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BANACH SPACES AND GROUPS — ORDER PROPERTIES AND UNIVERSAL MODELS

BY

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ABSTRACT

We deal with two natural examples of almost-elementary classes: the class of all Banach spaces (over $\mathbb{R}$ or $\mathbb{C}$) and the class of all groups. We show that both of these classes do not have the strict order property, and find the exact place of each one of them in Shelah's SOP_n (strong order property of order n) hierarchy. Remembering the connection between this hierarchy and the existence of universal models, we conclude, for example, that there are "few" universal Banach spaces (under isometry) of regular density characters.

1. Introduction and preliminaries

In this paper we investigate two natural classes—the class of all Banach spaces (real or complex) and the class of all groups, from the point of view of model theory, more precisely, of Shelah's classification theory for classes of logical structures. Classification theory is an attempt to classify theories/classes of models according to their model-theoretical complexity—number of nonisomorphic models, number of nonisomorphic relatively "nice" models, existence of "nice" (saturated, universal) models, etc. In order to analyze structure of classes, certain syntactical properties were defined, some well-known, like the order property (equivalent to non-stability, see [Sh:c], chapter I) and the tree property (equivalent to non-simplicity, see [Sh93], [KimPil]), some less known, e.g., the

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strict/strong order properties (SOP), see Definition 1.1 below. In this article we test the classes in question with respect to these properties, as well as a semantic property of existence of universal models, which is closely connected to SOP.

It follows from results proven here that Banach spaces and groups are not stable and even not simple, but on the other hand, neither of these classes has the strict order property. We do more: we find the exact position of each one of these classes in the SOP_n complexity hierarchy defined in [Sh500], namely we show that Banach spaces have SOP_n for all n, and even $FSOP$ (finitary strong order property), but do not have the strict order property. This is done in sections 2 and 3. In section 4 we show that groups have SOP_3, but not SOP_4. In section 2 we also derive a conclusion considering nonexistence of universal Banach spaces in many cardinalities.

Apart from giving particular information about Banach spaces or groups, this work also gives a certain intuition on the SOP_n hierarchy by exhibiting natural, "mathematically concrete" examples of classes that occupy nontrivial positions in it. For instance, no "interesting" example of a theory/class with $FSOP$ but without the strict order property was known before.

Note that in this paper we are not investigating the first order theory of a particular Banach space or group. Rather than that, we are interested in the class of all groups and the class of all Banach spaces, with the natural notion of submodel: subgroup in the case of groups, linear subspace with induced norm in the case of normed spaces. So types in these classes are determined by the natural embeddings/isomorphisms: isomorphisms of groups in the first case, linear isometries in the second one. In other words, the appropriate notion of type for us is the *quantifier free* type. So when we say "formula", we mean quantifier free formula, the same with types (unless specified otherwise).

In short, we are treating both classes as classes of models of universal theories, or (as both have amalgamation) as so-called *Robinson theories.* Groups is an example of a *classical* Robinson theory, see [Sh54] (called "classes of kind II" there), [Hr]. Banach spaces form a *positive* Robinson theory, see [Ben02]. Robinson theories are characterized (see [Hr], [Ben02]) by existence of *universal domains*, i.e. universal (for the class) homogeneous models, compact for quantifier free types (in the positive case—positive q.f. types, see a more detailed explanation below). We will fix such "monster" models for each class in question in "big enough" cardinalities, and all our discussions will take place inside these universal domains.

We recall now the definitions of strong/strict order properties ([Sh500]):

Definition 1.1:

(1) We say that a formula $\varphi(\bar{x},\bar{y})$ (or a type $p(\bar{x},\bar{y})$) exemplifies the *strict order property* (strictOP) in a model M if it defines a partial order on M with infinite indiscernible chains.

(2) We say that a formula $\varphi(\bar{x},\bar{y})$ (or a type $p(\bar{x},\bar{y})$) exemplifies the *strong order property of order n* (SOP_n) (for $n \geq 3$) in M if it defines on M a graph with infinite indiscernible chains and no cycles of size n.

(3) We say that a formula $\varphi(\bar{x},\bar{y})$ (or a type $p(\bar{x},\bar{y})$) exemplifies the *strong order property of order $\leq n$* ($SOP_{\leq n}$) (for $n \geq 3$) in M if it defines on M a graph with infinite indiscernible chains and no cycles of size smaller than or equal to n.

(4) We say that a type $p(\bar{x},\bar{y})$ ($\bar{x}$, $\bar{y}$ can be of infinite length) exemplifies the *strong order property* (SOP) in M if it defines on M a graph with infinite indiscernible chains and no cycles at all.

(5) We say that a type $p(\bar{x},\bar{y})$ exemplifies the *finitary strong order property* ($FSOP$) in M if it exemplifies the SOP in M and $\bar{x}$, $\bar{y}$ are of *finite* length.

(6) We say that a formula/type exemplifies $SOP_n/SOP_{\leq n}/SOP/FSOP/$ strictOP in a (positive) Robinson theory T if it exemplifies it in the universal domain of T. We say that T has one of the above properties if some (positive) quantifier free formula/type exemplifies it.

Remark 1.2:

(1) Obviously, strict$OP \Rightarrow FSOP \Rightarrow SOP \Rightarrow SOP_n$ for all n. All these implications are known to be proper (see [Sh500]).

(2) $SOP_{n+1} \Rightarrow SOP_n$ for all n (this is less obvious, but still easy—see [Sh500]). This implication is also proper.

(3) A first order theory T has $SOP_n \Longleftrightarrow$ it has $SOP_{\leq n}$ — once again, see [Sh500]. The right-to-left direction is, of course, obvious; in order to go from left to right, one might have to change the formula that exemplifies the property.

(4) Once we enlarge our logic so that it is possible to define transitive closure of a (type-)definable relation (for example, going to $\mathbb{L}_{\omega_1,\omega}$), the strict order property loses its meaning and becomes equivalent to the SOP. But in the first order logic, and even in other logics which are still compact, these two properties might differ. We shall see that the class of Banach spaces equipped with any logic satisfying Loś theorem (in particular, with Henson's logic, see below) exemplifies this.

The SOP_n hierarchy is connected in the following way to the well-known classes of stable and simple theories:

FACT 1.3 ([Sh500]): Any theory/class with SOP_3 is not simple (and therefore not stable). This implication is proper, i.e. there exists a non-simple $NSOP_3$ theory.

So SOP_n hierarchy is "above" simple theories from the point of view of classification theory (i.e., more complicated), but yet below theories with strict order property. So by [Sh:c], theorem 4.7, all theories in the SOP_n hierarchy (properly, i.e. unstable and without the strict order property) have the independence property ([Sh:c], definition 4.3). In particular, this holds for Banach spaces and groups.

As most people are unfamiliar with positive model theory, we will discuss now the class of Banach spaces in more detail. In order to turn Banach spaces into a positive Robinson theory, one has to equip it with the positive strongly bounded logic. One way to do this is introducing (in addition to the language of a vector space over a fixed field) predicates $P_{[q_1,q_2]}(x)$ for every $0 \leq q_1 < q_2$ rationals (interpreted in the models as "$q_1 \leq \| x \| \leq q_2$") and allowing only positive formulae (i.e. the only admissible connectives are disjunction and conjunction).

These formulae are in particular positive bounded in the sense of Henson, therefore results proven in [HenIov] hold for them. For instance, this logic is preserved under taking ultraproducts; see the ultraproduct theorem (1.4) below. In fact, this theorem is one way of showing compactness in the universal domain of the class of Banach spaces.

See [HenIov] for the basics of positive model theory and [Ben02] on positive Robinson theories. One can enrich the language, but this basic one will do for our purposes.

In section 2 we show that there exist normed spaces with SOP_n exemplified by quantifier free basic positive strongly bounded formulae, and in fact there exists a normed space with "uniform" definition of those properties, i.e. $FSOP$. This is a strong nonstructure result for the class of all normed spaces, and the logic exemplifying it is the most simple and basic one. No special property of Banach spaces as a Robinson theory (i.e. compactness) is used in that proof except the fact that we don't have to care about indiscernibility of sequences exemplifying the order properties; see 1.12. We could have avoided this usage, but see no need of doing that, as all the formulae appearing in the proof are positive strongly bounded, therefore there is no harm in working in a universal domain (which is compact for these formulae).

In section 3 we show that the strict order property cannot be exemplified in the universal domain of the class of Banach spaces by a positive bounded formula. This is done using compactness of the monster model. More precisely, we use the following Henson's version of Loŝ theorem for positive formulae with bounded parameters. We state only one direction of the theorem, because no new notions have to be defined in order to formulate it, and this is the only direction we're using here. See [HenIov] for more versions of compactness in positive bounded logics, precise definitions of the ultraproduct and proofs.

