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THE PRIMAL FRAMEWORK I

J.T. BALDWIN*
Department of Mathematics, University of Illinois, Chicago, IL 60680, USA

S. SHELAH**
Department of Mathematics Hebrew University of Jerusalem, Jerusalem, Israel

Introduction

This is the first of a series of articles dealing with abstract classification theory. The apparatus to assign systems of cardinal invariants to models of a first-theory (or determine its impossibility) is developed in [5]. It is natural to try to extend this theory to classes of models which are described in other ways. Work on the classification theory for nonelementary classes [6] and for universal classes [9] led to the conclusion that an axiomatic approach provided the best setting for developing a theory of wider application. This approach is reminiscent of the early work of Fraïssé and Jónson on the existence of homogeneous-universal models. As this will be a long project it seems appropriate to report our progress as we go along.

In large part this series of articles will parallel the development in [9]. A survey of that paper which could serve as an introduction to this one is [1]. The first chapter of this article corresponds to Section 2 of [9]. In it we describe the axioms on which the remainder of the article depends and give some examples and context to justify this level of generality. As is detailed later the principal goal of this series is indicated by its title. The study of universal classes takes as a primitive the notion of closing a subset under functions to obtain a model. We replace that concept by the notion of a prime model. We begin the detailed discussion of this idea in Chapter 2. One of the important contributions of classification theory is the recognition that large models can often be analyzed by means of a family of small models indexed by a tree of height at most ω. More precisely, the analyzed model is prime over such a tree. Chapter 3 provides

sufficient conditions for prime models over such trees to exist. The discussion of properties of a class which guarantee that each model in the class is prime over such a tree will appear later in the series.

We introduce in Chapters 1 and 2 a number of principles which we loosely refer to as axioms. At the beginning of Chapter 3 we define the notion of an adequate class — a class which satisfies those axioms that we assume in the mainline of the study. This notion of an adequate class will be embellished by further axioms in later papers of this series. Our use of the word axiom in this content is somewhat inexact; postulate might be better. In exploring an unknown area we list certain principles which appear to make important distinctions. In our definition of an adequate class we collect a family of these principles that is sufficient to establish a coherent collection of results. We thank N. Shi for carefully reading the paper and making a number of helpful suggestions.

1. The abstract framework

Shelah developed in [9] several frameworks for studying aspect of classification theory. In each case he studied a triple

$$\mathbf{K} = \langle K, \leqslant_{\mathbf{K}}, \mathrm{NF}_{\mathbf{K}} \rangle;$$

K is a collection of structures, $\leqslant_{\mathbf{K}}$ denotes elementary submodel with respect to $\mathbf{K}$, and $\mathrm{NF}_{\mathbf{K}}$ is a 4-ary relation (nonforking) denoting that certain models are in stable amalgamation. The original paper primarily studied classes which admitted a fourth basic notion: 'generated submodel'. We generalize that context here by taking as a fourth basic component a predicate cpr. Intuitively, $\mathrm{cpr} = \mathrm{cpr}_{\mathbf{K}}$ holds of a structure $M \in \mathbf{K}$ and a chain of models $\mathfrak{M}$ (Section 1.2) if M is prime over $\mathfrak{M}$. Thus this paper studies quadruples

$$\mathbf{K} = \langle K, \leqslant_{\mathbf{K}}, \mathrm{NF}_{\mathbf{K}}, \mathrm{cpr}_{\mathbf{K}} \rangle.$$

Section 1.1 reviews the properties of elementary submodel and free amalgamation which carry over from [9]. In Section 1.2 we provide a number of examples of classes which satisfy the basic axioms.

1.1. Basic properties of $\leqslant_{\mathbf{K}}$ and NF

K always denotes a class of structures of a fixed similarly type. K and all relations that we define on it are assumed to be closed under isomorphism. Although technically both $\leqslant$ and NF should be subscripted with $\mathbf{K}$, we usually omit the subscript for ease of reading. M and N (with subscripts) denote members of K unless we explicitly say otherwise. A and B will denote subsets of members of K. We write M is contained in N ($M \subseteq N$) if M is a substructure of N; that is, if the universe of M is a subset of that of N, the relations of N are those imposed by

M and M is closed under any functions in the language. We write M is a submodel or more explicitly a K-submodel of N for $M \leq N$.

The first group of axioms describe our notion of elementary submodel.

1.1.1. Axiom Group A: K-submodels

A0. If $M \in K$ then $M \leq M$.

A1. If $M \leq N$ then M is a substructure of N.

A2. $\leq$ is transitive.

A3. If $M_0 \subseteq M_1 \subseteq N$, $M_0 \leq N$ and $M_1 \leq N$ then $M_0 \leq M_1$.

It is sometimes essential to distinguish between $M \leq_K N$ which implies $M \subseteq N$ and the existence of an embedding f of M into N whose image is a **K**-submodel of N.

1.1.2. Definition. A **K**-*embedding* is an isomorphism f from an M in K to an N in K such that rng $f \leq_K N$. N is then a **K**-*extension* of M via f.

If f is not mentioned explictly then it is the identity.

We require one further important property of **K** and $\leq_K$.

1.1.3. Definition. **K** has the λ-Löwenheim–Skolem property (λ-LSP) or (the Löwenheim–Skolem property down to λ) if $A \subseteq M \in \mathbf{K}$ and $|A| \leq \lambda$ implies there is an $N \in \mathbf{K}$ with $A \subseteq N \leq M$ and $|N| \leq \lambda$. The Löwenheim–Skolem number of **K** is the least λ such that **K** has the λ-Löwenheim–Skolem property. We write $\mathbf{LS}(\mathbf{K}) = \lambda$.

Note that the set of cardinals for which **K** has the Löwenheim–Skolem property may not be convex. Moreover the related requirement on λ, for any A there is an $N \supseteq A$ with $N \in \mathbf{K}$ and $|N| \leq |A| + \lambda$, is still different and will be investigated later.

1.1.4. Axiom Group A: K-submodels

A4. $\mathbf{LS}(\mathbf{K}) < \infty$.

Our axioms differ from those of Fraissé and Jónsson in that there is neither a joint embedding nor an amalgamation axiom. Our approach here is to assume in the next set of axioms a particularly strong form of amalgamation. The use of nonamalgamation as a source of nonstructure has been explored by Shelah in several places. See especially Chapter 1 of [9] and [4] and its progenitor [6]. We will obtain joint embedding by fiat (by restricting to a subclass that satisfies it). The Jónsson Fraissé constructions also require closure under unions of chains. This requirement is more subtle than it first appears; it is the major topic of [3].

We say two structures are *compatible* if they (isomorphic copies of them) have

a common **K**-extension. Any class **K** that satisfies Axiom C2 (below) is split into classes with the joint embedding property by the equivalence relation of compatibility and if **K** has a Löwenheim–Skolem number then there will be only a set of equivalence classes. Given any fixed model M (or diagram of models) this equivalence relation will be refined by 'compatibility over M'. We explore this refinement in Chapter 2.

The second group of axioms concern the independence relation. We begin by describing a relation of four members of K.

$$\text{NF}(M_0, M_1, M_2, M_3)$$

means that M_1 and M_2 are freely amalgamated over M_0 within M_3. The notation NF arises from the reading the type of M_1 over M_2 inside M_3 does not fork over M_1. We will eventually show a dichotomy between nonstructure results and the existence of a monster model $\mathcal{M}$. Thus, in trying to establish a structure theory we can introduce a 3-ary relation $\text{NF}(M_0, M_1, M_2)$ to abbreviate $\text{NF}(M_0, M_1, M_2, \mathcal{M})$. We usually write this relation as $M_1 \downarrow_{M_0} M_2$. Even before showing the existence of the monster model we will employ this notation if the choice of M_3 is either clear from context or there are several possibilities which serve equally well.

1.1.5. Notation. A 4-tuple $\langle M_0, M_1, M_2, M_3 \rangle$ is called a *full free amalgam* if it satisfies $\text{NF}(M_0, M_1, M_2, M_3)$. The three-tuple $\langle M_0, M_1, M_2 \rangle$ is called a *free amalgam* if it is an initial segment of a full free amalgam. We often write M_1 and M_2 are freely amalgamated over M_0 in M_3. We refer to such a diagram as a 'free vee'. An isomorphism between two free amalgams $\mathfrak{M} = \langle M_0, M_1, M_2 \rangle$ and $\mathfrak{M} = \langle M_0', M_1', M_2' \rangle$ is a triple of isomorphisms f_i mapping M_i to M_i' with f_0 contained in f_1 and f_2. There is no guarantee (until we asssume Axiom C5 below) that the isomorphisms f_1 and f_2 have a common extension to an M_3 which completes $\mathfrak{M}$. We extend this notion of isomorphism to arbitrary diagrams in Section 1.2.

1.1.6. Axiom Group C: Independence. The following axioms describe the independence relation. For convenience of comparison we have kept the numbering from [9] when we have just copied an axiom. Some axioms from that list (e.g. C4) are omitted here. In particular, the role of Axiom Group B from [9], which dealt with the notion of generation, is played here by Axiom Group D. (See Section 2.2.)

C1. If $\text{NF}(M_0, M_1, M_2, M_3)$ then $M_0 \leqslant M_2 \leqslant M_3$ and $M_0 \leqslant M_1 \leqslant M_3$. In particular, each $M_i \in K$. If $\text{NF}(M_0, M_1, M_2, M_3)$, we say M_1 and M_2 are freely amalgamated (or independent) over M_0 in M_3.

C2. Existence. If M_0 is a K-submodel of both M_1 and M_2 then there are copies (over M_0) M_1' and M_2' of M_1 and M_2 which are freely amalgamated in some $M_3 \in K$.

C3. Monotonicity. (i) If M_1 and M_2 are freely amalgamated over M_0 in M_3 then so are M_1 and M_2' for any M_2' with $M_0 \leqslant M_2' \leqslant M_2$.

(ii) If M_1 and M_2 are freely amalgamated over M_0 in M_3 then they are freely amalgamated in any $M_3' \geqslant M_0$.

(iii) If M_1 and M_2 are freely amalgamated over M_0 in M_3 then they are freely amalgamated in any M_3' containing $M_1 \cup M_2$ and with $M_3' \leqslant M_3$.

C5. Weak Uniqueness. Suppose $\langle \mathfrak{M}, M_3 \rangle$ and $\langle \mathfrak{M}', M_3' \rangle$ are full free amalgams. If $\mathfrak{M}$ and $\mathfrak{M}'$ are isomorphic free amalgams (via f) then there is an $N \in \mathbf{K}$ with $M_3' \leqslant N$, and an extension of f mapping M_3 isomorphically onto a $\mathbf{K}$-submodel of N.

C6. Symmetry. If $\mathrm{NF}(M_0, M_1, M_2, M_3)$ then $\mathrm{NF}(M_0, M_1, M_2, M_3)$.

C7. Disjointness. If $\mathrm{NF}(M_0, M_1, M_2, M_3)$ then $M_1 \cap M_2 = M_0$.

Axiom C7 is largely a matter of notational convenience. We will indicate in Chapter 3 how the major argument of this paper could be slightly revised to avoid this axiom. With it we obtain immediately the following monotonicity property.

1.1.7. Proposition. *Suppose* $\mathrm{NF}(M_0, M_1, M_2, M_3)$ *and* $M_0 \leqslant N$ *which is a* $\mathbf{K}$-*submodel of* M_1 *and* M_2. *Then* $\mathrm{NF}(N, M_1, M_2, M_3)$.

