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NUMBER OF STRONGLY $\aleph_\varepsilon$-SATURATED MODELS — AN ADDITION

Saharon SHELAH*
Institute of Mathematics, The Hebrew University, Jerusalem, Israel
Rutgers University, Mathematics Department, New Brunswick, NJ 08903, USA

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We here improve Theorem 2.1 of [2].

2.1′. Theorem. *Suppose T is unsuperstable, $\lambda \geqslant \lambda(T) + \aleph_1$. Then T has 2^λ pairwise non-isomorphic strongly $\aleph_\varepsilon$-saturated models of cardinality λ.*

2.1.A. Remark. (1) $\lambda(T) = |\{\mathrm{stp}(\bar a, \emptyset) : \bar a \in {}^{\omega>}\mathfrak{C}\}|$ (counted up to equivalence).

(2) For most cases we get 2^λ such models, no one elementarily embeddable into another.

Proof. If T unstable use [3, III 3.10(3)] (see proof of [2, 2.1]). So w.l.o.g. T is stable.

Let $\varphi_n(\bar x, \bar y_n)$ $(n < \omega)$, $\bar a_\eta$ $(\eta \in {}^{\omega\geqslant}\lambda)$ be as in [1, III, §3] so $\langle \bar a_\eta : \eta \in {}^{\omega>}\lambda\rangle$ is a nonforking tree, and for $\eta \in {}^{\omega}\lambda$, $\mathrm{tp}(\bar a_\eta, \bigcup\{\bar a_\nu : \nu \in {}^{\omega>}\lambda\})$ does not fork over $\bigcup_{l<\omega} \bar a_{\eta \restriction l}$ and $\mathrm{tp}(\bar a_\eta, \bigcap_{l\leqslant k} \bar a_{\eta\restriction l})$ forks over $\bigcup_{l<k} \bar a_{\eta \restriction l}$. Let $I \subseteq {}^{\omega\geqslant}\lambda$ be closed under initial segments, $|I| = \lambda$ and we shall construct a model M_I. We work in $\mathfrak{C}^{\mathrm{eg}}$.

We define $\langle A_i : i < \alpha\rangle$ and $\langle f^i_{c,d} : c, d \in A_i\rangle$.

(1) $\langle A_i : i \leqslant \alpha\rangle$ is increasing continuous:

$$|A_i| = \lambda, \qquad A_i \subseteq \mathfrak{C}.$$

(2) $f^i_{c,d}$ is an elementary mapping, $f^i_{c,d}(c) = d$, $f^i_{d,c} = (f^i_{c,d})^{-1}$, $\langle f^i_{c,d} : i \leqslant \alpha\rangle$ is increasing continuous, and for $c \in A_0$, $\mathrm{Dom}\, f^i_{c,d} = \{c\}$.

(3) For each i: **either**

(i) $A_{i+1} = A_i \cup \{a_i\}$, $\mathrm{tp}(a_i, A(i))$ does not fork over some finite subset B_i of A_i, **or**

(ii) for some $c(i), d(i) \in A(i)$, $A_{i+1} = A_i \cup f^{i+1}_{c(i),d(i)}(A_i)$ and $(\exists j < i)$ $[\mathrm{Dom}\, f^i_{c,d} = A_j] \vee [\mathrm{Rang}\, f^i_{c,d} = \{d\}]$.

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(4) For every $c, d \in A_{i+1}$:

(i) If $\{c, d\}$ is not a subset of $A(i)$, then $\mathrm{Dom} f^i_{c,d} = \{c\}$.

(ii) If $c, d \in A(i)$, case (i) in (3) holds **or** case (ii) of (3) holds but $\langle c, d\rangle \neq \{c(i), d(i)\rangle$, $\langle c, d\rangle \neq \langle d(i), c(i)\rangle$, **then** $f^{i+1}_{c,d} = f^i_{c,d}$.

(iii) If $c = c(i)$, $d = d(i)$ and case (ii) of (3) holds, then $\mathrm{tp}(f^{i+1}_{c(i),d(i)}(A_i), A_i)$ does not fork over $\mathrm{Rang} f^i_{c(i),d(i)}$.

(5) $A_0 = \bigcup \{\bar{a}_\eta : \eta \in I\}$.

We can clearly find $\alpha < \lambda^+$ and A_i, $f^i_{c,d}$ satisfying (1)–(5) such that:

(*) (i) For every finite $B \subseteq A_\alpha$ and $b \in \mathfrak{C}$, $\mathrm{stp}(b, B)$ is realized by some $a \in A$.

(ii) For every $c, d \in A_\alpha$, $\mathrm{Dom} f^\alpha_{c,d} = A_\alpha = \mathrm{Rang} f^\alpha_{c,d}$.