THEOREM 1.4: *Let $\bar{V} = \langle V_i : i \in I \rangle$ be a sequence of normed spaces, I infinite. Let $\mathfrak{U}$ be a non-principal countably incomplete ultrafilter on I, and let V be the ultraproduct of $\bar{V}$ modulo $\mathfrak{U}$, or, more precisely, the Banach space which is (a completion of) the normed space we get by "throwing away" all the elements with infinite norm from the ultraproduct and dividing by the infinitesimals. Let $\bar{v}_i \in V_i$, $\bar{v}_i = \langle v_{i,j} : j < j^* \rangle$ (j^* does not depend on i). Suppose there exists a uniform bound for $\langle \| v_{i,j} \| : i \in I, j < j^* \rangle$. Denote $\bar{v} = \langle \bar{v}_i : i \in I \rangle / \mathfrak{U}$. Then for every positive bounded formula $\varphi(\bar{x})$, $\mathrm{len}(\bar{x}) = j^*$, $\{i : V_i \models \varphi(\bar{v}_i)\} \in \mathfrak{U} \Rightarrow V \models \varphi(\bar{v})$.*

Proof: This follows from a combination of theorems and propositions proved in [HenIov]. The proofs (as well as formulations) of those theorems use the notion of "approximate satisfaction" which we do not define here. Those who would like to see the whole proof will have to follow the following lead: first, recall that if a formula holds, then certainly so do all its approximations. Now combine [HenIov] (9.2) with the fact that the ultraproduct is $\aleph_1$-saturated by [HenIov] (9.18) and [HenIov] (9.20). ∎

Remark 1.5: Readers who are worried about the notion of approximate satisfaction that was ignored, should note that we will use the theorem for quantifier free positive bounded formulae, for which notions of approximate satisfaction and regular satisfaction coincide.

Another "test" line for classification of classes of models we consider here is whether or not a class can admit a universal model in certain cardinals. The questions discussed are the following (note that (2) seems more natural for our context than (1)):

QUESTION 1.6:

(1) *In which regular λ can the class of normed spaces have universal models, i.e. in which regular cardinalities can there exist a normed space that*

embeds (isometrically) any normed space of the same cardinality (and less)?

(2) *For which regular powers λ can there exist a Banach space B of density λ, which is universal (under isometries) for spaces of density $\leq \lambda$ (that is, every Banach space of density $\leq \lambda$ is isometrically embeddable to B)?*

These questions are certainly interesting for logicians investigating model theory of Banach spaces, but they can also be of some interest to functional analysts. Unlike saturated, big, compact models, etc., the concept and the importance of universal objects are well understood outside logic as well. Universal Banach spaces, for example, were studied by Banach himself in [Ban], and Szlenk in [Sz] showed there is no universal reflexive separable Banach space. Logicians and analysts do not always agree, though, on the question what a universal Banach space is. In the theory of Banach spaces more kinds of embeddings than just isometries (which are the natural functions between normed spaces as logical structures) are allowed. Therefore, the results here may not be of highest interest to non-logicians. In any case, we will present almost a full answer to 1.6.

Although questions concerning the universality spectrum seem no less natural (or maybe even more so, especially to non-logicians) than the spectrum of saturation, calculating it is much harder. Every theory T has a saturated model in a regular $\lambda = \lambda^{<\lambda} > |T|$. Considering other $\lambda \geq |T|$, T has a saturated model in such λ if and only if T is stable in λ. There is no similar result for universal models. Moreover, the universal spectrum is not absolute, so we have to deal with consistency results, as will be explained below.

Every saturated model is universal, therefore there exists a universal model (of every theory T) in every regular $\lambda = \lambda^{<\lambda}$. In fact, existence of a universal model can be shown by a straightforward construction for every first order theory and many non-elementary classes, in particular those discussed in this paper, in every regular $\lambda = 2^{<\lambda}$. But what about other regular cardinals? It is not hard to see that it is consistent for every T not to have a universal in a regular $\lambda < 2^{<\lambda}$ (see the appendix of [KjSh409]). Therefore, the only question one can ask for a given theory T is: for which regular $\lambda < 2^{<\lambda}$ is it *consistent* that T has a universal model in λ?

A group including the first author as well as Grossberg, Kojman, Džamonja and others has been interested in this question for particular theories and classes of theories. In [GrSh174], [Sh175], [Sh457], [DzSh614], [Dz] some positive (consistency) results can be found.

In [KjSh409], Kojman and Shelah showed a negative result very important for our discussion here:

THEOREM 1.7 ([KjSh409]): *Suppose λ is a regular cardinal such that for some cardinal κ, $\kappa^+ < \lambda < 2^\kappa$. Then there is no universal linear order of cardinality λ. Moreover, no theory with the strict order property (see Definition 1.1) has a universal model in λ.*

Remark 1.8: We will call a regular λ which satisfies the assumption of 1.7, a cardinal which is *far from the GCH.*

Remark 1.9: See [KjSh409] for results on singular cardinalities.

In [Sh500] the first author generalized 1.7 to theories which have a weaker property than the strict order property—SOP_4 (see 1.1). This is the most general known negative result, and the conjecture is that it cannot be generalized to bigger classes. We will use this result in the paper, so it should be formalized explicitly in a way that will come handy later:

THEOREM 1.10 ([Sh500]): *No theory/class with SOP_4 has a universal model in a cardinal which is far from the GCH. Moreover, suppose λ is far from the GCH, M is a model, $\varphi(\bar{x}, \bar{y})$ exemplifies SOP_4 in M. Then there exists a linear order J of cardinality λ such that there is no sequence of length$(\bar{x})$-tuples of order type J in M ordered by $\varphi(\bar{x}, \bar{y})$.*

So the result that the class of Banach spaces has SOP_4 gives a partial answer to Question 1.6. Together with the fact that there exists a universal Banach space in every regular $\lambda = 2^{<\lambda}$, we get a pretty good idea about the universality spectrum of Banach spaces.

Lastly, we point out that the requirement of indiscernibility of the infinite chains in the definition of the strong/strict order properties can be omitted. This will make our life easier when we prove those properties (in sections 2 and 4).

Suppose there exists in a "monster" model M an infinite sequence $\langle \bar{a}_i : i < \omega \rangle$ satisfying $i < j \Longrightarrow M \models \varphi(\bar{a}_i, \bar{a}_j)$. Then by compactness, there is such a sequence of any length in M. Therefore, using the Erdős–Rado theorem, without loss of generality the original ω-sequence $\langle \bar{a}_i : i < \omega \rangle$ is also indiscernible. Now, again by compactness, there is an indiscernible chain as required of any length and order type (just a chain of the same type as $\langle \bar{a}_i : i < \omega \rangle$).

To be more precise, we are using the following:

FACT 1.11: Suppose there exists an infinite sequence $\langle \bar{a}_i : i \in I \rangle$ (I—some infinite ordered set) in the universal domain M of some (positive) Robinson theory T, and let Δ be a finite set of formulae in the logic of T (note that the logic of T is compact). Then for any infinite ordered set J, there exists an *indiscernible* sequence $\langle \bar{b}_i : i \in J \rangle$ such that for all n there exist $i_1 < \cdots < i_n$ in I satisfying $\mathrm{tp}_\Delta(\bar{b}_0, \ldots, \bar{b}_{n-1}) \supseteq \mathrm{tp}_\Delta(\bar{a}_{i_1}, \ldots, \bar{a}_{i_n})$. In particular, if $\Delta = \{\varphi(x, y)\}$ and $\varphi(\bar{a}_i, \bar{a}_j)$ holds for all $i < j \in I$, the same thing holds for $\langle \bar{b}_i : i \in J \rangle$.

Proof: Use Erdős–Rado and compactness, exactly like the case of M a big model of a first order theory. For more details, see [Ben03]. ∎

We will use the following immediate corollary:

COROLLARY 1.12:

(1) *If M is the universal domain of a (positive) Robinson theory and we are interested in strict/strong order properties exemplified in it, indiscernibility can be omitted from all the items of Definition 1.1.*

(2) *If T is a (positive) Robinson theory with the universal domain M (or M is just a homogeneous model compact in the logic $\mathcal{L}$), T has $SOP/FSOP/SOP_n/SOP_{\leq n}/$ strict OP in $\mathcal{L} \iff$ it is exemplified in M by some formula in $\mathcal{L}$ with indiscernible infinite chains of any order type $\iff$ it is exemplified in M by some formula in $\mathcal{L}$ with an infinite (not necessarily indiscernible) chain.*

2. Banach spaces have FSOP

Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$.

Notation 2.1: We denote the "monster" Banach space (the universal domain of the positive Robinson theory of all Banach spaces) over $\mathbb{F}$ by $\mathcal{B}$. We identify the class (the Robinson theory) of Banach spaces with $\mathcal{B}$. So saying "$\mathcal{B}$ has or does not have property X" implies that the class has or does not have it.

THEOREM 2.2: *$\mathcal{B}$ has SOP_n for all $n \geq 3$. Moreover, there is a positive strongly bounded quantifier free formula $\varphi_n(\bar{x}, \bar{y})$ exemplifying $SOP_{\leq n}$ in $\mathcal{B}$ with $\operatorname{len} \bar{x} = \operatorname{len} \bar{y} = 2$, such that $\varphi_{n+2}(\bar{x}, \bar{y}) \vdash \varphi_n(\bar{x}, \bar{y})$.*

Proof: Choose $n > 2$.

