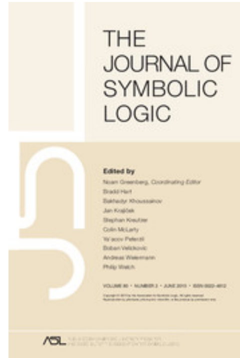

Ladder gaps over stationary sets

Uri Abraham and Saharon Shelah

The Journal of Symbolic Logic / Volume 69 / Issue 02 / June 2004, pp 518 - 532
DOI: 10.2178/jsl/1082418541, Published online: 12 March 2014

THE JOURNAL OF SYMBOLIC LOGIC
Volume 69, Number 2, June 2004

LADDER GAPS OVER STATIONARY SETS

URI ABRAHAM AND SAHARON SHELAH†

Abstract. For a stationary set $S \subseteq \omega_1$ and a ladder system C over S, a new type of gaps called C-Hausdorff is introduced and investigated. We describe a forcing model of ZFC in which, for some stationary set S, for every ladder C over S, every gap contains a subgap that is C-Hausdorff. But for every ladder E over $\omega_1 \setminus S$ there exists a gap with no subgap that is E-Hausdorff.

A new type of chain condition, called polarized chain condition, is introduced. We prove that the iteration with finite support of polarized c.c.c. posets is again a polarized c.c.c. poset.

§1. Introduction. We first review some notations and definitions related to Hausdorff gaps. In fact we follow here the terminology given by M. Scheepers in his monograph [3] on Hausdorff gaps, but since we restrict ourselves to (ω_1, ω_1^*) gaps our nomenclature is somewhat simpler. The collection of all infinite subsets of ω is denoted $[\omega]^\omega$, and for $a, b \in [\omega]^\omega$, $a \subseteq^* b$ means that $a \setminus b$ is finite. In this case $X(a, b)$ (the "excess" number) is defined to be the least k such that $a \setminus b \subseteq k$. Thus $a \setminus X(a, b) \subseteq b$, but if $X(a, b) > 0$ then $X(a, b) - 1 \in a \setminus b$.

A pre-gap is a pair of sequences $g = \{(a_i \mid i \in I), (b_j \mid j \in J)\}$ where I, $J \subseteq \omega_1$ are uncountable and $a_i, b_j \in [\omega]^\omega$ are such that

$$a_{i_0} \subseteq^* a_{i_1} \subseteq^* b_{j_1} \subseteq^* b_{j_0}$$

whenever $i_0 < i_1$ are in I and $j_0 < j_1$ in J. In most cases $I = J = \omega_1$. Given a pre-gap as above, and uncountable subsets $I' \subseteq I$ and $J' \subseteq J$, the restriction $g \restriction (I', J')$ of g is the pre-gap $\{(a_i \mid i \in I'), (b_j \mid j \in J')\}$. We write $g \restriction I$ for $g \restriction (I, I)$.

An interpolation for a pre-gap g is a set $x \in [\omega]^\omega$ such that

$$a_i \subseteq^* x \subseteq^* b_j$$

for every i and j. A pre-gap with no interpolation is called a gap. A famous construction of Hausdorff produces gaps in ZFC (which are now called Hausdorff gaps). Specifically, a Hausdorff gap is a pre-gap $g = \{(a_i \mid i \in \omega_1), (b_j \mid j \in \omega_1)\}$ such that for every $\alpha \in \omega_1$ and $n \in \omega$ the set

$$\{\beta \in \alpha \mid a_\beta \setminus n \subseteq b_\alpha\}$$

is finite.

Received July 29, 2002; revised February 9, 2004.
†The author would like to thank the Israel Science Foundation, founded by the Israel Academy of Science and Humanities. Publication # 598.

0022-4812/04/6902-0013/$2.50

A *special* (or Kunen) gap is a pre-gap $g = \{(a_i \mid i \in \omega_1), (b_j \mid j \in \omega_1)\}$ such that for some $n_0 \in \omega$:

1. $a_\alpha \setminus n_0 \subseteq b_\alpha$ for every $\alpha \in \omega_1$, and
2. for all $\alpha < \beta < \omega_1$,

$$(a_\alpha \cup a_\beta) \setminus n_0 \not\subseteq b_\alpha \cap b_\beta$$

(equivalently, $a_\alpha \setminus n_0 \not\subseteq b_\beta$ or $a_\beta \setminus n_0 \not\subseteq b_\alpha$).

The interest in these definitions arises from the fact (not too difficult to prove) that Hausdorff and Kunen pre-gaps are gaps and remain gaps as long as ω_1 is not collapsed: they have no interpolation in any extension in which ω_1 remains uncountable.

In this paper we define two additional types of "special" gaps: S-Hausdorff gaps where $S \subseteq \omega_1$ is a stationary set, and C-Hausdorff gaps, where C is a ladder system over S.

The motivation for this work is the desire to find an example with gaps of the phenomenon in which ω_1 is "split" in a certain behavior on a stationary set $S \subset \omega_1$ and an opposite behavior on its complement $\omega_1 \setminus S$.

DEFINITION 1.1. Let $S \subseteq \omega_1$ be a stationary set. A pre-gap $g = \{(a_i \mid i \in \omega_1), (b_j \mid j \in \omega_1)\}$ is S-Hausdorff iff for some closed unbounded (club) set $D \subseteq \omega_1$, for every $\delta \in S \cap D$ and for every sequence of ordinals $(i_n \in \delta \mid n \in \omega)$ increasing and cofinal in δ

$$\lim_{n\to\infty} X(a_{i_n}, b_\delta) = \infty. \tag{1}$$

That is, for every k, there is only a finite number of $n \in \omega$ for which $a_{i_n} \setminus k \subseteq b_\delta$. Since, for $\delta < \delta'$, $b_{\delta'} \subseteq^* b_\delta$, it follows that eventually $X(a_{i_n}, b_\delta) \le X(a_{i_n}, b_{\delta'})$, and hence (1) holds for every $\delta' \ge \delta$ in ω_1. If the pre-gap $g = \{(a_i \mid i \in I), (b_j \mid j \in J)\}$ is defined only on uncountable sets $I, J \subseteq \omega_1$, we can still define it to be S-Hausdorff if for some closed unbounded set $D \subseteq \omega_1$, for every $\delta \in S \cap D$, for every $j \in J \setminus \delta$, and for every increasing sequence of ordinals $(i_n \in \delta \cap I \mid n \in \omega)$ cofinal in δ,

$$\lim_{n\to\infty} X(a_{i_n}, b_j) = \infty \tag{2}$$

Clearly, every Hausdorff gap is an ω_1-Hausdorff gap, and the closed unbounded set D can be taken to be ω_1. The converse of this also holds, in the sense that every ω_1-Hausdorff gap contains a Hausdorff gap. For suppose that $g = \{(a_i \mid i \in I), (b_j \mid j \in J)\}$ is some ω_1-Hausdorff gap, and let $D \subseteq \omega_1$ be the closed unbounded set given by the definition of g as an ω_1-Hausdorff gap. Define $I' \subseteq I$ such that every two members of I' contain a point from D in between. We claim that $g' = \{(a_i \mid i \in I'), (b_j \mid j \in J)\}$, is a Hausdorff gap. Indeed, if $\alpha \in J$ then for every $n \in \omega$ the set $E = \{\beta \in \alpha \cap I' \mid a_\beta \setminus n \subseteq b_\alpha\}$ is necessarily finite. For if not, then let δ be an accumulation point of E, and let $\beta_i \in E$, for $i \in \omega$, be increasing and converging to δ. Necessarily $\delta \in D$, and $a_{\beta_i} \setminus n \subseteq b_\alpha$ shows that (2) does not hold.

