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Absolute E-rings ☆,☆☆

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Abstract

A ring R with 1 is called an *E-ring* if $\mathrm{End}_{\mathbb{Z}} R$ is ring-isomorphic to R under the canonical homomorphism taking the value 1σ for any $\sigma \in \mathrm{End}_{\mathbb{Z}} R$. Moreover R is an *absolute E-ring* if it remains an E-ring in every generic extension of the universe. E-rings are an important tool for algebraic topology as explained in the introduction. The existence of an E-ring R of each cardinality of the form $\lambda^{\aleph_0}$ was shown by Dugas, Mader and Vinsonhaler (1987) [9]. We want to show the existence of absolute E-rings. It turns out that there is a precise cardinal-barrier $\kappa(\omega)$ for this problem: (The cardinal $\kappa(\omega)$ is the first ω-Erdős cardinal defined in the introduction. It is a relative of measurable cardinals.) We will construct absolute E-rings of any size $\lambda < \kappa(\omega)$. But there are no absolute E-rings of cardinality $\geqslant \kappa(\omega)$. The non-existence of huge, absolute E-rings $\geqslant \kappa(\omega)$ follows from a recent paper by Herden and Shelah (2009) [24] and the construction of absolute E-rings R is based on an old result by Shelah (1982) [31] where families of absolute, rigid colored trees (with no automorphism between any distinct members) are constructed. We plant these trees into our potential E-rings with the aim to prevent unwanted endomorphisms of their additive group to survive. Endomorphisms will recognize the trees which will have branches infinitely often divisible by primes. Our main result provides the existence of absolute E-rings for all infinite cardinals $\lambda < \kappa(\omega)$, i.e. these E-rings remain E-rings in all generic extensions of the universe (e.g. using forcing arguments). Indeed *all* previously known E-rings (Dugas, Mader and Vinsonhaler, 1987 [9]; Göbel and Trlifaj, 2006 [23]) of cardinality $\geqslant 2^{\aleph_0}$ have a free additive group R^+ in some extended universe, thus are no longer E-rings, as

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explained in the introduction. Our construction also fills all cardinal-gaps of the earlier constructions (which have only sizes $\lambda^{\aleph_0}$). These E-rings are domains and as a by-product we obtain the existence of absolutely indecomposable abelian groups, compare Göbel and Shelah (2007) [22].

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1. Introduction

We want to investigate E-rings and their absolute behavior. E-rings appeared while studying rings R with the property that the endomorphism ring $\mathrm{End}_{\mathbb{Z}} R$ of the underlying additive structure is ring-isomorphic to R. (These rings are now called generalized E-rings.) However, Schultz [30] was able to isolate in 1973 an important class of rings which since then are called E-rings: R is an E-ring if the evaluation map $\mathrm{End}_{\mathbb{Z}} R \longrightarrow R$ ($\sigma \mapsto 1\sigma$) is an isomorphism. (The name E-ring refers to this particular mapping.) E-rings can also be defined dually: The homomorphism $R \longrightarrow \mathrm{End}_{\mathbb{Z}} R$ ($r \mapsto \rho_r$) (with ρ_r scalar multiplication by $r \in R$ on the right) is an isomorphism. Moreover, it is not hard to see that R is an E-ring if and only if $\mathrm{End}_Z R \cong R$ and R is commutative; see [23, pp. 468, 469, Proposition 13.1.9]. Thus R is an E-ring if and only if it is a commutative generalized E-ring. (This, of course, suggests the question about the existence of proper generalized E-rings, first noticed 50 years ago by Fuchs [14] and answered recently by providing (in ordinary set theory, ZFC) the existence of a proper class of such non-commutative rings in [21].) The first examples of E-rings are the $2^{\aleph_0}$ subrings of $\mathbb{Q}$.

The class of E-rings was in the focus of many papers since then. The algebraic properties were considered in fundamental papers by Mader, Pierce and Vinsonhaler [27–29] and the existence of arbitrarily large E-rings was first shown by examples of rank $\aleph_0$ in Faticoni [11] (extended to ranks $\leqslant 2^{\aleph_0}$ in [23, p. 471, Corollary 13.2.3]) and above $2^{\aleph_0}$ in Dugas, Mader and Vinsonhaler [9] using Shelah's Black Box as outlined in Corner and Göbel [4]. The existence of related E-modules as a natural by-product appeared soon after in [7]. From [30] also follows that the torsion-part of an E-ring can be classified; the same holds for the cotorsion-part as shown in [17]. In contrast the quotients of the ring modulo the ideal of torsion-elements and the ideal generated by the cotorsion submodules can be arbitrarily large as shown in [1,17], respectively.

The existence of E-rings contributes to algebraic topology: We rephrase the definition by the diagram

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\ \eta\ } & R \\ {\scriptstyle\varphi}\downarrow & \swarrow{\scriptstyle\overline{\varphi}} & \\ R, & & \end{array}$$

where η is the inclusion $\langle 1\rangle \subseteq R$ and for any φ there is a unique $\overline{\varphi}$ such that the diagram holds. However, this is the definition of a localization R of $\mathbb{Z}$, see [3]. This notion makes sense in many categories, and in particular can be studied in homotopy theory, as discussed in Dror Farjoun [5]. He raised the question if for a fixed compact space X, the distinct homotopy types of the form $L_f X$ form a set, where $f : Y \longrightarrow Z$ is running through all possible maps between topological spaces and L_f denotes homotopical localization with respect to these maps f. The following result is not hard to see, but is an important observation in the context of localizations of abelian Eilenberg–Mac Lane spaces. It will appear in Casacuberta, Rodríguez and Tai [3]: If a space X is a homotopical localization of the circle S^1 (i.e. $X \cong L_f S^1$), then X is the Eilenberg–Mac Lane space $K(R, 1)$ with R an E-ring and any E-ring appears this way (take $f : S^1 \longrightarrow K(R, 1)$ induced by the inclusion of 1 into R). (The Eilenberg–Mac Lane space $K(R, 1)$ is the connected space which has (abelian) fundamental group R and trivial higher homotopy groups. It is unique up to homotopy and it is well known how to construct such cellular models.) Thus the existence of a proper class of E-rings provides a negative answer to Dror Farjoun's question. Below we will discuss an 'absolute version' of this result.

Note that E-rings constructed earlier and here have also impact to other areas of algebra. They are useful for constructing nilpotent groups of class 2 (see Dugas and Göbel [8]) and build the core for investigating abelian groups with automorphism groups acting uniquely transitive, see [18–20]. Surveys and classical results on E-rings can be found in [12,13,23,32].

The second ingredient of this paper is the notion of absolute structures. The recent activity on this topic was initiated by Eklof and Shelah [10], who studied the existence of absolutely indecomposable abelian groups. Here a property of a structure is called absolute if it is preserved under generic extensions of the given universe (of set theory), in particular it is preserved under forcing. Absolute formulas are discussed in detail in a classical monograph by Levy [26], examples are the subset relation, or the property to be an ordinal. A quick survey on absolute formulas is given in [2, pp. 408–412]. However, the powerset relation is not absolute. Here is a more striking algebraic counterexample. The following statement (i) is not absolute.

(i) $A \neq \mathbb{Z}$ is an indecomposable abelian group and its subgroups of finite rank are free.

First we note that the freeness condition by Pontryagin's theorem (Fuchs [15, Vol. 1, p. 93]) is equivalent to say that all countable subgroups of A are free, i.e. A is $\aleph_1$-free. We can find a generic extension of the underlying model of set theory (the Levy collapse) such that $|A|$ becomes countable, hence $A \neq \mathbb{Z}$ is free and definitely not indecomposable. We immediately note, that all E-rings constructed in the past (and of size $\geqslant 2^{\aleph_0}$) are $\aleph_1$-free and thus can be treated the same way. They become free in an extended model and thus are no longer E-rings. The problem settled in this paper becomes obvious.

Can we find absolute E-rings?

As a by-product of these considerations we obtain new, very useful methods for the construction of 'complicated' structures. The crucial point is, that often the old constructions use stationary sets or tools which are not that friendly from a constructive point of view: the new methods are based on inductive arguments and thus provide a more elementary approach to the desired complicated structures.

Surprisingly, there is a precise cardinal bound $\kappa(\omega)$ for the construction of absolute E-rings. Here $\kappa(\omega)$ denotes the first ω-Erdős cardinal defined in Section 2. We note immediately that $\kappa(\omega)$ (like the first measurable cardinal) is a large inaccessible cardinal which may not exist in any universe; see [25]. Any model of set theory contains a submodel of ZFC which has no first ω-Erdős cardinal and it is also well known that Gödel's universe has no first ω-Erdős cardinal. In a recent paper Herden and Shelah [24] have shown that there are no absolute E-rings of size $\geqslant \kappa(\omega)$. We want to prove the converse.

Main Theorem 1.1. *If λ is any infinite cardinal $< \kappa(\omega)$ (the first ω-Erdős cardinal), then there is an absolute E-ring R of cardinality λ. Moreover $\mathbb{Z}[X] \subseteq R \subseteq \mathbb{Q}[X]$ with X a family of λ commuting free variables.*

The new method of constructing E-rings differs from those described in the references and above. For example, the construction in [9] (which does not provide any absolute E-rings) – due to the Black Box – also does not allow to show the existence of E-rings of cardinality cofinal with ω. However, Theorem 1.1 gives an answer for all infinite cardinals $< \kappa(\omega)$. In Corollary 5.2 we explain how to extend this result to obtain rigid families of (absolute) E-rings.