This result could be easily deduced from the base extension axiom 2.2.2 without using C7.

By taking 'formal copies' of M_2' and M_3, one derives a variant of C2 where M_1' can be demanded to be M_1. We refer to Axiom C5 as weak uniqueness because it simply demands that any two amalgamations of a given vee be compatible. Thus, it is making the compatibility class of the diagram unique, not the amalgamating model.

1.1.8. Lemma. *Suppose* $\langle M_0, M_1, M_2, M_3 \rangle$, $\langle M_0, M_1, M_2', M_3' \rangle$ *are full free amalgams. If* M_2' *is isomorphic to* M_2 *over* M_0 *by a map* g *then* g *is an isomorphism between* M_2' *and* M_2 *over* M_1.

Proof. Apply the weak uniqueness axiom to the map f that is the union of the identity on M_1 and the given g from M_2 to M_2'.

1.1.9. Smoothness. Does the class $\mathbf{K}$ admit a 'limit' of an ascending chain (or more generally a directed system) of $\mathbf{K}$-structures? There are several different variants on this question and the answers determine signficant differences in the behaviour of $\mathbf{K}$. We discuss the variants in detail in [3]; we now just mention a couple of possibilities and some of the consequences.

The strongest requirement is to deal directly with unions of chains. But even here there are several variations. One can demand that any union of a (continuous) increasing chain of $\mathbf{K}$ structures be a member of $\mathbf{K}$. More subtly,

one can ask that if each member of the chain is **K**-submodel of a fixed M then the union is also.

Shelah has shown that a class **K** satisfying the axioms A0 through A4 enumerated here and stringent requirements for closure under unions can be presented as the collection of models in a pseudoelementary class in an infinitary logic which omit a family of types. (See Section 1 of [6] and [4].)

Beginning in Chapter 2 we discuss the ways in which unions of chains can be replaced by demanding the existence of a prime model over the union. Again, there are a number of smoothness properties that can be discussed in this context. One obvious application is attempts to improve the Löwenheim–Skolem property to demand that each set can be imbedded in a model of roughly the same size.

1.2. Examples

We describe in this section a number of examples of classes and notions of amalgamation which satisfy at least some of the axioms that we are discussing. Of course, the prototype is the collection of models of a stable first-order theory where a free amalgamation is one that is independent in the sense of nonforking.

In this section we first discuss some contrived examples which although they lack any intrinsic interest make it easy to exhibit some of the pathologies that we are investigating. Then we place in context some classes which naturally arise in the attempt to extend classification theory to, e.g., infinitary classes.

1.2.1. Contrived examples. Let **B** be the class of all structures of the following sort. We work in a language with two unary predicates, U, V and a binary relation E. Now M is in **B** if via E, each member a of V is the name of a subset $X_a = \{m \in U(M) : E(m, a)\}$ of U and every subset of U has one and only one name. Thus each member M of **B** is determined up to isomorphism by the cardinality of $U(M)$. NF(M_0, M_1, M_2, M_3) holds just if for each $a \in M_1$ (M_2), X_a in the sense of M_3 is a subset of M_1 (M_2). We can illustrate the axioms by defining $\leqslant_{\mathbf{B}}$ in two different ways.

(i) Define $M \leqslant_{\mathbf{B}} N$ if $U(M) \subseteq U(N)$, $V(M) \subseteq V(N)$ and each element of $V(M)$ names in N the same subset of $U(M)$ that it names in M. Under this definition if $M_0 \leqslant_{\mathbf{B}} M_1 \leqslant_{\mathbf{B}} M_3$ and $M_0 \leqslant_{\mathbf{B}} M_2 \leqslant_{\mathbf{B}} M_3$ then NF(M_0, M_1, M_2, M_3). Moreover, if $M_0 \leqslant_{\mathbf{B}} M_1$ and $M_0 \leqslant_{\mathbf{B}} M_2$, we can find a common **K**-extension for them letting M_3 be $M_1 \cup M_2$ together with a collection of names for sets that intersect both $U(M_1)$ and $U(M_2)$.

(ii) On the other hand, let $M \leqslant_{\mathbf{B}} N$ just mean that M is a substructure of N. Now it is still possible to verify Axiom C2. If a subset X of M_0 is named by elements a of M_1 and b of M_2 then a and b can be identified by the embeddings into M_3. Using this strategy to amalgamate Axiom C7 fails; however the strategy outlined below when we consider an additional predicate Q, which is needed then to obtain even C2, will also show that C7 is verified.

Some of the problems with amalgamations become clear if we add a unary predicate Q and demand that for any subset W of $U(M)$ with power less than κ there exist both a and b in $V(M)$ with a satisfying Q but b not satisfying Q so that X_a and X_b both contain W. (That is, both Q and its complement are dense.) Axioms C2 and C5 hold under the first definition of $\leq_\mathbf{B}$. To make C2 hold under the second interpretation of $\leq$ we must deal with a subset X of M_0 that has one name in $Q(M_1)$ and another in $\neg Q(M_2)$. Now another strategy works. Add points to the set attached to one of the names and then fill out the model as freely as possible. It is easy to see (cf. Chapter 2) that there are extensions of chains which are incompatible.

$\mathbf{B}_\kappa$ is defined in the same way but with the additional restriction that each $|X_a| < \kappa$.

Another variant arises by replacing the single binary relation E by a family of binary relations E_i such that for each $i < \kappa$ and each $a \in V(M)$, there is a unique $m \in U(M)$ with $R_i(a, m)$. Thus we code κ sequences rather than sets.

1.2.2. $\aleph_1$-saturated models. Let T be a strictly stable first-order theory and let K be the class of $\aleph_1$-saturated models of T. Take $\mathrm{NF}_\mathbf{K}$ as nonforking and $\leq_K$ as elementary submodel. Then the basic axioms are clearly satisfied but K is not closed under unions of chains of small cofinality (not fully smooth). But there are prime models ($\mathbf{F}^s_{\aleph_1}$ in the notation of [5] or $\mathbf{SET}_{\aleph_1}$ in the notation of [2]) over such chains.

The theory REF_ω of countably many refining equivalence relations has K-prime models over chains of cofinality ω but they are not minimal. This leads to 2^λ K-models of power λ when $\lambda^\omega = \lambda$. This argument is treated briefly in [7] (the didip) and will be reported at more length in the current series of papers.

On the other hand if T is a two-dimensional stable theory (cf. Theorem V.5.8 of [5]) then $\mathbf{I}(\aleph_\alpha, \mathbf{K}) \leq |\alpha + 1|$.

1.2.3. Universal classes. See II.2.2 of [9].

1.2.4. Finite diagrams stable in power. See II.2.3 of [9].

1.2.5. Infinitary classes. See [6].

1.2.6. Banach spaces. See [8].

Question. *Are there any stable universal theories of Banach spaces beyond the L^p-spaces?*

2. Prime models over diagrams

In Section 2.1 we discuss diagrams of models and the basic properties of prime models over diagrams. Section 2.2 concerns prime models over independent pairs

of models. We also discuss several possibilities for the relation of a model M_1 which is independent from an M_4 over M_0 with an M_2 with $M_0 \subseteq M_2 \subseteq M_4$. Surprisingly, 2.2.3 deduces a property of the dependence relation but needs the properties of prime models for the proof. In Section 2.3 we discuss prime models over chains.

2.1. Diagrams

We consider two kinds of diagrams. The first is more abstract because the partial ordering among the structures is witnessed by **K**-embeddings. In the (second) case of a concrete diagram the partial ordering is witnessed by actual set-theoretic containments. The existence axioms for free amalgamations have the general form: Given an abstract diagram, there is a concrete stable diagram which is isomorphic to it (in the category of diagrams).

The discussion of abstract diagrams is essential at this stage in the development of theory. Once a 'monster' model or global universe of discourse has been posited, one can assume that all diagrams are concretely realized as subsets of the monster model. The monster model is easily justified in the first-order context so this subtlety does not seriously arise. In the more general case one must worry about it until 'smoothness' and thus the monster model are obtained. (See [9] and [3].)

We will assume the existence of prime models over certain simple diagrams and certain properties relating prime models over more complicated diagrams. In a later paper when considering prime model over still more complicated diagrams (tall trees) we will only be able to obtain a dichotomy between the existence of prime models and the existence of many models.

We call a triple $\langle M_0, M_1, M_2 \rangle$ with embeddings of M_0 into M_1 and M_2 a 'vee' diagram. Thus, Axiom C2 asserts the existence of free amalgamations over vee's.

2.1.1. Definition. An abstract **K**-*diagram* (indexed by a partial order $\langle I, \leq \rangle$) is a pair: a sequence of models and collection of maps. The models $\{M_x : x \in I\}$ and the maps $\{f_{xy} : x, y \in I, x \leq y\}$ must satisfy the following conditions.

- $M_x \in \mathbf{K}$.
- If $x \leq y$ then there is a **K**-embedding f_{xy} from M_x into M_y.
- if $x \leq y \leq z$ then $f_{xz} = f_{yz} \circ f_{xy}$.

We will often suppress mention of the maps. When we need to refer to them we will often describe the family $\{f_{xy} : x, y \in I\}$ by f and write $f \mid Y$ for $\{f_{xy} : x, y \in Y\}$ when Y is a subset of I.

2.1.2. Definition. $\mathfrak{M}$ and $\mathfrak{N}$ are *isomorphic* **K**-*diagrams* if they are both indexed by the same partial order I and for each $x \in I$ there is an isomorphism α_x between M_x and N_x such that the α_x and f_{xy} in $\mathfrak{M}$ and $\mathfrak{N}$ commute in the natural way. We will write α for the family of the α_x.

Note that no single function f is a morphism for **K** unless its domain is in **K**. In particular, we cannot speak of a single map whose domain is a vee.

2.1.3. Definition. A concrete **K**-diagram $\mathfrak{M}$ inside M (indexed by I) is a collection, $\mathfrak{M} = \{M_x : x \in I\}$, of members of K such that each $M_x \leqslant M$ and if $x \leqslant y$ then $M_x \leqslant M_y$.

We extend the notion of a free amalgam to more general diagrams.

2.1.4. Definition. A *stable* **K**-*diagram* $\mathfrak{M}$ inside M (indexed by the lower semilattice $\langle I, \wedge \rangle$) is a *concrete* **K**-diagram inside M satisfying the additional condition

- For any $x, y \in I$, M_x and M_y are freely amalgamated over $M_{x \wedge y}$ within M.

2.1.5. Definition. (i) N is a **K**-*extension* of $\mathfrak{M}$ if there is an isomorphism f between $\mathfrak{M}$ and a concrete **K**-diagram $\mathfrak{N}$ inside N. f is called a **K**-*embedding* of $\mathfrak{M}$ into N.

(ii) N is a *stable* **K**-*extension* of $\mathfrak{M}$ if there is an isomorphism f between $\mathfrak{M}$ and a stable **K**-diagram $\mathfrak{N}$ inside N. f is called a *stable* **K**-*embedding* of $\mathfrak{M}$ into N.

The phrase '$\mathfrak{M}$ is a stable **K**-diagram' means $\mathfrak{M}$ is stable in some M which is suppressed for convenience. A vee $\langle M_0, M_1, M_2 \rangle$ is stable if there is an M_3 completing it to a full free amalgam. Note that an isomorphism between diagrams need not preserve stability. It only guarantees isomorphism between the individual M_x and M'_x. In particular an isomorphic copy of a free amalgam need not be free. So we cannot say a vee diagram is free without specifying an ambient model and an embedding into it. This is true even in the first-order case. In this case we usually think of the monster model as the ambient model and identity as the embedding.