PROPOSITION 1.2. *If $S \subseteq \omega_1$ is stationary, then any S-Hausdorff pre-gap is a gap. (So that any pre-gap containing an S-Hausdorff pre-gap is a gap.)*

PROOF. Assume that this not so and let x be an interpolation of an S-Hausdorff pre-gap $g = \{(a_i \mid i \in I), (b_j \mid j \in J)\}$. Then there is a fixed $n_0 \in \omega$ such that for unbounded sets of indices $I' \subseteq I$ $J' \subseteq J$, for every $\alpha \in I'$, $\beta \in J'$

$$a_\alpha \setminus n_0 \subseteq x \setminus n_0 \subseteq b_\beta.$$

And as a consequence

$$a_\alpha \setminus n_0 \subseteq b_\beta \tag{3}$$

holds. Since g is assumed to be S-Hausdorff there exists a closed unbounded set $D \subseteq \omega_1$ as in the definition. We may assume that every $\delta \in D$ is an accumulation point of I'. Take now a limit $\delta \in S \cap D$ and any sequence $i_n \in I' \cap \delta$ increasing to δ. Take any $j \in J' \setminus \delta$. Then equation (3) implies that $X(a_{i_n}, b_\delta) \leq n_0$, which is a contradiction. ⊣

A seemingly stronger notion than that of being S-Hausdorff can be defined if the rate at which the sequences in (1) tend to infinity is uniform. For this we must recall the definition of a scale or ladder system on a stationary set.

If $S \subseteq \omega_1$ is a stationary set, then a ladder (system) over S is a sequence $C = (c_\alpha \mid \alpha \in S \textit{ is a limit ordinal})$ such that every $c_\alpha = (c_\alpha(n) \mid n \in \omega)$ is an increasing, cofinal in α, ω-sequence.

DEFINITION 1.3. For a ladder system C over S, we say that a pre-gap $g = \{(a_i \mid i \in I), (b_j \mid j \in J)\}$ is C-Hausdorff iff for some closed-unbounded set $D \subseteq \omega_1$ for all $\delta \in S \cap D$ and $j \in J \setminus \delta$ there is $k \in \omega$ such that for every $n \geq k$ in ω, if $i \in I$, $c_\delta(n) < i < \delta$, then $X(a_i, b_j) > n$.

Every C-Hausdorff gap (where C is a ladder system over a stationary set S) is S-Hausdorff. Every S-Hausdorff gap is actually C-Hausdorff for some ladder over S (we shall not use this). Our aim in this paper is to prove the following consistency result concerning the possibility that a stationary subset $S \subseteq \omega_1$ exhibits combinatorial properties that are opposed to its complement $\omega_1 \setminus S$.

THEOREM 1.4. *Assume G.C.H for simplicity. Suppose that κ is a cardinal such that* $\mathrm{cf}(\kappa) > \aleph_1$. *Let S be a stationary co-stationary subset of ω_1. Then there is a c.c.c. poset of size κ such that in every generic extension made via P $2^{\aleph_0} = \kappa$ and the following hold.*

1. *For every ladder system C over S, every gap contains a subgap that is C-Hausdorff.*
2. *For every ladder system E over $\omega_1 \setminus S$ there is a gap with no subgap that is E-Hausdorff.*

We can say that a ladder C over S has the *gap property* if every gap contains a subgap that is C-Hausdorff. So our consistency result is that every ladder over S has the gap property, but no ladder over $\omega_1 \setminus S$ has that property.

§2. Gaps introduced by forcing. Gaps can be created by forcing with finite conditions (a method due to Hechler [1]). These gaps are not S-Hausdorff for any stationary set, as we are going to see.

If $f \in 2^n$ (f is a function defined on n with range included in $\{0, 1\}$) then f is a characteristic function and we let $[f] = \{k \mid f(k) = 1\}$ be the subset of n represented by f.

Let $(I, <_I)$ be any ordering isomorphic to $\omega_1 + \omega_1^*$. For example take $I = (\omega_1 \times \{0\}) \cup (\omega_1 \times \{1\})$ with $\langle\alpha, 0\rangle <_I \langle\beta, 0\rangle <_I \langle\beta, 1\rangle <_I \langle\alpha, 1\rangle$ whenever $\alpha < \beta < \omega_1$.

Define the poset P by $p \in P$ iff p is a finite function defined on I and such that:

1. For some n (called the "height" of p) $p(i) \in 2^n$ for every $i \in \text{dom}(p)$. (The height of the empty function is defined to be 0.)
2. For every $\alpha \in \omega_1$, $\langle\alpha, 0\rangle \in \text{dom}(p)$ iff $\langle\alpha, 1\rangle \in \text{dom}(p)$, and in this case $[p(\langle\alpha, 0\rangle)] \subseteq [p(\langle\alpha, 1\rangle)]$.

The intuition behind this definition is that for $\alpha \in \omega_1$, $p(\langle\alpha, 0\rangle)$ will "grow" to become a_α, and $p(\langle\alpha, 1\rangle)$ will finally become b_α, as p runs over the generic filter. So that $(\langle a_\alpha \mid \alpha \in \omega_1\rangle, \langle b_\alpha \mid \alpha \in \omega_1\rangle)$ will be the generic gap with the additional property that $a_\alpha \subseteq b_\alpha$ for every α. The ordering of P reflects this intuition as follows.

For $p_1, p_2 \in P$ define $p_1 \leq p_2$ (p_2 extends p_1) iff

1. $d_1 = \text{dom}(p_1) \subseteq d_2 = \text{dom}(p_2)$, and for every $i \in d_1$, $p_1(i) \subseteq p_2(i)$ (so $\text{height}(p_1) \leq \text{height}(p_2)$).
2. For every $i, j \in \text{dom}(p_1)$, if $i <_I j$ then

$$[p_2(i)] \setminus [p_1(i)] \subseteq [p_2(j)].$$

The reader should check that $\leq$ is transitive.

It is easy to see that any condition in P has extensions with arbitrarily large height and with domains that extend arbitrarily over I. In fact, given $i \in \text{dom}(p)$ and $k \in \omega$ above height p, we can require that the extension p' puts k in $[p'(i)]$.

If $\alpha \in \omega_1$, we can write $\alpha \in \text{dom}(p)$ instead of $\langle\alpha, 0\rangle \in \text{dom}(p)$ (which is equivalent to $\langle\alpha, 1\rangle \in \text{dom}(p)$). So $\text{dom}(p)$ has two meanings, and the context decides if it means a set of ordinals or a set of pairs.

Suppose that $A \subseteq I$ is such that $\langle\alpha, 0\rangle \in A$ iff $\langle\alpha, 1\rangle \in A$. Let P_A be the subposet of P consisting of all conditions p such that $\text{dom}(p) \subseteq A$. If $p \in P$ then $p \restriction A \in P_A$ and $p \restriction A \leq p$. We prove some additional properties of this restriction map taking p to $p \restriction A$.

In the definition of $p \leq q$ what really counts is the restriction of q to the domain of p. That is, $p \leq q$ iff $p \leq q \restriction \text{dom}(p)$. It follows that $p \leq q$ implies that $p \restriction A \leq q \restriction A$. It also follows that if p and q are conditions such that for $C = \text{dom}(p) \cap \text{dom}(q)$, $p \restriction C = q \restriction C$, then p and q are compatible. In fact, in this case, if $\text{height}(p) = \text{height}(q)$ then $p \cup q$ is the minimal extension of p and q. (Observe that if $C \neq \emptyset$ then $p \restriction C = q \restriction C$ implies that $\text{height}(p) = \text{height}(q)$, yet if $C = \emptyset$ then $r = p \cup q$ is not a condition if $\text{height}(p) \neq \text{height}(q)$, since in defining p we required that each condition has a uniform height.)

Suppose that $\text{dom}(p) = \text{dom}(q)$. Then p and q are compatible in P iff $p \leq q$ or $q \leq p$.

For compatible conditions p and q, we define below a canonical extension $p \vee q$ of both p and q. However, P is not a lattice and $p \vee q$ is not the minimum of all extensions of p and q. To define it, we first make an observation. Consider $C = \text{dom}(p) \cap \text{dom}(q)$. Then $p \restriction C$ and $q \restriction C$ are comparable in P_C (since they are compatible and have the same domain), and hence we can assume without loss of generality that $q \geq p \restriction C$ and $n = \text{height}(q) \geq m = \text{height}(p)$ (the restriction on the heights is needed only in case $C = \emptyset$ since it follows from $q \geq p \restriction C$ otherwise).

