\theta}$. Then $NS^{[\sigma]^{<\theta}}_{\kappa,\lambda} \neq NS^{[\nu]^{<\theta}}_{\kappa,\lambda}|A$ for every $A \in (NS^{[\nu]^{<\theta}}_{\kappa,\lambda})^+$.*

Proof Fix $f : [\sigma]^{<\theta} \to [\lambda]^{<\kappa}$. Select $A \in I^+_{\sigma,\lambda}$ so that $A \subseteq \{a \in [\lambda]^{<\sigma} : \kappa \subseteq a\}$ and $|A| \leq \lambda^{<\theta}$. From Lemma 8.3 we get $\lambda^{<\theta} = \sigma^{<\theta}$, so we can find a one-to-one $j : A \to [\sigma]^{<\theta}$. Notice that if $a \in A$, then setting $\mu = |a \cup j(a)|$, we have $|\Gamma_f(a \cup j(a))| \leq \mu^{<\theta}$ since by Proposition 3.24 $(\mu^{<\theta})^{<\theta} = \mu^{<\theta}$. Define $h : [\sigma]^{<\theta} \to [\lambda]^{<2}$ so that for any $a \in A$, $h(j(a)) = \{\xi_a\}$, where ξ_a is the least element of the set $\lambda \backslash \Gamma_f(a \cup j(a))$. Now given $k : [\nu]^{<\theta} \to [\lambda]^{<\kappa}$, pick $a \in A$ so that $\bigcup ran(k) \subseteq a$, and put $b = \Gamma_{\varphi(f,k)}(\theta \cup j(a))$. Then $h(j(a)) \backslash b \neq \emptyset$ since $b \subseteq \Gamma_f(a \cup j(a))$, hence $b \notin C_h^{\kappa,\lambda}$. □

Corollary 8.9 *Assume that $\theta = (\mathrm{cf}(\lambda))^+ < \kappa$, and $\mu^{<\theta} < \lambda$ for every cardinal $\mu < \lambda$. Then for any cardinal ν with $\kappa \leq \nu < \lambda$, and any $A \in (NS^{[\nu]^{<\theta}}_{\kappa,\lambda})^+$, $NS^{[\lambda]^{<\theta}}_{\kappa,\lambda} \neq NS^{[\nu]^{<\theta}}_{\kappa,\lambda}|A$.*

9 Projections

Definition *Let ρ be a cardinal with $\kappa \leq \rho \leq \lambda$, and f be a function from $[\lambda]^{<\kappa}$ to $[\rho]^{<\kappa}$. Then we let $f(J)$ denote the collection of all $B \subseteq [\rho]^{<\kappa}$ such that $f^{-1}(B) \in J$.*

Menas [19] showed that for any cardinal ρ with $\kappa \leq \rho \leq \lambda$, $NS_{\kappa,\rho} = p(NS_{\kappa,\lambda})$, where $p : [\lambda]^{<\kappa} \to [\rho]^{<\kappa}$ is the projection defined by $p(z) = z \cap \rho$. Our aim in this section is to generalize this result, that is to prove that $NS_{\kappa,\rho}^{[\min(\delta,\rho)]^{<\theta}} = p(NS_{\kappa,\lambda}^{[\delta]^{<\theta}})$. Using Proposition 4.5, this is readily checked in case $\delta < \kappa$, since $I_{\kappa,\rho} = p(I_{\kappa,\lambda})$. So in the remainder of the section we concentrate on the case $\delta \geq \kappa$.

Lemma 9.1 *Suppose that $\delta \geq \kappa$, $\bar{\theta} \geq \omega$ and ρ is a cardinal with $\kappa \leq \rho < \lambda$. Suppose further that either $\delta \leq \rho$, or $\bar{\theta}$ is regular. Then $\{y \cap \rho : y \in C_h^{\kappa,\lambda}\} \in (NS_{\kappa,\rho}^{[\min(\delta,\rho)]^{<\theta}})^*$ for any $h : [\delta]^{<\bar{\theta}} \to [\lambda]^{<\kappa}$.*

Proof Fix $h : [\delta]^{<\bar{\theta}} \to [\lambda]^{<\kappa}$. Define $\psi : [\min(\delta,\rho)]^{<\bar{\theta}} \to \kappa$, $f : [\min(\delta,\rho)]^{<\bar{\theta}} \to [\lambda]^{<\kappa}$ and $g : [\min(\delta,\rho)]^{<\bar{\theta}} \to [\rho]^{<\kappa}$ by:

- $\psi(a) = \bar{\theta}$ if $\bar{\theta} < \kappa$, and $\psi(a) = |\omega \cup a|^+$ otherwise.
- $f(a) = \bigcap\{x \in C_h^{\kappa,\lambda} : a \cup \psi(a) \subseteq x\}$.
- $g(a) = f(a) \cap \rho$.

Notice that for any $a \in [\min(\delta,\rho)]^{<\bar{\theta}}$, $\psi(a) \subseteq f(a)$ and $a \in [f(a)]^{<|a\cap\bar{\theta}|}$.

Claim $ran(f) \subseteq C_h^{\kappa,\lambda}$.

Proof of the claim Fix $a \in [\min(\delta,\rho)]^{<\bar{\theta}}$ and $e \in [f(a) \cap \delta \cap \rho]^{<|f(a)\cap\bar{\theta}|}$. Then for any $x \in C_h^{\kappa,\lambda}$ with $a \cup \psi(a) \subseteq x$, $e \in [x \cap \delta \cap \rho]^{<|x\cap\bar{\theta}|}$ and consequently $h(e) \subseteq x$. It follows that $h(e) \subseteq f(a)$, which completes the proof of the claim. □

Let D be the set of all $d \in C_g^{\kappa,\rho}$ such that $\bar{\theta} \subseteq d$ if $\bar{\theta} < \kappa$, and $d \cap \kappa$ is a weakly inaccessible cardinal otherwise. Note that by Lemmas 3.13 and 3.17, $D \in (NS_{\kappa,\rho}^{[\min(\delta,\rho)]^{<\theta}})^*$. We will show that $D \subseteq \{y \cap \rho : y \in C_h^{\kappa,\lambda}\}$. Thus fix $d \in D$. Set $y = d \cup (\bigcup\{f(a) : a \in [d \cap \delta]^{<|d\cap\bar{\theta}|}\})$. Note that $y \cap \rho = d$. Moreover by Proposition 3.19, $y \in [\lambda]^{<\kappa}$. Let us prove that $y \in C_h^{\kappa,\lambda}$. Thus let $e \in [y \cap \delta]^{<|y\cap\bar{\theta}|}$. First assume that $e \subseteq \rho$. Then $e \in [d \cap \delta]^{<|d\cap\bar{\theta}|}$, and therefore $f(e) \subseteq y$. Furthermore $h(e) \subseteq f(e)$, since $e \in [f(e) \cap \delta]^{<|f(e)\cap\bar{\theta}|}$ and $f(e) \in C_h^{\kappa,\lambda}$. Hence $h(e) \subseteq y$.

Next assume that $e \backslash \rho \neq \emptyset$. Then clearly $\delta > \rho$. For $\xi \in e\backslash\rho$, select $b_\xi \in [d \cap \delta]^{<|d\cap\bar{\theta}|}$ with $\xi \in f(b_\xi)$. Set $t = (e \cap \rho) \cup (\bigcup_{\xi\in e\backslash\rho} b_\xi)$, and put $a = t$ if $\bar{\theta} < \kappa$, and $a = t \cup |e|$ otherwise. Then clearly $a \in [d \cap \delta]^{<|d\cap\bar{\theta}|}$, so $f(a) \subseteq y$. It is simple to see that $|e| < |f(a) \cap \bar{\theta}|$. Moreover $e \subseteq f(a)$ since $e \cap \rho \subseteq a \subseteq f(a)$ and for any $\xi \in e\backslash\rho$, $\xi \in f(b_\xi) \subseteq f(a)$. Thus $e \in [f(a)]^{<|f(a)\cap\bar{\theta}|}$, and consequently $h(e) \subseteq f(a)$ since $f(a) \in C_h^{\kappa,\lambda}$. Hence, $h(e) \subseteq y$. □

Proposition 9.2 *Suppose that $\delta \geq \kappa$, $\bar{\theta} \geq \omega$ and ρ is a cardinal with $\kappa \leq \rho < \lambda$. Suppose further that either $\delta \leq \rho$, or $\bar{\theta}$ is regular. Then $NS_{\kappa,\rho}^{[\min(\delta,\rho)]^{<\theta}} = p(NS_{\kappa,\lambda}^{[\delta]^{<\theta}})$, where $p : [\lambda]^{<\kappa} \to [\rho]^{<\kappa}$ is defined by $p(z) = z \cap \rho$.*

Proof Fix $B \subseteq [\rho]^{<\kappa}$. Let us first assume that $B \in (NS_{\kappa,\rho}^{[\min(\delta,\rho)]^{<\theta}})^*$. Then by Lemma 3.13 there is $k : [\min(\delta,\rho)]^{<\bar{\theta}} \to [\rho]^{<\kappa}$ such that $C_k^{\kappa,\rho} \subseteq B$. Pick $l : [\delta]^{<\bar{\theta}} \to [\rho]^{<\kappa}$ with $k \subseteq l$.