The following application to algebraic topology is immediate by the above remarks.

Corollary 1.2. *The family $L_f S^1$ (for any map f) of absolute localizations of the circle S^1 (based on Theorem 1.1) is a proper class, if and only if there is no ω-Erdős cardinal.*

Thus, in models of ZFC without ω-Erdős cardinals the negative answer to Dror Farjoun's problem is absolute.

Some absolute constructions for other categories of modules, trees and graphs can be seen in [22,16,31,6]. In these cases it also follows that the upper bound $\kappa(\omega)$ is sharp. However, it is still an open problem, if for the family of absolutely indecomposable abelian groups the upper bound can be larger than $\kappa(\omega)$, see also [10]. The strategy for the construction of absolute E-rings utilizes the existence of absolutely rigid, colored trees from Shelah [31], which we will describe in Section 2. In fact, in order to apply this to E-rings, we first must strengthen [31] in Theorem 2.8.

Finally we explain the strategy of this paper in the simpler case of Theorem 1.1 when X is (non-empty and) countable. In this case we can replace the existence of absolutely rigid trees by a countable family of primes automatically resulting in an absolute construction. Consider the family $\mathcal{F} = \{x - z, x^n \mid x \in X,\ z \in \mathbb{Z},\ 0 < n < \omega\} \subseteq \mathbb{Z}[X]$ of polynomials. For each $f \in \mathcal{F}$ we choose a distinct prime p_f. If $a \in A$ and A is a torsion-free abelian group, then recall that $p^{-\infty}a \subseteq \mathbb{Q} \otimes A$ is the family of unique quotients $p^{-n}a$ $(n < \omega)$ and $p^\infty A = \bigcap_{n<\omega} p^n A$ is the first Ulm subgroup of A. The group A is p-reduced if $p^\infty A = 0$. Finally $U_* \subseteq A$ denotes the unique minimal pure subgroup of A containing $U \subseteq A$.

Proposition 1.3. *The subring*

$$R = \left\langle \mathbb{Z}[X], p_f^{-\infty} f \;\middle|\; f \in \mathcal{F} \right\rangle$$

of the polynomial ring $\mathbb{Q}[X]$ in countably many variables is an E-ring.

Proof. It is easy to show that $p_f^\infty R = (fR)_*$ holds for all $f \in \mathcal{F}$. So by linearity the purification of the principal ideal fR of R is fully invariant for all polynomials $f = (x - z)m$ with m a monomial in $\langle X \rangle$, $x \in X$, $z \in \mathbb{Z}$. Since R now has visibly many fully invariant ideals it will also be easy to show the proposition:

Consider any $(x - z)m$ with m a monomial in $\langle X \rangle$, $x \in X$, $z \in \mathbb{Z}$ and $\varphi \in \operatorname{End}_{\mathbb{Z}} R$.

From the invariance of the pure ideals related to m, xm and $(x - z)m$ follows the existence of $g_m, g_{xm}, g_{(x-z)m} \in \mathbb{Q}[X]$ such that $m\varphi = mg_m$, $(xm)\varphi = xmg_{xm}$ and

$$\big((x - z)m\big)\varphi = (x - z)mg_{(x-z)m} = (xm)\varphi - z(m\varphi) = xmg_{xm} - zmg_m.$$

Thus $(x-z)mg_{(x-z)m} = xmg_{xm} - zmg_m$ or seen as functions depending on x

$$(x-z)\cdot m(x)\cdot g_{(x-z)m}(x) = x\cdot m(x)\cdot g_{xm}(x) - zm(x)\cdot g_m(x)$$

holds for every integer $z \in \mathbb{Z}$. Substituting $x := z$ we get $0 = zm(z)\cdot(g_{xm}(z) - g_m(z))$. Hence $h(z) = 0$ follows for $h(x) = g_{xm}(x) - g_m(x)$ and for all $0 \neq z \in \mathbb{Z}$ as $zm(z) \neq 0$. Thus $x - z$ is a factor of $h(x)$ for infinitely many $z \in \mathbb{Z}$, which is only possible if h is the zero-polynomial and $g_m = g_{xm}$. Beginning with $g = g_1 = 1\varphi \in R$ we get by recursion that φ acts by multiplication with g on the set of all monomials. But the monomials generate R additively, hence $\varphi = g\,\mathrm{id}_R$ and R is an E-ring. □

Proposition 1.3 is a new proof of the main result in [11] and the problem we must settle becomes also obvious: Even if we search for an (absolute) E-ring of size $\aleph_1$, then we must find a suitable substitute for primes, and this is how the large family of absolutely rigid trees comes into play.

2. Constructing strongly rigid colored trees

In this section we strengthen an earlier result by Shelah [31] on better quasi-orders which will be applied for E-rings. Thus we must first state one of the main results on colored trees from this paper. The reader should keep in mind that in the following tree maps will act on the left and module homomorphisms will act on the right of the argument, so as usual the order of the composition of two maps $\varphi\pi$ depends on the domain which is a tree or a module, respectively.

Let $\kappa(\omega)$ denote the *first ω-Erdős cardinal*. This is defined as the smallest cardinal κ such that $\kappa \to (\omega)^{<\omega}$ holds, i.e. for every function f from the finite subsets of κ to 2 there exist an infinite subset $X \subset \kappa$ and a function $g : \omega \to 2$ such that $f(Y) = g(|Y|)$ for all finite subsets Y of X. This well-studied cardinal $\kappa(\omega)$ is strongly inaccessible; see Jech [25, p. 392]. Thus $\kappa(\omega)$ is a very large cardinal. We should also emphasize that $\kappa(\omega)$ may not exist in any universe of ZFC. In this case the restriction $\lambda < \kappa(\omega)$ on a cardinal λ will be irrelevant.

If $\lambda < \kappa(\omega)$, then let

$$\mathcal{T} = {}^{\omega>}\lambda = \{f : n \longrightarrow \lambda\text{: with } n < \omega \text{ and } n = \mathrm{Dom}\, f\}$$

be the tree of all finite sequences f (of length or level $\lg f$) in λ. Since $n = \{0, \dots, n-1\}$ as ordinal, we also write $f = f(0)^\wedge f(1)^\wedge \dots ^\wedge f(n-1)$. By restriction $g = f \restriction m$ for any $m \leqslant n$ we obtain all *initial segments* of f. We will write $g \lhd f$ to denote that g is an initial segment of f. Thus

$$g \subseteq f \text{ as graphs} \quad\Longleftrightarrow\quad g \lhd f.$$

We denote the empty map by the symbol $\perp$ and call it the root of the tree. A subtree $\mathcal{T}'$ of $\mathcal{T}$ is a non-empty subset which is closed under initial segments and a homomorphism between two subtrees $\mathcal{T}_1, \mathcal{T}_2$ of $\mathcal{T}$ is a map $\varphi : \mathcal{T}_1 \to \mathcal{T}_2$ $(\eta \mapsto \varphi(\eta))$ that preserves levels and initial segments, i.e. $\lg \eta = \lg \varphi(\eta)$ and $\varphi(\nu) \lhd \varphi(\eta)$ for all $\nu \lhd \eta \in \mathcal{T}_1$. (Note that a homomorphism does not need to be injective or preserve $\ntriangleleft$.) If $\mathcal{T}'$ comes with a coloring map $c : \mathcal{T}' \longrightarrow \omega$ $(\eta \mapsto c(\eta))$ we call this tree an ω-colored (or just a colored) tree and write $(\mathcal{T}', c)$. Colored trees in this paper will always be ω-colored, and we often omit ω. Now, $\mathrm{Hom}((\mathcal{T}_1, c_1), (\mathcal{T}_2, c_2))$ will denote the homomorphisms φ between two such colored trees which are ordinary tree homomorphisms $\varphi : \mathcal{T}_1 \to \mathcal{T}_2$ that in addition preserve colors, i.e. $c_2(\varphi(\eta)) = c_1(\eta)$ for all $\eta \in \mathcal{T}_1$. Shelah [31] showed the existence of an *absolutely rigid* family of 2^λ colored subtrees of $\mathcal{T} = {}^{\omega>}\lambda$.

Theorem 2.1. *If $\lambda < \kappa(\omega)$ is infinite and $\mathcal{T} = {}^{\omega>}\lambda$, then there is a family $(\mathcal{T}_\alpha, c_\alpha)$ $(\alpha < 2^\lambda)$ of ω-colored subtrees of $\mathcal{T}$ (of size λ) such that for $\alpha, \beta < 2^\lambda$ and in any generic extension of the universe the following holds*:

$$\operatorname{Hom}\big((\mathcal{T}_\alpha, c_\alpha), (\mathcal{T}_\beta, c_\beta)\big) \neq \emptyset \quad \Longrightarrow \quad \alpha = \beta.$$

Remark 2.2. Such a family of colored trees $(\mathcal{T}_\alpha, c_\alpha)$ $(\alpha < 2^\lambda)$ is called an *absolutely rigid* family of trees of size λ. In the following we will show how to implement such a family to construct absolute E-rings of any infinite cardinality $< \kappa(\omega)$. For $\lambda > \kappa(\omega)$ such an absolutely rigid family of trees does not exist.