Then $r = p \vee q$ is defined as follows on $\mathrm{dom}(p) \cup \mathrm{dom}(q)$, and it will be evident that $p \vee q$ is an extension of p and q.

For $i \in \mathrm{dom}(q)$ define $r(i) = q(i)$. For $i \in \mathrm{dom}(p) \setminus C$ define $r(i) \in 2^n$ by the following two conditions:

$$p(i) \subseteq r(i). \tag{4}$$

$$[r(i)] \setminus [p(i)] = \bigcup\{[q(k)] \setminus m \mid k <_I i \text{ and } k \in C\}. \tag{5}$$

This definition makes sense since $C \subseteq \mathrm{dom}(q)$.

It is clear that $r \in P$, $\mathrm{dom}(r) = \mathrm{dom}(p) \cup \mathrm{dom}(q)$ and $r \restriction C = q$. We prove that $r \geq p$. Clause 1 in the definition of extension is obvious, and we have to check clause 2. Suppose that $i, j \in \mathrm{dom}(p)$ and $i <_I j$. We have to show that

$$[r(i)] \setminus [p(i)] \subseteq [r(j)]. \tag{6}$$

So consider any $a \in [r(i)] \setminus [p(i)]$.

CASE 1: $i \in C$. Then $r(i) = q(i)$. If $j \in C$ as well, then (6) follows from our assumption that $q \geq p \restriction C$, and since $r(i) = q(i)$, $r(j) = q(j)$ in this case. If, on the other hand, $j \notin C$, then

$$[r(j)] \setminus [p(j)] = \bigcup\{[q(k)] \setminus m \mid k <_I j \text{ and } k \in C\}$$

by the definition of r. Since $i \in C$, $i <_I j$, and $a \in q(i) \setminus m$, it follows that $a \in [r(j)] \setminus [p(j)]$ as required.

CASE 2: $i \notin C$. Then $i \in \mathrm{dom}(p) \setminus C$ and (5) implies that for some $k \in C$ such that $k <_I i$, $a \in [q(k)] \setminus m$. Then $k <_I j$, both indices are in $\mathrm{dom}(p)$, and $k \in C$, which brings us back to Case 1. ⊣

This argument is summed up in the following lemma.

LEMMA 2.1. *Suppose that $p_1, p_2 \in P$ and $C = \mathrm{dom}(p_1) \cap \mathrm{dom}(p_2)$ are such that $p_1 \restriction C \geq p_2 \restriction C$ and* $\mathrm{height}(p_1) \geq \mathrm{height}(p_2)$. *Then $p_1 \vee p_2$ can be formed (an extension of p_1 and p_2).*

LEMMA 2.2. *P satisfies the c.c.c. In fact if $\{p_\alpha \mid \alpha \in S\} \subseteq P$ where $S \subseteq \omega_1$ is stationary, then for some stationary set $S' \subseteq S$, every finite set of conditions in $\{p_\alpha \mid \alpha \in S'\}$ is compatible. (This is Talayaco's condition* [4].)

PROOF. If $p, q \in P$ have the same height and for $C = \mathrm{dom}(p) \cap \mathrm{dom}(q)$ it happens that $p \restriction C = q \restriction C$, then $p \cup q$ is an extension of p and q. Hence a Δ-system argument works here. ⊣

If $G \subset P$ is some generic filter over P, define for every $\alpha \in \omega_1$ $a_\alpha = \bigcup\{[p(\langle \alpha, 0\rangle)] \mid p \in G\}$, and $b_\alpha = \bigcup\{[p(\langle \alpha, 1\rangle)] \mid p \in G\}$. A standard density argument shows that g is a pre-gap, and we denote it as g.

LEMMA 2.3. *The generic pre-gap g is a gap.*

PROOF. Suppose that $x \in V^P$ is a name, forced to be an interpolation for the generic pre-gap g. For every $\alpha \in \omega_1$ find a condition $p_\alpha \in P$ with α in its domain and a number $n_\alpha \in \omega$ such that

$$p_\alpha \Vdash_P a_\alpha \setminus n_\alpha \subseteq x \setminus n_\alpha \subseteq b_\alpha. \tag{7}$$

Then for some stationary set $S \subseteq \omega_1$, and some fixed $n \in \omega$, $n = n_\alpha$ for every $\alpha \in S$, and the sets $\mathrm{dom}(p_\alpha)$ form a Δ-system with finite core C. We also assume that $p_\alpha \restriction C$ is fixed for $\alpha \in S$. For $\alpha < \beta$, both in S and above the ordinals involved in C, consider p_α and p_β. Pick any $k \geq n$ such that $k \geq \text{height}(p_\alpha)$ as well. Let $i = \langle \alpha, 0 \rangle$, and $j = \langle \beta, 1 \rangle$. We shall find an extension r of p_α and p_β such that $r(i)(k) = 1$ and $r(j)(k) = 0$. Then $r \Vdash k \in a_\alpha \wedge k \notin b_\beta$. But this contradicts (7).

To define r, define first an extension $p'_\alpha \geq p_\alpha$ by requiring that $p'_\alpha(i)(k) = 1$ and $[p'_\alpha(\langle \gamma, 0 \rangle)] = [p_\alpha(\langle \gamma, 0 \rangle)]$ for every $\langle \gamma, 0 \rangle \in C$. This is possible since i is never $<_I$ below $\langle \gamma, 0 \rangle \in C$. Now p'_α extends $p_\beta \restriction C$ and hence $r = p'_\alpha \vee p_\beta$ can be formed. Since the only members of C below j (in $<_I$) are of the form $\langle \gamma, 0 \rangle$, it follows that $[r(j)] = [p_\beta(j)]$. Thus $r(j)(k) = 0$. ⊣

The following lemma implies that if G is a (V, P)-generic filter, g the generic gap, and $U \in V[G]$ is any stationary subset of ω_1 in the extension, then no uncountable restriction of g is U-Hausdorff.

LEMMA 2.4. *The following holds in V^P for the generic gap $g = \{(a_i \mid i \in \omega_1), (b_j \mid j \in \omega_1)\}$. If $J, K \subseteq \omega_1$ are unbounded, then there is a club set $D_0 \subseteq \omega_1$ such that for every $\delta \in D_0$ and $k \in K \setminus \delta$ there are $m \in \omega$ and a sequence $j(n) \in \delta \cap J$ increasing and cofinal in δ such that $a_{j(n)} \setminus m \subset b_k$ for all $n \in \omega$.*

PROOF. Let $J, K \in V^P$ be names forced by every condition in P to be unbounded subsets of ω_1. Define in V^P the following set $D_0 \subseteq \omega_1$: $\delta \in D_0$ if and only if $\delta \in \omega_1$ is a limit ordinal such that:

for all $k \in K \setminus \delta$ there is some $m \in \omega$ and an increasing, cofinal in δ sequence $j(n) \in \delta \cap J$ with $a_{j(n)} \setminus m \subseteq b_k$ for all $n \in \omega$.

We want to prove that D_0 contains a closed unbounded subset of ω_1, and assume that it does not. So $R = \omega_1 \setminus D_0$ is (forced by some condition to be) stationary in V^P, and hence the set, defined in V, of ordinals that are potentially in R is stationary in V. Namely, the set $R_0 \subset \omega_1$ of ordinals forced by some condition to be in R is stationary. For every $\delta \in R_0$ pick a condition p_δ that forces $\delta \notin D_0$. By extending p_δ we can find some $k_\delta \geq \delta$ such that

$$p_\delta \Vdash_P k_\delta \in K \text{ shows that } \delta \notin D_0.$$

So, for every $m \in \omega$, p_δ forces that the set of $j \in \delta \cap J$ with $a_j \setminus m \subseteq b_{k_\delta}$ is bounded in δ.