Claim $C_l^{\kappa,\lambda} \subseteq \{z \in [\lambda]^{<\kappa} : z \cap \rho \in C_k^{\kappa,\rho}\}$.

Proof of the claim Fix $z \in C_l^{\kappa,\lambda}$. Then for any $e \in [(z \cap \rho) \cap (\delta \cap \rho)]^{<|(z\cap\rho)\cap\bar{\theta}|}$, we have $e \in [z \cap (\delta \cap \rho)]^{<|z\cap\bar{\theta}|}$, and consequently $k(e) = l(e) \subseteq z \cap \rho$. Hence $z \cap \rho \in C_k^{\kappa,\rho}$, which completes the proof of the claim. □

It follows from the claim that $\{z \in [\lambda]^{<\kappa} : z \cap \rho \in B\} \in (NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^*$.

For the converse, assume that $C \in (NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^*$, where $C = \{z \in [\lambda]^{<\kappa} : z\cap\rho \in B\}$. Then by Lemma 3.13 we may find $h : [\delta]^{<\bar{\theta}} \to [\lambda]^{<\kappa}$ such that $C_h^{\kappa,\lambda} \subseteq C$. Put $D = \{y \cap \rho : y \in C_h^{\kappa,\lambda}\}$. Then by Lemma 9.1, $D \in (NS_{\kappa,\rho}^{[\min(\delta,\rho)]^{<\theta}})^*$. It follows that $B \in (NS_{\kappa,\rho}^{[\min(\delta,\rho)]^{<\theta}})^*$, since clearly $D \subseteq B$. □

10 Dominating numbers

Throughout the section μ will denote a cardinal greater than 0.

The dominating numbers we will consider now are three-dimensional generalizations of the cardinal invariant $\mathfrak{d}_\kappa$. The connection with the notion of $[\delta]^{<\theta}$-normality will be established in the next section.

Definition *We let $\mathfrak{d}_{\kappa,\lambda}^\mu$ denote the smallest cardinality of any $F \subseteq {}^\mu([\lambda]^{<\kappa})$ such that for any $g \in {}^\mu([\lambda]^{<\kappa})$, there is $f \in F$ with $|\{\alpha \in \mu : g(\alpha)\setminus f(\alpha) \neq \emptyset\}| < \mu$.*

Recall from Sect. 2 that $u(\kappa, \lambda)$ denotes the least size of any $A \in I_{\kappa,\lambda}^+$.

Proposition 10.1 $\mathfrak{d}_{\kappa,\lambda}^\mu \geq u(\kappa, \lambda)$.

Proof Given $F \subseteq {}^\mu([\lambda]^{<\kappa})$ with $|F| < u(\kappa, \lambda)$, it is easy to define $g \in {}^\mu([\lambda]^{<\kappa})$ so that $g(\alpha)\setminus f(\alpha) \neq \emptyset$ for all $\alpha \in \mu$ and $f \in F$. □

Corollary 10.2 $\mathfrak{d}_{\kappa,\lambda}^\mu \geq \lambda$.

Proof By Propositions 10.1 and 2.5 (i). □

Proposition 10.3 $cf(\mathfrak{d}_{\kappa,\lambda}^\mu) > \mu$.

Proof We can assume that $\mu \geq \omega$, since otherwise the result is immediate from Corollary 10.2. Suppose toward a contradiction that we may find $F_\gamma \subseteq {}^\mu([\lambda]^{<\kappa})$ for $\gamma < \mu$ such that

- $|F_\gamma| < \mathfrak{d}_{\kappa,\lambda}^\mu$ for all $\gamma < \mu$.
- $F_\gamma \cap F_\xi = \emptyset$ for any two distinct members γ, ξ of μ.
- For each $g \in {}^\mu([\lambda]^{<\kappa})$, there is $f \in \bigcup_{\gamma<\mu} F_\gamma$ with $|\{\alpha \in \mu : g(\alpha)\setminus f(\alpha) \neq \emptyset\}| < \mu$.

Select a bijection $j : \mu \times \mu \to \mu$. For each $\gamma < \mu$, there is $g_\gamma \in {}^\mu([\lambda]^{<\kappa})$ such that $|\{\alpha < \mu : g_\gamma(\alpha)\setminus f(\alpha) \neq \emptyset\}| = \mu$ for every $f \in F_\gamma$. Define $h \in {}^\mu([\lambda]^{<\kappa})$ by: $h(j(\gamma, \alpha)) = g_\gamma(\alpha)$ whenever $(\gamma, \alpha) \in \mu \times \mu$. There must be $\gamma < \mu$ and $f \in F_\gamma$ such that $|\{\alpha \in \mu : g(\alpha)\setminus f(\alpha) \neq \emptyset\}| < \mu$. Then $|\{\alpha \in \mu : h(j(\gamma, \alpha))\setminus f(j(\gamma, \alpha)) \neq \emptyset\}| < \mu$, a contradiction. □

Definition *$F \subseteq {}^{\mu}([\lambda]^{<\kappa})$ is ${}^{\mu}([\lambda]^{<\kappa})$-dominating if for any $g \in {}^{\mu}([\lambda]^{<\kappa})$, there is $f \in F$ such that $g(\alpha) \subseteq f(\alpha)$ for all $\alpha < \mu$.*

Definition *$\delta^{\mu}_{\kappa,\lambda}$ is the least cardinality of any ${}^{\mu}([\lambda]^{<\kappa})$-dominating $F \subseteq {}^{\mu}([\lambda]^{<\kappa})$.*

Proposition 10.4 *Assume $\mu < \kappa$. Then $\delta^{\mu}_{\kappa,\lambda} = u(\kappa, \lambda)$.*

Proof Since clearly $\delta^{\mu}_{\kappa,\lambda} \geq \mathfrak{d}^{\mu}_{\kappa,\lambda}$, we get $\delta^{\mu}_{\kappa,\lambda} \geq u(\kappa, \lambda)$ by Proposition 10.1. For the reverse inequality, observe that given $g \in {}^{\mu}([\lambda]^{<\kappa})$, we have $g(\alpha) \subseteq \bigcup ran(g)$ for all $\alpha < \mu$. □

Proposition 10.5 $\mathfrak{d}^{\mu}_{\kappa,\lambda} = \delta^{\mu}_{\kappa,\lambda}$.

Proof It is immediate that $\delta^{\mu}_{\kappa,\lambda} \geq \mathfrak{d}^{\mu}_{\kappa,\lambda}$. In case $\mu < \kappa$, the reverse inequality follows from Propositions 10.1 and 10.4. Now assume that $\mu \geq \kappa$. Let $F \subseteq {}^{\mu}([\lambda]^{<\kappa})$ be such that for any $g \in {}^{\mu}([\lambda]^{<\kappa})$, there is $f \in F$ with $|\{\alpha \in \mu : g(\alpha)\setminus f(\alpha) \neq \emptyset\}| < \mu$. Select a bijection $j : \mu \times \mu \to \mu$. For $f \in F$ and $\beta < \mu$, define $f_\beta \in {}^{\mu}([\lambda]^{<\kappa})$ by $f_\beta(\xi) = f(j(\beta, \xi))$. Notice that by Proposition 10.3, $|\{f_\beta : \beta < \mu \text{ and } f \in F\}| \leq |F|$. Given $h \in {}^{\mu}([\lambda]^{<\kappa})$, define $g \in {}^{\mu}([\lambda]^{<\kappa})$ by: $g(j(\beta, \xi)) = h(\xi)$ whenever $(\beta, \xi) \in \mu \times \mu$. Pick $f \in F$ with $|\{\alpha \in \mu : g(\alpha)\setminus f(\alpha) \neq \emptyset\}| < \mu$. There exists $\beta < \mu$ such that

$$\{\alpha < \mu : g(\alpha)\setminus f(\alpha) \neq \emptyset\} \cap \{j(\beta, \xi) : \xi < \mu\} = \emptyset.$$

Then

$$h(\xi) = g(j(\beta, \xi)) \subseteq f(j(\beta, \xi)) = f_\beta(\xi)$$

for every $\xi < \mu$. □

Let us consider another variation on the definition of $\mathfrak{d}^{\mu}_{\kappa,\lambda}$.