We fix such a family and write

$$\big(\mathcal{T}''_\alpha, c''_a\big)\ \big(\alpha < 2^\lambda\big) \text{ for an absolutely rigid family of trees } \big(\text{for a fixed } \lambda < \kappa(\omega)\big). \tag{2.1}$$

2.1. A shift map for trees

In order to modify the family (2.1) we introduce two coding maps, which are bijections:

$$\mathrm{cd} : {}^{\omega>}\omega \to \omega$$

and

$$\mathrm{cd}_\lambda : {}^{\omega>}\big(\lambda \cup \{*\}\big) \to \lambda,$$

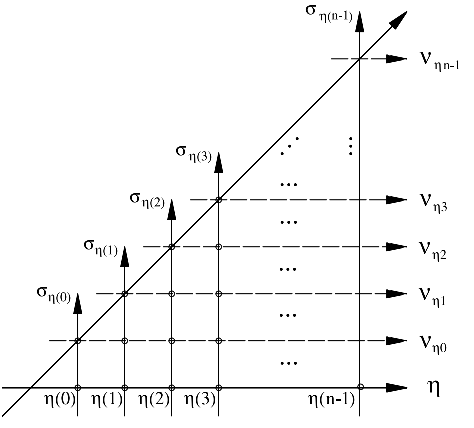
where $*$ denotes a new symbol (which does not appear in the set λ).

If $\alpha < 2^\lambda$, then let $\sigma_\alpha := \mathrm{cd}_\lambda^{-1}(\alpha)$ and define a subset $\mathcal{T}'_\alpha \subseteq {}^{\omega>}\lambda$ consisting of all elements $\eta \in {}^{\omega>}\lambda$ satisfying to the following two conditions. We let $\lg \eta = n$.

(a) For $\ell < n$ let $\lg \sigma_{\eta(\ell)} = \ell + 1$.
(b) For any $\ell < n$ there is $\nu_{\eta\ell} \in \mathcal{T}''_\alpha$ such that

$$\sigma_{\eta(\ell+m)}(\ell) = \begin{cases} \nu_{\eta\ell}(m) & \text{for } m < \lg \nu_{\eta\ell}, \\ * & \text{for } \lg \nu_{\eta\ell} \leqslant m \leqslant n - \ell - 1. \end{cases}$$

This in particular implies that $\lg \nu_{\eta\ell} \leqslant n - \ell - 1$ must hold. Given η, then the choice of elements $\nu_{\eta\ell}$ is illustrated by the following diagram.



The triangular shape of the diagram is a direct consequence of the above conditions (a) and (b). The ℓth line of the diagram is of the form $\nu_{\eta\ell}{}^\wedge\langle *, *, *, \ldots\rangle$, where the element $\nu_{\eta\ell} \in \mathcal{T}''_\alpha$ is uniquely determined by η and condition (b). Conversely, any choice of elements $\nu_\ell \in \mathcal{T}''_\alpha$ $(\ell < n)$ with $\lg \nu_\ell \leqslant n - \ell - 1$ by (b) determines some $\eta \in \mathcal{T}'_\alpha$ of length n with $\nu_{\eta\ell} = \nu_\ell$ $(\ell < n)$, thus $\mathcal{T}'_\alpha \neq \emptyset$. We get an immediate

Proposition 2.3. *Let $\aleph_0 \leqslant \lambda < \kappa(\omega)$. The above set $\mathcal{T}'_\alpha$ is a colored subtree of ${}^{\omega>}\lambda$ with the coloring map*

$$c'_\alpha : \mathcal{T}'_\alpha \longrightarrow \omega, \quad \eta \mapsto c'_\alpha(\eta) = \mathrm{cd}\big(\langle \lg \nu_{\eta\ell} \mid \ell < \lg \eta\rangle^\wedge \big\langle c''_\alpha(\nu_{\eta\ell}) \mid \ell < \lg \eta\big\rangle^\wedge \big\langle c''_\alpha(\perp)\big\rangle\big).$$

Hence we have a family $\langle (\mathcal{T}'_\alpha, c'_\alpha) \mid \alpha < 2^\lambda\rangle$ of colored trees.

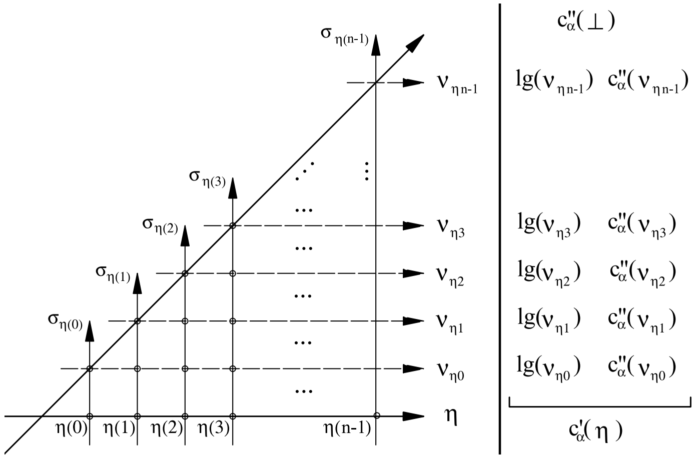
The relevant point here is that the color $c'_\alpha(\eta)$ encodes the length and the color of the branches $\nu_{\eta\ell}$ for all $\ell < \lg \eta$.

Proof. We have $\mathcal{T}'_\alpha \neq \emptyset$ from above. Let $\eta' \lhd \eta \in \mathcal{T}'_\alpha$ be an initial segment. We must show that η' satisfies the two conditions (a), (b) above. Condition (a) is obvious. Condition (b) is satisfied with

$$\nu_{\eta'\ell} = \begin{cases} \nu_{\eta\ell} \restriction (\lg \eta' - \ell - 1) & \text{for } \lg \eta' - \ell - 1 < \lg \nu_{\eta\ell}, \\ \nu_{\eta\ell} & \text{otherwise.} \end{cases}$$

Hence $\eta' \in \mathcal{T}'_\alpha$. It is clear that c'_α defines a coloring of $\mathcal{T}'_\alpha$. □

In particular, $\nu_{\eta'\ell} \lhd \nu_{\eta\ell}$ holds for $\eta' \lhd \eta \in \mathcal{T}'_\alpha$ and $\ell < \lg \eta'$. Finally we illustrate the above coloring $c'_\alpha(\eta)$ for a tree.



Next we will show that these trees are strongly rigid (in the sense of Theorem 2.5 below). We will use the following natural definition.

Definition 2.4. If $\alpha < 2^\lambda$ and $\eta \in \mathcal{T}'_\alpha$, then let $(\mathcal{T}'_\alpha)_{\geqslant\eta} := \{\nu \in \mathcal{T}'_\alpha \mid \eta \lhd \nu\}$ be the part of the tree $\mathcal{T}'_\alpha$ above η.

Using Theorem 2.1 we will establish the following

Theorem 2.5. *If $\alpha, \beta < 2^\lambda$ are distinct, and $\eta \in \mathcal{T}'_\alpha$, then there is no color preserving partial tree homomorphism $\varphi' : (\mathcal{T}'_\alpha)_{\geqslant\eta} \longrightarrow \mathcal{T}'_\beta$ in any generic extension of the universe.*

Proof. Suppose for contradiction that $\varphi' : (\mathcal{T}'_\alpha)_{\geqslant\eta} \longrightarrow \mathcal{T}'_\beta$ is a color preserving partial tree homomorphism. First we define an injective projection map

$$\pi : \mathcal{T}''_\alpha \longrightarrow \left(\mathcal{T}'_\alpha\right)_{\geqslant\eta},$$

which we then want to compose with φ'. If $\tau \in \mathcal{T}''_\alpha$ and $\ell < \lg\tau$, then we determine a branch $\tau'_\ell \in {}^{\omega>}(\lambda \cup \{*\})$ of length $\lg\tau'_\ell = \ell + \lg\eta + 1$. Let

$$\tau'_\ell(m) := \begin{cases} \tau(\ell) & \text{for } m = \lg\eta, \\ * & \text{otherwise.} \end{cases}$$

Then put $\pi(\tau) = \eta^\wedge\langle \mathrm{cd}_\lambda(\tau'_\ell) \mid \ell < \lg\tau\rangle$ which belongs to $(\mathcal{T}'_\alpha)_{\geqslant\eta}$ as required: Condition (a) above is clear and (b) can be seen directly from the diagram below. Thus $\lg\pi(\tau) = \lg\tau + \lg\eta$, and $\nu_{\pi(\tau),\lg\eta} = \tau$ holds for $\tau \neq \perp$. Moreover, if $\tau \lhd \tau'$, then also $\pi(\tau) \lhd \pi(\tau')$. So $\pi : \mathcal{T}''_\alpha \longrightarrow (\mathcal{T}'_\alpha)_{\geqslant\eta}$ is an injective map that preserves initial segments.