By extending p_δ again, we can find some $j_\delta \in \omega_1 \setminus \delta$ forced by p_δ to be in J (which is possible since J is supposed to be unbounded in ω_1). If necessary, a further extension ensures that both j_δ and k_δ are in the domain of p_δ. Now there exists some $m = m_\delta \in \omega$ such that

$$p_\delta \Vdash \ a_{j_\delta} \setminus m \subseteq b_{k_\delta}$$

(the height of p_δ will do). We can extend p_δ once again and find $f(\delta) < \delta$ such that

$$p_\delta \Vdash_P \text{ there is no } j \in J,\ f(\delta) < j < \delta, \text{ for which } a_j \setminus m \subseteq b_{k_\delta}. \tag{8}$$

We may assume that, for a stationary set $T \subseteq R_0$, the domains of p_α, for $\alpha \in T$, form a Δ system, that they all have the same height, say n, and the same restriction to the core. We also assume that the functions $p_\alpha(\langle j_\alpha, 0 \rangle) : n \to \{0, 1\}$ do not depend on α, and that $f(\alpha)$ and $m = m_\alpha$ are fixed on T ($m \leq n$). Now by Talayaco's chain condition for P, there is a stationary $T' \subseteq T$ such that for every $\alpha, \beta \in T'$, $p_\alpha \vee p_\beta$ is a common extension. Pick some $\alpha \in T'$ that is an accumulation point of T' (and such that for every $\beta < \alpha$, $j_\beta < \alpha$). Then find $\beta < \alpha$, $\beta \in T'$ such that $f(\alpha) < \beta$. Then (as we shall see)

$$p_\beta \vee p_\alpha \Vdash a_{j_\beta} \subseteq a_{j_\alpha},\ j_\beta \in J,\ and\ \alpha > j_\beta > f(\alpha).$$

Yet

$$p_\alpha \Vdash\ a_{j_\alpha} \setminus m \subseteq b_{k_\alpha},$$

and hence

$$p_\beta \vee p_\alpha \Vdash a_{j_\beta} \setminus m \subseteq b_{k_\alpha},$$

and this is a contradiction to (8). Why does $p_\beta \vee p_\alpha$ force $a_{j_\beta} \subset a_{j_\alpha}$? Because the functions $p_\alpha(\langle j_\alpha, 0\rangle)$ and $p_\beta(\langle j_\beta, 0\rangle)$ are the same, they describe $a_{j_\alpha} \cap n$ and $a_{j_\beta} \cap n$, so $p_\beta \vee p_\alpha$ forces $a_{j_\beta} \subset a_{j_\alpha}$. $\dashv$

§3. Specializing pre-gaps on a ladder system.

THEOREM 3.1. *For every ladder system C over a stationary set $S \subseteq \omega_1$, and gap g, there is a c.c.c. forcing notion $Q = Q_{g,C}$ such that in V^Q a restriction of g to some uncountable set is C-Hausdorff (and hence S-Hausdorff). In fact Q satisfies a stronger property than c.c.c., the polarized chain condition, which we shall define later.*

PROOF. Fix for the proof a ladder system $C = \langle c_\delta \mid \delta \in S\rangle$ over a stationary set $S \subseteq \omega_1$ consisting of limit ordinals, and a pre-gap $g = \{(a_i \mid i \in \omega_1), (b_j \mid j \in \omega_1)\}$. The forcing poset $Q = Q_{g,C}$ defined below is designed to make an uncountable restriction of g into a C-Hausdorff gap. We will prove that if g is a gap, then Q is a c.c.c. poset.

Define $p \in Q$ iff $p = (w, s)$ where

1. $w \in [\omega_1]^{<\aleph_0}$ (i.e., a finite subset of ω_1), and
2. $s \in [S]^{<\aleph_0}$.

If $p \in Q$ then we write $p = (w^p, s^p)$ for the two components of p.

The ordering $p \leq q$ (q extends p) is defined by

a. $w^p \subseteq w^q$, $s^p \subseteq s^q$, and
b. If $\delta \in s^p$ and $j \in w^p$ are such that $\delta \leq j$, then for every $i \in (w^q \setminus w^p) \cap \delta$,

$$a_i \setminus \mid c_\delta \cap i \mid \not\subseteq b_j.$$

Or, equivalently, $X(a_i, b_j) > |c_\delta \cap i|$. It is easy to check that this is indeed an ordering defined on Q.

Clearly, if $p = (w, s)$ is a condition, then for any $\sigma \in S$, $(w, s \cup \{\sigma\})$ extends p, and if $j \in \omega_1$ and $j > \max(w)$, then $(w \cup \{j\}, s)$ extends p. (If, however, $j < \max(w)$, then $(w \cup \{j\}, s)$ may be incompatible with (w, s).) If G is generic over Q, define $W = \bigcup\{w \mid \exists s (w, s) \in G\}$. By the c.c.c. (proved below), ω_1 is preserved. It follows that W is unbounded in ω_1 and $S = \bigcup\{s \mid \exists w\ (w, s) \in G\}$. It follows that $\{(a_i \mid i \in W), (b_j \mid j \in W)\}$ is C-Hausdorff.

So the generic filter over Q selects an unbounded in ω_1 restriction of g that is C-Hausdorff.

If $p = (w, s)$ is a condition then for every $\alpha \in \omega_1$ the restriction $p \restriction \alpha = (w \cap \alpha, s \cap \alpha)$ is defined. Clearly $p \restriction \alpha \leq p$.

If $p = (w, s)$ and $q = (v, r)$ are conditions in Q then define $p \cup q = (w \cup v, s \cup r)$. If p and q are compatible in Q, then $p \cup q \in Q$ is the least upper bound of p and q.

Sh:598

For any condition $p = (w, s)$, we write $\mathrm{dom}(p) = w \cup s$. $\dashv$

The following lemma describes a situation in which the compatibility of p_1 and p_2 can be deduced. This is the situation resulting when p_1 and p_2 come from a Δ-system, with core fixed below γ, and such that p_1 is bounded by some α such that the domain of p_2 has empty intersection with the ordinal interval $[\gamma, \alpha]$. The proof is straightforward.

LEMMA 3.2. *Suppose that*

1. $p_1 = (w_1, s_1)$, $p_2 = (w_2, s_2)$, *and* p^* *are in* P.
2. $\gamma < \alpha < \omega_1$ *are such that*
   (a) $\mathrm{dom}(p_1) \subseteq \alpha$, *and* $\mathrm{dom}(p^*) \subseteq \gamma$.
   (b) $w_2 \cap \alpha \subset \gamma$, *and* $s_2 \cap (\alpha + 1) \subset \gamma$. *So* $p_2 \restriction \alpha = p_2 \restriction \gamma$.
   (c) $p^* \geq p_1 \restriction \gamma, p_2 \restriction \gamma$.
   (d) p^* *is compatible with* p_1 *and is also compatible with* p_2.
3. *Define*

$$A = \bigcap \{a_i \mid i \in w_1 \setminus \gamma\}$$
$$B = \bigcup \{b_j \mid j \in w_2 \setminus \gamma\}$$

and suppose that there is $n \in A \setminus B$ *such that, for every* $\delta \in s_2 \setminus \alpha$, $n > |c_\delta \cap \alpha|$.

Then p_1 *and* p_2 *are compatible. In fact,* $p_1 \cup p_2 \cup p^*$ *is an extension of all three conditions.*

PROOF. Form $p = p_1 \cup p_2 \cup p^*$ and prove that $p_1, p_2, p^* \leq p$. Clearly, $p^* \leq p$ and $p_1 \leq p$ because p_1 and p_2 are compatible with p^*. As for $p_2 \leq p$, observe that $X(a_i, b_j) > |c_\delta \cap \alpha|$ for every $i \in w_1 \setminus \gamma$, $j \in w_2 \setminus \gamma$, and $\delta \in s_2 \setminus \gamma$. $\dashv$

The following simple lemma is used in proving that Q is a c.c.c. poset.