Definition *$\Delta^{\mu}_{\kappa,\lambda}$ is the least cardinality of any $F \subseteq {}^{\mu}([\lambda]^{<\kappa})$ with the property that for any $g \in {}^{\mu}\lambda$, there is $f \in F$ such that $g(\alpha) \in f(\alpha)$ for all $\alpha \in \mu$.*

Proposition 10.6 *$\Delta^{\mu}_{\kappa,\lambda} \leq \mathfrak{d}^{\mu}_{\kappa,\lambda} \leq \Delta^{\max(\mu,\tau)}_{\kappa,\lambda}$, where $\tau = \kappa$ if κ is a limit cardinal, and $\tau = \nu$ if $\kappa = \nu^+$.*

Proof It is immediate that $\Delta^{\mu}_{\kappa,\lambda} \leq \mathfrak{d}^{\mu}_{\kappa,\lambda}$. Let us prove the other inequality. Select a bijection $j_a : |a| \to a$ for each $a \in [\lambda]^{<\kappa}$. Let $F \subseteq {}^{(\mu\times\tau)}([\lambda]^{<\kappa})$ be such that for any $g \in {}^{(\mu\times\tau)}([\lambda]^{<\kappa})$, there is $f \in F$ with the property that $g(\gamma, \xi) \in f(\gamma, \xi)$ whenever $(\gamma, \xi) \in \mu \times \tau$. For $f \in F$, define $k_f \in {}^{\mu}([\lambda]^{<\kappa})$ by

$$k_f(\gamma) = \bigcup\{f(\gamma, 1 + \xi) : \xi \leq \sup(\kappa \cap f(\gamma, 0))\}.$$

Given $h \in {}^{\mu}([\lambda]^{<\kappa})$, define $g \in {}^{(\mu\times\tau)}\lambda$ as follows:

- $g(\gamma, 0) = |h(\gamma)|$.

- $g(\gamma, 1+\xi) = j_{h(\gamma)}(\xi)$ if $\xi < g(\gamma, 0)$, and $g(\gamma, 1+\xi) = 0$ otherwise.

There is $f \in F$ such that $g(\gamma, \xi) \in f(\gamma, \xi)$ whenever $(\gamma, \xi) \in \mu \times \tau$. We have that $h(\gamma) \subseteq k_f(\gamma)$ for all $\gamma \in \mu$. Hence $\{k_f : f \in F\}$ is ${}^{\mu}([\lambda]^{<\kappa})$-dominating, and so $\mathfrak{d}^{\mu}_{\kappa,\lambda} \leq |F|$. □

We will now see that $\mathfrak{d}^{\mu}_{\kappa,\lambda}$ is easy to compute if λ is large with respect to μ.

Lemma 10.7 (i) *Assume* $\mu < \kappa$. *Then* $\lambda^{<\kappa} = \max(\mathfrak{d}^{\mu}_{\kappa,\lambda}, 2^{<\kappa})$.
(ii) *Assume* $\mu \geq \kappa$. *Then* $\lambda^{\mu} = \max(\mathfrak{d}^{\mu}_{\kappa,\lambda}, 2^{\mu})$.

Proof (i) It is well-known (see e.g. [5]) that $\lambda^{<\kappa} = \max(u(\kappa, \lambda), 2^{<\kappa})$. By Propositions 10.4 and 10.5, the result follows.
(ii) By Proposition 10.5,

$$\lambda^{\mu} = |{}^{\mu}([\lambda]^{<\kappa})| \leq \max(\mathfrak{d}^{\mu}_{\kappa,\lambda}, |{}^{\mu}(2^{<\kappa})|) \leq |{}^{\mu}([\lambda]^{<\kappa})|.$$

□

Proposition 10.8 (i) *Assume that* $\mu < \kappa$ *and* $\lambda \geq 2^{<\kappa}$. *Then* $\mathfrak{d}^{\mu}_{\kappa,\lambda} = \lambda^{<\kappa}$.
(ii) *Assume that* $\mu \geq \kappa$ *and* $\lambda \geq 2^{\mu}$. *Then* $\mathfrak{d}^{\mu}_{\kappa,\lambda} = \lambda^{\mu}$.

Proof By Lemma 10.7 and Corollary 10.2. □

Proposition 10.9 *Assume* GCH. *Then the following hold.*
(i) $\mathfrak{d}^{\mu}_{\kappa,\lambda} = \mu^+$ *if* $\mu \geq \lambda$.
(ii) $\mathfrak{d}^{\mu}_{\kappa,\lambda} = \lambda^+$ *if* $\mu < \lambda$ *and* $\max(\mu^+, \kappa) > \mathrm{cf}(\lambda)$.
(iii) $\mathfrak{d}^{\mu}_{\kappa,\lambda} = \lambda$ *if* $\max(\mu^+, \kappa) \leq \mathrm{cf}(\lambda)$.

Proof (i) : By Lemma 10.7 (ii) and Proposition 10.3.
(ii) and (iii) : By Proposition 10.8. □

Notice that by Corollary 10.2 and Propositions 2.5 (ii), 10.3, 10.4 and 10.5, $\mathfrak{d}^{\mu}_{\kappa,\lambda} \geq \lambda$ and $\mathrm{cf}(\mathfrak{d}^{\mu}_{\kappa,\lambda}) \geq \max(\mu^+, \kappa)$. Thus Proposition 10.9 shows that $\mathfrak{d}^{\mu}_{\kappa,\lambda}$ assumes its least possible value under GCH. Let us now show that κ-c.c. forcing preserves this minimal value in case $\kappa > \omega$.

Proposition 10.10 *Assume* $\kappa > \omega$, *and let* $(P, <)$ *be a* κ*-c.c. notion of forcing. Then* $(\mathfrak{d}^{|\mu|}_{\kappa,\lambda})^{V^P} \leq (\mathfrak{d}^{\mu}_{\kappa,\lambda})^V$.

Proof Let G be P-generic over V. Given an ordinal ξ and $f : \xi \to \lambda$ in $V[G]$, there is by Lemma 6.8 in Chapter VII of [11], $F : \xi \to [\lambda]^{<\kappa}$ in V with the property that $f(\alpha) \in F(\alpha)$ for every $\alpha < \xi$. It immediately follows that $(\Delta^{|\mu|}_{\kappa,\lambda})^{V[G]} \leq (\mathfrak{d}^{\mu}_{\kappa,\lambda})^V$, which by Proposition 10.6 gives $(\mathfrak{d}^{|\mu|}_{\kappa,\lambda})^{V[G]} \leq (\mathfrak{d}^{\mu}_{\kappa,\lambda})^V$ if $\mu \geq \kappa$.

Now assume $\mu < \kappa$. Then by Propositions 10.4 and 10.5, $(\mathfrak{d}^{|\mu|}_{\kappa,\lambda})^{V[G]} = (u(\kappa, \lambda))^{V[G]}$ and $(\mathfrak{d}^{\mu}_{\kappa,\lambda})^V = (u(\kappa, \lambda))^V$. In V, let $A \in I^+_{\kappa,\lambda}$ with $|A| = (u(\kappa, \lambda))^V$. In $V[G]$, let $b \in [\lambda]^{<\kappa}$, and select a bijection $j : |b| \to b$. There is $F : |b| \to [\lambda]^{<\kappa}$ in V such that $j(\alpha) \in F(\alpha)$ for all $\alpha < |b|$. Pick $a \in A$ with $\bigcup ran(F) \subseteq a$. Then $b \subseteq a$. Thus it is still true in $V[G]$ that $A \in I^+_{\kappa,\lambda}$. It follows that $(u(\kappa, \lambda))^{V[G]} \leq (u(\kappa, \lambda))^V$. □

We will present a few identities and inequalities that can be used to evaluate $\mathfrak{d}_{\kappa,\lambda}^{\mu}$ in the absence of GCH. The following is immediate.