2.1.6. Examples. Of course a (full) free amalgam is a stable **K**-diagram. So is any chain of **K**-models which have a common **K**-extension. Another natural example is an 'L'-diagram (see Fig. 2.1) where for each $i \leqslant \delta$, $M_i \downarrow_{M_0} N$ inside the ambient model.

Compatibility of two extensions of a single model is a straightforward notion.

2.1.7. Definition. Two members N_1, N_2 of K are said to be *compatible* over the **K**-embeddings f_1, f_2 of N_0 into N_1, N_2 if there is an N_3 in K with both N_1 and N_2 **K**-embeddible in N_3 by **K**-embeddings g_1, g_2 such that $g_1 \circ f_1 = g_2 \circ f_2$.

In view of the existence axiom for free amalgamations any two members M_1, M_2 of K with an $M_0 \in \mathbf{K}$ that is a **K**-substructure of each of them are compatible over M_0. When N_0 is replaced by a diagram $\mathfrak{M}$ involving infinitely many members of K the situation is more complicated.
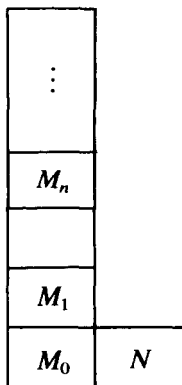
Fig. 2.1. An 'L' diagram.

2.1.8. Definition. Suppose $\mathfrak{M}$ is a **K**-diagram. Let f_1, f_2 be **K**-embeddings of $\mathfrak{M}$ into M_1 and M_2 respectively. M_1 and M_2 are **K**-*compatible over* $\mathfrak{M}$ via f_1, f_2 if there exists an $N \in K$ and **K**-embeddings g_1, g_2 of M_1, M_2 into N such that $g_1 \circ f_1$ and $g_2 \circ f_2$ agree on $\mathfrak{M}$.

(The f_i are families of maps and the g_i are single maps but the meaning of the composition should be clear.)

Clearly over each $\mathfrak{M}$, compatibility defines a reflexive and symmetric relation. Axiom C2 shows that the relation is also transitive so for any diagram $\mathfrak{M}$ we have an equivalence relation, compatibility over $\mathfrak{M}$.

2.1.9. Definition. The abstract **K**-diagram $\mathfrak{M}$ is *stably univalent* if all stable **K**-extensions of $\mathfrak{M}$ are compatible.

In this language Axiom C5 asserts that every vee is stably univalent.

There is little hope to find a 'prime' model (in the usual categorical sense) over an arbitrary diagram $\mathfrak{M}$. For, if a diagram isomorphism from $\mathfrak{M}$ into N collapsed two elements of models $M \in \mathfrak{M}$ then there could be no isomorphism from a structure containing all members of $\mathfrak{M}$ into N. Thus, we restrict the following definitions to stable embeddings. We first define the notion of a 'prime' model within a compatibility class. Remember that the image of a stable embedding is required to be stable.

2.1.10. Definition. (i) Let f be a stable **K**-embedding of the abstract diagram $\mathfrak{M}$ in M. We say M is *compatibility prime* over $(\mathfrak{M}, f)$ (or over $\mathfrak{M}$ via f) if for every $M' \in \mathbf{K}$ and stable embedding f' of $\mathfrak{M}$ into M' such that M and M' are compatible via f, f' there is an embedding of $g : M \mapsto M'$ with $g \circ f = f'$.

(ii) We omit f if it is an identity map.

Strictly speaking, we should refer to a triple $\langle \mathfrak{M}, f, M \rangle$. A little looseness to ease reading seems acceptable here. We are more precise when we introduce the notion of canonically prime in Section 2.3 since a new basic relation is added to the system.

Note that if the diagram $\mathfrak{M}$ has a unique maximum element then that element is compatability prime over $\mathfrak{M}$. Note that if M is compatibility prime over $\mathfrak{M}$ via f then M is isomorphic to an M' which is compatibility prime over $\mathfrak{M}$ via the identity.

There are various ways in which a compatability prime model can fail to be unique. There could be more than one compatibility class; within a given compatability class there could be nonisomorphic compatibility prime models.

2.1.11. Definition. M is *absolutely prime* over $(\mathfrak{M}, f)$ if M can be embedded over $(\mathfrak{M}, f)$ into any stable **K**-extension N of $\mathfrak{M}$.

Clearly, if M is compatibility prime over $\mathfrak{M}$ and $\mathfrak{M}$ is univalent then M is absolutely prime. In view of the weak uniqueness axiom if there is a prime model over a stable diagram $\mathfrak{M}$ then $\mathfrak{M}$ is stably univalent. We will see that with strong enough hypotheses on **K** the various notions of prime coalesce. We have introduced the notion of 'absolutely prime' to emphasize the distinction with compatibility prime; 'absolutely prime' is the natural extension of the usual model theoretic notion to 'prime over a diagram'. When we write 'prime' with no adjective we mean 'absolutely prime'.

2.2. Prime models over vees

In the next two sections we discuss axioms concerning the existence and properties of prime models over certain specific diagrams. We begin by assuming the existence of prime models over vee's. In the light of C5 (weak uniqueness) two completions of an amalgam are compatible so it would be equivalent to replace absolutely prime by compatibility prime in the following axiom. We will often just say 'prime' for the absolutely prime model over a vee. Again because of weak uniqueness we don't have to specify M_3 is prime over $M_1 \cup M_2$ in M_4.

2.2.1. Prime models over vees
D1. There is an absolutely prime model over any free amalgam $\langle M_0, M_1, M_2 \rangle$.

For compactness we write M is prime over $M_1 \cup M_2$ instead of M is prime over $\langle M_0, M_1, M_2 \rangle$.

Now we describe properties which relate independence and prime models. The next axiom of this group corresponds to Axiom C4 of [9]. Diagram 2.2 illustrates each of the principles discussed below.

M_4	M_5
M_2	M_3
M_0	M_1

Fig. 2.2. Base extension axioms.

2.2.2. The base extension axiom

D2. Suppose M_1 and M_4 are freely amalgamated over M_0 in M_5. If $M_0 \leqslant M_2 \leqslant M_4$ and M_3 is prime over $M_1 \cup M_2$ in M_5 then M_3 and M_4 are freely amalgamated over M_2 in M_5.

To understand the base extension axiom in the first-order context, think of M_3 as $M_2[\dot{M}_1]$. (For this notation see [2].) Then the axiom is implied by the fact that for any X and any model M (in the appropriate category, e.g. ω-stable and the normal notion of prime model), X dominates $M[X]$ over M.

Axiom 2.2.2 yields a somewhat surprising consequence. We can obtain the following 'transitivity of nonforking' from 2.2.2 and the weak uniqueness we posited in Axiom C5. This result is remarkable because we are establishing a property of the nonforking relation which makes no reference to prime or generated models. But, the proof uses properties of either generated models (the version in [9]) or prime models (here).

Theorem 2.2.3 (Transitivity of independence). *If* $\mathrm{NF}(M_0, M_1, M_2, M_3)$ *and* $\mathrm{NF}(M_2, M_3, M_4, M_5)$ *then* $\mathrm{NF}(M_0, M_1, M_4, M_5)$.

Proof. By the existence axiom there are M_4'' and M_5'' and an isomorphism g of M_4 and M_4'' over M_0 such that $M_1 \downarrow_{M_0} M_4''$ in M_5''. By monotonicity (C3(iii)) we may assume M_5'' is prime over $M_1 \cup M_4''$. Let M_2'' denote $g(M_2)$ so $M_0 \leqslant M_2'' \leqslant M_4''$. We want to show the existence of an isomorphism with domain M_5'' which fixes M_1 and maps M_4'' to M_4.

Let M_3' be prime over $M_1 \cup M_2$ and contained in M_3. By the monotonicity axioms (C3(iii) and C3(i)) we have $M_1 \downarrow_{M_0} M_2$ in M_3' and $M_3' \downarrow_{M_2} M_4$ in M_5. Similarly, if M_5' is chosen prime over $M_3' \cup M_4$ in M_5 then $M_5' \leqslant M_5$ and $M_3' \downarrow_{M_2} M_4$ in M_5' using axiom C3(iii).

Since M_2 and M_2'' are isomorphic over M_0 and both are independent from M_1 over M_0 there is an isomorphism f taking M_3' into M_5'' which extends $g \mid M_2 \cup 1_{M_1}$. (This follows because M_3' is prime over $M_2 \cup M_1$ and applying the weak uniqueness axiom.) Let M_3'' denote $f(M_3')$. Then M_3'' is prime over $M_2'' \cup M_1$ since f is an isomorphism. By the base extension axiom 2.2.2, $M_4'' \downarrow_{M_2''} M_3''$ in M_5''. Let M_5''' be prime over $M_4'' \cup M_3''$ and let h be a map from M_5''' into some N which extends $g \cup f^{-1}$. Now $M_4' = h(M_4'')$ and M_4 are isomorphic over M_2 and both are

independent from M'_3 over M_2. So by the weak uniqueness axiom there is an h' defined on N which takes M'_4 to M_4 and fixes M'_3. Now $h' \circ h$ takes M''_4 to M_4 and fixes M_1 as required. $\quad\square$

Theorem 2.2.4 (Transitivity of primality). *Suppose M_1 and M_4 are freely amalgamated over M_0 in M_5. If $M_0 \leqslant M_2 \leqslant M_4$, M_3 is prime over $M_1 \cup M_2$ in M_5 and M_5 is prime over $M_3 \cup M_4$ in M_5 then M_5 is prime over $M_1 \cup M_4$ in M_5.*

Proof. Let N be a stable **K**-extension of $\langle M_0, M_1, M_4 \rangle$. That is, there exist maps f_1, f_4 from M_1, M_4 to **K**-submodels M'_1, M'_4 of N that agree on M_0; let M'_0 denote the common image of the f_i on M_0. Then since the embedding is stable $M'_1 \downarrow_{M'_0} M'_4$ inside N. We must find a common extension of the f_i to M_5. Let f_2 denote the restriction of f_4 to M_2 and M'_2 its image. Now, since M_3 is prime over $M_1 \cup M_2$ there is a map f_3 with domain M_3 which extends $f_1 \cup f_2$. Denote the image of f_3 by M'_3. By the base extension axiom 2.2.2, $M'_3 \downarrow_{M'_2} M'_4$ in N. So there is an f_5 mapping M_5 into N and extending $f_3 \cup f_4$. A fortiori, f_5 extends $f_3 \cup f_4$ and we finish. $\quad\square$

There are some further properties of prime models which both arise in some natural situations and are useful tools. We describe them now but they do not play an important role in the theory until we reach some rather special cases.

2.2.5. Definition. The concrete **K**-extension M of $\mathfrak{M}$ is *minimal* over the diagram $\mathfrak{M}$ if there is no proper **K**-submodel of M that contains $\mathfrak{M}$.

2.2.6. Some further axioms
D3. The prime model over a free amalgam $\mathfrak{M}$ is minimal over $\mathfrak{M}$.

D4. The prime model over a free amalgam $\mathfrak{M}$ is unique up to isomorphism over $\mathfrak{M}$.

D5. Suppose M_1 and M_4 are freely amalgamated over M_0 in M_5 and M_5 is prime over $M_1 \cup M_4$. If $M_0 \leqslant M_2 \leqslant M_4$ and M_3 is prime over $M_2 \cup M_1$ then M_5 is prime over $M_4 \cup M_3$.