LEMMA 3.3. *Suppose* $g = \{(A_i \mid i \in I), (B_j \mid j \in J)\}$ *is a pre-gap such that for every* $i \in I$ *and* $j \in J$, $i < j$ *implies that* $A_i \subseteq B_j$. *Then* g *is not a gap.*

PROOF. By throwing away a countable set of indices from J we can assume for every $n \in \omega$ that if $n \notin B_j$ for some j, then $n \notin B_j$ for uncountably many j's. Define then $x = \bigcup_{i \in I} A_i$. Then $x \subseteq B_j$ for every j, because otherwise there are some $i \in I$, $j \in J$, and $n \in \omega$ such that $n \in A_i \setminus B_j$. But then we may find uncountably many indices j' such that $n \notin B_{j'}$ and in particular there is such $j' > i$. Thence $A_i \not\subseteq B_{j'}$, contradicting our assumption. $\dashv$

THEOREM 3.4. *Suppose that the domain of our ladder system* C, *namely* S, *is co-stationary. Assume that* g *is a gap.*

1. $Q = Q_{g,C}$ *satisfies the c.c.c.*
2. *Suppose that* $T_1, T_2 \subseteq \omega_1 \setminus S$ *are stationary sets and* $\overline{p} = (p^\ell_\delta \mid \delta \in T_\ell)$, *for* $\ell = 1, 2$, *are two sequences of conditions in* Q *such that some fixed* $p^* \in Q$ *extends all* $p^1_\delta \restriction \delta$ *and* $p^2_\mu \restriction \mu$, *for every* $\delta \in T_1$ *and* $\mu \in T_2$, *and is such that* p^* *is compatible with every* p^1_δ *and with every* p^2_μ. *Then there are stationary subsets* $T_1' \subseteq T_1$ *and* $T_2' \subseteq T_2$ *such that, for every* $\alpha_1 \in T_1'$ *and* $\alpha_2 \in T_2'$, *if* $\alpha_1 < \alpha_2$ *then* $p^1_{\alpha_1}$ *and* $p^2_{\alpha_2}$ *are compatible in* Q.

PROOF. We prove 2 since 1 can be derived from it. For any condition $p = (w, s)$ define $\mathrm{dom}(p) = w \cup s$. Suppose that $\mathrm{dom}(p^*) \subseteq \gamma$. Then $\mathrm{dom}(p^1_\delta) \cap \delta \subseteq \gamma$, and $\mathrm{dom}(p^2_\mu) \cap \mu \subseteq \gamma$, for every $\delta \in T_1$ and $\mu \in T_2$. We may assume (by shrinking T_1 and T_2) that if $i < j$ are in T_ℓ and T_m (for $\ell, m \in \{1,2\}$) then $\mathrm{dom}(p_i^\ell) \subset \cap\, (\mathrm{dom}(p_j^m) \setminus \gamma)$

Since $\delta \in T_\ell$ implies that $\delta \notin S$, it follows for $p_\delta^\ell = (w, s)$ that the ladder sequence c_i for any $i \in s$ is bounded below δ. So the finite union

$$\delta \cap \bigcup \{c_i \mid i \in s \setminus \delta\}$$

is bounded below δ. Using Fodor's lemma, we may even assume that this intersection is bounded below γ (extend γ if necessary) and has a fixed finite cardinality.

For every $\delta \in T_1$ define

$$A_\delta = \bigcap \{a_i \mid i \in w^{p_\delta^1} \setminus \gamma\}$$

Similarly, for $\delta \in T_2$ define

$$B_\delta = \bigcup \{b_i \mid i \in w^{p_\delta^2} \setminus \gamma\}.$$

Clearly, any interpolation for $G = \{(A_\delta \mid \delta \in T_1\}, \{B_\delta \mid \delta \in T_2\})$ is also an interpolation for g, and hence G is a gap.

Let $k \in \omega$ be such that for every $\delta \in T_\ell$, if $p_\delta^\ell = (w, s)$ and $\alpha \in s \setminus \gamma$, then $| c_\alpha \cap \delta | < k$.

Now we find a stationary set $T_1' \subset T_1$ such that for every $n \in \omega$ if $n \in A_\delta$ for some $\delta \in T_1'$ then $n \in A_\delta$ for a stationary set of δ's in T_1'. Simply throw away countably many non-stationary sets from T_1. Similarly, find a stationary $T_2' \subseteq T_2$ such that if $n \notin B_\delta$ for some $\delta \in T_2'$ then $n \notin B_\delta$ for a stationary set of $\delta \in T_2'$.

Now Lemma 3.3 gives $\alpha_1 \in T_1'$ and $\alpha_2 \in T_2'$ with $\alpha_1 < \alpha_2$ such that $A_{\alpha_1} \setminus k \not\subseteq B_{\alpha_2}$. If we pick $n \in A_{\alpha_1} \setminus B_{\alpha_2}$ such that $n \geq k$ then there are stationary sets $T_1'' \subseteq T_1'$ and $T_2'' \subseteq T_2'$ such that $n \in A_{\alpha_1} \setminus B_{\alpha_2}$ *for every* $\alpha_1 \in T_1''$ and $\alpha_2 \in T_2''$. Hence if $\alpha_1 \in T_1''$, $\alpha_2 \in T_2''$, and $\alpha_1 < \alpha_2$, then $p_{\alpha_1}^1$ and $p_{\alpha_2}^2$ are compatible in Q by Lemma 3.2. $\dashv$

3.1. Polarized chain condition. Theorem 3.4 shows that the poset $Q_{g,C}$ for a gap g and ladder C over a stationary co-stationary set S satisfies some kind of a chain condition, which refers to two sequences of conditions indexed by stationary subsets of $\omega_1 \setminus S$. We formulate this property in general and later prove that it is preserved under finite support iteration.

DEFINITION 3.5. Let $T \subseteq \omega_1$ be a stationary set. A c.c.c. poset P satisfies the polarized chain condition (*p.c.c.*) for T if it satisfies the following requirement. Suppose that

1. $\overline{p}^\ell = (p_\delta^\ell \mid \delta \in T_\ell)$ for $\ell = 1, 2$
   are two sequences of conditions in P, where $T_\ell \subseteq T$ are stationary for $\ell = 1, 2$.
2. $p^* \in P$ is such that for each $\ell = 1, 2$

$$p^* \Vdash_P \{\delta \in T_\ell \mid p_\delta^\ell \in G\} \text{ is stationary in } \omega_1,$$

where G is the name of the generic filter over P.

Then there are stationary sets $T_\ell' \subseteq T_\ell$ for $\ell = 1, 2$ such that $p_{\alpha_1}^1$ and $p_{\alpha_2}^2$ are compatible in P whenever $\alpha_1 < \alpha_2$ are in T_1' and T_2' respectively.

We want to prove that if g is a gap and C a ladder over a stationary set S such that $T = \omega_1 \setminus S$ is also stationary, then $Q = Q_{g,C}$ satisfies the p.c.c. for T. The problem is that if p^* is as in the p.c.c. definition then it is not necessarily of the form to which Theorem 3.4 is applicable, and so we need some argument to deduce that Q is p.c.c.

Recall that every club subset of ω_1 in a generic extension of V made via a c.c.c. poset contains a club subset in V. The following property of c.c.c. posets is also needed.

LEMMA 3.6. *Let P be a c.c.c. poset. Suppose that $T \subseteq \omega_1$ is stationary, and $\langle p_\alpha \mid \alpha \in T\rangle$ is a sequence of conditions in P indexed along T. Then there exists some p_α such that*

$$p_\alpha \Vdash \{\beta \in T \mid p_\beta \in G\} \textit{ is stationary.} \tag{9}$$

In fact, the set of these α's is stationary in ω_1. By reducing T, we may assume that p_α is compatible with every p_β ($\beta \in T$).

PROOF. Assume that this is not the case and, for some club $D \subseteq \omega_1$, for every $\alpha \in T \cap D$ there is a club set C_α (necessarily in V) and an extension p'_α of p_α such that

$$p'_\alpha \Vdash \forall\beta\,(\beta \in C_\alpha \cap T \rightarrow p_\beta \notin G). \tag{10}$$

Let $C = \{\beta \in \omega_1 \mid (\forall\alpha < \beta)\beta \in C_\alpha\}$ be the diagonal intersection of these club sets. Then C is closed unbounded in ω_1. Take a maximal antichain (surely countable) from the set of extensions $\{p'_\alpha \mid \alpha \in T \cap D\}$, and let α_0 be an index in $T \cap C \cap D$ higher than all indexes of this countable antichain. Then p'_{α_0} is compatible with some p'_α with $\alpha < \alpha_0$. But $\alpha_0 \in C_\alpha$ leads to a contradiction since p'_{α_0} forces that $p_{\alpha_0} \in G$, and p'_α forces that $p_{\alpha_0} \notin G$ (by 10).