Lemma 10.11 *Let τ and ν be two cardinals such that $\tau \geq \lambda$ and $\nu \geq \mu$. Then $\mathfrak{d}_{\kappa,\tau}^{\nu} \geq \mathfrak{d}_{\kappa,\lambda}^{\mu}$.*

Proposition 10.12 *Assume that $\lambda > \kappa$ and* $\mathrm{cf}(\lambda) \geq \max(\kappa, \mu^+)$. *Then* $\mathfrak{d}_{\kappa,\lambda}^{\mu} = \max(\lambda, \sup(\{\mathfrak{d}_{\kappa,\rho}^{\mu} : \kappa \leq \rho < \lambda\}))$.

Proof $\leq$: Use that ${}^{\mu}([\lambda]^{<\kappa}) = \bigcup_{\kappa \leq \alpha < \lambda} {}^{\mu}([\alpha]^{<\kappa})$.
$\geq$: By Corollary 10.2 and Lemma 10.11. □

Definition *We let $\mathfrak{d}_{\kappa}^{\mu}$ denote the least cardinality of any $F \subseteq {}^{\mu}\kappa$ with the property that for any $g \in {}^{\mu}\kappa$, there is $f \in F$ such that $g(\alpha) \leq f(\alpha)$ for all $\alpha < \mu$.*

Note that $\mathfrak{d}_{\kappa}^{\kappa} = \mathfrak{d}_{\kappa}$.

Lemma 10.13 *Assume* $\mathrm{cf}(\lambda) \geq \kappa$. *Then $\Delta_{\kappa,\lambda}^{\mu} \geq \mathfrak{d}_{\kappa}^{\mu}$.*

Proof Let $F \subseteq {}^{\mu}([\lambda]^{<\kappa})$ be such that for any $g \in {}^{\mu}\lambda$, there is $f \in F$ with the property that $g(\alpha) \in f(\alpha)$ for all $\alpha < \mu$. To each $f \in F$, assign the function $\alpha \longmapsto \bigcup f(\alpha)$, and note that these assigned functions witness $\mathfrak{d}_{\kappa}^{\mu}$. □

Proposition 10.14 $\mathfrak{d}_{\kappa,\kappa}^{\mu} = \mathfrak{d}_{\kappa}^{\mu}$.

Proof By Proposition 10.6 and Lemma 10.13, $\mathfrak{d}_{\kappa,\kappa}^{\mu} \geq \mathfrak{d}_{\kappa}^{\mu}$. Now let $F \subseteq {}^{\mu}\kappa$ be such that for any $g \in {}^{\mu}\kappa$, there is $f \in F$ with the property that $g(\alpha) \leq f(\alpha)$ for every $\alpha \in \mu$. Given $h \in {}^{\mu}([\kappa]^{<\kappa})$, select $f \in F$ so that $\sup(h(\alpha) < f(\alpha)$ for all $\alpha \in \mu$. Then $h(\alpha) \subseteq f(\alpha)$ for every $\alpha \in \mu$. Hence $\mathfrak{d}_{\kappa,\kappa}^{\mu} \leq \mathfrak{d}_{\kappa}^{\mu}$. □

The following is very useful.

Proposition 10.15 (i) $\mathfrak{d}_{\kappa,\lambda}^{\mu} \leq \max(\mathfrak{d}_{\kappa,\rho}^{\mu}, \mathfrak{d}_{\rho^+,\lambda}^{\mu}) \leq \mathfrak{d}_{\kappa,\lambda}^{\max(\mu,\rho)}$ *for every cardinal ρ with $\kappa \leq \rho < \lambda$.*
(ii) $\mathfrak{d}_{\kappa,\lambda}^{\mu} \leq \max(\mathfrak{d}_{\kappa,\rho}^{\mu}, \mathfrak{d}_{\rho,\lambda}^{\mu}) \leq \mathfrak{d}_{\kappa,\lambda}^{\max(\mu,\rho)}$ *for every regular cardinal ρ with $\kappa \leq \rho \leq \lambda$.*

Proof Fix a cardinal ρ with $\kappa \leq \rho \leq \lambda$, and let τ be a regular cardinal with $\rho \leq \tau \leq \min(\lambda, \rho^+)$. Select a bijection $j_a : |a| \to a$ for each $a \in [\lambda]^{<\tau}$.

Let us first show that $\mathfrak{d}_{\kappa,\lambda}^{\mu} \leq \max(\mathfrak{d}_{\kappa,\rho}^{\mu}, \mathfrak{d}_{\tau,\lambda}^{\mu})$. Pick a ${}^{\mu}([\rho]^{<\kappa})$-dominating $F \subseteq {}^{\mu}([\rho]^{<\kappa})$ and a ${}^{\mu}([\lambda]^{<\tau})$-dominating $G \subseteq {}^{\mu}([\lambda]^{<\tau})$. Define $\psi : F \times G \to {}^{\mu}([\lambda]^{<\kappa})$ by $(\psi(f, g))(\alpha) = j_{g(\alpha)}``(f(\alpha) \cap |g(\alpha)|)$. We claim that $ran(\psi)$ is ${}^{\mu}([\lambda]^{<\kappa})$-dominating. Let $r \in {}^{\mu}([\lambda]^{<\kappa})$. Pick $g \in G$ so that $r(\alpha) \subseteq g(\alpha)$ for all $\alpha < \mu$. Then select $f \in F$ so that $j_{g(\alpha)}^{-1}(r(\alpha)) \subseteq f(\alpha)$ for every $\alpha < \mu$. Then $r(\alpha) \subseteq (\psi(f, g))(\alpha)$ for all $\alpha < \mu$, which proves our claim.

Let us next show that $\max(\mathfrak{d}_{\kappa,\rho}^{\mu}, \mathfrak{d}_{\tau,\lambda}^{\mu}) \leq \mathfrak{d}_{\kappa,\lambda}^{\max(\mu,\rho)}$. By Lemma 10.11, $\mathfrak{d}_{\kappa,\rho}^{\mu} \leq \mathfrak{d}_{\kappa,\lambda}^{\max(\mu,\rho)}$. Now let $H \subseteq {}^{(\mu\times\rho)}([\lambda]^{<\kappa})$ be such that for any $p \in {}^{(\mu\times\rho)}([\lambda]^{<\kappa})$, there is $h \in H$ with the property that $p(\alpha, \beta) \subseteq h(\alpha, \beta)$ whenever $(\alpha, \beta) \in \mu \times \rho$.

Given $q \in {}^{\mu}([\lambda]^{<\tau})$, select $h \in H$ so that $\{j_{q(\alpha)}(\beta)\} \subseteq h(\alpha, \beta)$ whenever $\alpha \in \mu$ and $\beta \in |q(\alpha)|$. If $\tau = \rho^+$, then $q(\alpha) \subseteq \bigcup_{\beta \in \rho} h(\alpha, \beta)$, and we can conclude that $\mathfrak{d}^{\mu}_{\tau,\lambda} \leq \mathfrak{d}^{\max(\mu,\rho)}_{\kappa,\lambda}$. Now assume $\tau = \rho$, and let $K \subseteq {}^{\mu}\tau$ be such that for any $i \in {}^{\mu}\tau$, there is $k \in K$ with the property that $i(\alpha) \leq k(\alpha)$ for all $\alpha < \mu$. Then there is $k \in K$ such that $|q(\alpha)| \leq k(\alpha)$ for every $\alpha < \mu$. We have $q(\alpha) \subseteq \bigcup_{\beta \in k(\alpha)} h(\alpha, \beta)$ for all $\alpha < \mu$. Thus $\mathfrak{d}^{\mu}_{\tau,\lambda} \leq \max(\mathfrak{d}^{\max(\mu,\rho)}_{\kappa,\lambda}, \mathfrak{d}^{\mu}_{\tau})$, which gives $\mathfrak{d}^{\mu}_{\tau,\lambda} \leq \mathfrak{d}^{\max(\mu,\rho)}_{\kappa,\lambda}$, since by Lemmas 10.11 and 10.3 and Proposition 10.6, $\mathfrak{d}^{\mu}_{\tau} \leq \mathfrak{d}^{\mu}_{\kappa,\tau} \leq \mathfrak{d}^{\max(\mu,\rho)}_{\kappa,\lambda}$. □

Corollary 10.16 (i) $\mathfrak{d}^{\mu}_{\kappa,\lambda} = \max(\mathfrak{d}^{\mu}_{\kappa,\rho}, \mathfrak{d}^{\mu}_{\rho^+,\lambda})$ *for every cardinal* ρ *with* $\kappa \leq \rho \leq \min(\mu, \lambda)$.