Finally we want to define a color preserving tree homomorphism

$$\varphi : \mathcal{T}''_\alpha \longrightarrow \mathcal{T}''_\beta \tag{2.2}$$

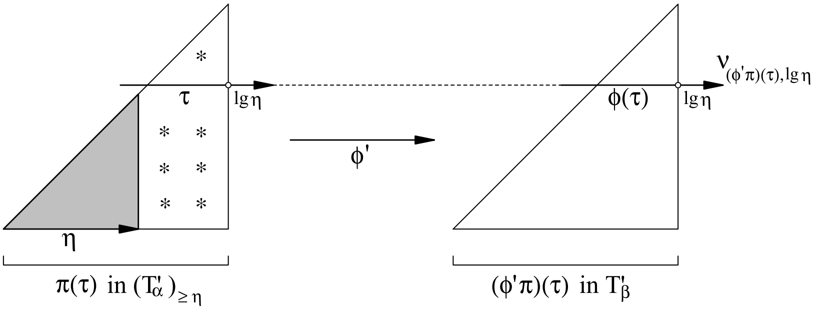
by setting

$$\varphi(\tau) = \nu_{(\varphi'\pi)(\tau),\lg\eta}$$

for $\tau \neq \perp$ and $\varphi(\perp) = \perp$. Note that $\varphi(\tau)$ is well-defined: For this we must show that $\lg \eta < \lg(\varphi'\pi)(\tau)$ for $\tau \neq \perp$. But note that by the above (using that φ' preserves length)

$$\lg\big(\varphi'\pi\big)(\tau) = \lg \pi(\tau) = \lg \tau + \lg \eta > \lg \eta$$

as desired.



It is clear that $\varphi(\tau) \in \mathcal{T}''_\beta$. Finally we have to show that φ preserves the length and color of branches as well as initial segments.

If $\tau \lhd \tau' \in \mathcal{T}''_\alpha$ then by the properties of π mentioned above we have $\pi(\tau) \lhd \pi(\tau') \in (\mathcal{T}'_\alpha)_{\geqslant \eta}$, and using that φ' is a tree homomorphism also $(\varphi'\pi)(\tau) \lhd (\varphi'\pi)(\tau')$ and

$$\varphi(\tau) = \nu_{(\varphi'\pi)(\tau),\lg \eta} \lhd \nu_{(\varphi'\pi)(\tau'),\lg \eta} = \varphi\big(\tau'\big)$$

holds. To show that φ preserves length and color we recall $c'_\alpha(\pi(\tau)) = c'_\beta((\varphi'\pi)(\tau))$ for $\tau \in \mathcal{T}''_\alpha$ as φ' preserves colors. However c' codes the length and color of the elements of the form $\nu_{\ldots \lg \eta}$ (if $\tau \neq \perp$) and it follows by definition of π and φ, respectively, that

$$\lg \tau = \lg \nu_{\pi(\tau),\lg \eta} = \lg \nu_{(\varphi'\pi)(\tau),\lg \eta} = \lg \varphi(\tau)$$

and similarly $c''_\alpha(\tau) = c''_\beta(\varphi(\tau))$. As c' always codes the color of the root $\perp$ we also have $c''_\alpha(\perp) = c''_\beta(\perp) = c''_\beta(\varphi(\perp))$, while $\lg \perp = \lg \varphi(\perp)$ is obvious.

Hence φ is a color preserving tree homomorphism, which by Theorem 2.1 cannot exist unless $\alpha = \beta$. This case however was excluded. □

2.2. *Strongly rigid trees*

In the final step of the tree construction we will modify the trees from Section 2.1 to prove the non-existence of color preserving partial tree homomorphisms on an even smaller domain. It helps to consider for branches $\eta \in {}^{\omega>}\lambda$ and $\sigma \in {}^{\omega>}\omega$ with $\lg \eta = \lg \sigma$ the induced branch

$$\eta \bullet \sigma := \big\langle \omega \cdot \eta(\ell) + \sigma(\ell) \mid \ell < \lg \eta \big\rangle \in {}^{\omega>}(\omega \cdot \lambda) = {}^{\omega>}\lambda.$$

If $\eta \in {}^{\omega>}\lambda$, then there is an obviously unique decomposition $\eta = \eta' \bullet \sigma$ with $\eta' \in {}^{\omega>}\lambda$, $\sigma \in {}^{\omega>}\omega$ and $\lg \eta' = \lg \sigma$. Furthermore, $\eta'_1 \bullet \sigma_1 \lhd \eta'_2 \bullet \sigma_2$ holds iff $\eta'_1 \lhd \eta'_2$ and $\sigma_1 \lhd \sigma_2$.

Using the trees $\mathcal{T}'_\alpha$ ($\alpha < 2^\lambda$) from Theorem 2.5 we put

$$\mathcal{T}_\alpha := \{\eta \in {}^{\omega>}\lambda \mid \eta = \eta' \bullet \sigma, \eta' \in \mathcal{T}'_\alpha, \sigma \in {}^{\omega>}\omega \text{ and } \lg \eta' = \lg \sigma\}$$

and define a coloring

$$c_\alpha(\eta) = \mathrm{cd}(\langle c'_\alpha(\eta' \restriction \ell) \mid \ell < \lg \eta' \rangle^\wedge \sigma) \quad \text{for } \eta = \eta' \bullet \sigma \in \mathcal{T}_\alpha.$$

Here cd is the coding map from the beginning of Section 2.1.

Our final tree-results are the following two theorems.

Theorem 2.6. *Let* $(\mathcal{T}_\alpha, c_\alpha)$ $(\alpha < 2^\lambda)$ *be as above. Then the following holds.*

(i) $\mathcal{T}_\alpha \subseteq {}^{\omega>}\lambda$ *is a subtree.*
(ii) $c_\alpha : \mathcal{T}_\alpha \longrightarrow \omega$ *is a coloring.*
(iii) *For* $\eta \in \mathcal{T}_\alpha$ *and* $\nu \in \mathcal{T}_\beta$ *with* $c_\alpha(\eta) = c_\beta(\nu)$ *follows*
  (a) $\lg \eta = \lg \nu$.
  (b) $c_\alpha(\eta \restriction \ell) = c_\beta(\nu \restriction \ell)$ *for all* $\ell < \lg \eta$.
  (c) *If* $\eta = \eta' \bullet \sigma$, $\nu = \nu' \bullet \tau$ *then* $\sigma = \tau$.

Proof. It is clear that $\mathcal{T}_\alpha \neq \emptyset$, and conditions (i) and (ii) are obvious. For (iii) we consider $c_\alpha(\eta) = c_\beta(\nu)$. Thus $\langle c'_\alpha(\eta' \restriction \ell) \mid \ell < \lg \eta' \rangle^\wedge \sigma = \langle c'_\beta(\nu' \restriction \ell) \mid \ell < \lg \nu' \rangle^\wedge \tau$ by definition of the coloring. We get $\lg \eta = \lg \eta' = \lg \nu' = \lg \nu$, $\sigma = \tau$ and $c'_\alpha(\eta' \restriction \ell) = c'_\beta(\nu' \restriction \ell)$ for all $\ell < \lg \eta$. Now (a), (b) and (c) are obvious. □

In preparation of the next theorem we define a special closure property.

Definition 2.7. We will also use the following *closure condition* for subsets $\mathcal{T}^*_\alpha \subseteq \mathcal{T}_\alpha$ and $\eta = \eta' \bullet \sigma \in \mathcal{T}^*_\alpha$:

(1) If $\nu = \nu' \bullet \tau \in \mathcal{T}^*_\alpha$ then $\nu' \in (\mathcal{T}'_\alpha)_{\geqslant \eta'}$.
(2) If $\nu = \nu' \bullet \tau \in \mathcal{T}^*_\alpha$ and $\nu' \lhd \xi' \in \mathcal{T}'_\alpha$ and $\lg \xi' = \lg \nu' + 1$, then there is $\tau \lhd \upsilon \in {}^{\omega>}\omega$ with $\lg \upsilon = \lg \tau + 1$ and $\xi' \bullet \upsilon \in \mathcal{T}^*_\alpha$.

Theorem 2.8. *If* $(\mathcal{T}_\alpha, c_\alpha)$ $(\alpha < 2^\lambda)$ *is as above and* $\mathcal{T}^*_\alpha \subseteq \mathcal{T}_\alpha$ *satisfies the closure condition from Definition* 2.7 *for* $\eta = \eta' \bullet \sigma \in \mathcal{T}^*_\alpha$ *and* $\alpha \neq \beta < 2^\lambda$, *then there is no color preserving partial tree homomorphism* $\mathcal{T}^*_\alpha \longrightarrow \mathcal{T}_\beta$ *in any generic extension of the universe.*

Proof. Let $\eta = \eta' \bullet \sigma$ be as in the theorem and suppose for contradiction that $\varphi : \mathcal{T}^*_\alpha \longrightarrow \mathcal{T}_\beta$ is a color preserving partial tree homomorphism. We want to define a color preserving partial tree homomorphism

$$\varphi' : (\mathcal{T}'_\alpha)_{\geqslant \eta'} \longrightarrow \mathcal{T}'_\beta.$$

In the first step we define recursively a partial tree homomorphism

$$g : (\mathcal{T}'_\alpha)_{\geqslant \eta'} \longrightarrow {}^{\omega>}\omega$$

such that $\nu' \bullet g(\nu') \in \mathcal{T}^*_\alpha$ for all $\nu' \in (\mathcal{T}'_\alpha)_{\geqslant \eta'}$. The (relative) bottom element is $\eta' \in (\mathcal{T}'_\alpha)_{\geqslant \eta'}$ and we put $g(\eta') = \sigma$ and note that $\eta = \eta' \bullet \sigma \in \mathcal{T}^*_\alpha$ by assumption of the theorem. For the inductive step we consider $\nu' \in (\mathcal{T}'_\alpha)_{\geqslant \eta'}$, $\nu' \bullet g(\nu') \in \mathcal{T}^*_\alpha$, let $\nu' \lhd \xi'$ be with $\lg \xi' = \lg \nu' + 1$ and define $g(\xi')$ with the help of Definition 2.7(2). In particular $g(\nu') \lhd g(\xi')$ and $\lg g(\xi') = \lg g(\nu') + 1 = \lg \nu' + 1$. Hence g is well defined on $(\mathcal{T}'_\alpha)_{\geqslant \eta'}$ and preserves lengths and initial segments.