First we define a seminormed space B_0. As a vector space over $\mathbb{F}$, its basis is $\{a_\alpha : \alpha < \omega\} \cup \{b_\alpha : \alpha < \omega\}$. The seminorm is defined by $s(v) = \sup_{\gamma < \omega} \{|f_\gamma(v)|\}$, where f_γ is a functional defined on the basis as follows:

$$f_\gamma(a_\alpha)=1 \quad \text{if } \alpha<\gamma, \quad f_\gamma(a_\alpha)=0 \quad \text{if } \alpha\geq\gamma,$$
$$f_\gamma(b_\alpha)=0 \quad \text{if } \alpha<\gamma, \quad f_\gamma(b_\alpha)=1 \quad \text{if } \alpha\geq\gamma,$$

and extended to every $v \in B_0$ in the only possible way. Note that in fact $s(v) = \max_{\gamma<\omega}\{|f_\gamma(v)|\}$ (i.e. the seminorm is finite). It is not a norm: it's easy to see that, for example, $s(a_0-a_1-b_1+b_0)=0$. So we define B_1 as the normed space $B_0/\{v : s(v)=0\}$. Note that $\{a_\alpha : \alpha<\omega\}\cup\{b_\alpha : \alpha<\omega\}$ is no longer a basis for B_1, though it certainly still is a set that generates the vector space. The following easy fact will be important for us:

⊗ 2.2.1 $\{a_\alpha : \alpha<\omega\}\cup\{b_\alpha : \alpha<\omega\}$ is a sequence of distinct non-zero elements in B_1.

Now denote the completion of B_1 by B (of which we can think as of a subspace of the "monster" $\mathcal{B}$).

Now we define a *term* in the language of Banach spaces (a positive bounded term) $\tau_{n,\ell}(x,y)$ by

$$\tau_{n,\ell}(x,y)=(n-2\ell)x+(n-2\ell+1)y.$$

Now define $c_{n,\ell,\alpha}=\tau_{n,\ell}(a_\alpha,b_\alpha)$, i.e.

$$c_{n,\ell,\alpha}=(n-2\ell)a_\alpha+(n-2\ell+1)b_\alpha.$$

It is clear from the definitions that

$$|f_\gamma(c_{n,\ell,\alpha})|=n-2\ell, \quad \text{if } \alpha<\gamma$$

and

$$|f_\gamma(c_{n,\ell,\alpha})|=n-2\ell+1, \quad \text{if } \alpha\geq\gamma.$$

Therefore (check the calculation),

⊗ 2.2.2

$$\bigwedge_{\alpha<\beta<\omega}\ \bigwedge_{\ell<n}\|c_{n,\ell+1,\beta}-c_{n,\ell,\alpha}\|=2.$$

In this case the maximum is achieved when either $\gamma\leq\alpha<\beta$ or $\alpha<\beta<\gamma$. For example, if $\alpha<\beta<\gamma$, then $f_\gamma(c_{n,\ell,\alpha})=n-2\ell$, while $f_\gamma(c_{n,\ell+1,\beta})=n-2(\ell+1)=n-2\ell-2$.

Also,

⊗ 2.2.3

$$\bigwedge_{\alpha<\beta<\omega}\ \bigwedge_{m\leq n}\|c_{n,m,\alpha}-c_{n,0,\beta}\|=2m+1.$$

In this case the maximum is achieved when $\alpha < \gamma \leq \beta$, then $f_\gamma(c_{n,m,\alpha}) = n - 2m$, while $f_\gamma(c_{n,0,\beta}) = n+1$.

Now we define $\varphi_n = \varphi_n(x_1y_1, x_2y_2)$:

$$\varphi_n = \bigwedge_{\ell<n} (\|\tau_{n,\ell+1}(x_2,y_2) - \tau_{n,\ell}(x_1,y_1)\| \leq 2) \;\&$$

$$\bigwedge_{m\leq n} (\|\tau_{n,m}(x_1,y_1) - \tau_{n,0}(x_2,y_2)\| \geq 2m+1) \;\&$$

$$\bigwedge_{m\leq n} (\|\tau_{n,m}(x_1,y_1) - \tau_{n,0}(x_2,y_2)\| \leq 2m+2).$$

Remark 2.2.4: The last demand is not needed; as the reader will see in the proof, its only purpose is to make the formula strongly bounded. Readers who are interested only in Henson and Iovino's logic can just omit it.

Now we shall show that φ_n exemplifies $SOP_{\leq n}$ in $\mathcal{B}$. First, by 2.2.2, 2.2.3, and of course 2.2.1, the sequence $\langle a_\alpha b_\alpha : \alpha < \omega\rangle$ verifies the first part of the definition (in B, of which we think as of a subspace of $\mathcal{B}$), i.e. it is an infinite chain of the graph defined by φ_n on $\mathcal{B}$ (by 1.12, we don't need to prove indiscernibility).

The only thing that is left to verify is that there are no cycles of length $\leq n$ in this graph, and this is an immediate consequence of the triangle inequality (well hidden under the cover of long formulae):

Suppose $2 < m \leq n$, and suppose there are $\langle c_i, d_i : i \leq m\rangle$ in $\mathcal{B}$ such that $\mathcal{B} \models \varphi_n(c_id_i, c_{i+1}d_{i+1})$ for $i < m$ and $\mathcal{B} \models \varphi_n(c_md_m, c_0d_0)$. Then in particular, from $\mathcal{B} \models \varphi_n(c_id_i, c_{i+1}d_{i+1})$ for $i < m$ follows (taking only the "first component" of φ_n)

$\otimes$ 2.2.5

$$\bigwedge_{i<m} (\|\tau_{n,i+1}(c_{i+1},d_{i+1}) - \tau_{n,i}(c_i,d_i)\| \leq 2).$$

On the other hand, $\mathcal{B} \models \varphi_n(c_md_m, c_0d_0)$ implies (taking only the "second component" of φ_n)

$\otimes$ 2.2.6

$$\|\tau_{n,m}(c_m,d_m) - \tau_{n,0}(c_0,d_0)\| \geq 2m+1.$$

But from 2.2.5 it follows that

$$\begin{aligned}\|\tau_{n,m}(c_m,d_m) - \tau_{n,0}(c_0,d_0)\| \leq & \|\tau_{n,m}(c_m,d_m) - \tau_{n,m-1}(c_{m-1},d_{m-1})\| \\ & + \cdots + \|\tau_{n,1}(c_1,d_1) - \tau_{n,0}(c_0,d_0)\| \\ \leq & 2m,\end{aligned}$$

which contradicts 2.2.6.

Remark 2.2.7: Careful readers have probably noted that we actually showed that φ_n exemplifies $SOP_{\leq(n+1)}$ in $\mathcal{B}$, but it doesn't matter for our discussion.

We know now that φ_n exemplifies $SOP_{\leq n}$. In order to complete the proof of the theorem, we need to show that $\varphi_{n+2} \vdash \varphi_n$ for all $n \geq 3$. For this, just note that $\tau_{n,\ell}(x,y) = \tau_{n+2,\ell+1}(x,y)$, and the rest follows immediately from the definition of φ_n. ∎

As corollaries, we derive several nonstructure results. The first one appears also in [Ben03]:

COROLLARY 2.3: *The class of Banach spaces is not simple, i.e. there exists a Banach space, which is not simple, therefore not stable.*

Proof: See 1.3. ∎

The following corollary can be summarized as "universal normed spaces in regular cardinals exist only if they have to", i.e. there are "few" universal normed spaces (under isometry).

COROLLARY 2.4: *Suppose λ is a regular cardinal far from the GCH (see 1.8). Then there is no universal normed space of cardinality λ (under isometries).*

Proof: SOP_4 is enough for this result—see 1.10 ([Sh500], Theorem 2.13). ∎

Remark 2.5: This almost answers 1.6(1).

In fact, if we look closer at 1.10, we'll find out that a more interesting (for our context) result can be formalized:

COROLLARY 2.6: *Suppose λ is a regular cardinal far from the GCH (see 1.8). Then there is no universal model for the class of Banach spaces of density λ (under isometries).*

Proof: Let M be a candidate. By universality, it certainly embeds B from the proof of 2.2, and on the other hand is itself embedded in the "monster". So there is $\varphi(\bar{x}, \bar{y})$ exemplifying SOP_4 in it. Let J be as in 1.10. Now the Banach space B_J, built in a similar way as B with basis of order type J (so its density is at most λ), cannot be embedded into M, which is a contradiction to universality. ∎

Remark 2.7: This almost answers 1.6(2).

COROLLARY 2.8: *There exists a positive strongly bounded quantifier free type* $p(\bar{x}, \bar{y})$ *with* $\operatorname{len}(\bar{x}) = \operatorname{len}(\bar{y}) = 2$, *exemplifying* $FSOP$ *in* $\mathcal{B}$.

Proof: Choose

$$p(\bar{x}, \bar{y}) = \bigwedge_{n \in \omega} \varphi_{2n+3}(\bar{x}, \bar{y});$$

p is consistent by compactness, as $\varphi_{n+2}(\bar{x}, \bar{y})$ implies $\varphi_n(\bar{x}, \bar{y})$. Now, as $\varphi_n(\bar{x}, \bar{y})$ exemplifies $SOP_{\leq n}$ in $\mathcal{B}$, and $\langle a_\alpha b_\alpha : \alpha < \omega \rangle$ from the proof of 2.2 is an infinite sequence ordered by φ_n for every n, the result is clear. ∎

3. Banach spaces do not have the strict order property

A natural question after we have shown 2.8 is: does $\mathcal{B}$ have the strict order property? Or, a more general question: does having a (type-definable) graph as in 2.8 imply the strict order property (maybe also type-definable)? Suppose we gave up compactness and allowed ourselves $\mathbb{L}_{\omega_1, \omega}$ formulae, i.e. infinite disjunctions as well as infinite conjunctions. Then the answer to the second question is certainly positive, as one can define the transitive closure of a relation using an infinite disjunction, and the transitive closure of $p(\bar{x}, \bar{y})$ is easily seen to be a partial order on $\mathcal{B}$. But in our case the implication is not clear, and in fact turns out to be false—we will give a negative answer to the first question (and therefore to the second one). So the positive Robinson theory of Banach spaces turns out to be an example of a class having a "uniform" definition of SOP_n, but yet without the strict OP.