The final remark of the lemma is that we may assume in (9) that p_α is compatible with every p_β for $\beta \in T$. For this, let $T_0 \subseteq T$ be the set of $\beta \in T$ such that p_α and p_β are compatible, and prove that (9) holds for T_0. $\dashv$

Now we prove that Q is p.c.c. for $T = \omega_1 \setminus S$.

LEMMA 3.7. *If C is a ladder system over a stationary set S, and $T = \omega_1 \setminus S$ is stationary, then, for any gap g, $Q_{g,C}$ is p.c.c. over T.*

PROOF. Suppose that $T_1, T_2 \subseteq T$ are stationary, and $\overline{p}^1, \overline{p}^2$ are two sequences of conditions indexed along T_1 and T_2. Let $q^* \in Q$ be such that for $\ell = 1, 2$

$$q^* \Vdash_Q \{\delta \in T_\ell \mid p^\ell_\delta \in G\} \text{ is stationary in } \omega_1. \tag{11}$$

In order to prove the polarized chain condition we are free to reduce T_1 and T_2 to stationary subsets and to increase the conditions p^ℓ_δ (for if the extensions are compatible then so are the given conditions). Observe that, whenever (11) holds, the set of $\delta \in T_\ell$ for which p^ℓ_δ and q^* are compatible is stationary, and (11) holds for this set substituting T_ℓ. Hence, if q^* extends all $p^\ell_\delta \restriction \delta$ (for $\ell = 1, 2$) and (11) holds then Theorem 3.4 can be applied and its conclusion, the p.c.c., is obtained. So assume that (11) holds with all p^ℓ_δ being compatible with p^*.

We claim first that we may assume that $q^* \geq p^*_\delta \restriction \delta$ for every $\delta \in T_1$. Apply Fodor's theorem to fix $p^1_\delta \restriction \delta$, rename T_1 to the resulting stationary set, and redefine p^1_δ as $p^1_\delta \cup q^*$. Apply Lemma 3.6 to obtain δ_0 such that

$$p^1_{\delta_0} \Vdash_Q \{\delta \in T_1 \mid p^1_\delta \in G\} \text{ is stationary in } \omega_1.$$

Redefine $q^* = p^1_{\delta_0}$. Then $q^* \geq p^1_\delta \restriction \delta$ for $\delta \in T_1$, and since q^* extends the original q^*, it still satisfies (11) with respect to T_2. Repeat this procedure for T_2, and obtain two sequences and an extended q^* so that (11) holds and $q^* \geq p^\ell_\delta \restriction \delta$ for all δ as required. ⊣

We note here for a possible future use a stronger form of polarized chain condition (strong-p.c.c.) which is not used in this paper.

DEFINITION 3.8. Let $T \subseteq \omega_1$ be stationary. A poset P is said to satisfy the strong-p.c.c. over T if whenever two sequences are given

$$\overline{p}^\ell = (p^\ell_\delta \mid \delta \in T_\ell) \text{ for } \ell = 1, 2$$

of conditions in P, where $T_\ell \subseteq T$ are stationary for $\ell = 1, 2$, and for some $p^* \in P$, for every $\ell = 1, 2$,

$$p^* \Vdash_P \{\delta \in T_\ell \mid p^\ell_\delta \in G_P\} \text{ is stationary in } \omega_1,$$

then there are stationary subsets $T'_\ell \subseteq T_\ell$ for $\ell = 1, 2$, and conditions $q_\delta \geq p^2_\delta$ for $\delta \in T'_2$ such that:

For every $\delta \in T'_2$ and $q \in P$ such that $q_\delta \leq q$ there exists $\alpha < \delta$ such that for every β that satisfies $\alpha < \beta \in T'_1 \cap \delta$

$$q \text{ and } p^1_\beta \text{ are compatible in } P.$$

§**4. Iteration of p.c.c. posets.** Our aim in this section is to prove that the iteration with finite support of p.c.c. posets is again p.c.c. It is well known (by Martin and Solovay [2]) that since each of the iterands satisfies the countable chain condition the iteration is again c.c.c., but we have to prove the preservation of the polarized property.

A poset is separative iff $p \nleq q$ implies that some extension of q is incompatible with p. Our posets defined above are not necessarily separative, but a well-known transformation yields equivalent separative posets. So we iterate separative posets.

LEMMA 4.1. *Suppose that P is a p.c.c. poset, and that $Q \in V^P$ is (forced by every condition in P to be) a p.c.c. poset for some fixed stationary set T. Then the iteration $P * Q$ satisfies the polarized chain condition too.*

PROOF. Suppose that $(p^\ell_\delta, q^\ell_\delta) \in P * Q$ are given for $\delta \in T_\ell \subseteq T$ and for $\ell = 1, 2$, such that for some condition $(p, q) \in P * Q$

$$(p, q) \Vdash \{\delta \in T_\ell \mid (p^\ell_\delta, q^\ell_\delta) \in G_{P*Q}\} \text{ is stationary} \tag{12}$$

for $\ell = 1, 2$. Since forcing with $P * Q$ can be done in two stages:

$$p \Vdash_P \{\delta \in T_\ell \mid p^\ell_\delta \in G_P\} \text{ is stationary.}$$

Let $G \subset P$ be V-generic, with $p \in G$. In $V[G]$ form the interpretations $q[G]$ (interpretation of q) and $Q[G]$ (interpretation of Q). Then $q[G] \in Q[G]$. Define the sets

$$T'_\ell = \{\delta \in T_\ell \mid p^\ell_\delta \in G\}, \ \ell = 1, 2$$

(which are stationary) and define the sequences

$$\langle q^\ell_\delta[G] \mid \delta \in T'_\ell \rangle, \quad \text{for } \ell = 1, 2.$$

Then in $V[G]$

$$q[G] \Vdash_{Q[G]} \{\delta \in T'_\ell \mid q^\ell_\delta[G] \in H\} \text{ is stationary}$$

where H is the name for the $V[G]$ generic filter over $Q[G]$. (This follows from (12) and since forcing with $P * Q$ is equivalent to the iteration of forcing with P and then with $Q[G]$.)

Since $Q[G]$ satisfies the polarized chain condition for T, there are stationary sets $T''_\delta \subseteq T'_\delta$ such that:

if $\delta_1 \in T''_1$, $\delta_2 \in T''_2$, and $\delta_1 < \delta_2$, then $q^1_{\delta_1}[G]$ and $q^2_{\delta_2}[G]$ are compatible in $Q[G]$.

Back in V, let S_1 and S_2 be V^P names of T''_1 and T''_2 respectively, forced to have these properties. The following short lemma will be applied to S_1 and to S_2.

LEMMA 4.2. *Suppose that $\langle p_\delta \mid \delta \in T\rangle$ is a sequence in P, S is a name of a subset of ω_1 and $p \in P$ a condition such that*

$$p \Vdash_P S \subseteq \{\alpha \in T \mid p_\delta \in G\} \text{ and } S \text{ is stationary in } \omega_1.$$

Then there is a stationary subset $T^* \subseteq T$, and conditions p^*_δ extending both p_δ and p for each $\delta \in T^*$ such that $p^*_\delta \Vdash \delta \in S$.