(ii) $\mathfrak{d}^{\mu}_{\kappa,\lambda} = \max(\mathfrak{d}^{\mu}_{\kappa,\rho}, \mathfrak{d}^{\mu}_{\rho,\lambda})$ *for every regular cardinal* ρ *with* $\kappa \leq \rho \leq \min(\mu, \lambda)$.

Corollary 10.17 *Suppose* $\mu > \omega$, *and let* χ *be an uncountable cardinal. Then there is a regular infinite cardinal* $\sigma < \min(\mu, \chi)$ *such that* $\mathfrak{d}^{\mu}_{\rho,\chi} = \mathfrak{d}^{\mu}_{\sigma,\chi}$ *for every regular cardinal* ρ *with* $\sigma < \rho < \min(\mu, \chi)$.

Proof Suppose otherwise. Then, using Corollary 10.16, we may define an increasing sequence $\langle \sigma_n : n < \omega \rangle$ of regular cardinals less than $\min(\mu, \chi)$ such that

- $\sigma_0 = \omega$.
- $\mathfrak{d}^{\mu}_{\sigma_n,\chi} > \mathfrak{d}^{\mu}_{\sigma_{n+1},\chi}$.

Contradiction. □

Corollary 10.18 *Suppose that* $u(\kappa, \lambda) = \lambda$. *Then* $\mathfrak{d}^{\mu}_{\sigma,\lambda} = \max(\mathfrak{d}^{\mu}_{\sigma,\kappa}, \mathfrak{d}^{\mu}_{\kappa,\lambda})$ *for every regular infinite cardinal* $\sigma \leq \kappa$.

Proof Let $\sigma \leq \kappa$ be a regular infinite cardinal. If $\kappa \leq \mu$, then by Corollary 10.16 (ii), $\mathfrak{d}^{\mu}_{\sigma,\lambda} = \max(\mathfrak{d}^{\mu}_{\sigma,\kappa}, \mathfrak{d}^{\mu}_{\kappa,\lambda})$. Let us now assume that $\kappa < \mu$. Then by Lemma 10.11, $\mathfrak{d}^{\mu}_{\sigma,\lambda} \geq \mathfrak{d}^{\mu}_{\sigma,\kappa}$. Moreover, by Corollary 10.2 and Proposition 10.4, $\mathfrak{d}^{\mu}_{\sigma,\lambda} = \mathfrak{d}^{\mu}_{\sigma,u(\kappa,\lambda)} \geq u(\kappa, \lambda) = \mathfrak{d}^{\mu}_{\kappa,\lambda}$. Hence by Proposition 10.15 (ii), $\mathfrak{d}^{\mu}_{\sigma,\lambda} = \max(\mathfrak{d}^{\mu}_{\sigma,\kappa}, \mathfrak{d}^{\mu}_{\kappa,\lambda})$. □

Corollary 10.19 *Assume* $\kappa \leq \mu < \lambda$. *Then* $\mathfrak{d}^{\mu}_{\kappa,\lambda} = \max(\mathfrak{d}^{\mu}_{\kappa,\mu}, u(\mu^+, \lambda))$.

Proof By Propositions 10.4, 10.5 and 10.15 (i). □

Proposition 10.4 and Corollary 10.19 show that for $\mu \leq \lambda$, the value of $\mathfrak{d}^{\mu}_{\kappa,\lambda}$ is determined by the values taken by $\mathfrak{d}^{\tau}_{\kappa,\tau}$ and $u(\tau, \lambda)$ when τ ranges from κ to λ.

Let us next consider the relationship between $\mathfrak{d}^{\mu}_{\kappa,\lambda}$ and $\mathfrak{d}^{\mu}_{\kappa,\lambda^+}$.

Proposition 10.20 (i) $\mathfrak{d}^{\mu}_{\kappa,\lambda^{+n}} = \max(\mathfrak{d}^{\mu}_{\kappa,\lambda}, \prod_{i=1}^{n} \mathfrak{d}^{\mu}_{\lambda^{+i}})$ *for every* $n \in \omega \backslash \{0\}$.

(ii) *Assume* $\mu \leq \lambda$. *Then* $\mathfrak{d}^{\mu}_{\kappa,\lambda^{+n}} = \max(\mathfrak{d}^{\mu}_{\kappa,\lambda}, \lambda^{+n})$ *for every* $n \in \omega$.

Proof (i) By Propositions 10.14 and 10.15 (i), $\mathfrak{d}^{\mu}_{\kappa,\lambda^+} \leq \max(\mathfrak{d}^{\mu}_{\kappa,\lambda}, \mathfrak{d}^{\mu}_{\lambda^+,\lambda^+}) \leq \max(\mathfrak{d}^{\mu}_{\kappa,\lambda}, \mathfrak{d}^{\mu}_{\lambda^+})$. Moreover, $\mathfrak{d}^{\mu}_{\kappa,\lambda} \leq \mathfrak{d}^{\mu}_{\kappa,\lambda^+}$ by Lemma 10.11, and $\mathfrak{d}^{\mu}_{\lambda^+} \leq \Delta^{\mu}_{\kappa,\lambda^+} \leq \mathfrak{d}^{\mu}_{\kappa,\lambda^+}$ by Lemma 10.13 and Proposition 10.6. It follows that $\mathfrak{d}^{\mu}_{\kappa,\lambda^+} = \max(\mathfrak{d}^{\mu}_{\kappa,\lambda}, \mathfrak{d}^{\mu}_{\lambda^+})$. The desired result is then obtained by induction.

(ii) The result follows from (i) and Propositions 10.4, 10.5 and 10.14 if $n > 0$, and from Corollary 10.2 otherwise. □

Corollary 10.21 (i) $\mathfrak{d}^{\mu}_{\kappa,\kappa^{+n}} = \prod_{i=0}^{n} \mathfrak{d}^{\mu}_{\kappa^{+i}}$ *for every* $n \in \omega$.
(ii) $\mathfrak{d}^{\kappa}_{\kappa,\kappa^{+n}} = \max(\mathfrak{d}_{\kappa}, \kappa^{+n})$ *for every* $n \in \omega$.
(iii) $\mathfrak{d}^{\lambda}_{\kappa,\lambda^{+}} = \mathfrak{d}^{\lambda}_{\kappa,\lambda}$.

Proof (i) By Propositions 10.20 (i) and 10.14.
(ii) By Propositions 10.20 (ii) and 10.14.
(iii) By Propositions 10.20 (ii) and 10.3.

□

Let us now deal with the computation of $\mathfrak{d}^{\mu}_{\kappa,\lambda^{<\eta}}$.

Proposition 10.22 (i) $\mathfrak{d}^{\mu}_{\kappa,\lambda^{<\eta}} = \mathfrak{d}^{\mu}_{\kappa,\max(\lambda,\kappa^{<\eta})}$ *for every cardinal* η *with* $\omega < \eta < \kappa$.
(ii) $\mathfrak{d}^{\mu}_{\kappa,\lambda^{<\eta}} = \max(\mathfrak{d}^{\mu}_{\kappa,2^{<\eta}}, \mathfrak{d}^{\mu}_{\eta,\lambda})$ *for every regular cardinal* η *with* $\kappa \leq \eta \leq \lambda$.
(iii) $\mathfrak{d}^{\mu}_{\kappa,\lambda^{<\eta}} = \max(\mathfrak{d}^{\mu}_{\kappa,2^{<\eta}}, \mathfrak{d}^{\mu}_{\eta^{+},\lambda})$ *for every regular cardinal* η *such that* $\kappa \leq \eta < \lambda$ *and either* $\eta \leq \mu$, *or* $\eta^{+} = \lambda$.
(iv) $\mathfrak{d}^{\mu}_{\kappa,\lambda^{<\eta}} = \max(\mathfrak{d}^{\mu}_{\kappa,\eta^{<\eta}}, \mathfrak{d}^{\mu}_{\eta^{+},\lambda})$ *for every cardinal* η *such that* $\mathrm{cf}(\eta) < \kappa < \eta < \lambda$ *and either* $\eta \leq \mu$, *or* $\eta^{+} = \lambda$.