D5 is a kind of 'converse' to Theorem 2.2.4 that we may need later. It will only hold in very restricted cases. It is true for the notion of generation in universal classes; it fails for prime models in the first-order case (even ω-stable). However, if prime models over vees are minimal (e.g. for a first order theory without the dimensional order property) then an even stronger version of D5 holds easily. Namely the monotonicity requirement that if M_5 is minimal over $M_1 \cup M_4$ then M_5 is minimal over $M_3 \cup M_4$.

2.3. Prime models over chains

An abstract chain $\langle M_i, f_{i,j} : i, j \in I \rangle$ is an abstract diagram whose index set I is linearly ordered. Any closed initial segment of an abstract chain has a natural

representation as a concrete chain. To see this consider $\langle M_i, f_{i,j} : i, j \leq \alpha \rangle$. Let M'_i denote $f_{i,\alpha}(M_i)$. Then the M'_i are a concrete chain with inclusion maps $f'_{i,j} = f_{j,\alpha} \circ f_{i,j} \circ f^{-1}_{i,\alpha}$. Thus for any chain indexed by an ordinal γ and any limit ordinal $\delta < \gamma$ it makes sense to speak of $\bigcup_{i<\delta} M_i$ as we can concretely realize $\mathfrak{M} \mid \delta$ in M_β for any β with $\delta \leq \beta \leq \gamma$.

We discuss in this section the specification and existence of prime models over chains. In general, given an increasing chain of **K**-models there is no reason to assume that the chain has any common extension in **K**, let alone one that is prime. If we assume the existence but not the weak uniqueness of compatibility prime models, there may be two incompatible compatibility prime models over the same chain. (We say **K** is not smooth.) For this reason, we can not in the most general case just define 'prime' as compatibility prime and posit that 'prime' models exist. We would need to introduce another axiom asserting that there is only one compatability class over any chain. Justification of such an axiom is the main point of [3]. However, to reduce the set theoretic hypotheses of that argument we introduce a new predicate (cpr) with the intuitive meaning, 'M is canonically prime over $\mathfrak{M}$' and prescribe axioms describing the behavior of such prime models.

Consider the last example in Paragraph 1.2.1. We have a unary predicate U which is dense and codense. At any limit stage of cofinality κ, we have to decide whether the name of certain κ sequences are in U or not. Different answers correspond to different compatability classes.

We want to demand the existence of a 'prime' model over a union of a chain. If the chain has length longer than ω several possibilities arise for what we should demand of models at limit stages in the chain. It is unreasonable to demand the existence of 'prime' models over chains that are not **K**-continuous in the following sense.

2.3.1. Definition. (i) The chain $\langle M_i, f_{i,j} : i, j < \beta \rangle$ is **K**-*continuous* if for each limit ordinal $\delta < \beta$, $\text{cpr}(\mathfrak{M}_\delta, M_\delta, f \mid \delta)$.

(ii) the chain $\langle M_i, f_i : i < \beta \rangle$ is *continuous* if for each limit ordinal $\delta < \beta$, $M_\delta = \bigcup_{i<\delta} M_i$.

Axiom Ch1 gives an implicit definition of a canonically prime model; Axiom Ch2 asserts that such a model exists. Suppose $\mathfrak{M}$ is a 'long' chain. At each limit stage one has a choice of compatibility classes for a 'prime' model over that segment of the chain. The notion of a canonically prime model requires that these choices cohere.

Ch1. $\text{cpr}(\mathfrak{M}, M, f)$ implies
　　　(i) $\mathfrak{M}$ is a **K**-continuous chain.
　　　(ii) M is compatibility prime over $\mathfrak{M}$ via f.

We often write M_δ is canonically prime over $\mathfrak{M}_\delta = \langle M_i : i < \delta \rangle$ for the formal expression cpr($\mathfrak{M}_\delta$, M_δ, $f \mid \delta$). That is, for brevity we do not mention the specific embeddings $f_{i,j}$ unless they play an active role in the discussion.

Ch2. For any **K**-continuous chain $\mathfrak{M}$ there is a model M and a family of maps f that satisfy cpr($\mathfrak{M}$, M, f).

Ch3. The canonically prime model over an increasing chain $\mathfrak{M}$ is unique up to isomorphism over $\mathfrak{M}$.

Again the uniqueness axiom is regarded as a desirable property to prove and is not assumed in the general development.

2.3.2. Definition. The chain $\langle M_i, f_{i,j} : i < j < \beta \rangle$ is called *essentially* **K**-*continuous* if for each limit ordinal $\delta < \beta$, there is a model M'_δ which can be interpolated between the predecessors of M_δ and M_δ and is canonically prime over $\mathfrak{M}_\delta$.

More formally, we add to the index set a new δ' for each limit ordinal δ. There exists a system of embeddings g such that if α and β are successor ordinals $g_{\alpha,\beta} = f_{\alpha,\beta}$, and for each limit ordinal δ and each $\alpha < \delta g_{\alpha,\delta}$ factors as $g_{\alpha,\delta'} \circ g_{\delta',\delta}$. Finally cpr($\mathfrak{M}_\delta$, $M_{\delta'}$, $g \mid \delta'$).

The next axiom provides a local character for dependence of the prime model over a chain over another model. This is our only basic axiom connecting the independence of an infinite diagram with that of its constituents (see Fig. 2.3).

2.3.3. Axiom L1: Local dependence for L diagrams. If $\langle M_i : i \leq \delta \rangle$ is a **K**-continuous increasing sequence inside M' and for each $i < \delta$, $M_i \downarrow_{M_0} N$ in M' then $M_\delta \downarrow_{M_0} N$ in M'.
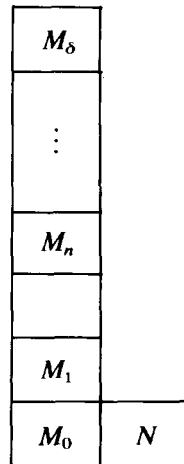

Fig. 2.3. Local dependence for L's.

3. Prime models over small loose trees

In this chapter we extend the existence of prime models over simple diagrams to obtain prime models over more complicated diagrams. Ideally we would show the existence of prime models over an arbitrary independent tree. This project runs into difficulties when considering trees of height greater than ω; as a substitute we show how to obtain prime models over 'loose trees' of models indexed by subsets of $\lambda^{<\omega}$. In a later paper we expect to reduce the discussion of prime models over trees with large height to treatment of loose trees of height ω.

While the original motivation for loose trees was the reduction of problems about tall trees to problems about loose trees of countable height, it turns out that loose trees are closed under several useful operations such as quotient and trees are not. Thus, by passing through loose trees we are able to obtain results about trees that, at least a priori, are otherwise unavailable.

Consider a concrete stable diagram. An initial strategy for building a prime model over $\mathfrak{M}$ is to enumerate $\mathfrak{M}$ say as M_i for $i < \alpha$ and choose a family of models N_i for $i < \alpha$ so that for each i, M_{i+1} is independent from N_i over the predecessors of M_{i+1} in the tree and then to take N_{i+1} prime over M_{i+1} and N_i. Take canonically prime models over the earlier N_i at limits. The resulting model clearly depends on the order of enumeration. Can one still prove that this model is compatability prime over the diagram? We show that the answer is yes if the diagram is a 'short' tree (and generalize to allow loose trees). Finite trees are considered in Section 3.1; trees of countable height in Section 3.2. In the case of finite trees we show fairly directly that if a loose tree is free under one enumeration then it is free under any enumeration. In the second case we pass to the ostensibly more general notion of locally free loose tree, show the existence of prime models over such a tree, and deduce from that the fact that a locally free loose tree is free under any enumeration.

'Assumption: An adequate class. We assume in this chapter axiom groups A and C, axiom D1 and D2 from group D, Ch1, Ch2 from group Ch, and (beginning with 3.2.6) L1. We call a class with these properties an *adequate class*.

3.1. Free loose trees

In this section we show that if $\mathfrak{M}$ is a *finite* free loose tree (definitions follow) of models from an adequate class then there is an absolutely prime model over $\mathfrak{M}$. More precisely, we say that N is *explicitly prime* over a finite loose tree $\mathfrak{M}$ if it is the last in a sequence of prime models over vee's satisfying crtain conditions. We show that if N is explicitly prime over a subdiagram of $\mathfrak{M}$ then the sequence witnessing this can be extended to one witnessing the existence of an explicitly prime model over $\mathfrak{M}$. This ostensibly technical result is essential for the discussion of prime models over *infinite* loose trees in the next section.

3.1.1. Notation. A tree **T** is a partially ordered set which is isomorphic to a subset of $\lambda^{<\omega}$ which is closed under initial segment. We will often deal directly with this representation. A tree is partially ordered by containment. If $s, t \in \mathbf{T}$ then $s \wedge t$ denotes the largest common initial segment of s and t and for any t other than the root, denoted $\langle \ \rangle$, t^- denotes the predecessor of t.

As in the study of stable diagrams we work only with embeddings that are inclusions, that is, with concrete diagrams.

3.1.2. Definition. A *loose tree of models* $\mathfrak{M} = \{M_t : t \in \mathbf{T}\}$ inside M indexed by a tree **T** is a collection of **K**-models such that if $t^- = s$ then $M_t \cap M_s \leqslant M_t$ and $M_t \cap M_s \leqslant M_s$.

Note that any stable diagram indexed by a tree is a loose tree.

We have *not* introduced a loose tree as a kind of partial order. All our index sets are trees in the normal sense. A loose tree *of models* is loose because of the inclusions amongst the models determined by the indexing.

In defining an independent loose tree (below) we will speak of independence over $M_t \cap M_{t^-}$. This use of intersection depends on Axiom C7 (which implies that $M_s \cap M_t \in \mathbf{K}$). However, at a cost in complexity of notation the results of this chapter could be obtained through modifying the definition of a loose tree and a free loose tree by replacing $M_t \cap M_s$ by a substructure $M_{t,s} \in \mathbf{K}$. This would allow us to extend the definition of loose trees to abstract diagrams. With such an extension the analogy would be closer between the notion of compatibility prime (over a stable embedding of an abstract diagram 2.1.10) and the definition 3.1.16 of a compatibility prime model over a loose tree.

3.1.3. Definition. An *isomorphism of loose trees* $f : \mathfrak{M} \mapsto \mathfrak{M}'$ is a family of isomorphisms f_t taking $M_t \in \mathfrak{M}$ to $M'_t \in \mathfrak{M}'$ such that for $t \leqslant s$, $f_t \mid (M_t \cap M_s) \subseteq f_s$. We say the loose tree $\mathfrak{M}$ can be **K**-embedded in N if there is such an isomorphism between $\mathfrak{M}$ and a loose tree $\mathfrak{M}'$ inside N.

3.1.4. Definition. A wellordering $\bar{t} = \{t_i : i < \beta\}$, where $|\beta| = |\mathbf{T}|$, such that if t_i precedes t_j in **T** then $i \leqslant j$ is called an *enumeration* of the tree **T**.

An enumeration of **T** induces an enumeration of any $\mathfrak{M}$ indexed by **T**.