PROOF. Define T^* by the condition that $\delta \in T^*$ iff $\delta \in T$ and there is a common extension of p and p_δ that forces $\delta \in S$. We must prove that T^* is stationary. If $C \subseteq \omega_1$ is any closed unbounded set, find $p' \geq p$ and $\delta \in C$ such that $p' \Vdash \delta \in S$. Then $\delta \in T$ and $p' \Vdash p_\delta \in G$. Hence $p_\delta \leq p'$ (because P is separative). So $\delta \in T^*$. ⊣

Apply the lemma to S_1 and find a stationary set $T^*_1 \subseteq T_1$ and conditions $p^{*1}_\delta \geq p^1_\delta, p$, for $\delta \in T^*_1$ such that

$$p^{*1}_\delta \Vdash \delta \in S_1.$$

Then (Lemma 3.6) find an extension of p, denoted p^*, such that

$$p^* \Vdash \{\delta \in T^*_1 \mid p^{*1}_\delta \in G\} \text{ is stationary.}$$

Apply the same argument to S_2, and find a stationary set $T^*_2 \subseteq T_2$ and conditions $p^{*2}_\delta \geq p^2_\delta, p^*$ for $\delta \in T^*_2$ such that $p^{*2}_\delta \Vdash \delta \in S_2$. Now $p^{**} \geq p^*$ can be found such that

$$p^{**} \Vdash \{\delta \in T^*_2 \mid p^{*2}_\delta \in G\} \text{ is stationary.}$$

Since P satisfies the p.c.c., there are stationary sets $T^{**}_1 \subseteq T^*_1$ and $T^{**}_2 \subseteq T^*_2$ such that for every $\delta_1 < \delta_2$ in T^{**}_1 and T^{**}_2 (respectively) $p^{*1}_{\delta_1}$ and $p^{*2}_{\delta_2}$ are compatible in P, say by some condition p' extending both. But then $p' \Vdash \delta_1 \in S_1$ and $\delta_2 \in S_2$. It follows that $(p^1_{\delta_1}, q^1_{\delta_1})$ and $(p^2_{\delta_2}, q^2_{\delta_2})$ are compatible in $P * Q$ showing that $P * Q$ satisfies the p.c.c. The point is that

$$p' \Vdash_P q^1_{\delta_1} \text{ and } q^2_{\delta_2} \text{ are compatible in } Q$$

and hence for some $q' \in V^P$, $p' \Vdash_P q' \geq q^1_{\delta_1}, q^2_{\delta_2}$.
That is, $(p', q') \geq (p^1_{\delta_1}, q^1_{\delta_1})$, $(p^2_{\delta_2}, q^2_{\delta_2})$. ⊣

THEOREM 4.3. *Let T be a stationary subset of ω_1. An iteration with finite support of p.c.c. for T posets is again p.c.c. for T.*

PROOF. The theorem is proved by induction on the length, δ, of the iteration. For δ a successor ordinal, this is essentially Lemma 4.1. So we assume that δ is a limit ordinal, and $\langle P_\alpha \mid \alpha \leq \delta\rangle$ is a finite support iteration, where $P_{\alpha+1} = P_\alpha * Q_\alpha$ is obtained with $Q_\alpha \in V^{P_\alpha}$ a p.c.c. poset for T. Thus conditions in P_δ are finite functions p defined on a finite subset $\mathrm{dom}(p) \subset \delta$, and are such that for every $\alpha \in \mathrm{dom}(p)$, $p \restriction \alpha \Vdash_{P_\alpha} p(\alpha) \in Q_\alpha$. It is well-known that P_δ satisfies the c.c.c., and we must prove the polarized property.

Suppose that $\overline{p}^\ell = \langle p_i^\ell \mid i \in T_\ell\rangle$ for $\ell = 1, 2$ are two sequences of conditions in P_δ, where $T_\ell \subseteq T$ are stationary, and suppose also that $p^* \in P_\delta$ is such that

$$p^* \Vdash_{P_\delta} \{i \in T_\ell \mid p_i^\ell \in G\} \text{ is stationary for } \ell = 1, 2. \tag{13}$$

We may assume that p^* is compatible with every p_i^ℓ (just throw away those conditions that are not). The case $\mathrm{cf}(\delta) > \omega_1$ is trivial, because the support of all conditions is bounded by some $\delta' < \delta$ to which induction is applied). So there are two cases to consider.

CASE 1: $\mathrm{cf}(\delta) = \omega$. Let $\langle \delta_n \mid n \in \omega\rangle$ be an increasing ω-sequence converging to δ. For every p_i^ℓ there is $n \in \omega$ such that $\mathrm{dom}(p_i^\ell) \subseteq \delta_n$. It follows from (13) that for some specific $n \in \omega$, for some extension $p^{**} \geq p^*$

$$p^{**} \Vdash \{i \in T_\ell \mid p_i^\ell \in P_{\delta_n} \cap G\} \text{ is stationary for } \ell = 1, 2.$$

Now we can apply the inductive assumption to P_{δ_n}.

CASE 2: $\mathrm{cf}(\delta) = \omega_1$. Let $\langle \delta_\alpha \mid \alpha \in \omega_1\rangle$ be an increasing, continuous, and cofinal in δ sequence. Intersecting T_1 and T_2 with a suitable closed unbounded set, we may assume that for every $\alpha < \beta$ $\alpha \in T_1$ and $\beta \in T_2$, $\mathrm{dom}(p_\alpha^1) \subset \beta$.

We claim that we may without loss of generality assume that, for some $\gamma < \delta$, $\mathrm{dom}(p_\alpha^\ell) \cap \delta_\alpha$ is bounded by γ for all $\alpha \in T_\ell$. We get this in two steps.

In the first step, find a stationary $T_1' \subseteq T_1$ such that the sets $\mathrm{dom}(p_\alpha^1) \cap \delta_\alpha$, for $\alpha \in T_1'$, are bounded by some $\gamma < \delta$. For each $\alpha \in T_1'$ let p_α^{1*} be a common extension of p_α^1 and p^*. Then (use Lemma 3.6) find an extension $p^{**} \geq p^*$ such that

$$p^{**} \Vdash \{\alpha \in T_1' \mid p_\alpha^1 \in G\} \text{ is stationary.}$$

Since p^{**} extends p^*, $p^{**} \Vdash \{i \in T_2 \mid p_i^2 \in G\}$ *is stationary*. We can again assume that each p_i^2 is compatible with p^{**} and get $T_2' \subseteq T_2$ stationary such that $\mathrm{dom}(p_\alpha^2) \cap \delta_\alpha$ is bounded by some $\gamma' < \delta$ (we rename γ to be the maximum of γ and γ'). Rename the stationary sets as T_1, T_2 and we have our assumption.

Apply induction to P_γ and to the conditions $p_\alpha^\ell \restriction \gamma$. This yields two stationary subsets which are as required. ⊣

§5. The model.

THEOREM 5.1. *Assuming the consistency of ZFC, the following property is consistent with ZFC. There is a stationary co-stationary set $S \subseteq \omega_1$ such that*

1. *For every ladder system C over S, every gap contains a C-Hausdorff subgap.*
2. *For every ladder system H over $T = \omega_1 \setminus S$ there is a gap g with no subgap that is H-Hausdorff.*

To obtain the required generic extension we assume that κ is a cardinal in V (the ground model) such that $\mathrm{cf}(\kappa) > \omega_1$ and even $\kappa^{\aleph_1} = \kappa$. We shall obtain a generic extension $V[G]$ in which $2^{\aleph_0} = \kappa$ and the two required properties of the theorem hold. For this we define a finite support iteration of length κ, iterating posets P

as in section 2, which introduce generic gaps, and posets of the form $Q_{g,C}$, as in Section 3, which are designed to introduce a C-Hausdorff subgap to g.

We denote this iteration $\langle P_\alpha \mid \alpha < \kappa \rangle$. So $P_{\alpha+1} \simeq P_\alpha * R(\alpha)$, where the α-th iterand $R(\alpha)$ is either some P or some $Q_{g,C}$. The rules to determine $R(\alpha)$ are specified below. For any limit ordinal $\delta \leq \kappa$, P_δ is the finite support iteration of the posets $\langle P_\alpha \mid \alpha < \delta \rangle$. We define P_κ as our final poset, and we shall prove that in V^{P_κ} the two properties of the theorem hold.

Recall that P satisfies Talayaco's condition (Lemma 2.2) and is hence a p.c.c. poset, and each $Q_{g,C}$ is p.c.c. over $T = \omega_1 \setminus S$ (by Lemma 3.7).