THEOREM 3.1: $\mathcal{B}$ *does not have the strict order property exemplified by a positive bounded type (in particular,* $\mathcal{B}$ *doesn't have the* strict OP *exemplified by a p.b. formula).*

Proof: Suppose towards a contradiction that $q(\bar{x}, \bar{y})$ is a positive bounded type which exemplifies strict OP in $\mathcal{B}$. So for every linear order I, there is an indiscernible sequence $\langle \bar{a}_i : i \in I \rangle$ which is linearly ordered by $q(\bar{x}, \bar{y})$. We will choose $I = \mathbb{Z}$ with the usual order.

Denote $\operatorname{len}(\bar{x}) = \operatorname{len}(\bar{y})$ in $q(\bar{x}, \bar{y})$ by n and assume wlog that there exists $n^* < n$ such that $\bigwedge_{\ell < n^*} (a_{i,\ell} = a^*_\ell)$ for all $i \in I$ and $\langle \bar{a}_{i,\ell} : n^* \leq \ell < n, i \in I \rangle$ is a linearly independent sequence over $\langle a^*_\ell : \ell < n^* \rangle$. In other words, we assume

Assumption 3.1.1: $\langle a^*_\ell : \ell < n^* \rangle \bigcup \langle \bar{a}_{i,\ell} : n^* \leq \ell < n, i \in I \rangle$ is a *basis* for $\langle \bar{a}_i : i \in I \rangle_{\mathcal{B}}$.

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Define for $k < \omega$, $B'_k = \langle \bar{a}_k, \bar{a}_{k+1} \rangle_{\mathcal{B}}$. Denote, for any $k_2 > k_1 + 1$, $B'_{k_1} \cap B'_{k_2}$ by V^- (generated by $\langle a^*_\ell : \ell < n^* \rangle$).

Pick $m < \omega$ and define V_m as a vector subspace (over $\mathbb{F}$) generated by $\langle \bar{a}_0, \bar{a}_1, \ldots, \bar{a}_m \rangle$ in $\mathcal{B}$. Note that by 3.1.1, V_m (as a vector space) is just a free amalgamation of $B'_0, \ldots, B'_{m-1}$ over $\langle \bar{a}_1 \rangle, \ldots, \langle \bar{a}_{k-1} \rangle$ and V^-. We shall define three different norms on V_m. In order not to get confused between the original indiscernible sequence and the new normed space that we are going to define, we'll write $\langle \bar{b}_i : i \leq m \rangle$ instead of $\langle \bar{a}_i : i \leq m \rangle$. Let $h_1 \colon V_m \to \mathcal{B}$ and $h_{-1} \colon V_m \to \mathcal{B}$ be natural (linear) embeddings such that for $0 \leq k \leq m$, $h_1(\bar{b}_k) = \bar{a}_k$ and $h_{-1}(\bar{b}_k) = \bar{a}_{-k}$. Let $g_{\ell,k} \colon \langle \bar{b}_\ell \rangle \to \langle \bar{b}_k \rangle$ be the natural isomorphism mapping $\bar{b}_\ell$ onto $\bar{b}_k$ and $g_k = g_{0,k}$. Let $h^\ell_k \colon \langle \bar{b}_k \rangle \to \langle \bar{a}_\ell \rangle_{\mathcal{B}}$ be the natural isomorphism mapping $\bar{b}_k$ onto $\bar{a}_\ell$, i.e. $h^\ell_k = h_1 \restriction \langle \bar{b}_\ell \rangle \circ g_{k,\ell}$.

Now we define three different norms on B'_k (for $k < m$). $\|\cdot\|_1$ is a norm induced by h_1 (which is in fact the identity), $\|\cdot\|_{-1}$ is induced by h_{-1}, lastly $\|\cdot\|_0$ is defined by $\max\{\|\cdot\|_1, \|\cdot\|_{-1}\}$. Now we expand these definitions to V_m: define for $t \in V_m$,

$$i \in \{0, 1, -1\}, \| t \|_i = \inf\{ \sum_{k<m} \| t_k \|_i : t_k \in B'_k, \sum_{k<m} t_k = t \}.$$

In other words, V_m as a normed space (with three different norms) is defined by induction on m: for $m = 1$ it is just B'_0, V_{m+1} is the amalgamation (in the regular sense of normed spaces) of V_m and B'_m over $\langle \bar{b}_m \rangle$. In particular, V_m and B'_k agree on the definitions of the norms, and so do V_{m_1} and V_{m_2} for $m_1 < m_2$. Therefore, when we use norms $\|\cdot\|_i$, we will not specify in which V_m they are calculated.

In fact, eventually we'll be interested only in $\|\cdot\|_1$. Our goal is to show that taking free amalgamations of $\langle \bar{a}_0, \bar{a}_1 \rangle_{\mathcal{B}} \ldots \langle \bar{a}_{m-1}, \bar{a}_m \rangle_{\mathcal{B}}$ leads (in the limit, when m tends to infinity — and here is where the compactness of our logic will be used) to a symmetric type. The two other norms are useful for showing the limit is symmetric, and their role will become clear in 3.1.4.

Note an easy but important observation:

Remark 3.1.2: For all $r \in V_m$, $i \in \{1, -1\}$, $\| r \|_i \geq \| h_i(r) \|_{\mathcal{B}}$.

Proof: By the definition of $\|\cdot\|_i$, given $\varepsilon > 0$, there are $t_p \in B'_p$ for $p < m$ such that $r = \sum_{p<m} t_p$ and $\| r \|_i + \varepsilon \geq \sum_{p<m} \| t_p \|_i = \sum_{p<m} \| h_i(t_p) \|_{\mathcal{B}}$. The last equality is true by the definition of $\|\cdot\|_i$ on B'_p. As h_i is linear, we conclude $\| h_i(r) \|_{\mathcal{B}} = \| h_i(\sum_{p<m} t_p) \|_{\mathcal{B}} \leq \sum_{p<m} \| h_i(t_p) \|_{\mathcal{B}} \leq \| r \|_i + \varepsilon$. As ε was arbitrary, we are done. ∎

Let $r', r'' \in \langle \bar{b}_0 \rangle$. Define for $0 < k \leq m$, $r_k = r' + g_k(r'')$. We will be interested in $\| r_k \|_i$ for $i \in \{0, 1, -1\}$. Note that by the definition of the norm $\| \cdot \|_i$, for each $\varepsilon > 0$, there are $t_p \in B'_p$ for $p < k$ such that $r_k = \sum_{p<k} t_p$ and $\| r_k \|_i + \varepsilon \geq \sum_{p<k} \| t_p \|_i \geq \| r_k \|_i$. In the following claim we will see what t_p as above look like.

Denote (for $k < m$) $V_k^+ = \langle b_{k,\ell} : n^* \leq \ell < n \rangle_{V_m}$. Denote (for $\ell < n^*$) $b_\ell^* = b_{k,\ell}$ for some/all $k < \omega$, so $b_\ell^* = a_\ell^*$. We will still write V^- for $\langle b_\ell^* : \ell < n^* \rangle_{V_m} = \langle a_\ell^* : \ell < n^* \rangle_B$.

Claim 3.1.3: *Let $t_p \in B'_p$ such that $r_k = \sum_{p<k} t_p$. Then there exist $r'_p \in \langle \bar{b}_p \rangle$ and $s_p \in V^-$ for $0 \leq p \leq k$ such that $t_p = -r'_p + r'_{p+1} + s_p$ and for $0 < p < k, r'_p \in V_p^+$. Moreover, we may assume $r'_0 = -r'$ and $r'_k = g_k(r'')$, therefore $\sum_{p<k} s_p = 0$.*

Proof: As $t_p \in B'_p$, we can write for every $p < k$, $t_p = \hat{r}_{p+1} - \check{r}_p + s_p$ for $\hat{r}_{p+1} \in V_{p+1}^+, \check{r}_p \in V_p^+$ and $s_p \in V^-$. So we get $r_k = r' + g_k(r'') = -\check{r}_0 + \sum_{0<p<k}(\hat{r}_p - \check{r}_p) + \hat{r}_k + \sum_{p<k} s_p$. By 3.1.1 and the definition of V_m, $\langle b_\ell^* : \ell < n^* \rangle \cup \langle \bar{b}_{i,\ell} : n^* \leq \ell < n, i \leq m \rangle$ is a basis of V_m. As $\hat{r}_p$ and $\check{r}_p$ are both elements of V_p^+, remembering the fact that $r_k = r' + g_k(r'')$, where $r' \in \langle \bar{b}_0 \rangle$ and $r'' \in \langle \bar{b}_k \rangle$, we get that necessarily $\hat{r}_p = \check{r}_p$. For $0 < p < k$, this is going to be r'_p. As the claim does not demand $r'_0, r'_k \notin V^-$, and we know that $r' + \check{r}_0 \in V^-$, as well as $g_k(r'') - \hat{r}_k$, by changing s_0 and s_{k-1}, we may assume $\check{r}_0 = -r'$ and $\hat{r}_k = g_k(r'')$. As $r_k = -r'_0 + r'_k + \sum_{p<k} s_p$, we get $\sum_{p<k} s_p = 0$. ∎

Now we shall show

Claim 3.1.4:

(1) *For each $i \in \{0, 1, -1\}$, $\| r_k \|_i$ is an increasing (with k) uniformly bounded sequence (the bound does not depend on m).*

(2) *For each $j > 1$, $m = j^2$, for each $i \in \{0, 1, -1\}$, $\| r_m \|_0 \geq \| r_m \|_i \geq (1 + 2/j)^{-1} \cdot \| r_j \|_0$.*

Proof:

(1) First we show the boundedness: $\| r' + g_k(r'') \|_i \leq \| r' \|_i + \| g_k(r'') \|_i = \| h_1(r') \|_B + \| h_1(g_k(r'')) \|_B = \| h_1(r') \|_B + \| h_1(r'') \|_B$. So as we see, the bound does not depend on m.