Recall that for any $\nu' \in (\mathcal{T}'_\alpha)_{\geqslant \eta'}$ we have $\nu = \nu' \bullet g(\nu') \in \mathcal{T}^*_\alpha$. In particular, $\varphi(\nu) = \nu'' \bullet \tau \in \mathcal{T}_\beta$ is well defined, and since φ preserves colors, we derive from Theorem 2.6(iii)(c) that $\tau = g(\nu')$; hence

$$\varphi\big(\nu' \bullet g\big(\nu'\big)\big) = \nu'' \bullet g\big(\nu'\big) \tag{2.3}$$

and we put $\varphi'(\nu') = \nu'' \in \mathcal{T}'_\beta$. Thus the map φ' above is defined and we must check that it preserves initial segments, lengths and colors.

Let $\nu' \lhd \xi'$ and recall that g preserves initial segments. Hence also $g(\nu') \lhd g(\xi')$ and $\nu' \bullet g(\nu') \lhd \xi' \bullet g(\xi')$, and since φ is a partial tree homomorphism we conclude $\varphi'(\nu') \bullet g(\nu') \lhd \varphi'(\xi') \bullet g(\xi')$ and $\varphi'(\nu') \lhd \varphi'(\xi')$ from (2.3).

From $\varphi(\nu' \bullet g(\nu')) = \varphi'(\nu') \bullet g(\nu')$ and the assumption that φ preserves colors [together with Theorem 2.6(iii)(b), (c)] we get $c'_\alpha(\nu') = c'_\beta(\varphi'(\nu'))$ and see that also φ' preserves colors. Moreover $\lg \nu' = \lg(\nu' \bullet g(\nu')) = \lg(\varphi'(\nu') \bullet g(\nu')) = \lg \varphi'(\nu')$ and φ' also preserves the length. Such a map φ' however is forbidden by Theorem 2.5 for $\alpha \neq \beta$, so Theorem 2.8 holds. □

3. The construction of *E*-rings

Let $\lambda < \kappa(\omega)$ be a fixed infinite cardinal and enumerate by

$$\Pi = \{p_{nki}, q_{nki} \mid n, k, i < \omega\}$$

some of the primes of $\mathbb{Z}$ without repetition. Let $\mathbb{Q}$ denote the field of rational numbers. If $p \in \Pi$ and a is an element of a torsion-free abelian group M, then we denote (as usual) by $p^{-\infty}a$ the family of unique elements $\{p^{-n}a \mid n < \omega\}$ of the divisible hull $\mathbb{Q}M = \mathbb{Q} \otimes M$ using $M \subseteq \mathbb{Q}M$. If $p^{-\infty}a \subseteq M$, we will also write $p^\infty \mid a$ (in M).

First we decompose λ into $\lambda = \bigcup_{n<\omega} U_n$ with equipotent subsets U_n of size λ, write $U_{<n} = \bigcup_{i<n} U_i$ and constitute a chain $\{X_n \mid n < \omega\}$ with the help of some of the absolute trees $\mathcal{T}_\alpha \subseteq {}^{\omega>}\lambda$ ($\alpha < 2^\lambda$) given by Theorem 2.8 as follows. Let

$$X_n = \big\{x_\gamma, x_{\alpha\eta} \bigm| \gamma < \lambda,\ \alpha \in U_{<n},\ \eta \in \mathcal{T}_\alpha \setminus \{\bot\}\big\} \quad \text{for all } n < \omega \text{ and } X = \bigcup_{n<\omega} X_n.$$

Note that $X_0 = \{x_\gamma \mid \gamma < \lambda\}$ (because $U_{<0} = \emptyset$).

By induction on n we define a chain $\{R_n \mid n < \omega\}$ of subrings R_n of $\mathbb{Q}[X_n]$ and let $R = \bigcup_{n<\omega} R_n$. Let $R_0 = \mathbb{Z}[x_\gamma \mid \gamma < \lambda]$ be the polynomial ring with integer coefficients in λ commuting variables. Given R_n, we will choose an enumeration

$$\mathcal{R}_n = \{r_{\alpha n} \mid \alpha \in U_n\}$$

(without repetition) of all polynomials from $R_n \setminus \{0\}$ to define R_{n+1}.

Let $x_{\alpha\perp} := r_{\alpha n} \in \mathcal{R}_n$ and put

$$R_{n+1} = \left\langle R_n, p_{nki}^{-\ell} x_{\alpha\eta}, q_{nki}^{-\ell}(x_{\alpha\eta} - x_{\alpha\nu}) \;\middle|\; \alpha \in U_n,\ \eta \in \mathcal{T}_\alpha \setminus \{\perp\},\ i, k, \ell < \omega \right\rangle \subseteq \mathbb{Q}[X_{n+1}]$$

subject to the conditions

$$c_\alpha(\eta) = i, \qquad \lg \eta = k + 1, \qquad \eta \upharpoonright k = \nu$$

where $\langle S \rangle$ denotes the ring generated by the set S.

Using the notation $p^{-\infty}a$ from above R_{n+1} is generated as a ring by the set

$$\left\{ R_0,\ p_{mki}^{-\infty} x_{\alpha\eta},\ q_{mki}^{-\infty}(x_{\alpha\eta} - x_{\alpha\nu}) \;\middle|\; \alpha \in U_m,\ \eta \in \mathcal{T}_\alpha \setminus \{\perp\},\ m \leqslant n,\ \text{and } i, k < \omega \right\}$$

with the restrictions of the last display.

The ring R is situated between the polynomial rings $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$. Our main result will then be the following

Main Theorem 3.1. *If λ is any infinite cardinal $< \kappa(\omega)$ (the first ω-Erdős cardinal), then R is an absolute E-ring of cardinality λ. Moreover $\mathbb{Z}[X] \subseteq R \subseteq \mathbb{Q}[X]$ with X a family of λ commuting free variables.*

The main step for a proof of this theorem is the central result of the next section.

4. Invariant principal ideals of *R*

Theorem 4.1. *If $\varphi \in \operatorname{End}_{\mathbb{Z}} R$ and $r \in R$, then $r\varphi \in (Rr)_*$.*

Here $(Rr)_*$ denotes the (unique) group purification of the principal ideal Rr of R, which is the smallest ideal I of R containing Rr with torsion-free abelian quotient R/I. Theorem 4.1 can be rephrased saying that all purified principal ideals of R are fully invariant under the action of $\operatorname{End}_{\mathbb{Z}} R$.

Recall that a submodule U of an R-module M is fully invariant if U is an $\operatorname{End}_R M$-submodule of M. We begin with a countable family of ideals which by arithmetical reasons are obviously fully invariant ideals of the ring R:

- If $p = p_{nki} \in \Pi$, then $I_{nki} := p^{-\infty}R = \bigcap_{\ell<\omega} p^{-\ell}R$ is a fully invariant ideal of R.
- If $q = q_{nki} \in \Pi$, then $J_{nki} := q^{-\infty}R = \bigcap_{\ell<\omega} q^{-\ell}R$ is a fully invariant ideal of R.

We want to characterize these ideals in different ways and define two families of ring homomorphisms accordingly.

Definition 4.2. Let $p = p_{nki} \in \Pi$ and $q = q_{nki} \in \Pi$, respectively.

(i) The ring homomorphism $F_{nki}^p : R \longrightarrow \mathbb{Q}[X]$ is defined by

$$x_{\alpha\eta} F_{nki}^p := \begin{cases} 0 & \text{if } \alpha \in U_n, \eta \in \mathcal{T}_\alpha, c_\alpha(\eta) = i, \lg \eta = k+1, \\ x_{\alpha\eta} & \text{otherwise.} \end{cases}$$

(ii) The ring homomorphism $F^q_{nki} : R \longrightarrow \mathbb{Q}[X]$ is defined by

$$x_{\alpha\eta} F^q_{nki} := \begin{cases} x_{\alpha\nu} & \text{if } \alpha \in U_n, \eta \in \mathcal{T}_\alpha, c_\alpha(\eta) = i, \lg \eta = k+1, \nu = \eta \restriction k, \\ x_{\alpha\eta} & \text{otherwise.} \end{cases}$$

The maps F^p_{nki}, F^q_{nki} extend uniquely from the free generators of $\mathbb{Q}[X]$ to ring endomorphisms of $\mathbb{Q}[X]$ that can be restricted to R. The following lemma characterizes the ideals J_{nki}.

Lemma 4.3. *For $q = q_{nki} \in \Pi$ the following holds.*

$$J_{nki} = \ker F^q_{nki} = \big\langle R(x_{\alpha\eta} - x_{\alpha\nu}) \bigm| \alpha \in U_n,\ \eta \in \mathcal{T}_\alpha \big\rangle_* \subseteq R$$

$$\textbf{subject to the conditions: } c_\alpha(\eta) = i, \quad \lg \eta = k+1, \quad \nu = \eta \restriction k.$$

Here $\langle S\rangle_*$ denotes the group purification in R of the ring $\langle S\rangle$ generated by the set S.