Proof (i), (ii) and (iv) : Let $\eta \leq \lambda$ be an uncountable cardinal. We assume that $\eta \neq \lambda$ in case η is singular. We define ρ and τ by:

- If $\eta < \kappa$, then $\rho = \kappa$ and $\tau = \kappa^{<\eta}$.
- If $\kappa \leq \eta = \mathrm{cf}(\eta)$, then $\rho = \eta$ and $\tau = 2^{<\eta}$.
- If $\mathrm{cf}(\eta) < \kappa < \eta$, then $\rho = \eta^{+}$ and $\tau = \eta^{<\eta}$.

Let $F \subseteq {}^{\mu}([\lambda]^{<\rho})$ be ${}^{\mu}([\lambda]^{<\rho})$-dominating, and $K \subseteq {}^{\mu}([\tau]^{<\kappa})$ be ${}^{\mu}([\tau]^{<\kappa})$-dominating. Fix a bijection $j : \lambda^{<\eta} \to [\lambda]^{<\eta}$. For $f \in F$ and $\alpha \in \mu$, select a one-to-one $i_{f,\alpha} : j^{-1}([f(\alpha)]^{<\eta}) \to \tau$. Given $h \in {}^{\mu}([\lambda^{<\eta}]^{<\kappa})$, pick $f \in F$ so that $\bigcup j``(h(\alpha)) \subseteq f(\alpha)$ for every $\alpha \in \mu$. Then pick $k \in K$ so that $i_{f,\alpha}``(h(\alpha)) \subseteq k(\alpha)$ for each $\alpha \in \mu$. Then $h(\alpha) \subseteq i^{-1}_{f,\alpha}(k(\alpha))$ for all $\alpha \in \mu$. Hence $\mathfrak{d}^{\mu}_{\kappa,\lambda^{<\eta}} \leq \max(\mathfrak{d}^{\mu}_{\kappa,\tau}, \mathfrak{d}^{\mu}_{\rho,\lambda})$.

Since $\tau \leq \lambda^{<\eta}$, we have $\mathfrak{d}^{\mu}_{\kappa,\lambda^{<\eta}} \geq \mathfrak{d}^{\mu}_{\kappa,\tau}$ by Lemma 10.11. If $\rho = \kappa$, then by Lemma 10.11, $\mathfrak{d}^{\mu}_{\kappa,\lambda^{<\eta}} \geq \mathfrak{d}^{\mu}_{\rho,\lambda}$. If $\rho = \lambda$, then by Lemmas 10.11 and 10.13 and Propositions 10.6 and 10.14, $\mathfrak{d}^{\mu}_{\kappa,\lambda^{<\eta}} \geq \mathfrak{d}^{\mu}_{\kappa,\lambda} \geq \mathfrak{d}^{\mu}_{\lambda} = \mathfrak{d}^{\mu}_{\rho,\lambda}$. If $\kappa < \rho < \min(\lambda, \mu^{+})$, then by Proposition 10.15 (ii) and Lemma 10.11, $\mathfrak{d}^{\mu}_{\kappa,\lambda^{<\eta}} = \mathfrak{d}^{\max(\mu,\rho)}_{\kappa,\lambda^{<\eta}} \geq \mathfrak{d}^{\mu}_{\rho,\lambda^{<\eta}} \geq \mathfrak{d}^{\mu}_{\rho,\lambda}$. Finally if $\eta = \rho > \mu$, then by Corollary 10.2 and Propositions 10.4 and 10.5, $\mathfrak{d}^{\mu}_{\kappa,\lambda^{<\eta}} \geq \lambda^{<\eta} \geq u(\rho,\lambda) = \mathfrak{d}^{\mu}_{\rho,\lambda}$.

(iii) : Let η be a regular cardinal with $\kappa \leq \eta < \lambda$. Let us first assume that $\eta \leq \mu$. Then by (ii) and Corollary 10.16 (i),

$$\mathfrak{d}^{\mu}_{\kappa,\lambda^{<\eta}} = \max\left(\mathfrak{d}^{\mu}_{\kappa,2^{<\eta}}, \mathfrak{d}^{\mu}_{\eta,\lambda}\right) = \max\left(\mathfrak{d}^{\mu}_{\kappa,2^{<\eta}}, \mathfrak{d}^{\mu}_{\eta,\eta}, \mathfrak{d}^{\mu}_{\eta^{+},\lambda}\right).$$

It follows that $\mathfrak{d}^{\mu}_{\kappa,\lambda^{<\eta}} = \max(\mathfrak{d}^{\mu}_{\kappa,2^{<\eta}}, \mathfrak{d}^{\mu}_{\eta^{+},\lambda})$, since by Lemmas 10.11 and 10.13 and Propositions 10.6 and 10.14,

$$\mathfrak{d}^{\mu}_{\kappa,2^{<\eta}} \geq \mathfrak{d}^{\mu}_{\kappa,\eta} \geq \Delta^{\mu}_{\kappa,\eta} \geq \mathfrak{d}^{\mu}_{\eta} = \mathfrak{d}^{\mu}_{\eta,\eta}.$$

Now assume that $\eta > \mu$ and $\eta^+ = \lambda$. Since $(\eta^+)^{<\eta} = \max(\eta^{<\eta}, \eta^+)$ and $\eta^{<\eta} = 2^{<\eta}$, we have by Lemma 10.11 and Proposition 10.20 (ii) that $\mathfrak{d}^{\mu}_{\kappa,\lambda^{<\eta}} = \mathfrak{d}^{\mu}_{\kappa,\max(2^{<\eta},\eta^+)} = \max(\mathfrak{d}^{\mu}_{\kappa,2^{<\eta}}, \mathfrak{d}^{\mu}_{\kappa,\eta^+}) = \max(\mathfrak{d}^{\mu}_{\kappa,2^{<\eta}}, \mathfrak{d}^{\mu}_{\kappa,\eta}, \eta^+) = \max(\mathfrak{d}^{\mu}_{\kappa,2^{<\eta}}, \eta^+)$. It remains to observe that by Propositions 10.4 and 10.5, $\eta^+ = \mathfrak{d}^{\mu}_{\eta^+,\lambda}$. □

Let us make the following remark concerning Proposition 10.22 (iii). Suppose that GCH holds and $\max(\kappa, \mu^+) \leq \mathrm{cf}(\lambda) < \lambda$. Set $\eta = \mathrm{cf}(\lambda)$. Then $\mathfrak{d}^{\mu}_{\kappa,\lambda^{<\eta}} \neq \max(\mathfrak{d}^{\mu}_{\kappa,2^{<\eta}}, \mathfrak{d}^{\mu}_{\eta^+,\lambda})$, since by Proposition 10.9, $\mathfrak{d}^{\mu}_{\kappa,\lambda^{<\eta}} = \lambda$ and $\mathfrak{d}^{\mu}_{\eta^+,\lambda} = \lambda^+$.

Corollary 10.23 *Let $n \in \omega$. Suppose that either $n \neq 0$, or $\mu \geq \omega$. Then for any cardinal $\sigma \geq \omega_n$, $\mathfrak{d}^{\mu}_{\omega_n,\sigma^{\aleph_0}} = \mathfrak{d}^{\mu}_{\omega_n,\max(\sigma,2^{\aleph_0})}$.*

Proof Fix a cardinal $\sigma \geq \omega_n$. The desired equality follows from Proposition 10.22 (i) if $n \geq 2$, and from Proposition 10.22 (ii) if $n = 1$. Let us now assume that $n = 0$. If $\sigma = \omega$, the result is obvious. Otherwise, by Propositions 10.22 (ii) and 10.16 (i) and Lemma 10.11, $\mathfrak{d}^{\mu}_{\omega,\sigma^{\aleph_0}} = \max(\mathfrak{d}^{\mu}_{\omega,2^{\aleph_0}}, \mathfrak{d}^{\mu}_{\omega_1,\sigma}) = \max(\mathfrak{d}^{\mu}_{\omega,2^{\aleph_0}}, \mathfrak{d}^{\mu}_{\omega,\omega}, \mathfrak{d}^{\mu}_{\omega_1,\sigma}) = \max(\mathfrak{d}^{\mu}_{\omega,2^{\aleph_0}}, \mathfrak{d}^{\mu}_{\omega,\sigma}) = \mathfrak{d}^{\mu}_{\omega,\max(\sigma,2^{\aleph_0})}$. □

Notice that if $n = 0$ and $\mu < \omega$, then by Propositions 10.4, 10.5 and 2.5 (i), $\mathfrak{d}^{\mu}_{\omega_n,\sigma^{\aleph_0}} = \sigma^{\aleph_0}$ and $\mathfrak{d}^{\mu}_{\omega_n,\max(\sigma,2^{\aleph_0})} = \max(\sigma, 2^{\aleph_0})$, and so $\mathfrak{d}^{\mu}_{\omega_n,\sigma^{\aleph_0}}$ and $\mathfrak{d}^{\mu}_{\omega_n,\max(\sigma,2^{\aleph_0})}$ are not necessarily equal.