Now suppose $k < \ell$. We aim to show that $\| r_k \|_i \leq \| r_\ell \|_i$. First we'll prove this for $i = 1$.

Choose $\varepsilon > 0$. By the definition, there exist $t_p \in B'_p$ such that $r_\ell = \sum_{p<\ell} t_p$ and $\| r_\ell \|_1 \leq \sum_{p<\ell} \| t_p \|_1 \leq \| r_\ell \|_1 + \varepsilon$.

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Let $r'_p \in \langle \bar{b}_p \rangle$ and $s_p \in V^-$ for $0 \leq p \leq \ell$ be as in 3.1.3.

Then

$$\begin{aligned} r_k &= r' + g_k(r'') = -r'_0 + g_k(r'') \\ &= -r'_0 + r'_1 - r'_1 + r'_2 - r'_2 + \cdots + r'_{k-1} - r'_{k-1} + g_k(r'') + \sum_{p<\ell} s_p \end{aligned}$$

(remember: $\sum_{p<\ell} s_p = 0$).

Therefore,

⊗ 3.1.5:

$$\begin{aligned} \| r_k \|_1 \leq \| -r'_0 + r'_1 + s_0 \|_1 + \| -r'_1 + r'_2 + s_1 \|_1 + \cdots \\ + \| -r'_{k-1} + g_k(r'') + \sum_{k-1 \leq p < \ell} s_p \|_1. \end{aligned}$$

Remembering that

$$\| -r'_{k-1} + g_k(r'') + s \|_1 = \| h_1(-r'_{k-1} + g_k(r'') + s) \|_{\mathcal{B}}$$

for $s \in V^-$ (as $-r'_{k-1} + g_k(r'') + s \in B'_{k-1}$) and using the fact that h_1 is linear, we get

$$\begin{aligned} &\left\| - r'_{k-1} + g_k(r'') + \sum_{k-1 \leq p < \ell} s_p \right\|_1 = \\ &\qquad \left\| h_1(-r'_{k-1}) + h_1(g_k(r'')) + h_1\Big(\sum_{k-1 \leq p < \ell} s_p \Big) \right\|_{\mathcal{B}}. \end{aligned}$$

Now by the indiscernibility of $\bar{a}_i$ in $\mathcal{B}$ (note that $k-1<k$, $k-1<\ell$),

$$\begin{aligned} &\left\| h_1(-r'_{k-1}) + h_1(g_k(r'')) + h_1\Big(\sum_{k-1 \leq p < \ell} s_p \Big) \right\|_{\mathcal{B}} \\ &\qquad = \left\| h_1(-r'_{k-1}) + h_1(g_\ell(r'')) + h_1\Big(\sum_{k-1 \leq p < \ell} s_p \Big) \right\|_{\mathcal{B}} \\ &\qquad = \left\| h_1(-r'_{k-1} + g_\ell(r'') + \sum_{k-1 \leq p < \ell} s_p) \right\|_{\mathcal{B}}. \end{aligned}$$

But by 3.1.2,

$$\begin{aligned} &\left\| h_1\Big(- r'_{k-1} + g_\ell(r'') + \sum_{k-1 \leq p < \ell} s_p \Big) \right\|_{\mathcal{B}} \\ &\qquad \leq \left\| - r'_{k-1} + g_\ell(r'') + \sum_{k-1 \leq p < \ell} s_p \right\|_1. \end{aligned}$$

Combining all the above,

$\otimes$ 3.1.6:

$$\Big\| -r'_{k-1}+g_k(r'')+\sum_{k-1\le p<\ell} s_p\Big\|_1 \le \Big\| -r'_{k-1}+g_\ell(r'')+\sum_{k-1\le p<\ell} s_p\Big\|_1 .$$

Now we will go back to 3.1.5 and use 3.1.6:

$$\begin{aligned}\| r_k \|_1 \le & \| -r'_0+r'_1+s_0 \|_1 + \| -r'_1+r'_2+s_1 \|_1+\cdots \\ & + \Big\| -r'_{k-1}+g_k(r'')+\sum_{k-1\le p<\ell} s_p\Big\|_1 \\ \le & \| -r'_0+r'_1+s_0 \|_1 + \| -r'_1+r'_2+s_1 \|_1+\cdots \\ & + \Big\| -r'_{k-1}+g_\ell(r'')+\sum_{k-1\le p<\ell} s_p\Big\|_1 \\ = & \| -r'_0+r'_1+s_0 \|_1 + \| -r'_1+r'_2+s_1 \|_1+\cdots \\ & + \Big\| -r'_{k-1}+r'_k-r'_k+\cdots+r'_{\ell-1}-r'_{\ell-1}+g_\ell(r'')+\sum_{k-1\le p<\ell} s_p\Big\|_1 \\ \le & \sum_{p<\ell} \| -r'_p+r'_{p+1}+s_p \|_1 = \sum_{p<\ell} \| t_p \|_1 \le \| r_\ell \|_1 + \varepsilon .\end{aligned}$$

As ε was arbitrary, we finish for the case $i=1$. The same argument is used for $i=-1$, and the case $i=0$ follows.

(2) Define (just for the proof) $\mathcal{B}_{i,j} = \langle \bar a_i, \bar a_j\rangle_{\mathcal{B}}$. Just as in case of $\langle \bar b_i, \bar b_j\rangle$, we can define three norms on $\mathcal{B}_{i,j}$: one is induced from the original norm on $\mathcal{B}$ (an analog of $\|\cdot\|_1$), the second one is induced from the norm on $\mathcal{B}_{j,i}$, using the isomorphism from $\mathcal{B}_{j,i}$ onto $\mathcal{B}_{i,j}$ taking $\bar a_i$ onto $\bar a_j$ and vice versa (an analog of $\|\cdot\|_{-1}$). The third norm on $\mathcal{B}_{i,j}$ (the one we will be actually interested in) will be denoted by $\|\cdot\|_{\mathcal{B}^{max}_{i,j}}$, and it is naturally an analog of $\|\cdot\|_0$, i.e. the maximum of the first two norms.

So we start the proof with the following

MAIN CLAIM 3.1.7: *Suppose $m>k+1$, $c_k\in\langle \bar b_k\rangle_{V_m}$, $c_m\in\langle \bar b_m\rangle_{V_m}$. Denote $r=c_m-c_k\in\langle \bar b_k,\bar b_m\rangle_{V_m}$. Then $\| r\|_1 \ge (1+\frac{2}{m-k})^{-1}\cdot \| h_1(r)\|_{\mathcal{B}^{max}_{k,m}}$.*

Proof of the Main Claim: First of all, wlog $k=0$. Pick $\varepsilon>0$. By the definition of $\|\cdot\|_1$, there are $t_p\in B'_p$ for $p<m$ such that $r=\Sigma t_p$ and $\| r\|_1+\varepsilon \ge \Sigma \| t_p\|_1 \ge \| r\|_1$. By the same argument as in 3.1.3, we can find $c_p\in\langle \bar b_p\rangle$ for $p<m$ such that for all p, $t_p=c_{p+1}-c_p$. Denote

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$\| t_p \|_1 = \| h_1(t_p) \|_{\mathcal{B}}$ by ϱ_p. So

$$\| r \|_1 + \varepsilon \geq \Sigma_{p<m}\varrho_p \geq \| r \|_1$$

and we aim to show

$$\| h_1(r) \|_{\mathcal{B}^{max}_{0,m}} \leq \left(1 + \frac{2}{m}\right) \cdot \| r \|_1.$$

Instead, we will show

$$\| h_1(r) \|_{\mathcal{B}^{max}_{0,m}} \leq \left(1 + \frac{2}{m}\right) \cdot \Sigma\varrho_p.$$

This will certainly suffice, as this will imply

$$\| h_1(r) \|_{\mathcal{B}^{max}_{0,m}} \leq \left(1 + \frac{2}{m}\right) \cdot \| r \|_1 + \left(1 + \frac{2}{m}\right) \cdot \varepsilon$$

and ε was arbitrary, while m here is fixed.