Proof. The inclusion $\langle R(x_{\alpha\eta} - x_{\alpha\nu}) \mid \alpha \in U_n,\ \eta \in \mathcal{T}_\alpha\rangle_* \subseteq J_{nki}$ holds because J_{nki} is pure in R. Next we show that $J_{nki} \subseteq \ker F^q_{nki}$:

Note that $(x_{\alpha\eta} - x_{\alpha\nu})F^q_{nki} = 0$ whenever $\alpha \in U_n, c_\alpha(\eta) = i, \lg \eta = k+1$ and $\nu = \eta \restriction k$, thus

$$RF^q_{nki} \subseteq \big\langle p^{-\infty}\mathbb{Z}[X] \bigm| q_{nki} \neq p \in \Pi \big\rangle \subseteq \mathbb{Q}[X].$$

If $r \in J_{nki}$, then $r \in q^{-\infty}_{nki} R$ and $rF^q_{nki} \in q^{-\infty}_{nki}(RF^q_{nki})$. But by the above inclusion $q^{-\infty}_{nki}(RF^q_{nki}) = 0$ is obvious, hence $r \in \ker F^q_{nki}$.

For $\ker F^q_{nki} \subseteq \langle R(x_{\alpha\eta} - x_{\alpha\nu}) \mid \alpha \in U_n,\ \eta \in \mathcal{T}_\alpha\rangle_*$ we consider any $0 \neq r \in \ker F^q_{nki}$. As an element from R there is an integer $a \neq 0$ such that $ar \in \mathbb{Z}[X]$ can be expressed as a finite sum, where we isolate the variables $x_{\alpha\eta} \in X$ that meet the restriction of the lemma. Thus we represent

$$ar = \sum_{j=1}^{s} m_j(x_{\alpha\eta}) \cdot f_j + g,$$

where the monomials $m_j \in \mathbb{Z}[X]$ contain only the $x_{\alpha\eta}$ with $c_\alpha(\eta) = i$, $\lg \eta = k+1$, $\nu = \eta \restriction k$ while the polynomials $f_j, g \in \mathbb{Z}[X]$ do not have contributions from this set. We can express $x_{\alpha\eta} = (x_{\alpha\eta} - x_{\alpha\nu}) + x_{\alpha\nu}$ and rewrite the sum as

$$ar = \sum_{\eta} (x_{\alpha\eta} - x_{\alpha\nu}) \cdot f'_\eta + g'$$

with suitable polynomials $f'_\eta \in \mathbb{Z}[X]$ and $g' = \sum_{j=1}^{s} m_j(x_{\alpha\nu}) \cdot f_j + g$ a polynomial without contributions from $x_{\alpha\eta}$. We now apply F^q_{nki} and get

$$\begin{aligned} 0 = (ar)F^q_{nki} &= \sum_{\eta} \big[(x_{\alpha\eta} - x_{\alpha\nu}) \cdot f'_\eta\big] F^q_{nki} + g' F^q_{nki} \\ &= \sum_{\eta} (x_{\alpha\eta} - x_{\alpha\nu}) F^q_{nki} \cdot f'_\eta F^q_{nki} + g' F^q_{nki} = g'. \end{aligned}$$

Thus $g' = 0$ and $ar = \sum_\eta (x_{\alpha\eta} - x_{\alpha\nu}) \cdot f'_\eta \in \langle R(x_{\alpha\eta} - x_{\alpha\nu}) \mid \alpha \in U_n, \eta \in \mathcal{T}_\alpha \rangle$. It follows that r belongs to the corresponding purification as claimed. Thus the three displayed sets of the lemma coincide. □

Similarly we can characterize the ideals I_{nki}. The proof follows the arguments of the previous lemma.

Lemma 4.4. *For $p = p_{nki} \in \Pi$ the following holds.*

$$I_{nki} = \ker F^p_{nki} = \left\langle Rx_{\alpha\eta} \;\middle|\; \alpha \in U_n,\ \eta \in \mathcal{T}_\alpha,\ c_\alpha(\eta) = i,\ \lg \eta = k+1 \right\rangle_* \subseteq R.$$

We now come to the proof of Theorem 4.1. Let $\varphi \in \operatorname{End}_{\mathbb{Z}} R$ and $0 \neq r \in R$ be as in the theorem. We fix some $n \in \omega$ and $\alpha \in U_n$ with $r = r_{\alpha n} = x_{\alpha\perp}$ and consider the family $X_\alpha = \{x_{\alpha\eta} \mid \eta \in \mathcal{T}_\alpha\}$ of generators from R and let $Y_\alpha = \{y_{\alpha\eta} := x_{\alpha\eta}\varphi \mid \eta \in \mathcal{T}_\alpha\}$. From the definition of the variables $x_{\alpha\eta}$, the ideals I_{nki}, J_{nki} and the observation that these ideals are fully invariant we get

- $x_{\alpha\perp} = r$ and $y_{\alpha\perp} = r\varphi$.
- If $\lg \eta > 0$, then $x_{\alpha\eta} \in X$.
- If $c_\alpha(\eta) = i$, $\lg \eta = k+1$, $\nu = \eta \restriction k$, then $x_{\alpha\eta} \in I_{nki}$ and $x_{\alpha\eta} - x_{\alpha\nu} \in J_{nki}$.
- If $c_\alpha(\eta) = i$, $\lg \eta = k+1$, $\nu = \eta \restriction k$, then $y_{\alpha\eta} \in I_{nki}$ and $y_{\alpha\eta} - y_{\alpha\nu} \in J_{nki}$.

The next definition helps to investigate Y_α.

Definition 4.5. If $\eta \in \mathcal{T}_\alpha$, then let $\Lambda_\alpha(\eta)$ be the set of all monomials from $\langle X \rangle$ which appear in the canonical representation of $y_{\alpha\eta}$ in $\mathbb{Q}[X]$. If $m \in \langle X \rangle$, then let $\text{active}_{\alpha\eta}(m)$ be the list of all variables $x_{\beta\nu}$ from m with $c_\alpha(\eta) = c_\beta(\nu)$.

The definition of $\text{active}_{\alpha\eta}(m)$ will mainly be used for $m \in \Lambda_\alpha(\eta)$. Note that a list is not a set: a variable $x_{\beta\nu}$ will appear with its multiplicity (for m) which in general may be > 1. We do not care about the ordering of this list.

Corollary 4.6. *For $\eta \in \mathcal{T}_\alpha$, $c_\alpha(\eta) = i$, $\lg \eta = k+1$ and $m \in \Lambda_\alpha(\eta)$ the following holds.*

(i) $\text{active}_{\alpha\eta}(m) \neq \emptyset$.
(ii) *If $k > 0$ and $\nu = \eta \restriction k$, then $m' := mF^q_{nki} \in \langle X \rangle$ and for the lists we have* $(\text{active}_{\alpha\eta}(m))F^q_{nki} \subseteq \text{active}_{\alpha\nu}(m')$.

From Corollary 4.6(ii) follows immediately $|\text{active}_{\alpha\eta}(m)| \leqslant |\text{active}_{\alpha\nu}(m')|$ for the sizes of the lists.

Proof. (i) From $x_{\alpha\eta} \in I_{nki}$ follows $y_{\alpha\eta} \in I_{nki}$ and by Lemma 4.4 we have $y_{\alpha\eta}F^p_{nki} = 0$, hence also $mF^p_{nki} = 0$. Note that the members of $\text{active}_{\alpha\eta}(m)$ have color $c_\alpha(\eta) = i$ and thus $\text{active}_{\alpha\eta}(m) \neq \emptyset$ as the map F^p_{nki} replaces $x_{\alpha'\eta'} \in \text{active}_{\alpha\eta}(m) \subseteq X$ with 0.

(ii) If $x_{\alpha'\eta'} \in \text{active}_{\alpha\eta}(m)$, then $\lg \eta' = \lg \eta = k+1$ by Theorem 2.6(iii)(a) and the map F^q_{nki} replaces $x_{\alpha'\eta'}$ by $x_{\alpha'\nu'} \in X$ (as $k > 0$) with $\nu' = \eta' \restriction k$. Thus $mF^q_{nki} \in \langle X \rangle$.

Furthermore, from $x_{\alpha'\eta'} \in \text{active}_{\alpha\eta}(m)$ follows $c_{\alpha'}(\eta') = c_\alpha(\eta)$ and $c_{\alpha'}(\nu') = c_\alpha(\nu)$ by Theorem 2.6(iii)(b). Thus $x_{\alpha'\nu'} = x_{\alpha'\eta'} F^q_{nki} \in \text{active}_{\alpha\nu}(m')$ and for the lists we have $(\text{active}_{\alpha\eta}(m)) F^q_{nki} \subseteq \text{active}_{\alpha\nu}(m')$. $\square$

Recall the trees $\mathcal{T}_\alpha, \mathcal{T}'_\alpha$ from Section 2.