3.1.5. Example. The following observation does not figure in our argument but illustrates one of the subtle differences between a loose tree and a tree of models. The partial order of a tree of models is determined by containments among the models. The partial order of a loose tree is artificially imposed (and thus cannot be ignored). Consider a collection of models $M_0 \subseteq M_1$, $M_0 \subseteq M_2 \subseteq M_3 \subseteq M_4$ with $M_1 \downarrow_{M_0} M_4$ and $M_1 \cap M_4 = M_0$. As a loose tree of models they can be indexed by the integers less than 5 with the following tree order; 0 is the root, 1 and 2 are incomparable successors of 0, 3 and 4 are *incomparable* successors of 2. Clearly $\langle 0, 1, 2, 3, 4 \rangle$ is an enumeration of the loose tree; so is $\langle 0, 1, 2, 4, 3 \rangle$.

Again, we could index these models by a tree with 0 as an initial element, 2, 3, 4 as incomparable succesors and 1 above 2. Now one can enumerate the tree as $\langle 0, 2, 3, 4, 1 \rangle$.

With this housekeeping out of the way we can introduce a more important concept.

3.1.6. Definition. The loose tree $\mathfrak{M}$ indexed by **T** is *free* or *independent* in M with respect to an enumeration $\bar{t}$ of **T** if there exists an ordinal β and a sequence of **K**-models $\langle N_i : i < \beta \rangle$ such that

(i) All the M_t, $N_i \leqslant M$.

(ii) $M_{t_0} = N_0$.

(iii) Fix $k < i$ such that $t_i^- = t_k$.

  • If $1 \leqslant i < \omega$ then $M_{t_i} \downarrow_{M_{t_i} \cap M_{t_k}} N_{i-1}$ in N_i.

  • If $i \geqslant \omega$ then $M_{t_i} \downarrow_{M_{t_i} \cap M_{t_k}} Ni$ in N_{i+1}.

(iv) $\langle N_i : i < \beta \rangle$ is **K**-continuous.

thus $|\beta| = |\mathbf{T}|$ but when **T** is infinite the ordinality of β will be $\mathrm{ord}(\bar{t}) + 1$ which may be greater than $|\mathbf{T}|$. If $\mathfrak{M}$ is actually a tree the $M_{t_i} \cap M_{t_k}$ in condition (iii) becomes M_{t_k}. We call $\mathfrak{N}$ a witnessing sequence for the freeness.

3.1.7. Remark. The two conditions in (iii) of 3.1.6 could be combined if we indexed the N_i by $1 \leqslant i \leqslant \beta$. We didn't make the change to avoid introducing errors in the proofs of the later theorems using finite trees indexed in accordance with the official definition.

A trivial induction shows the following refinement of the definition. We use this observation without comment below.

3.1.8. Lemma. *If $\mathfrak{M}$ is a loose tree in M indexed by the finite tree* **T**, *the witnessing sequence $\langle N_i : i < \beta \rangle$ can be chosen to satisfy N_i is prime over $M_{t_i} \cup N_{i-1}$ inside N.*

With this lemma in mind, suppose the loose tree $\mathfrak{M}$ indexed by **T** is *free* or *independent* in M with respect to the enumeration $\bar{t}$ of **T** and there is an initial sequence $\langle M_{t_i} : i < j \rangle$ of the enumeration with $M_{t_i} \subseteq M_{t_{i+1}}$. Then for $i < j$, N_i can be chosen as M_{t_i}. This is a likely possibility for a tree, an unlikely one for a properly loose tree of models.

3.1.9. Some generalizations. The following variant may turn out to be necessary. Define a loose tree to be *almost free in N* with respect to an enumeration $\bar{t}$ if it is free in some extension N' of N. Then a compatibility class of $\mathfrak{M}$ is almost free with respect to $\bar{t}$ if some $\mathfrak{M}$ is free with respect to $\bar{t}$ in some N in the compatibility class. We will finally show in this section that almost free implies free by showing the explicitly prime model over $\mathfrak{M}$ inside N' can be chosen inside N. This paragraph is analogous to the monotonicity conditions on freeness over a vee.

The following lemma is proved by a straightforward induction on the length of the enumeration using the existence of prime models over independent pairs and canonically prime models over chains.

3.1.10. Lemma. *Let $\mathfrak{M}$ be a loose tree in M enumerated by $\bar{t}$. There is an N and an isomorphic copy of $\mathfrak{M}$ inside N that is free with respect to the enumeration $\bar{t}$ in N.*

Note that the isomorphism in Lemma 3.1.10 is an isomorphism of loose trees. Example 3.1.5 is not free in either enumeration. We want to replace 'free with respect to an enumeration' in Lemma 3.1.10 by 'free'; that is to show that if $\mathfrak{M}$ is free with respect to one enumeration it is free with respect to any enumeration. Then we will show that compatibility prime models exist over free trees. We first handle the case of finite trees. For this we need some combinatorial tools to reduce the main result to a more manageable case.

3.1.11. Definition. Let $\bar{s}$ and $\bar{t}$ be enumerations of **T**. Then, $\bar{s}$ and $\bar{t}$ are *close neighbors* if for some i, $s_i = t_{i+1}$, $t_i = s_{i+1}$, and they agree on all other arguments. $\bar{t}$ and $\bar{s}$ are *neighbors* if there is a sequence $\bar{t} = \bar{t}^0, \bar{t}^1, \ldots, \bar{t}^k = \bar{s}$ such that for each $j < k$, $\bar{t}^j$ and $\bar{t}^{j+1}$ are close neighbors.

The next lemma reduces many problems about the relations between two enumerations to the relation between neighbors and thus by easy induction to relations between close neighbors.

3.1.12. Lemma. *If $\bar{s}$ and $\bar{t}$ are enumerations of **T** then they are neighbors.*

Proof. Fix $\bar{s}$ and $\bar{t}$. Choose k maximal such that for some neighbor $\bar{t}'$ of $\bar{t}$, $\bar{s} \mid k = \bar{t}' \mid k$. Choose the least l, necessarily greater than k, such that for some such $\bar{t}'$, $s_{k+1} = t_l'$. Note that $s_{k+1} = t_l'$ is incomparable with t_{l-1}', $l - 1 \geqslant k$. Now if we manufacture $\bar{t}''$ from $\bar{t}'$ by switching t_l' and t_{l-1}' we contradict the minimality of l. $\square$

3.1.13. Remark. Our notion of enumeration corresponds to the concept of 'linear extension' in the theory of partial orderings. We are not aware if this result has been proved in that context although it seems likely.

3.1.14. Lemma. *If **T** is a finite tree and $\mathfrak{M}$ is a loose tree indexed by **T** which is free inside N with respect to any enumeration $\bar{t}$ then $\mathfrak{M}$ is free inside N with respect to any enumeration $\bar{s}$ of $\mathfrak{M}$.*

Proof. By Lemma 3.1.12 it suffices to show that if $\mathfrak{M}$ is independent with respect to $\bar{t}$ then $\mathfrak{M}$ is independent with respect to the enumeration $\bar{t} \circ (i, i + 1)$. Let

$i = k + 1$. Since $\bar{t}$ and $s = \bar{t} \circ (i, i+1)$ are enumerations, t_i and t_{i+1} are incomparable. We will construct a witnessing sequence N_i' for the independence of $\mathfrak{M}$ with respect to $\bar{s}$ from the given sequence N_i witnessing the independence of $\mathfrak{M}$ with respect to $\bar{t}$. For $l \leqslant k$, let $N_l' = N_l$. To simplify notation, let M^a denote M_{t_i}, M_a denote $M_{t_i^-}$, M^b denote $M_{t_{i+1}}$, and M_b denote $M_{t_{i+1}^-}$. We have $M^a \downarrow_{M^a \cap M_a} N_{i-1}$ in N_i and $M^b \downarrow_{M^b \cap M_b} N_i$ in N_{i+1}. Since $M_b = M_{t_l}$ for some $l < i$, $M_b \leqslant N_{i-1}$. By monotonicity, $M^b \downarrow_{M_b \cap M^b} N_{i-1}$ in N_i. Now choose $N_i' \leqslant N_{i+1}$ prime over $M^b \cup N_{i-1}$. By the base extension axiom $N_i \downarrow_{N_{i-1}} N_i'$ in N_{i+1}. Since we also have $M^a \downarrow_{M_a \cap M^a} N_{i-1}$ in N_i and $M^a \subseteq N_i$ transitivity of independence, Theorem 2.2.3, yields $M^a \downarrow_{M_a \cap M^a} N_i'$ in N_{i+1}. Since $N_i' \leqslant N_{i+1}$ and if $l > i$, $M_{t_l} \downarrow_{M_{t_l^-}} N_i$ implies $M_{t_l} \downarrow_{M_{t_l^-}} N_i'$ we can let $N_l' = N_l$ for $l > i + 1$. $\square$

3.1.15. Definition. Henceforth we will say a finite loose tree of models is *free* if it is free under some enumeration.

We want to find an analogous result for certain loose trees indexed by subsets of $\lambda^{<\omega}$. This requires several further concepts. We begin by extending our notion of prime over a stable diagram to free loose trees. Compare the following definition to Definitions 2.1.8 and 2.1.10.

3.1.16. Definition. The model M is *compatibility prime over the free loose tree* $\mathfrak{M}$ with respect to the enumeration $\bar{t}$ via the embedding f of $\mathfrak{M}$ into M if the image of $\mathfrak{M}$ under f is free inside M with respect to $\bar{t}$ and for every $N \in K$ and every g embedding $\mathfrak{M}$ into N with the image of q free inside N such that M and N are compatible over $\mathfrak{M}$ via f and g, there is a $\hat{g}$ mapping M into N such that $g = \hat{g} \circ f$.

As in the definition of prime over a stable diagram, we need some 'freeness' condition on a loose tree before it makes sense to speak of a 'prime' model over it. If we strengthen the definition by allowing N to be arbitrary rather than requiring that N be compatible over $\mathfrak{M}$ with M we call M *absolutely prime over* $\mathfrak{M}$.

We need a new 'categorical' definition for prime over a loose tree because the hidden requirements on composition of maps are much looser when a family is indexed as a loose tree than in our notion of prime model over a stable diagram.

When we restrict to finite loose trees of models we can define absolutely prime models.

3.1.17. Lemma. *If* **T** *is a finite tree and* $\mathfrak{M}$ *is a loose tree of models indexed by* **T** *which is free inside N with respect to an enumeration $\bar{t} = \langle y_i : i < k \rangle$ then there is an absolutely prime model over the loose tree* $\mathfrak{M}$.

Proof. An easy induction on $|\mathbf{T}|$ shows N_{k-1} is the required model. $\square$

We christen the result of the last construction.

3.1.18. Definition. (i) If $\langle N_i : i < k \rangle$ witnesses that the finite loose tree $\mathfrak{M}$ is free then we say N_{k-1} is *explicitly prime* over $\mathfrak{M}$ (with respect to the embedding map and a specific enumeration).

(ii) If $\langle N_i : i < \beta \rangle$ witnesses that the loose tree $\mathfrak{M}$ is free and β is a limit ordinal then we say N_β is *explicitly prime* over $\mathfrak{M}$ (with respect to the embedding map and a specific enumeration) if N_β is canonically prime over $\{N_i : i < \beta\}$.

Now we invoke the prime models to strengthen the sense in which 'freeness' is independent of the enumeration of $\mathfrak{M}$. In order to state the result we need several further notations.

3.1.19. Definition. (i) An *ideal* of a tree $\mathbf{T}$ is a subset $\mathbf{T}_1$ of $\mathbf{T}$ that is closed under initial segment.