By 3.1.2, $\| h_1(r) \|_{\mathcal{B}} \leq \| r \|_1 \leq \Sigma\varrho_p \leq (1 + \frac{2}{m}) \cdot \Sigma\varrho_p$. Therefore it's left to show that

$$\| h_{-1}(r) \|_{\mathcal{B}} \leq \left(1 + \frac{2}{m}\right) \cdot \Sigma\varrho_p.$$

Denote for $p < m$ and $\alpha \in I$, $c_p^\alpha = h_p^\alpha(c_p)$. By the indiscernibility of $\bar{a}_\alpha$, for all $\alpha < \beta \in I$,

$$\varrho_p = \| c_{p+1}^\beta - c_p^\alpha \|_{\mathcal{B}}.$$

Also, denote for some/all $\alpha < \beta$

$$\varrho^* = \| c_0^\beta - c_m^\alpha \|_{\mathcal{B}}.$$

Note that $\varrho^* = \| h_{-1}(r) \|_{\mathcal{B}}$, so our goal is

$$\varrho^* \leq \left(1 + \frac{2}{m}\right) \cdot \Sigma\varrho_p.$$

For every $\alpha < \beta \in I$ there is a functional $f_{\alpha,\beta}\colon \mathcal{B} \to \mathbb{F}$ such that

$$\| f_{\alpha,\beta} \| = 1, f_{\alpha,\beta}(c_0^\beta - c_m^\alpha) = \varrho^*.$$

Choose ℓ such that ϱ_ℓ is minimal. In particular,

$\bigotimes$ 3.1.8:

$$\varrho_\ell \leq \frac{1}{m}\Sigma_{p<m}\varrho_p.$$

Choose $\alpha_0 < \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$ in I.

$$\begin{aligned}
\varrho^* =& \| c_0^{\alpha_3} - c_m^{\alpha_1} \| = |f_{\alpha_1,\alpha_3}(c_0^{\alpha_3} - c_m^{\alpha_1})| \\
=& |f_{\alpha_1,\alpha_3}(c_0^{\alpha_3} - c_1^{\alpha_4}) \\
&+ \sum_{p=1}^{\ell-1} f_{\alpha_1,\alpha_3}(c_p^{\alpha_4+p-1} - c_{p+1}^{\alpha_4+p}) + f_{\alpha_1,\alpha_3}(c_\ell^{\alpha_4+\ell-1} - c_{\ell+1}^{\alpha_0-m+\ell+1}) \\
&+ \sum_{p=\ell+1}^{m-2} f_{\alpha_1,\alpha_3}(c_p^{\alpha_0-m+p} - c_{p+1}^{\alpha_0-m+p+1}) + f_{\alpha_1,\alpha_3}(c_{m-1}^{\alpha_0-m+(m-1)} - c_m^{\alpha_1})| \\
\le& \varrho_0 + \cdots + \varrho_{\ell-1} + |f_{\alpha_1,\alpha_3}(c_\ell^{\alpha_4+\ell-1} - c_{\ell+1}^{\alpha_0-m+\ell+1})| + \varrho_{\ell+1} + \cdots + \varrho_{m-1}.
\end{aligned}$$

The last inequality is true as $\| f_{\alpha_1,\alpha_3} \| = 1$.

Denote $\beta_2 = \alpha_4 + \ell - 1$, $\beta_1 = \alpha_0 - m + \ell + 1$. Find $\beta_0 < \beta_1 < \beta_2 < \beta_3$ in I. Now note that

$$c_\ell^{\beta_2} - c_{\ell+1}^{\beta_1} = (c_\ell^{\beta_2} - c_{\ell+1}^{\beta_3}) - (c_\ell^{\beta_0} - c_{\ell+1}^{\beta_3}) - (c_{\ell+1}^{\beta_1} - c_\ell^{\beta_0}).$$

Therefore, $\| c_\ell^{\beta_2} - c_{\ell+1}^{\beta_1} \| \le 3\varrho_\ell$. So (as $\| f_{\alpha_1,\alpha_3} \| = 1$)

$$|f_{\alpha_1,\alpha_3}(c_\ell^{\alpha_4+\ell-1} - c_{\ell+1}^{\alpha_0-m+\ell+1})| \le \| c_\ell^{\beta_2} - c_{\ell+1}^{\beta_1} \| \le 3\varrho_\ell.$$

Putting all the inequalities together (including 3.1.8), we conclude

$$\begin{aligned}
\varrho^* &\le \varrho_0 + \cdots + \varrho_{\ell-1} + |f_{\alpha_1,\alpha_3}(c_\ell^{\alpha_4+\ell-1} - c_{\ell+1}^{\alpha_0-m+\ell+1})| + \varrho_{\ell+1} + \cdots + \varrho_{m-1} \\
&\le \left(\sum_{p<m} \varrho_p - \varrho_\ell \right) + 3\varrho_\ell = \sum_{p<m} \varrho_p + 2\varrho_\ell \le \sum_{p<m} \varrho_p + 2 \cdot \frac{1}{m} \sum_{p<m} \varrho_p \\
&= \left(1 + \frac{2}{m}\right) \sum_{p<m} \varrho_p,
\end{aligned}$$

which finishes the proof of the Main Claim. ∎

Now assume $m = j^2 > 1$. We aim to show $\| r_j \|_0 \le \| r_m \|_1 \cdot (1 + \frac{2}{j})$. As usual, we pick $\varepsilon > 0$ arbitrary and assume $\| r_m \|_1 + \varepsilon \ge \Sigma_{p<m} \| t_p \|_1$ for some $t_p \in B'_p$ satisfying $r_m = \Sigma_{p<m} t_p$, where $t_p = -r'_p + r'_{p+1} + s_p$ as in 3.1.3. Denote for $\ell \le j$, $\hat{r}_\ell = g_{\ell \cdot j, \ell}(r'_{\ell \cdot j})$, i.e. $\hat{r}_\ell$ is a copy of $r'_{\ell \cdot j}$ in $\langle \bar{b}_\ell \rangle$.

So by the definition of $r_j = r' + g_j(r'')$, we have

$$\| r_j \|_0 = \| r' + \hat{r}_1 - \hat{r}_1 + \hat{r}_2 - \hat{r}_2 + \cdots + \hat{r}_{j-1} - \hat{r}_{j-1} + g_j(r'') \|_0.$$

Now note that $g_j(r'') = \hat{r}_j$: $g_m(r'') = r'_m$ (by 3.1.3), therefore $g_j(r'') = g_{m,j}(g_m(r'')) = g_{m,j}(r'_m)$. Remembering that $m = j^2$ and the definition of

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$\hat{r}_j$, we get the desired result.

Also remember that $r' = -r'_0$ and $\Sigma_{p<m} s_p = 0$ (see 3.1.3). We get

$$\begin{aligned}
\| r_j \|_0 =& \Big\| - r'_0 + \hat{r}_1 - \hat{r}_1 + \cdots + \hat{r}_{j-1} - \hat{r}_{j-1} + \hat{r}_j + \sum_{p<m} s_p \Big\|_0 \\
=& \Big\| - \hat{r}_0 + \hat{r}_1 - \hat{r}_1 + \cdots + \hat{r}_{j-1} - \hat{r}_{j-1} + \hat{r}_j + \sum_{p<m} s_p \Big\|_0 \\
\leq& \Big\| - \hat{r}_0 + \hat{r}_1 + \sum_{p<j} s_p \Big\|_0 + \Big\| - \hat{r}_1 + \hat{r}_2 + \sum_{j \leq p < 2j} s_p \Big\|_0 \\
&+ \cdots + \Big\| - \hat{r}_{j-1} + \hat{r}_j + \sum_{j(j-1) \leq p < m} s_p \Big\|_0 \\
=& \Big\| h_1 \Big(- \hat{r}_0 + \hat{r}_1 + \sum_{p<j} s_p \Big) \Big\|_{\mathcal{B}^{max}_{0,1}} + \Big\| h_1 \Big(- \hat{r}_1 + \hat{r}_2 + \sum_{j \leq p < 2j} s_p \Big) \Big\|_{\mathcal{B}^{max}_{1,2}} \\
&+ \cdots + \Big\| h_1 \Big(- \hat{r}_{j-1} + \hat{r}_j + \sum_{j(j-1) \leq p < m} s_p \Big) \Big\|_{\mathcal{B}^{max}_{j-1,j}} \\
=& \Big\| h_1 \Big(- r'_0 + r'_j + \sum_{p<j} s_p \Big) \Big\|_{\mathcal{B}^{max}_{0,j}} + \Big\| h_1 \Big(- r'_j + r'_{2j} + \sum_{j \leq p < 2j} s_p \Big) \Big\|_{\mathcal{B}^{max}_{j,2j}} \\
&+ \cdots + \Big\| h_1 \Big(- r'_{(j-1)j} + r'_m + \sum_{j(j-1) \leq p < m} s_p \Big) \Big\|_{\mathcal{B}^{max}_{(j-1)j,m}}.
\end{aligned}$$

The last equality is true just by definition of $\hat{r}_\ell$ and indiscernibility of $\bar{a}_i$ in $\mathcal{B}$.