Corollary 4.7. *Let $\nu = \nu' \bullet \tau \in \mathcal{T}_\alpha$ with $\lg \nu' = k$ and $\nu' \lhd \eta'' \in \mathcal{T}'_\alpha$ with $\lg \eta'' = k + 1$. Then there is some branch $\sigma_{\alpha\nu}$ with the following properties.*

(i) *$\tau \lhd \sigma_{\alpha\nu} \in {}^{\omega>}\omega$ with $\lg \sigma_{\alpha\nu} = k + 1$.*
(ii) *If $x_{\beta\xi} \in X$ appears in the canonical representation of $y_{\alpha\nu}$ with $\xi = \xi' \bullet \upsilon$, then $\sigma_{\alpha\nu} \neq \upsilon$.*
(iii) *If $\eta := \eta' \bullet \sigma_{\alpha\nu} \in \mathcal{T}_\alpha$ for some $\nu' \lhd \eta' \in \mathcal{T}'_\alpha$ with $\lg \eta' = k + 1$, then $c_\alpha(\eta) \neq c_\beta(\xi)$ for all $x_{\beta\xi} \in X$ which appear in the canonical representation of $y_{\alpha\nu}$.*

Proof. On the one hand there are only finitely many $x_{\beta\xi}$ which may appear in $y_{\alpha\nu}$, on the other hand there are infinitely many choices for $\sigma_{\alpha\nu} \in {}^{\omega>}\omega$ with (i). So it is easy to choose $\sigma_{\alpha\nu}$ with (ii). Property (iii) is an immediate consequence of (i) and (ii):

The branch $\eta \in \mathcal{T}_\alpha$ is well defined by (i) and the definition of $\mathcal{T}_\alpha$, while from $c_\alpha(\eta) = c_\beta(\xi)$ follows $\sigma_{\alpha\nu} = \upsilon$ by Theorem 2.6(iii)(c), contradicting (ii). $\square$

Definition 4.8. If $\nu = \nu' \bullet \tau \in \mathcal{T}_\alpha$ with $\lg \nu' = k$, then set

$$\text{suc}^*_\alpha(\nu) = \left\{\eta = \eta' \bullet \sigma_{\alpha\nu} \;\middle|\; \nu' \lhd \eta' \in \mathcal{T}'_\alpha,\ \lg \eta' = k + 1\right\}$$

as a set of special successors of ν and

$$\mathcal{T}^*_\alpha = \left\{\eta \in \mathcal{T}_\alpha \;\middle|\; \eta \restriction (\ell + 1) \in \text{suc}^*_\alpha(\eta \restriction \ell) \text{ for all } \ell < \lg \eta\right\}$$

as the subtree of $\mathcal{T}_\alpha$ induced by these successors.

The next corollary shows that for any $\eta \in \mathcal{T}^*_\alpha$ the set $(\mathcal{T}^*_\alpha)_{\geqslant\eta}$ satisfies the closure condition from Definition 2.7 and thus qualifies for Theorem 2.8.

Corollary 4.9. *If $\eta = \eta' \bullet \sigma \in \mathcal{T}^*_\alpha$, then $(\mathcal{T}^*_\alpha)_{\geqslant\eta}$ satisfies the closure condition from Definition* 2.7 *for η. In particular $\mathcal{T}^*_\alpha \subseteq {}^{\omega>}\lambda$ is a subtree with the closure condition for $\eta = \bot \bullet \bot \in \mathcal{T}^*_\alpha$.*

Proof. It is clear that $\mathcal{T}^*_\alpha$ and $(\mathcal{T}^*_\alpha)_{\geqslant\eta}$ are closed under initial segments, thus subtrees of ${}^{\omega>}\lambda$. The closure condition is also immediate from Corollary 4.7. Note that $\nu = \nu' * \tau \in \mathcal{T}_\alpha$ with $\lg \nu' = k$ and $\nu' \lhd \eta' \in \mathcal{T}'_\alpha$ with $\lg \eta' = k + 1$ implies that ν' has successors in $\mathcal{T}'_\alpha$, hence some $\sigma_{\alpha\nu}$ exists and also $\text{suc}^*_\alpha(\nu) \neq \emptyset$. $\square$

Corollary 4.10. *If $\eta \in \mathcal{T}^*_\alpha$, $c_\alpha(\eta) = i$, $\lg \eta = k + 1$ and $\nu = \eta \restriction k$, then the following holds.*

(a) *If $m \in \Lambda_\alpha(\nu)$, then $m F^q_{nki} = m$.*
(b) *$y_{\alpha\eta} F^q_{nki} = y_{\alpha\nu}$.*
(c) *If $k > 0$, $m \in \Lambda_\alpha(\nu)$, then there is $m' \in \Lambda_\alpha(\eta)$ with $m' F^q_{nki} = m$.*

Proof. (a) Suppose that some $x_{\beta\xi} \in X$ appears in m which is not a fix-point of F^q_{nki}. Then necessarily $c_\alpha(\eta) = i = c_\beta(\xi)$ which contradicts Corollary 4.7(iii).

(b) By the choice of $x_{\alpha\eta} - x_{\alpha\nu} \in J_{nki}$ we also have $y_{\alpha\eta} - y_{\alpha\nu} \in J_{nki}$ and thus $(y_{\alpha\eta} - y_{\alpha\nu})F^q_{nki} = 0$ by Lemma 4.3. It follows $0 = y_{\alpha\eta}F^q_{nki} - y_{\alpha\nu}F^q_{nki} = y_{\alpha\eta}F^q_{nki} - y_{\alpha\nu}$ which is (b).

(c) We write $y_{\alpha\nu} = \sum_{m_i \in \Lambda_\alpha(\nu)} a_i m_i$ and $y_{\alpha\eta} = \sum_{m'_j \in \Lambda_\alpha(\eta)} a'_j m'_j$; by (b) follows

$$\sum_{m'_j \in \Lambda_\alpha(\eta)} a'_j m'_j F^q_{nki} = y_{\alpha\eta} F^q_{nki} = y_{\alpha\nu} = \sum_{m_i \in \Lambda_\alpha(\nu)} a_i m_i .$$

The summands on the left-hand side are monomials in $\langle X \rangle$ by Corollary 4.6(ii) and $k > 0$. Comparing the two sides, for any $m_i \in \Lambda_\alpha(\nu)$ there must be an $m'_j \in \Lambda_\alpha(\eta)$ with $m'_j F^q_{nki} = m_i$. So $m_i = m$ demonstrates (c). □

Definition 4.11. If $\eta \in \mathcal{T}^*_\alpha$, $c_\alpha(\eta) = i$, $\lg \eta = k + 1$, $\nu = \eta \restriction k$ and $k > 0$. Then let

$$g_{\alpha\nu} : \Lambda_\alpha(\nu) \longrightarrow \Lambda_\alpha(\eta) \quad \text{with } m g_{\alpha\nu} F^q_{nki} = m \text{ for all } m \in \Lambda_\alpha(\nu).$$

The map $g_{\alpha\nu}$ is well defined by Corollary 4.10(c) and the following holds by Corollary 4.6(ii).

Proposition 4.12. *If $g_{\alpha\nu}$ is defined as in Definition 4.11, then*

$$F^q_{nki} \restriction \mathrm{active}_{\alpha\eta}(m g_{\alpha\nu}) : \mathrm{active}_{\alpha\eta}(m g_{\alpha\nu}) \longrightarrow \mathrm{active}_{\alpha\nu}(m)$$

is an injective map of the lists.

The following innocent looking lemma collects most of the earlier results and is the platform for the final stage of the proof of Theorem 4.1.

Lemma 4.13. *If $\eta \in \mathcal{T}^*_\alpha$, $\lg \eta = k + 1$ and $m \in \Lambda_\alpha(\eta)$, then there is $\xi \in \mathcal{T}_\alpha$ such that $x_{\alpha\xi} \in \mathrm{active}_{\alpha\eta}(m)$.*

Proof. If η is as in the lemma, then we want to define inductively a family $\{m_{\eta'} \mid \eta' \in (\mathcal{T}^*_\alpha)_{\geqslant\eta}\}$ with

(i) $m_{\eta'} \in \Lambda_\alpha(\eta')$,
(ii) $m_\eta := m$,
(iii) $m_{\eta'} := m_{\nu'} g_{\alpha\nu'}$ for $\eta \lhd \eta' \in \mathcal{T}^*_\alpha$ with $c_\alpha(\eta') = i'$, $\lg \eta < \lg \eta' = k' + 1$ and $\nu' = \eta' \restriction k'$.