(ii) If I is an ideal of $\mathbf{T}$ then $\mathbf{T}_I$ denotes the (quotient) tree whose elements are $(\mathbf{T} - I) \cup \{\langle \; \rangle\}$ with the same meets as $\mathbf{T}$ if possible but, if $x \wedge y$ in the sense of $\mathbf{T}$ is in I, then $x \wedge y = \langle \; \rangle$ in the sense of $\mathbf{T}_I$.

Any diagram can naturally be restricted to a subset X of its index set $\mathbf{T}$ by forgetting the models attached elements $\mathbf{T} - X$. We describe several conditions when the restriction $\mathfrak{M} \,|\, I$ of a (loose) tree $\mathfrak{M}$ is a (loose) tree. For loose trees (but not trees) there is a natural complement or quotient structure $\mathfrak{M}_I$ to $M \,|\, I$ which we describe in (iii) of the next definition.

3.1.20. Notation. (i) If $X \subseteq \mathbf{T}$ then for any set of models $\mathfrak{M}$ indexed by $\mathbf{T}$ there is a natural notion of the restriction $\mathfrak{M} \,|\, X$ of $\mathfrak{M}$ to $I : \mathfrak{M} \,|\, X = \{M_t : t \in X\}$.

(ii) M_I denotes some compatibility prime model over $\mathfrak{M} \,|\, I$.

(iii) Suppose $\mathfrak{M} = \langle M_t : t \in \mathbf{T} \rangle$ is a free loose tree inside M. If I is an ideal of $\mathbf{T}$ then $\mathfrak{M}_I$ denotes $\langle M_t : t \in \mathbf{T}_I \rangle$.

The term M_I is an abuse of notation since we are choosing one of a number of possible compatibility prime models. If I is an ideal of $\mathbf{T}$ and $\mathfrak{M}$ is a free loose tree of models indexed by $\mathbf{T}$ it is easy to see that the quotient tree is also free. Formally,

3.1.21. Lemma. *If $\mathfrak{M}$ is a free loose tree inside M and I is an ideal of $\mathbf{T}$ then $\mathfrak{M}_I$ is a free loose tree inside M indexed by $\mathbf{T}_I$.*

Now we consider substitutions for a model in a loose tree.

3.1.22. Notation. Suppose $\mathfrak{M} = \langle M_t : t \in \mathbf{T} \rangle$ is an indexed family of models. Then $\mathfrak{M}(N/s)$ denotes the indexed family of models, obtained by replacing M_s by N. $\mathfrak{M}(N_1/s_1, N_2/s_2)$ is the natural extension of this notation to allow two substitutions.

We are interested in a number of substitutions in indexed trees. Most will be approximations to the result of replacing the root in the tree $\mathfrak{M}_I$ by the prime model over I: $\mathfrak{M}_I(M_I/\langle\ \rangle)$. The next lemma describes two slightly less simple ways of deriving new loose trees from a given one. The fact that loose trees are obtained by the constructions is an immediate verification: the more important fact that the second construction preserves freeness is proved in Lemma 3.1.24.

3.1.23. Lemma. *Suppose $\mathfrak{M} = \langle M_t : t \in \mathbf{T}\rangle$ is a loose tree inside M.*

(i) *Fix $s \in \mathbf{T}$ and a model $N \leqslant M$ such that for each t with $t^- = s$ or $s^- = t$, $N \cap M_t \leqslant M_t$ and $N \cap M_t \leqslant N$. Then $\mathfrak{M}' = \mathfrak{M}(N/s)$ is a loose tree inside M.*

(ii) *Fix $s, r \in \mathbf{T}$ with $r^- = s$ and $M_s \leqslant M_r$. Let $\mathbf{T}'$ denote $\mathbf{T} - \{r\}$. Then $\mathfrak{M}' = \mathfrak{M} \mid \mathbf{T}'(M_r/s)$ is a loose tree inside M.*

We have established the notation to state the following lemma.

3.1.24. Lemma (The Omission Lemma). *Suppose $\mathfrak{M} = \langle M_t : t \in \mathbf{T}\rangle$ is finite and free loose tree inside M, $s, r \in \mathbf{T}$ with $r^- = s$ and $M_s \leqslant M_r$. Let $\mathbf{T}'$ denote $\mathbf{T} - \{r\}$. Then $\mathfrak{M}' = \mathfrak{M} \mid \mathbf{T}'(M_r/s)$ is free.*

Proof. Let $\bar{t}$ enumerate $\mathbf{T}$ with $r = t_{j+1}$ and $s = t_j$. Suppose $\mathfrak{N}$ witnesses the freedom of $\mathfrak{M}$ and $|\mathbf{T}| = k$. Now define

$$t_i' = \begin{cases} t_i & \text{if } i < j, \\ t_{i+1} & \text{if } j \leqslant i < k - 1, \end{cases}$$

$$N_i' = \begin{cases} N_i & \text{if } i < j - 1, \\ N_{i+1} & \text{if } j - 1 \leqslant i < k - 1. \end{cases}$$

It is straightforward that $\mathfrak{N}'$ witnesses the freeness of $\mathfrak{M}'$ with respect to the enumeration $\bar{t}'$. □

The following theorem asserts that the freeness of a finite loose tree is independent not only of the enumeration but for a given enumeration of the choice of the prime models N_i witnessing the freeness. This theorem allows us to decompose $\mathfrak{M}$ into $\mathfrak{M} \mid I$ and $\mathfrak{M}_I$. An advantage of loose trees in performing this construction is that they allow the mixing of the models in the tree and the witnessing sequence.

3.1.25. Theorem. *Suppose $\mathfrak{M}$ is a free loose tree of models inside M indexed by the finite tree $\mathbf{T}$ and I is an ideal of $\mathbf{T}$. Suppose also that N is explicitly prime over $\mathfrak{M} \mid I$ for an enumeration $\bar{t}$ of I. Then $\mathfrak{M}_I(N/\langle\ \rangle)$ is free inside M.*

Proof. Suppose $|I| = j$. Extend the given enumeration of I to an enumeration of $\mathbf{T}$ and also denote the extension by $\bar{t}$. Let $\mathfrak{N}$ witness that $\mathfrak{M} \mid I$ is free. Noting that any initial segment of an enumeration is an ideal, let $\mathbf{T}_i$ be the ideal composed of

the first i elements in the enumeration. Let $\mathfrak{M}^i = \mathfrak{M}_{\mathbf{T}_i}(N_i/\langle\ \rangle)$. We finish by showing by induction on $i < |I|$ that $\mathfrak{M}^i$ is free inside M.

If $i = 0$, $\mathfrak{M}^i = \mathfrak{M}$ and the result is clear. Suppose we know $\mathfrak{M}^i$ is free and consider $\mathfrak{M}^{i+1}$. In $\mathbf{T}_{\mathbf{T}_i}$, t_{i+1} is an immediate successor of $\langle\ \rangle$. Lemma 3.1.26 below shows $\mathfrak{M}^i(N_{i+1}/t_{i+1})$ is free inside M. This implies $\mathfrak{M}^{i+1}$ is free by the omission lemma and the induction hypothesis. Thus, it remains only to prove Lemma 3.1.26.

Note that any finite tree can be enumerated so that $M_{t_{i+1}^-} \leqslant M_{t_i}$ for any i.

3.1.26. Lemma. *Suppose $\mathfrak{M}$ is a free loose tree of models inside M indexed by the finite tree $\mathbf{T}$. Let $N \leqslant M$ and suppose N is prime over $M_v \cup M_{v-}$ for some $v \in \mathbf{T}$. Then $\mathfrak{M}' = \mathfrak{M}(N/v)$ is free inside M.*

Proof. Fix an enumeration $\bar{t}$ of $\mathfrak{M}$ with $t_j = v$. Suppose that $\mathfrak{N} = \langle N_i : i < |\mathbf{T}|\rangle$ witnesses the freedom of $\mathfrak{M}$. Before constructing the sequence N_i' which will witness the freedom of the new tree we need an auxiliary sequence. We define for $j \leqslant i < |\mathbf{T}|$ an increasing chain of models N_i^* beginning with $N_j^* = M_v = M_{t_j}$ such that

    (i) each $N_i^* \leqslant N_i$,
    (ii) $M_{t_{i+1}} \downarrow_{M_{t_{i+1}^-} \cap M_{t_{i+1}}} N_i^*$ inside N_{i+1}^*,
    (iii) $N_{i+1}^* \downarrow_{N_i^*} N_i$ inside N_{i+1},
    (iv) N_{i+1}^* is prime over $M_{t_{i+1}} \cup N_i^*$.

Base Step. First note setting $N_j^* = M_{t_j}$ satisfies conditions (ii) and (iii). Choosing N_{j+1}^* to satisfy (iv), we must verify (iii). This follows by the base extension axiom applied as in Fig. 3.1. Recall $M_{t_j} = M_v = N_j^*$.

Induction Step. Suppose that for some $k \geqslant j$ we have chosen N_k^* to satisfy (ii) and (iii) (with $i = k - 1$). Choose N_{k+1}^* inside N_{k+1} prime over $M_{t_{k+1}} \cup N_k^*$ to satisfy (i) and (iv). Now N_{k+1}^* satisfies (ii) by monotonicity and (iii):

$$N_{k+1}^* \downarrow_{N_k^*} N_k \quad \text{in } N_{k+1}$$

follows from the base extension axiom as in the induction step diagram, Fig. 3.2. This completes the construction of the N_i^*.

Let u denote v^-; note that since $N_j^* = M_v$ we have $N_{j-1} \downarrow_{M_v \cap M_u} N_j^*$. With this as

N_j	N_{j+1}
M_v	N_{j+1}^*
$M_{t_{j+1}} \cap M_v$	$M_{t_{j+1}}$

Fig. 3.1. The base step.

N_k	N_{k+1}
N_k^*	N_{k+1}^*
$M_{t_{k+1}} \cap M_{t_{k+1}^-}$	$M_{t_{k+1}}$

Fig. 3.2. The induction step.

the base an easy induction on l for $j \leq l \leq k$ shows $N_{j-1} \downarrow_{M_v \cap M_u} N_l^*$. We will rely on the case $l = k$.

We have from the free enumeration that $M_v \downarrow_{M_u \cap M_v} N_{j-1}$. The base extension axiom then gives, since N is prime over $M_v \cup M_{v^-}$, $N \downarrow_{M_v} N_{j-1}$. Choose $M_v' \leq N_k$ prime over $N \cup N_{j-1}$. By transitivity of primeness, Lemma 2.2.4, M_v' is prime over $M_v \cup N_{j-1}$.

From the base extension axiom and the conclusions of the last two paragraphs, we deduce $M_v' \downarrow_{M_v} N_k^*$. This allows us to perform the $i = j$ step of the following construction. Choose N_i' so that

$$N_i' = \begin{cases} N_i & \text{if } i < j, \\ M_v' & \text{if } i = j \end{cases}$$

and for $j < i$ so that $N_i' \leq M$ is prime over $N_i^* \cup N_{i-1}'$ and $N_i' \downarrow_{N_i^*} N_k^*$. The choice of the N_i' for $i > j$ is a straightforward induction.

Now to complete the proof we must observe that the N_i' witness the freeness of $\mathfrak{M}'$. That is we must show that N_{i+1}' is prime over $M_{t_{i+1}} \cup N_i'$. This follows from transitivity of primeness, Lemma 2.2.4, since for each i we have N_{i+1}^* is prime over $M_{t_{i+1}} \cup N_i^*$ and N_{i+1}' is prime over $N_{i+1}^* \cup N_i'$. $\square$

We restate Theorem 3.1.25 in a more applicable form.