Corollary 10.24 *If $\lambda \geq 2^{<\kappa}$, then $\mathfrak{d}^{\mu}_{\kappa,\lambda^{<\kappa}} = \mathfrak{d}^{\mu}_{\kappa,\lambda}$.*

Proof By Proposition 10.22 (ii) and Lemma 10.11. □

Corollary 10.25 *Let σ be an infinite cardinal such that $\mathrm{cf}(\sigma) < \kappa$ and $\kappa^{\mathrm{cf}(\sigma)} < \sigma < \lambda \leq \sigma^{\mathrm{cf}(\sigma)}$. Then $\mathfrak{d}^{\mu}_{\kappa,\lambda} = \mathfrak{d}^{\mu}_{\kappa,\sigma}$.*

Proof If $(\mathrm{cf}(\sigma))^+ < \kappa$, then by Lemma 10.11 and Proposition 10.22 (i),

$$\mathfrak{d}^{\mu}_{\kappa,\lambda} \leq \mathfrak{d}^{\mu}_{\kappa,\sigma^{\mathrm{cf}(\sigma)}} = \mathfrak{d}^{\mu}_{\kappa,\max(\sigma,\kappa^{\mathrm{cf}(\sigma)})} = \mathfrak{d}^{\mu}_{\kappa,\sigma} \leq \mathfrak{d}^{\mu}_{\kappa,\lambda}.$$

If $(\mathrm{cf}(\sigma))^+ = \kappa$, then by Lemma 10.11 and Proposition 10.22 (ii),

$$\mathfrak{d}^{\mu}_{\kappa,\lambda} \leq \mathfrak{d}^{\mu}_{\kappa,\sigma^{\mathrm{cf}(\sigma)}} = \max(\mathfrak{d}^{\mu}_{\kappa,2^{\mathrm{cf}(\sigma)}}, \mathfrak{d}^{\mu}_{\kappa,\sigma}) = \mathfrak{d}^{\mu}_{\kappa,\sigma} \leq \mathfrak{d}^{\mu}_{\kappa,\lambda}.$$

□

We conclude the section with a look at $\mathfrak{d}^{\mu^{<\rho}}_{\kappa,\lambda}$.

Proposition 10.26 (i) *Let $\rho \leq \mu$ be an infinite cardinal. Then $\mathfrak{d}^{\mu^{<\rho}}_{\kappa,\lambda}$ is the least cardinality of any $F \subseteq {}^{([\mu]^{<\rho})}([\lambda]^{<\kappa})$ with the property that for any $g \in {}^{([\mu]^{<\rho})}([\lambda]^{<\kappa})$, there is $f \in F$ with $\{d \in [\mu]^{<\rho} : g(d) \subseteq f(d)\} \in I^*_{\rho,\mu}$.*

(ii) *Let $\rho \leq \mu$ be an infinite cardinal such that $2^{\tau} < \kappa$ for every cardinal $\tau < \rho$. Then $\mathfrak{d}^{\mu^{<\rho}}_{\kappa,\lambda} = \mathfrak{d}^{u(\rho,\mu)}_{\kappa,\lambda}$.*

Proof (i) Let $F \subseteq {}^{([\mu]^{<\rho})}([\lambda]^{<\kappa})$ be such that for any $g \in {}^{([\mu]^{<\rho})}([\lambda]^{<\kappa})$, there is $f \in F$ with the property that $\{d \in [\mu]^{<\rho} : g(d) \subseteq f(d)\} \in I^*_{\rho,\mu}$. By Corollary 2.7, we may find $A_e \in P(\widehat{e}) \cap I^+_{\rho,\mu}$ for $e \in [\mu]^{<\rho}$ such that

- $|A_e| = \mu^{<\rho}$ for all $e \in [\mu]^{<\rho}$.
- $A_e \cap A_{e'} = \emptyset$ whenever e, e' are two distinct members of $[\mu]^{<\rho}$.
- $\bigcup_{e\in[\mu]^{<\rho}} A_e = [\mu]^{<\rho}$.

Select a bijection $j_e : A_e \to [\mu]^{<\rho}$ for each $e \in [\mu]^{<\rho}$. Given $h \in {}^{([\mu]^{<\rho})}([\lambda]^{<\kappa})$, define $g \in {}^{([\mu]^{<\rho})}([\lambda]^{<\kappa})$ so that $g(d) = h(j_e(d))$ whenever $d \in A_e$. Pick $f \in F$ and $e \in [\mu]^{<\rho}$ so that $\widehat{e} \subseteq \{d \in [\mu]^{<\rho} : g(d) \subseteq f(d)\}$. Then $h(j_e(d)) \subseteq (f \circ j_e^{-1})(j_e(d))$ for all $d \in A_e$. Thus $\mathfrak{d}^{\mu^{<\rho}}_{\kappa,\lambda} \leq \max(|F|, \mu^{<\rho})$, and therefore by Proposition 10.3, $\mathfrak{d}^{\mu^{<\rho}}_{\kappa,\lambda} \leq |F|$.

(ii) By Lemma 10.11, $\mathfrak{d}^{u(\rho,\mu)}_{\kappa,\lambda} \leq \mathfrak{d}^{\mu^{<\rho}}_{\kappa,\lambda}$. For the reverse inequality, fix $A \in I^+_{\rho,\mu}$ with $|A| = u(\rho, \mu)$, and $F \subseteq {}^{A}([\lambda]^{<\kappa})$ with the property that for any $g \in {}^{A}([\lambda]^{<\kappa})$, there is $f \in F$ such that $g(a) \subseteq f(a)$ for all $a \in A$. For $f \in F$, define $f' \in {}^{([\mu]^{<\rho})}([\lambda]^{<\kappa})$ as follows. Given $b \in [\mu]^{<\rho}$, pick $a \in A$ with $b \subseteq a$, and set $f'(b) = f(a)$. Now given $h \in {}^{([\mu]^{<\rho})}([\lambda]^{<\kappa})$, define $g \in {}^{A}([\lambda]^{<\kappa})$ by $g(a) = \bigcup_{b\subseteq a} h(b)$. Select $f \in F$ so that $g(a) \subseteq f(a)$ for all $a \in A$. Then $h(b) \subseteq f'(b)$ for all $b \in [\mu]^{<\rho}$. □

11 Cofinality of *J*

This section is devoted to the computation of $\mathrm{cof}(NS^{[\delta]^{<\theta}}_{\kappa,\lambda})$.

Lemma 11.1 *Assume* $\nabla^{[\delta]^{<\theta}} I_{\kappa,\lambda} \subseteq J$. *Then* $\mathrm{cof}(J) \geq \mathfrak{d}^{|[\delta]^{<\bar{\theta}}|}_{\kappa,\lambda}$.

Proof Fix $S \subseteq J$ with $J = \bigcup_{B\in S} P(B)$. For $B \in S$, define $h_B : [\delta]^{<\bar{\theta}} \to [\lambda]^{<\kappa}\setminus B$ so that $e \in [h_B(e)]^{<|\bar{\theta}\cap h_B(e)|}$ for all $e \in [\delta]^{<\bar{\theta}}$. Given $g : [\delta]^{<\bar{\theta}} \to [\lambda]^{<\kappa}$, there is by Proposition 3.3 (i) and Corollary 3.8 ((iv) → (ii)) $B \in S$ with $[\lambda]^{<\kappa}\setminus B \subseteq \Delta_{e\in[\delta]^{<\bar{\theta}}} \widehat{g(e)}$. Then $g(e) \subseteq h_B(e)$ for every $e \in [\delta]^{<\bar{\theta}}$. □

Proposition 11.2 $\mathrm{cof}(NS^{[\delta]^{\theta}}_{\kappa,\lambda}|A) = \mathfrak{d}^{|[\delta]^{<\bar{\theta}}|}_{\kappa,\lambda}$ *for each* $A \in (NS^{[\delta]^{<\theta}}_{\kappa,\lambda})^+$.