Now (remembering that $j > 1$) we can apply the Main Claim (3.1.7) and get for each $0 \leq \ell \leq j$ the following inequality:

$$\begin{aligned}
\Big\| h_1 \Big(- r'_{\ell j} + r'_{(\ell+1)j} + \sum_{\ell j \leq p < (\ell+1)j} s_p \Big) \Big\|_{\mathcal{B}^{max}_{\ell j,(\ell+1)j}} \\
\leq \Big(1 + \frac{2}{j}\Big) \Big\| - r'_{\ell j} + r'_{(\ell+1)j} + \sum_{\ell j \leq p < (\ell+1)j} s_p \Big\|_1.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\| r_j \|_0 \leq& \Big(1 + \frac{2}{j}\Big) \Big(\Big\| - r'_0 + r'_j + \sum_{p<j} s_p \Big\|_1 + \Big\| - r'_j + r'_{2j} + \sum_{j \leq p < 2j} s_p \Big\|_1 \\
&+ \cdots + \Big\| - r'_{(j-1)j} + r'_m + \sum_{j(j-1) \leq p < m} s_p \Big\|_1 \Big).
\end{aligned}$$

Rewriting the last inequality in a different way, we get

$$\begin{aligned}\left(1+\frac{2}{j}\right)^{-1}\| r_j \|_0 \leq & \left\| -r'_0 + r'_j + \sum_{p<j} s_p \right\|_1 + \left\| -r'_j + r'_{2j} + \sum_{j\leq p<2j} s_p \right\|_1 \\ & + \cdots + \left\| -r'_{(j-1)j} + r'_m + \sum_{j(j-1)\leq p<m} s_p \right\|_1 \\ \leq & \| -r'_0 + r'_1 + s_0 \|_1 + \| -r'_1 + r'_2 + s_1 \|_1 \\ & + \cdots + \| -r'_{m-1} + r'_m + s_{m-1} \|_1 \\ \leq & \| r_m \|_1 + \varepsilon\end{aligned}$$

which finishes the proof of 3.1.4 (2) for the case $i = 1$. A similar argument is used for $i = -1$, and we are done. ∎

By 3.1.4 (1), each one of the three sequences $\langle \| r_m \|_i : m < \omega\rangle$ converges. By 3.1.4 (2), all of them converge to the same limit. Let us denote this limit by $\rho(r', r'') \in \mathbb{R}$.

Let V be an ultraproduct of all the V_m modulo some non-principal ultrafilter $\mathfrak{U}$ on ω (where V_m is a normed space with the norm $\| \cdot \|_1$):

$$V = \prod_{m<\omega} V_m/\mathfrak{U}$$

(see 1.4 for a precise meaning of an ultraproduct of Banach spaces).

Remark 3.1.9:

(1) Certainly, this is where the compactness becomes important. We will use several times the analog of Loś theorem for positive bounded formulae (see 1.4), claiming $V \models \varphi(\langle \xi_i : i < \omega\rangle)$ if $V_i \models \varphi(\xi_i)$ for "almost all" i.

(2) Instead of looking at V_m and V, we should have looked at their completions, which are Banach spaces, and not just normed spaces, but it doesn't matter (see [HenIov] (6.7), or recall that we deal with quantifier free formulae).

(3) We will think of V as embedded into our "monster" $\mathcal{B}$.

(4) Note that there is a natural embedding i_m of V_m into V:

$$i_m(r) = (0, \ldots, 0, r, \ldots, r, \ldots),$$

i.e. $i_m(r) = g\colon \omega \to \bigcup V_m$ s.t. $g(k) = 0$ for $k < m$ and $g(k) = r$ for $k \geq m$. Moreover, for $k < m$ we get $i_m \restriction V_k = i_k$.

So we will not distinguish between elements of V_m for some m (in fact, for all $k \geq m$) and the appropriate elements of V.

The following discussion will be conducted inside V (and therefore inside $\mathcal{B}$). Let $\bar{b}_\omega \in V$ be the "limit" of the sequence $\langle \bar{b}_m : m \in \omega\rangle$, i.e. $\bar{b}_\omega = \langle \bar{b}_m : m \in \omega\rangle/\mathfrak{U}$. Let g_ω be the "limit" of $\langle g_m : m \in \omega\rangle$ taking $\bar{b}_0$ onto $\bar{b}_\omega$.

Remember that given $r', r'' \in \langle \bar{b}_0\rangle$, we defined $r_m = r' + g_m(r'')$ for $1 < m < \omega$. We expand our definitions: let $r_m = r' + g_m(r'')$, $r_{-m} = r'' + g_m(r')$, for $1 < m \leq \omega$. Also remember that for r', r'' as above, we denoted the limit of the sequences (of real numbers) $\langle \| r_m \|_i : m < \omega\rangle$ (for $i \in \{1, -1, 0\}$) by $\rho(r', r'')$. We are going to show now that in the limit model (V), one can exchange the roles of r' and r''.

Claim 3.1.10: *Let $r', r'' \in \langle \bar{b}_0\rangle$, $r_m = r' + g_m(r'')$, $r_{-m} = r'' + g_m(r')$, for $1 < m \leq \omega$. Let $\rho = \rho(r', r'')$. Then:*

(1) *For every m, $\| r_{-m} \|_1 = \| r_m \|_{-1}$;*
(2) $\| r_\omega \|_V = \| r_{-\omega} \|_V = \rho$.

Proof:

(1) First, we show $\| r_m \|_{-1} \geq \| r_{-m} \|_1$. Pick $\varepsilon > 0$ and let (as usual) $r_m = \sum_{p<m} t_p$ such that $\| r_m \|_{-1} + \varepsilon \geq \sum_{p<m} \| t_p \|_{-1}$, where $t_p = -r'_p + r'_{p+1} + s_p$, $r'_p \in \langle \bar{b}_p\rangle$, $r' = -r'_0$, $g_m(r'') = r'_m$ (see 3.1.3). So

$$\begin{aligned}\| r_m \|_{-1} + \varepsilon &\geq \sum_{p<m} \| -r'_p + r'_{p+1} + s_p \|_{-1} \\ &= \sum_{p<m} \| h_{-1}(-r'_p + r'_{p+1} + s_p) \|_{\mathcal{B}}.\end{aligned}$$

By indiscernibility, for all $p < m$,

$$\begin{aligned}&\|h_{-1}(-r'_p + r'_{p+1} + s_p)\|_{\mathcal{B}} \\ &\quad = \| h_{-1}(g_{p,m-(p+1)}(-r'_p) + g_{p+1,m-p}(r'_{p+1}) + s_p) \|_{\mathcal{B}}.\end{aligned}$$

So

$$\begin{aligned}\| r_m \|_{-1} + \varepsilon &\geq \sum_{p<m} \| h_{-1}(g_{p,m-(p+1)}(-r'_p) + g_{p+1,m-p}(r'_{p+1}) + s_p) \|_{\mathcal{B}} \\ &= \sum_{p<m} \| h_1(g_{p,m-p}(-r'_p) + g_{p+1,m-(p+1)}(r'_{p+1}) + s_p) \|_{\mathcal{B}} \\ &= \sum_{p<m} \| g_{p,m-p}(-r'_p) + g_{p+1,m-(p+1)}(r'_{p+1}) + s_p \|_1.\end{aligned}$$

The equality between the second and third terms is true because of the definition of h_1, h_{-1} on V_m and indiscernibility. Now we conclude the following:

$$\begin{aligned}\| r_m \|_{-1} + \varepsilon &\geq \sum_{p<m} \| g_{p,m-p}(-r'_p) + g_{p+1,m-(p+1)}(r'_{p+1}) + s_p \|_1 \\ &\geq \| g_{0,m}(-r'_0) + g_{m,0}(r'_m) \|_1 \\ &= \| g_m(r') + r'' \|_1 = \| r_{-m} \|_1.\end{aligned}$$

As ε was arbitrary, we finish. The inequality in the other direction is proven in the exact same way.

(2) Remember that by 3.1.4, both $\| r_m \|_1$ and $\| r_m \|_{-1}$ are increasing sequences converging to ρ. So on the one hand, $\| r_m \|_1 \leq \rho$ and $\| r_{-m} \|_1 \leq \rho$ for all $m < \omega$ (remember that by part (1), $\| r_{-m} \|_1 = \| r_m \|_{-1}$). Therefore,

3.1.11:

$$\| r_\omega \|_V \leq \rho, \quad \| r_{-\omega} \|_V \leq \rho.$$

On the other hand, for every real $\varepsilon > 0$, for almost all m,

$$\| r_m \|_1 \geq \rho - \varepsilon, \quad \| r_{-m} \|_1 = \| r_m \|_{-1} \geq \rho - \varepsilon.$$

Therefore, for all real $\varepsilon > 0$,

3.1.12:

$$\| r_\omega \|_V \geq \rho - \varepsilon, \quad \| r_{-\omega} \|_V \geq \rho - \varepsilon.$$

Combining 3.1.11 with 3.1.12, we get the desired equalities. ∎

The following claim can be viewed as the heart of the proof we've been working hard for:

CLAIM 3.1.13: *$\mathrm{tp}(\bar{b}_0, \bar{b}_\omega)$ is symmetric, i.e. $\mathrm{tp}(\bar{b}_0, \bar{b}_\omega) = \mathrm{tp}(\bar{b}_\omega, \bar{b}_0)$.*

Proof: Define the obvious mapping Φ from $\langle \bar{b}_0, \bar{b}_\omega \rangle$ onto itself, extending $g_\omega \cup g_\omega^{-1}$ ("exchanging" $\bar{b}_0$ and $\bar{b}_\omega$ and respecting the linear structure). It is obviously an isomorphism of vector spaces, so we just have to show it is also an isometry. Take $r \in \langle \bar{b}_0, \bar{b}_\omega \rangle$; then for some $r', r'' \in \langle \bar{b}_0 \rangle$, $r = r' + g_\omega(r'')$. Therefore $\Phi(r) = r'' + g_\omega(r')$. Now, by 3.1.10, $\| r \|_V = \| \Phi(r) \|_V = \rho(r', r'')$. ∎

Now we have obviously reached a contradiction. Why? First of all, note that as $\mathrm{tp}(\bar{b}_i, \bar{b}_{i+1}) = \mathrm{tp}(\bar{a}_i, \bar{a}_{i+1})$ for all $i \in \omega$, we get $q(\bar{b}_i, \bar{b}_{i+1})$ for all i (remember: $q(\bar{x}, \bar{y})$ defines a partial order on $\mathcal{B}$, $\langle \bar{a}_i : i < \omega \rangle$ is ordered by q). As $q(\bar{x}, \bar{y})$ is a partial order, it is in particular transitive, so $q(\bar{b}_0, \bar{b}_m)$ holds for all $m > 0$, and therefore $\mathcal{B} \models q(\bar{b}_0, \bar{b}_\omega)$ (Łoś theorem (1.4)+the fact that q is a positive bounded type). But by 3.1.13, $\mathcal{B} \models q(\bar{b}_\omega, \bar{b}_0)$, a contradiction to q being a partial order! ∎

4. Groups

Let $\mathcal{G}$ be the "monster" group (the universal domain). As in the previous sections, we identify the class (of all groups) with its monster. Our first theorem in this section is a non-structure result.