3.1.27. Corollary. *Suppose $\mathfrak{M}$ is a free loose tree of models inside M indexed by the finite tree* **T** *and I is an ideal of* **T**. *If $\bar{\iota}$ is an enumeration of I and $\mathfrak{N}$ witnesses that $\mathfrak{M} \mid I$ is free inside M then $\mathfrak{N}$ can be extended to a sequence witnessing that $\mathfrak{M}$ is free inside M.*

We have shown that if **K** is an adequate class and $\mathfrak{M}$ is a finite (loose) tree $\mathfrak{M}$ which is free inside N under some enumeration then $\mathfrak{M}$ is free under any enumeration and there is an absolutely prime model over $\mathfrak{M}$. With more difficulty we showed that if I is an ideal of $\mathfrak{M}$ and N_0 is explicitly prime over I then there is a sequence defining an explicitly prime model over $\mathfrak{M}$ which includes N_0.

3.2. Locally free loose trees

We now extend our analysis to infinite trees. Recall that in the absence of the monster model we can only speak of a diagram being free when we have an

embedding into an ambient model in mind. And only then we can we discuss the possibility of any sort of 'prime' model over the diagram. This freeness can be verified by an enumeration of the loose tree and if the index tree is finite the freeness is independent of the enumeration. We do not at first claim so much for an infinite tree. Rather we introduce a notion of a locally free (loose) tree that has the following properties. If $\mathfrak{M}$ is free under some enumeration then $\mathfrak{M}$ is locally free. The property of being locally free does not depend on the enumeration. We will establish the existence of compatibility prime models over locally free loose trees (which have height at most ω). From this we deduce that if such a tree is free is under one enumeration then it is free under any enumeration.

3.2.1. Definition. A loose tree of models $\mathfrak{M}$ is *locally free in N* if $\mathfrak{M} = \langle M_t : t \in \mathbf{T} \rangle$ is contained in N and for every finite subtree $\mathbf{T}_1 \subseteq \mathbf{T}$ the finite loose tree $\mathfrak{M} \mid \mathbf{T}_1 = \langle M_t : t \in \mathbf{T}_1 \rangle$ is free inside N.

3.2.2. Remark. This definition relies on our restriction to subtrees of $\lambda^{<\omega}$. Since subtrees are closed under predecessor trees of greater height cannot be covered by finite subtrees. Thus, when we deal with trees of greater height we will modify the definition of locally free but leave the meaning the same on the low trees considered here.

The next proposition is obvious.

3.2.3. Proposition. *If the loose tree $\mathfrak{M}$ is free inside for N for some enumeration then $\mathfrak{M}$ is locally free inside N.*

Now we define 'prime' models over locally free loose trees. Definition 3.1.16 is almost a special case of this. (Ostensibly, a map could take a free tree to a locally free tree so prime over locally free is more restrictive than prime over free.)

3.2.4. Definition. The model M is *compatibility prime over the locally free loose tree* $\mathfrak{M}$ via the embedding f of $\mathfrak{M}$ into M if the image of $\mathfrak{M}$ under f is locally free inside M and for every $N \in K$ and g embedding $\mathfrak{M}$ into N with the image of g locally free inside N such that M and N are compatible over $\mathfrak{M}$ via f and g there is a $\hat{g}$ mapping M into N such that $g = \hat{g} \circ f$.

The name of this notion is unconscionably long so we resort to the following abbreviation: LFP($\mathfrak{M}, M, f$). We omit the f if $\mathfrak{M}$ is concretely realized in M (whence f is family of inclusions).

We are going to show that if a loose tree $\mathfrak{M}$ is locally free there is a 'prime' model over the loose tree $\mathfrak{M}$. We need the following notation (extending 3.1.19 and 3.1.22) for trees obtained from a representation of given tree as a union of (smaller) trees to carefully state and prove this proposition.

3.2.5. Notation. Suppose $T = \bigcup_{\alpha < \lambda} I_\alpha$ where the I_α form an increasing continuous sequence of ideals in T.

(i) $\mathfrak{M}_\alpha$ denotes the quotient of $\mathfrak{M}$ indexed by the quotient tree $\{t : t \in T_{I_\alpha}\}$.

(ii) For $\beta < \alpha$, $\mathfrak{M}_{\alpha,\beta}$ denotes the quoteient of $\mathfrak{M} \mid (I_\alpha)$ by I_β. That is, $\mathfrak{M}_{\alpha,\beta}$ is indexed by the quotient tree $\{t : t \in (I_\alpha)_{I_\beta}\}$. (Remember I_α is a subtree of T so this is a natural extension of our previous notation 3.1.22).

Thus $\mathfrak{M}_\alpha$ just abbreviates $\mathfrak{M}_{I_\alpha}$. Although $\mathfrak{M}_\alpha$ is a quotient of $\mathfrak{M}$ and $\mathfrak{M}_{\alpha,\beta}$ is a quotient of $\mathfrak{M} \mid (I_\alpha)$, as sets of models $\mathfrak{M}_\alpha$ and $\mathfrak{M}_{\alpha,\beta}$ are contained in $\mathfrak{M}$.

Here is the main result of this section. Condition 1 is the result we really want (see Conclusion 3.2.10) but to establish it we must prove Conditions 1 and 2 by a simultaneous induction. A first try to prove this theorem by induction would decompose a tree T as a union of properly smaller ideals and then choose a 'prime' model over each of these ideals and take the canonically prime model over this chain of models. The difficulty is to guarantee that the sequence of models forms a chain. The double induction accomplishes this end.

3.2.6. Theorem. *Suppose $\mathfrak{M}$ is a locally free loose tree indexed by T_0 in N. Conditions 1 and 2 will be proved by simultaneous induction on λ.*

Condition 1. *If $|T_0| = \lambda$ then there is a compatibility prime model for locally free loose trees over $\mathfrak{M}$. That is, there is an N satisfying $LFP(\mathfrak{M}, N)$.*

Condition 2. *If I is an ideal of T_0 and $|I| = \lambda$ then $\mathfrak{M}_I(M_I/\langle \ \rangle)$ is locally free inside M.*

Note that the cardinality of T_0 is not bounded in Condition 2. The following lemma provides most of the technicalities of the proof.

Lemma 3.2.7 (The key lemma). *Suppose $\mathfrak{M}$ is a free loose tree in N' indexed by the finite tree T and suppose s is a minimal $(\neq \langle \ \rangle)$ element of T. Suppose also $\langle M_s^\alpha : \alpha \leqslant \delta \rangle$ is an increasing K-continuous sequence satisfying the following conditions.*

(i) *For each $\alpha < \delta$, $\mathfrak{M}^\alpha = \mathfrak{M}(M_s^\alpha/s)$ is a free loose tree inside N'.*

(ii) *For each $\alpha < \delta$, $M_{\langle \ \rangle} \leqslant M_s^\alpha$.*

(iii) *If $t^- = s$ then $M_t \downarrow_{M_t \cap M_s^0} M_s^\alpha$.*

Then $\mathfrak{M}(M_s^\delta/s)$ is free inside N'.

Proof. Fix an enumeration $\bar{t}$ of T such that the elements not in the cone with vertex s come first. Suppose $s = t_k$ and $|T| = n$. Let $\mathfrak{N}^\alpha$ witness that $\mathfrak{M}^\alpha$ is free. By Corollary 3.1.27 we may assume that $N_i^\alpha = N_i^0$ if $i < k$. (This use of Corollary 3.1.27 only simplifies notation; the later use is essential.) In order to discuss uniformly the trees $\mathfrak{M}^\alpha$, we refer below to models M_t^α. Unless $t = s$, $M_t^\alpha = M_t^0 = M_t$.

Expand the tree **T** to $\hat{\mathbf{T}}$ by adding a new element r with $r^- = s$ but with no elements above r. We define for each $\alpha \leqslant \delta$ a loose tree of models $\hat{\mathfrak{M}}^\alpha$ and a sequence of models $\hat{\mathfrak{N}}^\alpha$:

$$\hat{M}_x^\alpha = \begin{cases} M_x^0 & \text{if } x \in \mathbf{T}, \\ M_s^\alpha & \text{if } x = r, \end{cases}$$

$$\hat{N}_i^\alpha = \begin{cases} N_i^0 & \text{if } i \leqslant k, \\ N_{i-1}^\alpha & \text{if } i > k. \end{cases}$$

Consider the following enumeration $\bar{u}$ of $\hat{\mathbf{T}}$ which we obtain from $\bar{t}$:

$$u_i = \begin{cases} t_i & \text{if } i \leqslant k, \\ r & \text{if } i = k+1, \\ t_{i-1} & \text{if } k+1 < i \leqslant n-1. \end{cases}$$

Recall that if $i \neq k$, $M_{t_i}^0 = M_{t_i}^\alpha$. Note that both M_s^0 and M_s^α occur in $\hat{\mathfrak{M}}^\alpha$ ($\hat{M}_s^\alpha = M_s^0$, $\hat{M}_r^\alpha = \hat{M}_{u_{k+1}}^\alpha = M_s^\alpha$). We will show below that for each α, $\hat{\mathfrak{N}}^\alpha$ witnesses that $\hat{\mathfrak{M}}^\alpha$ is free inside N'. Assuming this fact we now complete the proof of the lemma.

Since we proved that the freeness of a loose tree does not depend on the enumeration, each $\hat{\mathfrak{M}}^\alpha$ is also free by an enumeration that places r last. That is, $\hat{M}^\alpha$ is free with respect to the enumeration $\bar{v}$ defined as follows:

$$v_i = \begin{cases} t_i & \text{if } i \leqslant n-1, \\ r & \text{if } i = n. \end{cases}$$

Let $I = \{v_i : i < n\}$. By Corollary 3.1.27 any sequence, in particular $\hat{\mathfrak{N}}^0 = \langle N_0^0 \cdots N_{n-1}^0 \rangle$, which witnesses the freeness of $\hat{\mathfrak{M}}^\alpha \,|\, I$ can be extended to a sequence witnessing the freeness of $\hat{\mathfrak{M}}^\alpha$. Let N denote N_{n-1}^0. (It was to make the choice of N independent of α that we needed to prove Theorem 3.1.25 and Corollary 3.1.27.) Now for each α since $\hat{\mathfrak{M}}^\alpha$ is free with respect to $\bar{v}$ we have

$$N \underset{\hat{M}_r^\alpha \cap \hat{M}_{r^-}^\alpha}{\downarrow} \hat{M}_r^\alpha$$

inside $\hat{N}_n^\alpha$. That is,

$$N \underset{M_s^\alpha \cap M_s^0}{\downarrow} M_s^\alpha$$

inside some N_n^α. This implies since $M_s^0 \subseteq M_s^\alpha$ that for each α

$$N \underset{M_s^0}{\downarrow} M_s^\alpha$$

inside N'. Therefore, by L1,

$$N \underset{M_s^0}{\downarrow} M_s^\delta$$

inside N'. Choosing N_n prime over $N \cup M_s^\delta$, we finish the proof of the key lemma from the assumption.