If $m_{\eta'}$ is from this list, then $m_{\eta'} F^q_{nk'i'} = m_{\nu'}$. First we consider the family

$$\{ |\mathrm{active}_{\alpha\eta'}(m_{\eta'})| \mid \eta' \in (\mathcal{T}^*_\alpha)_{\geqslant\eta} \}.$$

If $\eta \lhd \eta_1 \lhd \eta_2$, then $|\mathrm{active}_{\alpha\eta_2}(m_{\eta_2})| \leqslant |\mathrm{active}_{\alpha\eta_1}(m_{\eta_1})|$ by Corollary 4.6(ii). Choose $\mu \in (\mathcal{T}^*_\alpha)_{\geqslant\eta}$ with $|\mathrm{active}_{\alpha\mu}(m_\mu)|$ minimal. Then $|\mathrm{active}_{\alpha\eta'}(m_{\eta'})|$ is constant for all $\eta' \in (\mathcal{T}^*_\alpha)_{\geqslant\mu}$. If now $\mu \lhd \eta' \in \mathcal{T}^*_\alpha$ with $c_\alpha(\eta') = i'$, $\lg \mu < \lg \eta' = k' + 1$, $\nu' = \eta' \restriction k'$, then

$$F^q_{nk'i'} : \mathrm{active}_{\alpha\eta'}(m_{\eta'}) \longrightarrow \mathrm{active}_{\alpha\nu'}(m_{\nu'}) \text{ is a bijection of lists.} \tag{4.1}$$

By Corollary 4.6(i) we also have that $\text{active}_{\alpha\mu}(m_\mu) \neq \emptyset$. So we can choose

$$x_{\beta\mu'} \in \text{active}_{\alpha\mu}(m_\mu) \tag{4.2}$$

and we define inductively a color preserving partial tree homomorphism

$$\Psi : (\mathcal{T}_\alpha^*)_{\geqslant\mu} \longrightarrow \mathcal{T}_\beta \quad \text{such that} \quad x_{\beta\Psi(\eta')} \in \text{active}_{\alpha\eta'}(m_{\eta'}).$$

First we choose $\Psi(\mu) = \mu' \in \mathcal{T}_\beta$. Since $x_{\beta\mu'} \in \text{active}_{\alpha\mu}(m_\mu)$ we get $c_\beta(\mu') = c_\alpha(\mu)$ and Ψ preserves the color at this stage. Moreover, since the colors code the branches from ${}^{\omega>}\omega$ and the lengths of branches, also $\lg\mu' = \lg\mu$ and Ψ preserves the length at this stage. In the inductive step we consider $\mu \lhd \eta' \in \mathcal{T}_\alpha^*$ with $c_\alpha(\eta') = i'$, $\lg\mu < \lg\eta' = k' + 1$, $\nu' = \eta' \restriction k'$ and $\Psi(\nu') = \xi \in \mathcal{T}_\beta$ such that $x_{\beta\xi} \in \text{active}_{\alpha\nu'}(m_{\nu'})$. By (4.1) there is $x_{\beta'\xi'} \in \text{active}_{\alpha\eta'}(m_{\eta'})$ with $x_{\beta'\xi'}F^q_{nk'i'} = x_{\beta\xi}$. We put $\Psi(\eta') = \xi'$. By definition of $F^q_{nk'i'}$ follows $\beta = \beta'$, and $\xi' \in \mathcal{T}_\beta$ with $\xi' \restriction (\lg\xi' - 1) = \xi$. Thus $\Psi(\eta') \in \mathcal{T}_\beta$ preserves lengths and initial segments; moreover $x_{\beta\Psi(\eta')} \in \text{active}_{\alpha\eta'}(m_{\eta'})$. Finally $x_{\beta\Psi(\eta')} \in \text{active}_{\alpha\eta'}(m_{\eta'})$ implies that $c_\beta(\Psi(\eta')) = c_\alpha(\eta')$, so Ψ also preserves the color and thus is as required above. We are ready to apply Theorem 2.8 (together with Corollary 4.9) and derive that $\alpha = \beta$. By (4.2) there is $\mu' \in \mathcal{T}_\alpha$ such that $x_{\alpha\mu'} \in \text{active}_{\alpha\mu}(m_\mu)$. Applying Corollary 4.6(ii) and $\eta \lhd \mu$ we also find some $x_{\alpha\xi} \in \text{active}_{\alpha\eta}(m_\eta) = \text{active}_{\alpha\eta}(m)$ and the crucial lemma is shown. □

The final stage of the proof of Theorem 4.1. We now chose any $\eta \in \mathcal{T}_\alpha$ with $\lg\eta = 1$, $c_\alpha(\eta) = i$. By Lemma 4.13 we can write

$$y_{\alpha\eta} = \sum_{m_i \in \Lambda_\alpha(\eta)} a_i m_i = \sum_{m_i \in \Lambda_\alpha(\eta)} a_i x_{\alpha\eta_i} m_i',$$

where $m_i = x_{\alpha\eta_i} m_i'$ with $x_{\alpha\eta_i} \in \text{active}_{\alpha\eta}(m_i)$. It follows that $c_\alpha(\eta_i) = c_\alpha(\eta)$ and thus $\lg\eta_i = \lg\eta = 1$. By the earlier choice of $x_{\alpha\perp} = r$, the definition of $y_{\alpha\perp}$ and Corollary 4.10(b) we get from the above that

$$r\varphi = x_{\alpha\perp}\varphi = y_{\alpha\perp} = y_{\alpha\eta}F^q_{n0i} = \sum_{m_i \in \Lambda_\alpha(\eta)} a_i \left(x_{\alpha\eta_i}F^q_{n0i}\right)\left(m_i' F^q_{n0i}\right) = r \sum_{m_i \in \Lambda_\alpha(\eta)} a_i \left(m_i' F^q_{n0i}\right)$$

is an element from $(Rr)_*$. □

5. The Main Theorem and consequences

5.1. Proof of Main Theorem 3.1

Lemma 5.1. *Let $\varphi \in \text{End}_{\mathbb{Z}}\,\mathbb{Q}[X]^+$ with*

$$f\varphi \in \mathbb{Q}[X] \cdot f \quad \textit{for all } f \in \mathbb{Q}[X],$$

then φ is multiplication by an element of $\mathbb{Q}[X]$.

Proof. By hypothesis on φ we find for each $f \in \mathbb{Q}[X]$ an element $g_f \in \mathbb{Q}[X]$ such that $f\varphi = f \cdot g_f$. If $m \in \langle X \rangle$ is a monomial and $x \in X$, then $m\varphi = m \cdot g_m = m(x) \cdot g_m(x)$ and $(xm)\varphi = xm \cdot g_{xm} = x \cdot m(x) \cdot g_{xm}(x)$ seen as functions $g(x)$ depending on x. Now we fix $r \in \mathbb{Q}$ and use $\mathrm{End}_{\mathbb{Z}}\, \mathbb{Q}[X]^+ = \mathrm{End}_{\mathbb{Q}}\, \mathbb{Q}[X]^+$ to compute $(rm - xm)\varphi = r \cdot m\varphi - (xm)\varphi = r \cdot m(x) \cdot g_m(x) - x \cdot m(x) \cdot g_{xm}(x)$, while by hypothesis also $(rm - xm)\varphi = (rm - xm) \cdot g_{rm-xm}(x)$ holds. Thus

$$(rm - xm) \cdot g_{rm-xm}(x) = r \cdot m(x) \cdot g_m(x) - x \cdot m(x) \cdot g_{xm}(x).$$

We now substitute $x := r$ into this polynomial equation and get

$$0 = r \cdot m(r) \cdot g_m(r) - r \cdot m(r) \cdot g_{xm}(r) \tag{5.1}$$

which holds for all $r \in \mathbb{Q}$. If $r \neq 0$ also $rm(r)$ is a non-zero element of the integral domain $\mathbb{Q}[X]$, so (5.1) gives

$$h(r) = 0 \quad \text{for } h(x) = g_m(x) - g_{xm}(x) \text{ and for all } 0 \neq r \in \mathbb{Q}.$$

Thus $x - r$ is a factor of $h(x)$ for infinitely many $r \in \mathbb{Q}$, which is only possible if h is the zero-polynomial and $g_m = g_{xm}$. We apply this recursively for all monomials $m \in \langle X \rangle$ to get $g_m = g_1$ for all $m \in \langle X \rangle$, and it is now clear (by linearity) that also $g_f = g_1$ for all $0 \neq f \in \mathbb{Q}[X]$. We conclude $\varphi = g_1 \cdot \mathrm{id}$, where id denotes the identity map on $\mathbb{Q}[X]$. □

Proof of Main Theorem 3.1. Let $\varphi \in \mathrm{End}_{\mathbb{Z}}\, R$ for the ring R constructed in Section 3. Since the additive group of $\mathbb{Q}[X]$ is divisible, φ can be lifted to a group endomorphism of $\mathbb{Q}[X]^+$ and satisfies by Theorem 4.1 the hypothesis of Lemma 5.1. Thus $\varphi = g \cdot \mathrm{id}$ for some polynomial $g \in \mathbb{Q}[X]$. However $1\varphi = g \in R$ which completes the proof. □

5.2. *Large families of E-rings*

The Main Theorem 3.1 can easily be extended to a family of rigid E-rings. For this decompose the family of trees given by Theorem 2.1 into 2^λ families of trees $\{(T_\alpha, c_\alpha) \mid \alpha \in \lambda_i\}$ of size 2^λ $(i < 2^\lambda)$ and apply the earlier arguments for the corresponding families of trees. We get E-rings R_i $(i < 2^\lambda)$ and the following holds.

Corollary 5.2. *If λ is any infinite cardinal $< \kappa(\omega)$ (the first ω-Erdős cardinal), there is a family R_i $(i < 2^\lambda)$ of absolute E-rings of cardinality λ. If* $\mathrm{Hom}_{\mathbb{Z}}(R_i^+, R_j^+) \neq 0$ *in some generic extension of the universe for some $i, j < 2^\lambda$, then $i = j$; thus $\{R_i \mid i < 2^\lambda\}$ is absolutely rigid, and also $\mathbb{Z}[X] \subseteq R_i \subseteq \mathbb{Q}[X]$ for all $i < 2^\lambda$ for a set X of λ commuting free variables.*

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