PROPOSITION 4.1: *$\mathcal{G}$ has SOP_3*

Proof: Consider the formula $\varphi(x, y)$ defined by "$(xyx^{-1} = y^2) \wedge (x \neq y)$".

* First, we have to show that there is a sequence $\langle a_i : i < \omega \rangle$ such that $i < j \Rightarrow \varphi(a_i, a_j)$. But this is trivial by using HNN extensions (see [Rot], p. 407) and compactness.
* Secondly, we have to make sure there is no "triangle", but this is actually a well-known example in geometric group theory (see [Grp], p. 493) of a triangle $X = \langle a, b : aba^{-1} = b^2 \rangle$, $Y = \langle b, c : bcb^{-1} = c^2 \rangle$, $Z = \langle a, c : cac^{-1} = a^2 \rangle$ that generates a *trivial* group when put together. Therefore,

  $$\mathcal{G} \models (\forall x, y, z)(xyx^{-1} = y^2 \wedge yzy^{-1} = z^2 \wedge zxz^{-1} = x^2 \longrightarrow x = y = z = e)$$

  (where e is the group identity). Therefore

  $$\mathcal{G} \models \neg(\exists x, y, z)(\varphi(x, y) \wedge \varphi(y, z) \wedge \varphi(z, x)),$$

  as required. ∎

The proof uses the fact that there cannot be a triangle of a certain kind. A natural question now is — what about quadrangles? In particular, is the group $H = \langle a, b, c, d : aba^{-1} = b^2, bcb^{-1} = c^2, cdc^{-1} = d^2, dad^{-1} = a^2 \rangle$ also trivial? Once again, it's a well-known fact that it is actually infinite, and the proof is even more interesting than the fact itself, as it seems very general — it doesn't speak at all about the relations between the generators. In fact, the proof suggests a generalization that roughly speaking says that it is impossible to "collapse" a group with four generators by forcing relations between only adjacent pairs. Model theoretically, this leads to the following *structure* result.

THEOREM 4.2: *$\mathcal{G}$ does not have SOP_4, not even type-definable.*

Proof: Suppose towards a contradiction that a type $p(\bar{x},\bar{y})$ exemplifies SOP_4 in $\mathcal{G}$. In particular, there exists an indiscernible sequence $\langle \bar{a}_i : i < \omega \rangle$ such that $i < j \Rightarrow p(\bar{a}_i, \bar{a}_j)$. Define for all $i \in \omega$, $H_i = \langle \bar{a}_i \rangle_{\mathcal{G}}$. We denote $\mathrm{len}(\bar{x}) = \mathrm{len}(\bar{y})$ in $p(\bar{x},\bar{y})$ by α (not necessarily finite) and assume wlog (by indiscernibility) that there exists $\alpha^* < \alpha$ such that $\bigwedge_{\ell<\alpha^*}(a_{i,\ell} = a^*_\ell)$ for all $i < \omega$ and $\langle a_{i,\ell} : \alpha^* \leq \ell < \alpha, i < \omega \rangle$ is a sequence of distinct elements. Define $H^- = \langle a^*_\ell : \ell < \alpha^* \rangle$. As α may be infinite, we also assume that $\bar{a}_i$ in fact lists H_i, and therefore $H^- = H_i \cap H_j$ for all $i < j < \omega$.

By the indiscernibility, there exists for $i \neq j \in \omega$, an isomorphism $f_{i,j} : H_i \to H_j$ mapping $\bar{a}_i$ onto $\bar{a}_j$. Define for all $i < j \in \omega$, $H_{i,j} = \langle \bar{a}_i, \bar{a}_j \rangle_{\mathcal{G}}$. For $j < i \in \omega$ we define $H_{i,j}$ by "relabelling", changing the roles of $\bar{a}_i$ and $\bar{a}_j$, i.e. as a set $H_{i,j}$ equals $H_{j,i}$, and the group action is defined on it such that there exists $f^{i,j}_{j,i} : H_{j,i} \to H_{i,j}$ an isomorphism extending $f_{i,j} \cup f_{j,i}$. So for $j < i$, $H_{i,j}$ does not have to be a subgroup of $\mathcal{G}$ (but we can embed it into $\mathcal{G}$, as $\mathcal{G}$ is universal, although not necessarily over $H_i \cup H_j$).

Given two groups G_1 and G_2 and a subgroup of both, G_0, we shall denote the free amalgamation (amalgam) of the two over G_0 (see [Rot], p. 401) by $G_1 *_{G_0} G_2$. Now let us concentrate on H_0, H_1, H_2, H_3. Define $K_0 = H_0 *_{H^-} H_2$, $K_1 = H_{0,1} *_{H_1} H_{1,2}$, $K_2 = H_{2,3} *_{H_3} H_{3,0}$. Once again, those groups do not have to be subgroups of $\mathcal{G}$. It is obvious, though, that K_0 is a subgroup of both K_1 and K_2 (by chasing diagrams, see [Rot], p. 401). So we define $K = K_1 *_{K_0} K_2$.

$\mathcal{G}$ is universal, so we can embed K into $\mathcal{G}$. Denote the image of $\bar{a}_i$ under this embedding by $\bar{b}_i \in \mathcal{G}$. Now we note

CLAIM 4.2.1: $\mathrm{tp}(\bar{b}_0\bar{b}_1,\mathcal{G}) = \mathrm{tp}(\bar{b}_1\bar{b}_2,\mathcal{G}) = \mathrm{tp}(\bar{b}_2\bar{b}_3,\mathcal{G}) = \mathrm{tp}(\bar{b}_3\bar{b}_0,\mathcal{G}) = \mathrm{tp}(\bar{a}_0\bar{a}_1,\mathcal{G})$.

Proof: $\mathrm{tp}(\bar{b}_0\bar{b}_1,\mathcal{G}) = \mathrm{tp}(\bar{a}_0\bar{a}_1,K) = \mathrm{tp}(\bar{a}_0\bar{a}_1,K_1) = \mathrm{tp}(\bar{a}_0\bar{a}_1,H_{0,1}) = \mathrm{tp}(\bar{a}_0\bar{a}_1,\mathcal{G})$. The first equality is true because types are preserved under group isomorphisms ("embeddings"), and the rest — just the definitions of the groups. Using the same arguments for $\mathrm{tp}(\bar{b}_1\bar{b}_2,\mathcal{G})$, we get $\mathrm{tp}(\bar{b}_1\bar{b}_2,\mathcal{G}) = \mathrm{tp}(\bar{a}_1\bar{a}_2,\mathcal{G})$, but the latter equals $\mathrm{tp}(\bar{a}_0\bar{a}_1,\mathcal{G})$ by indiscernibility. The same argument (replacing K_1 by K_2) shows $\mathrm{tp}(\bar{b}_2\bar{b}_3,\mathcal{G}) = \mathrm{tp}(\bar{a}_0\bar{a}_1,\mathcal{G})$. Now $\mathrm{tp}(\bar{b}_3\bar{b}_0,\mathcal{G}) = \mathrm{tp}(\bar{a}_3\bar{a}_0,K) = \mathrm{tp}(\bar{a}_3\bar{a}_0,K_2) = \mathrm{tp}(\bar{a}_3\bar{a}_0,H_{3,0})$, but the latter equals $\mathrm{tp}(f^{3,0}_{0,3}(\bar{a}_3)f^{3,0}_{0,3}(\bar{a}_1),\mathcal{G}) = \mathrm{tp}(\bar{a}_0\bar{a}_3,\mathcal{G})$ by the definition of $H_{3,0}$ and $f^{3,0}_{0,3}$, and by indiscernibility we're done. ∎

Now we obviously get a contradiction, as by (4.2.1),

$$\mathcal{G} \models p(\bar{b}_0, \bar{b}_1) \wedge p(\bar{b}_1, \bar{b}_2) \wedge p(\bar{b}_2, \bar{b}_3) \wedge p(\bar{b}_3, \bar{b}_0),$$

which contradicts the fact that $p(\bar{x}, \bar{y})$ exemplifies SOP_4 in $\mathcal{G}$. ∎

We derive structure and nonstructure corollaries, using model-theoretical properties that people are more familiar with than the SOP_n hierarchy:

COROLLARY 4.3: *The class of groups does not have the strict order property. In other words, no group has the strict order property (in the pure quantifier free language of groups).*

Proof: Obvious, as 4.2 states that the class of groups does not have SOP_4, which follows from the strict order property. ∎

COROLLARY 4.4: *The class of groups is not simple, i.e. there exists a group with the tree property.*

Proof: We constructed a group with SOP_3, which implies the tree property; see 1.3. ∎

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