We are left with verifying the assumption; that is, showing that for each α, $\hat{\mathfrak{N}}^\alpha$ witnesses that $\hat{\mathfrak{M}}^\alpha$ is free inside N' with respect to $\bar{u}$. We will rely on the following fact. For each α and each $i < n$

$$M^\alpha_{t_i} \underset{M^\alpha_{t_i} \cap M^\alpha_{t_i^-}}{\downarrow} N^\alpha_{i-1} \quad \text{inside } N^\alpha_i. \tag{3.1}$$

We must show that for each $i < n+1$,

$$\hat{M}^\alpha_{u_i} \underset{\hat{M}^\alpha_{u_i} \cap \hat{M}^\alpha_{u_i^-}}{\downarrow} \hat{N}^\alpha_{i-1} \quad \text{inside } \hat{N}^\alpha_i. \tag{3.2}$$

For $i \leq k$ this is an immediate translation of property (3.1) (with $\alpha = 0$). For $i = k+1$ property (3.2) translates to

$$M^\alpha_s \underset{M^\alpha_s \cap M^0_s}{\downarrow} N^0_k \quad \text{inside } N^\alpha_k \tag{3.3}$$

since $\hat{M}^\alpha_{u_{k+1}^-} = \hat{M}^\alpha_{u_k} = M^0_{t_k} = M^0_s$. Remember that N^0_k is prime over $N^0_{k-1} \cup M^0_s$ and since $N^0_{k-1} = N^\alpha_{k-1}$, N^α_k is prime over $N^0_{k-1} \cup M^\alpha_s$. Moreover, $M^\alpha_s \cap M^\alpha_{s^-} = M^\alpha_s \cap M^\alpha_{\langle\rangle} = M^\alpha_{\langle\rangle}$. Now property (3.3) follows by base extension from

$$M^\alpha_s \underset{M^\alpha_s \cap M^0_{\langle\rangle}}{\downarrow} N^\alpha_{k-1} \quad \text{inside } N^\alpha_k \tag{3.4}$$

which is an instance of property (3.1).

For $i = k+2$, since $u_{k+2}^- = t_{k+1}^- = t_k = s$ property (3.2) becomes

$$M^\alpha_{t_{k+1}} \underset{M^\alpha_{t_{k+1}} \cap M^0_s}{\downarrow} N^\alpha_k \quad \text{inside } N^\alpha_{k+1}. \tag{3.5}$$

Now by assumption (iii) of the lemma, $M^\alpha_{t_{k+1}} \downarrow_{M^\alpha_{t_{k+1}} \cap M^0_s} M^\alpha_s$ so we can deduce property (3.5) from property (3.1) and transitivity of independence (Theorem 2.2.3).

For $i > k+2$, property (3.2) translates to

$$M^\alpha_{t_{i-1}} \underset{M^\alpha_{t_{i-1}} \cap M^\alpha_{t_{i-1}^-}}{\downarrow} N^\alpha_{i-2} \quad \text{inside } N^\alpha_i \tag{3.6}$$

which follows immediately from property (3.1).

We will apply the following consequence of this lemma (with I_δ a proper ideal of **T**) in the main proof.

3.2.8. Corollary. *Suppose $\mathfrak{M}$ is a locally free loose tree in some N which is indexed by* **T** *and $\langle I_\alpha : \alpha \leq \delta \rangle$ is a continuous increasing sequence of ideals in* **T**. *Suppose further that there is a* **K**-*continuous sequence $\langle N_\alpha : \alpha \leq \delta \rangle$ such that for each $\alpha < \delta$, $\bigcup_{t \in I_\alpha} M_t \subseteq N_\alpha \leq N$ and $\mathfrak{M}_\alpha(N_\alpha/\langle\rangle)$ is a locally free loose tree inside N. Then $\mathfrak{M}_\delta(N_\delta/\langle\rangle)$ is a locally free loose tree inside N.*

Proof. Without loss of generality, (since we are trying to establish local freeness), $\mathbf{T} - I_\delta$ is finite. Let $\{t_i : i < k\}$ be the minimal elements of $\mathbf{T} - I_\delta$ and let $s_i \in I_\delta$ be

the predecessor of t_i in I_δ. Again, without loss of generality, we may assume all the s_i are in I_0. For each $\alpha \leq \delta$, we define a loose tree $\tilde{\mathfrak{M}}^\alpha$ (not following our previous conventions) as follows. $\tilde{\mathfrak{M}}^\alpha$ is indexed by $(\mathbf{T} - I_\delta) \cup \{s, \langle \ \rangle\}$ where s is interpolated above $\langle \ \rangle$ and below each member of $\mathbf{T} - I_\sigma$. $\bar{M}^\alpha_s = M_t$ if $t \in (\mathbf{T} - I_\delta)$, N_0 if $t = \langle \ \rangle$ and N_α if $t = s$. $\tilde{\mathfrak{M}}^\alpha$ is a free loose tree inside N. (Just enumerate $\langle \ \rangle$, s and then the rest using $\mathfrak{M}_\alpha(N_\alpha/\langle \ \rangle)$ is a locally free loose tree inside N.) By the choice of the t_i and s_i, and the local freeness of $\mathfrak{M}_\alpha(N_\alpha/\langle \ \rangle)$ inside N we verify the third asssumption of Lemma 3.2.7. In the present context (since $M_{s_i} \subseteq N_0$ and $M_{t_i} \cap M_{s_i} \leq N_0$) it translates as for each $i < k$, $M_{t_i} \downarrow_{M_{t_i} \cap N_0} N_\alpha$. Now by Lemma 3.2.7, $\tilde{\mathfrak{M}}^\delta$ is free. Applying the omission lemma to $\tilde{\mathfrak{M}}^\delta$ we conclude that $\mathfrak{M}_\delta(N_\delta/\langle \ \rangle)$ is a locally free loose tree inside N.

Now we continue with the proof of the main theorem (Theorem 3.2.6).

3.2.9. Proof of 3.2.6. We assume by induction that Condition 1 of the theorem holds for any tree with cardinality less than λ and Condition 2 of the theorem holds for any tree and any ideal of cardinality less than λ.

Fix a subtree $\mathbf{T}_1 \subseteq \mathbf{T}_0$ with $|\mathbf{T}_1| = \lambda$. We will describe a construction relative to $\mathbf{T}_1$. Then to verify each of Conditions 1 and 2 we use a different choice of $\mathbf{T}_1$.

As in 3.2.5, suppose $\mathbf{T}_1 = \bigcup_{\alpha < \lambda} I_\alpha$ where the I_α are an increasing continuous sequence of ideals in $\mathbf{T}_1$ and $|I_\alpha| < \lambda$. We define by induction on $\alpha < \lambda$, an increasing $\mathbf{K}$-continuous sequence $\langle N_\alpha : \alpha < \lambda \rangle$ of $\mathbf{K}$-submodels of N satisfying the following conditions.

  (i) If $\alpha = \beta + 1$ then N_α is prime for loose trees over $\mathfrak{M}_{\alpha,\beta}(N_\beta/\langle \ \rangle)$.
  (ii) $\mathfrak{M}_\alpha(N_\alpha/\langle \ \rangle)$ is locally free.

Recall that $\mathfrak{M}_{\alpha,\beta}$ denotes the quotient of $\mathfrak{M} \mid (I_\alpha)$ by I_β. Note that $\mathfrak{M}_\alpha$ is indexed by $(\mathbf{T}_0 - I_\alpha) \cup \{\langle \ \rangle\}$ (not $(\mathbf{T}_1 - I_\alpha) \cup \{\langle \ \rangle\}$). This is crucial for the verification of Condition 2.

There are three cases in the construction.

$\alpha = 0$. Condition (i) doesn't apply. Condition (ii) holds by the induction hypothesis applied to Condition 2 of the main theorem.

α *is a limit ordinal.* Let N_α be canonically prime over $\langle N_\beta : \beta < \alpha \rangle$. Condition (i) does not apply and condition (ii) is immediate from the last result: Corollary 3.2.8.

$\alpha = \gamma + 1$. Applying condition (ii) γ, $\mathfrak{M}_\gamma(N_\gamma/\langle \ \rangle)$ is locally free in N. So the subdiagram $\mathfrak{M}_{\alpha,\gamma}$ is locally free in N. So by induction applied to Condition 1 of the main theorem with I_α as I there is an N_α satisfying condition (i). That is, N_α is prime for loose trees over $\mathfrak{M}_{\alpha,\gamma}(N_\gamma/\langle \ \rangle)$. In particular, note N_γ is embedded into N_α. Applying the induction hypothesis for Condition 2 (with $\mathbf{T}_0 = \mathfrak{M}_{I_\alpha}$) $\mathfrak{M}_\alpha(N_\alpha/\langle \ \rangle)$ is locally free and we satisfy condition (ii).

This completes the construction.

To see that Condition 1 holds of $\mathbf{T}_0$, take $\mathbf{T}_1$ as $\mathbf{T}_0$. We must show that if N_λ is taken canonically prime over $\langle N_\beta : \beta < \lambda \rangle$ then N_λ satisfies LFP($\mathfrak{N}, N_\lambda$). Suppose $f = \langle f_t : t \in \mathbf{T} \rangle$ maps the loose tree $\mathfrak{M}$ isomorphically to a loose tree $\mathfrak{M}'$ which is locally free in some M'. We must extend f to an embedding of N_λ into M'.

For this, repeat the preceding argument constructing a sequence N'_α for $\alpha < \lambda$ inside M' which satisfy conditions (i) and (ii) (for $\mathfrak{M}'$) and simultaneously construct a continuous increasing sequence of maps g_α for $\alpha < \delta$ which map N_α isomorphically onto N'_α and such that g_α extends each f_t with $t \in I_\alpha$ We conclude Condition 1.

For Condition 2 we must show that if I is an ideal of $\mathbf{T}$ and $|I| = \lambda$ then $\mathfrak{M}_I$ is locally free. Applying the construction above with I as $\mathbf{T}_1$, we have constructed the N_α so that $\mathfrak{M}_\alpha(N_\alpha / \langle \ \rangle)$ is locally free. Thus, by Corollary 3.2.8 $\mathfrak{M}_\alpha(N_\delta \langle \ \rangle)$ is locally free and we can conclude Condition 2.

3.2.10. Conclusion. Let $\mathbf{K}$ be an adequate class and suppose $\mathfrak{M}$ is a loose tree of $\mathbf{K}$-models indexed by a subtree of $\lambda^{<\omega}$. If $\mathfrak{M}$ is locally free then there is a compatibility prime model M over $\mathfrak{M}$ (i.e. LFP($\mathfrak{M}, M, f$), see 3.2.4.)

In fact, if $\mathfrak{M}$ (indexed by a subtree of $\lambda^{<\omega}$) is free under one enumeration then it is free under any enumeration. For, by Proposition 3.2.3 it is locally free. Let $\bar{t} = \langle t_i : i < |\mathbf{T}| \rangle$ be an arbitrary enumeration of $\mathfrak{M}$. For each $\gamma < |\mathbf{T}|$ let I_γ be the ideal containing each t_i with $i < \gamma$. Now construct a family of models $\langle N_\gamma : \gamma < |\mathbf{T}| \rangle$ as in the proof of Theorem 3.2.6. By the basic properties of independence (using L1 at limit stages) these models witness that $\mathfrak{M}$ is free under the given enumeration.

We have shown that if $\mathfrak{M}$ is indexed by a subtree of $\lambda^{<\omega}$ and if $\mathfrak{M}$ is free with respect to some enumeration then there is a compatibility prime model over $\mathfrak{M}$. In [3] we will show that if $\mathbf{K}$ is at all manageable then there is only one compatibility class over $\mathfrak{M}$. We are attempting to show that models in manageable classes are 'tree-decomposable' by trees of small height. The existence of such prime models is an essential step in this program.

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