Proof Let us first observe that if $f : [\delta]^{<\max(3,\bar{\theta})} \to [\lambda]^{<\kappa}$ and $g : [\delta]^{<\max(3,\bar{\theta})} \to [\lambda]^{<\kappa}$ are such that $f(e) \subseteq g(e)$ for all $e \in [\delta]^{<\max(3,\bar{\theta})}$, then $C^{\kappa,\lambda}_g \subseteq C^{\kappa,\lambda}_f$. Hence $\mathrm{cof}(NS^{[\delta]^{<\theta}}_{\kappa,\lambda}) \leq \mathfrak{d}^{|[\delta]^{<\bar{\theta}}|}_{\kappa,\lambda}$ by Lemma 3.13. So given $A \in (NS^{[\delta]^{<\theta}}_{\kappa,\lambda})^+$, we have $\mathrm{cof}(NS^{[\delta]^{<\theta}}_{\kappa,\lambda}|A) \leq \mathfrak{d}^{|[\delta]^{<\bar{\theta}}|}_{\kappa,\lambda}$ by Fact 2.2. The reverse inequality holds by Lemma 11.1 since $NS^{[\delta]^{<\theta}}_{\kappa,\lambda}|A$ is $[\delta]^{<\theta}$-normal. □

The following is well-known.

Fact 11.3 $\mathrm{cof}(I_{\kappa,\lambda}|A) = u(\kappa, \lambda)$ *for each* $A \in I^+_{\kappa,\lambda}$.

Proof By Propositions 4.5, 3.11 and 3.19 (i), $I_{\kappa,\lambda} = NS_{\kappa,\lambda}^{[1]^{<2}}$, so the result follows from Propositions 11.2 and 10.3. □

It follows from Proposition 4.5 and Fact 11.3 that if $\delta < \kappa$, then $\mathrm{cof}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A) = u(\kappa,\lambda)$ for all $A \in (NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^+$. For $\delta \geq \kappa$ we have the following.

Proposition 11.4 *Assume $\delta \geq \kappa$. Then $\mathrm{cof}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A) = \mathfrak{d}_{\kappa,\lambda}^{u(\max(\omega,\bar{\theta}),|\delta|)}$ for every $A \in (NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^+$.*

Proof By Propositions 11.2, 10.26 (ii) and 3.19. □

Under GCH we obtain the following values.

Proposition 11.5 *Assume that the* GCH *holds and $\delta \geq \kappa$, and let $A \in (NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^+$. Then the following hold.*

(i) $\mathrm{cof}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A) = \lambda^{++}$ *if* $\delta = \lambda$ *and* $\mathrm{cf}(\lambda) < \bar{\theta}$.
(ii) $\mathrm{cof}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A) = \lambda^{+}$ *if* $\mathrm{cf}(\lambda) \leq |\delta|^{<\bar{\theta}} < \lambda$, *or* $\lambda = |\delta|^{<\bar{\theta}}$ *and* $\mathrm{cf}(\lambda) \geq \bar{\theta}$.
(iii) $\mathrm{cof}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A) = \lambda$ *if* $|\delta|^{<\bar{\theta}} < \mathrm{cf}(\lambda)$.

Proof By Propositions 10.9 and 11.2. □

Proposition 11.6 *Let δ' be an ordinal with $\kappa \leq \delta' \leq \lambda$, and θ' be a cardinal with $2 \leq \theta' \leq \kappa$. Suppose that either $\lambda^{|\delta'|^{<\bar{\theta}'}} = |\delta|^{<\bar{\theta}}$, or $\lambda^{|\delta'|^{<\bar{\theta}'}} = \lambda$ and $\mathrm{cf}(\lambda) \leq |\delta|^{<\bar{\theta}}$. Then there is no $A \in (NS_{\kappa,\lambda}^{[\delta]^{<\theta}})^+ \cap (NS_{\kappa,\lambda}^{[\delta']^{<\theta'}})^+$ such that $NS_{\kappa,\lambda}^{[\delta]^{<\theta}}|A = NS_{\kappa,\lambda}^{[\delta']^{<\theta'}}|A$.*

Proof If $\lambda^{|\delta'|^{<\bar{\theta}'}} = |\delta|^{<\bar{\theta}}$, then by Proposition 10.3, $\mathfrak{d}_{\kappa,\lambda}^{|\delta'|^{<\bar{\theta}'}} \leq \lambda^{|\delta'|^{<\bar{\theta}'}} = |\delta|^{<\bar{\theta}} < \mathfrak{d}_{\kappa,\lambda}^{|\delta|^{<\bar{\theta}}}$. If $\lambda^{|\delta'|^{<\bar{\theta}'}} = \lambda$ and $\mathrm{cf}(\lambda) \leq |\delta|^{<\bar{\theta}}$, then by Propositions 10.1 and 10.3, $\mathfrak{d}_{\kappa,\lambda}^{|\delta'|^{<\bar{\theta}'}} \leq \lambda^{|\delta'|^{<\bar{\theta}'}} = \lambda < \mathfrak{d}_{\kappa,\lambda}^{|\delta|^{<\bar{\theta}}}$. The result now follows from Proposition 11.2. □

Proposition 11.7 *Assume $\delta \geq \kappa$. Then*

$$\mathrm{cof}(NS_{\kappa,\lambda}^{[\delta]^{<\theta}}) = \max(\mathrm{cof}(NS_{\kappa,|\delta|}^{[|\delta|]^{<\theta}}), \mathrm{cov}(\lambda, (|\delta|^{<\bar{\theta}})^+, (|\delta|^{<\bar{\theta}})^+, 2)).$$

Proof If $\bar{\theta} < \kappa$, then by Propositions 10.22 (i) and 3.24, $\mathfrak{d}_{\kappa,|\delta|^{<\bar{\theta}}}^{|\delta|^{<\bar{\theta}}} = \mathfrak{d}_{\kappa,\max(|\delta|,\kappa^{<\bar{\theta}})}^{|\delta|^{<\bar{\theta}}} = \mathfrak{d}_{\kappa,|\delta|}^{|\delta|^{<\bar{\theta}}}$. It $\bar{\theta} = \kappa$, then by Lemma 10.11 and Propositions 10.22 (ii) and 3.19 (ii), $\mathfrak{d}_{\kappa,|\delta|^{<\bar{\theta}}}^{|\delta|^{<\bar{\theta}}} = \max(\mathfrak{d}_{\kappa,2^{<\kappa}}^{|\delta|^{<\bar{\theta}}}, \mathfrak{d}_{\kappa,|\delta|}^{|\delta|^{<\bar{\theta}}}) = \mathfrak{d}_{\kappa,|\delta|}^{|\delta|^{<\bar{\theta}}}$. Thus in each case, $\mathfrak{d}_{\kappa,|\delta|^{<\bar{\theta}}}^{|\delta|^{<\bar{\theta}}} = \mathfrak{d}_{\kappa,|\delta|}^{|\delta|^{<\bar{\theta}}}$. Hence if $|\delta|^{<\bar{\theta}} < \lambda$, we may infer from Corollary 10.19 that

$$\mathfrak{d}_{\kappa,\lambda}^{|\delta|^{<\bar{\theta}}} = \max(\mathfrak{d}_{\kappa,|\delta|}^{|\delta|^{<\bar{\theta}}}, u((|\delta|^{<\bar{\theta}})^+, \lambda)) = \max(\mathfrak{d}_{\kappa,|\delta|}^{|\delta|^{<\bar{\theta}}}, \mathrm{cov}(\lambda, (|\delta|^{<\bar{\theta}})^+, (|\delta|^{<\bar{\theta}})^+, 2)).$$

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If $|\delta|^{<\bar{\theta}} \geq \lambda$, Lemma 10.11 tells us that $\mathfrak{d}_{\kappa,|\delta|}^{|\delta|^{<\bar{\theta}}} \leq \mathfrak{d}_{\kappa,\lambda}^{|\delta|^{<\bar{\theta}}} \leq \mathfrak{d}_{\kappa,|\delta|^{<\bar{\theta}}}^{|\delta|^{<\bar{\theta}}}$, so

$$\mathfrak{d}_{\kappa,\lambda}^{|\delta|^{<\bar{\theta}}} = \mathfrak{d}_{\kappa,|\delta|}^{|\delta|^{<\bar{\theta}}} = \max(\mathfrak{d}_{\kappa,|\delta|}^{|\delta|^{<\bar{\theta}}}, \mathrm{cov}(\lambda, (|\delta|^{<\bar{\theta}})^+, (|\delta|^{<\bar{\theta}})^+, 2)).$$

The result now follows from Proposition 11.2. □

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