Since the iterand posets satisfy the p.c.c. over T, each P_α is a p.c.c. poset over T (and in particular a c.c.c. poset). It follows from our assumption that $\mathrm{cf}(\kappa) > \omega_1$ that every ladder system and every gap in V^{P_κ} are already in some V^{P_α} for $\alpha < \kappa$. It is obvious that if $g \in V^{P_\alpha}$ is forced by $p \in P_\kappa$ to be a gap in V^{P_κ}, then $p \restriction \alpha$ forces it to be a gap already in V^{P_α}.

To determine the iterands, we assume a standard bookkeeping scheme which ensures two things:

1. For every ladder system C over S and gap g in V^{P_κ}, there exists a stage $\alpha < \kappa$ so that $C, g \in V^{P_\alpha}$, and $R(\alpha)$ is $Q_{g,C}$.
2. For some unbounded set of ordinals $\alpha \in \kappa$ the iterand $R(\alpha)$ is P, which produces a generic gap g, and the subsequent iterand $R(\alpha+1)$ is $Q_{g,C}$ for some ladder sequence C over S.

The first item ensures that, in V^{P_κ}, for every ladder system C over S, every gap contains a C-Hausdorff subgap. (A C-Hausdorff subgap in $V_{\alpha+1}$ remains C-Hausdorff at every later stage and in the final model).

The second item ensures that every ladder H over $T = \omega_1 \setminus S$ has a gap g with no H-Hausdorff subgap. The gap is introduced by a pair of forcings of the form $P * Q_{g,C}$ which introduces a generic gap and immediately seals it to ensure that it remains a gap in the remaining forcing.

To prove this, suppose that H is a ladder over $T = \omega_1 \setminus S$. Then H appears in some V^{P_α} such that $R(\alpha)$ is the poset P, and $R(\alpha+1)$ is the poset $Q_{g,C}$ where g is the generic gap introduced by $R(\alpha)$, and C is some ladder sequence over S. We want to prove that g is a gap in $V^{P_\kappa appa}$ that has no H-Hausdorff subgap there. We first prove that g remains a gap in V^{P_κ}. It is clearly a gap in $V^{P_{\alpha+1}}$ by Lemma 2.3. Since g is C-Hausdorff in $V^{P_{\alpha+2}}$, it remains a gap in V^{P_κ} (by Lemma 1.2).

This generic gap g satisfies the conclusion of Lemma 2.4 in $V^{P_{\alpha+1}}$:

(14) If $J, K \subseteq \omega_1$ are unbounded, then there is a club set $D_0 \subseteq \omega_1$ such that for every $\delta \in D_0$ and $k \in K \setminus \delta$ there are $m \in \omega$ and a sequence $j(n) \in \delta \cap J$ increasing and cofinal in δ such that $a_{j(n)} \setminus m \subset b_k$ for all $n \in \omega$.

Since $P_\kappa \simeq P_{\alpha+1} * R$, where the remainder $R \simeq P_\kappa / P_{\alpha+1}$ is interpreted in $V^{P_{\alpha+1}}$ as a finite support iteration of p.c.c. posets over T, we can view P_κ as a two-stage iteration in which the second stage is a p.c.c. poset over T. Thus, for simplicity of expression, we can assume that $V^{P_{\alpha+1}}$ is the ground model. The following lemma then ends the proof.

LEMMA 5.2. *Suppose in the ground model V a ladder system H over a stationary set $T \subseteq \omega_1$, and a gap g that has the property* (14) *quoted above. Suppose also a poset R that is p.c.c. over T. Then in V^R the gap g contains no H-Hausdorff subgap.*

PROOF. Let $g = \{(a_i \mid i \in \omega_1), (b_j \mid j \in \omega_1)\}$ and assume (for the sake of a contradiction) that some condition q in R forces that $g' = \{(a_\alpha \mid \alpha \in A), (b_\beta \mid \beta \in B)\}$ is a H-Hausdorff subgap, where A and B are names forced by q to be unbounded in ω_1. Since every club subset of ω_1 in a c.c.c. generic extension contains a club subset in the ground model, we may assume that the club, D, which appears in Definition 1.3 (of g' being H-Hausdorff) is in V.

For every $\delta \in T \cap D$ define two conditions in R (extending the given condition q):

1. $p_\delta \in R$ is such that for some $\alpha(\delta) \in \omega_1 \setminus \delta$, $p_\delta \Vdash_R \alpha(\delta) \in A$. (This is possible since A is forced to be unbounded.)
2. $q_\delta \in R$ extending p_δ is such that, for some $\beta(\delta) \in \omega_1 \setminus \delta$, $q_\delta \Vdash_R \beta(\delta) \in B$. Moreover, as g' is forced to be H-Hausdorff, we can assume that for some $m_\delta \in \omega$,

   $q_\delta \Vdash_R$ *for every* $n \geq m_\delta$, *if* $i \in A \cap (\delta \setminus c_\delta(n) + 1)$, *then* $X(a_i, b_{\beta(\delta)}) > n$.

By Lemma 3.6 some condition forces that $q_\delta \in G$ (and hence $p_\delta \in G$) for a stationary set of indices $\delta \in T \cap D$. Since R is p.c.c. for T, there are stationary subsets $T_1, T_2 \subseteq T$ such that any p_{δ_1} is compatible with q_{δ_2} if $\delta_1 \in T_1, \delta_2 \in T_2$ and $\delta_1 < \delta_2$.

Consider now the two unbounded sets $J = \{\alpha(\delta) \mid \delta \in T_1\}$, and $K = \{\beta(\delta) \mid \delta \in T_2\}$. Apply (14) to J and K, and let D_0 be the club set that appears there. Pick any $\delta \in D \cap T_2 \cap D_0$. Consider $k = \beta(\delta)$. Then $k \in K \setminus \delta$, and so there are $m \in \omega$ and a sequence $j(n) \in \delta \cap J$ cofinal in δ such that

$$a_{j(n)} \setminus m \subset b_k \text{ for all } n \in \omega. \tag{15}$$

Yet every $j(n)$ is of the form $\alpha(\delta_n)$ for some $\delta_n \in T_1 \cap \delta$, and the δ_n's tend to δ. So (15) can be written as

$$X(a_{\alpha(\delta_n)}, b_k) \leq m. \tag{16}$$

It follows from the definition of T_1 and T_2 that, for every $n \in \omega$, p_{δ_n} and q_δ are compatible in R. It suffices now to take $m_0 = \max\{m, m_\delta\}$ and $n \geq m_0$ with $\delta_n > c_\delta(m_0)$ to get a contradiction to (16). Because if q' is a common extension of p_{δ_n} and q_δ, then it forces (as $m_0 \geq m_\delta$) for every $i \in A \cap (\delta \setminus c_\delta(m_0) + 1)$ that $X(a_i, b_k) > m_0$. In particular for $i = \alpha(\delta_n)$, $q' \Vdash X(a_i, b_k) > m_0$. $\dashv$

REFERENCES

[1] S. H. HECHLER, *Short nested sequences in $\beta N \setminus N$ and small maximal almost disjoint families*, ***General Topology and its Applications***, vol. 2 (1972), pp. 139–149.

[2] D. A. MARTIN and R. SOLOVAY, *Internal Cohen extensions*, ***Annals of Mathematical Logic***, vol. 2 (1970), pp. 143–178.

[3] M. SCHEEPERS, *Gaps in ω^ω*, ***Set Theory of the Reals***, Israel Mathematical Conference Proceedings (Ramat Gan, 1991), vol. 6, Bar-Ilan University, Ramat Gan, 1993, pp. 439–561.

[4] D. TALAYACO, *Applications of cohomology to set theory I. Hausdorff gaps*, ***Annals of Pure and Applied Logic***, vol. 71 (1995), no. 1, pp. 69–106.

DEPARTMENT OF MATHEMATICS
BEN-GURION UNIVERSITY
BEÉR-SHEVA, 84105, ISRAEL
E-mail: abraham@math.bgu.ac.il

INSTITUTE OF MATHEMATICS
THE HEBREW UNIVERSITY
JERUSALEM, ISRAEL
E-mail: shelah@math.huji.ac.il