This is easy by reasonable bookkeeping and (3) above. Hence A_α is the universe of a strongly $\aleph_\varepsilon$-saturated model (of cardinality λ) (remember we work in $\mathfrak{C}^{\mathrm{eq}}$). We call it M_I (and should have written $\alpha_I < \lambda^+$, A^I_i, etc). Note that we can prove by induction

(**) If $\mathrm{Dom} f^i_{c,d} \neq \{c\}$, then

(i) $(\exists \delta \leq i)[\mathrm{Dom} f^i_{c,d} = A_\delta = \mathrm{Rang} f^i_{c,d}]$,

or (ii) $(\exists \alpha < \beta \leq i)[\mathrm{Dom} f^i_{c,d} = A_\beta \;\&\; \mathrm{Rang} f^i_{c,d} = A_\alpha \cup (A_{\beta+1} - A_\beta)]$,

or (iii) $(\exists \alpha < \beta \leq i)[\mathrm{Rang} f^i_{c,d} = A_\beta \;\&\; \mathrm{Dom} f^i_{c,d} = A_\alpha \cup (A_{\beta+1} - A_\beta)]$.

Our next note that we can prove by induction on i that:
(remember: A is $\boldsymbol{F}^f_{\aleph_0}$-atomic over B if: for $\bar{a} \in A$, $\mathrm{tp}(\bar{a}, \beta)$ does not fork over some finite subset of B). (We use [1, III §3] for $\boldsymbol{F}^f_{\aleph_0}$, see table in [1, III §2].)

(***) (i) For $j < i$, A_i is $\boldsymbol{F}^f_{\aleph_0}$-atomic over A_j, and

(ii) for $j \leq i$, $c \in A_j$, $d \in A_j$ we have: A_i is $\boldsymbol{F}^f_{\aleph_0}$-atomic over $\mathrm{Dom} f^j_{c,d}$ and over $\mathrm{Rang} f^j_{c,d}$.

Now we define by induction on i, a well ordering $<^i$ of $A_i - A_0$ such that: for $j < i$, $<^i \restriction (A_j - A_0) = <^j$, and $A_j - A_0$ is an initial segment of $(A_i - A_0, <^i)$, and for $x \in A_i - A_0$, A_i is $\boldsymbol{F}^f_{\aleph_0}$-atomic over $A_0 \cup \{y \in A_i : y < x\}$. In other words M_I is $\boldsymbol{F}^f_{\aleph_0}$-constructible over $\bigcup_{\eta \in I} \bar{a}_\eta$. So for every $b \in M_I$ we can find finite $B_b \subseteq M_I - \bigcup_{\eta \in I} \bar{a}_\eta$, $\mu_b \subseteq I$ such that: if $b \in A_0$, $B_b = \emptyset$; if $b \notin A_0$, b is the maximal (by $<^{I,\alpha}$) member of B_b and for $c \in B_b$, $\mathrm{tp}(c, A_0 \cup \{d \in M_I - A_0 : b <^{I,\alpha} c\})$ does not fork over $\{d \in B_b : d <^{I,\alpha} c\} \cup \{\bar{a}_\eta : \eta \in \mu_b\})$ (so if $b \in A_0$, then $b \in \bigcup_{\eta \in \mu_b} \bar{a}_\eta$).

W.l.o.g. $[c \in B_b \Rightarrow B_c \subseteq B_b]$.

Now we can note that the proof of [1, VIII 2.7] works when λ is regular; when λ is singular combine the proof of [1, VIII 2.7] with the suitable proofs of [1, VIII §2]. Alternatively, let

$$R_{n_1,n_2,n_3} = \{\bar{a}^\frown \bar{b}^\frown \bar{c} : \bar{a} \in {}^{n_1}M_I,\ \bar{b} \in {}^{n_2}M_I,\ \bar{c} \in {}^{n_3}M_I \text{ and } \mathrm{tp}(\bar{a}, \bar{b}^\frown \bar{c}) \text{ does not fork over } \bar{b}\}.$$

$$\Delta^* = \{R_{n_1,n_2,n_3} : n_1, n_2, n_3 \in \omega\}.$$

Claim. *M_I is semi Δ^*-representable in $\mathfrak{M}_{\aleph_0,\aleph_0}(I)$.*

Remark. See the second version of [3, Ch. III 2.2] for the definition.

Proof. W.l.o.g. the $\bar{a}_\eta$ $(\eta \in I)$ are pairwise disjoint with no repetition. Let $F_l(x_\eta)$ represent the l-th element of $\bar{a}_\eta$, and $b \in M_I - \bigcup_{\eta \in I} \bar{a}_\eta$ will be represented by $F_n(\sigma_1, \ldots, \sigma_k, \eta_1, \ldots, \eta_k)$ where $\{\eta_1, \ldots, \eta_k\} = \mu_b$ (in increasing lexicographic order of I), and $\{\sigma_1, \ldots, \sigma_k\}$ are the representations of $\{d : d \in B_d\}$ (in $<^{I,\alpha}$-increasing order).

The rest is by the nonforking calculus.

Now by the second version of [3, Ch. III] we get our conclusion.

References

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