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A WEAK GENERALIZATION OF MA TO HIGHER CARDINALS

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ABSTRACT

We generalized MA e.g., to $\aleph_1$-complete forcing, by strengthening the $\aleph_2$-C.C. condition which occurs in many proofs. We show some consequences of MA generalized, and show that we get a model of ZFC in which the modadic theory of ω_2 is decidable.

§1. A weak Martin's axiom for uncountable cardinals

We prove here the consistency of a weak form of Martin's axiom generalized from $\aleph_1$ to $\aleph_2$, namely, the consistency of $2^{\aleph_0} = \aleph_1 + 2^{\aleph_1} > \aleph_2$ and the following:

(*) Let P be a set of conditions of cardinality less than $2^{\aleph_1}$ satisfying the following:

(a) if $p, q \in P$ are compatible then they have a least upper bound $p \cup q \in P$,

(b) if $p_0 \leqq p_1 \leqq \cdots$ is an increasing sequence of length ω then the least upper bound $\bigcup_{n<\omega} p_n$ is in P,

(c) if $p_i \in P$, $i < \omega_2$ then there is a closed unbounded set $C \subseteq \omega_2$ and a regressive function $f: \omega_2 \to \omega_2$ such that for $\alpha, \beta \in C$ if $cf(\alpha), cf(\beta) > \aleph_0$ and $f(\alpha) = f(\beta)$ then p_α and p_β are compatible.

Then letting $D_i \subseteq P$, $i < \lambda < 2^{\aleph_1}$ be dense subsets of P there is a filter $G \subseteq P$ which intersects every D_i, $i < \lambda$.

REMARK. (1) The main condition (c) is a strengthening of $\aleph_2$-C.C.

(2) In (c), instead of $f: \omega_2 \to \omega_2$ regressive we can ask $f: \omega_2 \to A$, $A = \bigcup_{i<\omega_2} A_i$, A_i increasing, and continuous (at least at ordinals of cofinality ω_1) and

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$f(i) \in A_i$, $|A_i| < \aleph_2$. Also we can wave in (c) the demand $cf(\alpha) = cf(\beta) = \omega_1$, if we have $\aleph_0$ functions f.

THEOREM 1.1. *Suppose C.H. and* $2^{<\kappa} \leqq \kappa$, κ *a regular cardinal, then there is a set of conditions* P *satisfying* (b), (c) *of* $(*)$ *such that in* V^P, *cofinality is preserved,* $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \kappa$ *and* $(*)$. *Moreover, if* $(\forall \mu < \kappa)\ \mu^{\aleph_0} < \kappa$ *we can wave in* $(*)$ *the demand* $|P| < 2^{\aleph_1}$ *(but not* $\lambda < 2^{\aleph_1}$*).*

PROOF. The proof is modelled after Solovay and Tennenbaum [4], i.e., we iterate (with extensions satisfying (a), (b), (c)) κ times, at limit stages of cofinality ω we take the inverse limit. (Notice that the set of forcing conditions we get does not satisfy (a).)

More specifically, we define a set of conditions P_α, $\alpha < \kappa$ such that:

(1) P_α satisfy (b), (c) and is of cardinality $\leqq \kappa$.

(2) For every α we choose a name $\mathbf{A}_\alpha$ such that in V^{P_α}, $\mathbf{A}_\alpha$ is a partially ordered set satisfying (a), (b), (c) of cardinality less than κ.

(3) The elements of P_α are the function with domain a countable subset of α's such that for $\xi \in \operatorname{Dom} f$, $\| f(\xi) \in A \|^{P_\xi} = 1$. The order is: $f \geqq g$ iff for any $\xi \in \operatorname{Dom} g$ also $\xi \in \operatorname{Dom} f$ and $f \restriction \xi \Vdash^{P_\xi} f(\xi) \geqq g(\xi)$.

(4) The definition of $\mathbf{A}_\alpha$ is done in such a way that eventually every possible set of conditions will be treated. (This is possible by Lemma 2.)

To prove the theorem it is enough to prove the following two lemmas.

LEMMA 1.2. *If* P *satisfies conditions* (b), (c) *and* Q *is a partially ordered set (in* V*) such that* $\emptyset \vDash^P$ "Q *satisfy* (c)", *then* Q *satisfy* (c).

PROOF. Let $\{q_i \mid i < \omega_2\} \subseteq Q$ (in V). As in V^P Q satisfy (c), we have in V^P a closed unbounded set $\mathbf{E} \subseteq \omega_2$ and an appropriate function $\mathbf{g}$. Because P satisfies the $\aleph_2$-C.C. we can find in V a closed unbounded subset of $\mathbf{E}$, call it E. Now for any $\alpha < \omega_2$ we find $p_\alpha \in P$ such that $p_\alpha \vDash$ "$\mathbf{g}(\alpha) = \zeta(\alpha)$" for some ordinal $\zeta(\alpha) < \alpha$. As P satisfies condition (c) we have a closed unbounded $F \subseteq \omega_2$ and a regressive h appropriate to $\{p_\alpha : \alpha < \omega_2\}$. Now there is a closed unbounded $C \subseteq E \cap F$ and a decreasing function f such that for $\alpha, \beta \in C$, $f(\alpha) = f(\beta)$ iff $\xi(\alpha) = \xi(\beta)$ and $h(\alpha) = h(\beta)$. C, f are as required for $\{q_i : i < \omega_2\}$.

LEMMA 1.3. P_δ *satisfy* (c).

PROOF. Let $f_i \in P_\delta$, $i < \omega_2$; we construct for each $i < \omega_2$ an increasing ω sequence $f_i^n \in P_\delta$, $n < \omega$. $f_i^0 = f_i$. Suppose for some $n < \omega$, $f_i^n : i < \omega_2$ are defined. Now for each $\xi < \delta$, $\{f_i^n(\xi) : i < \omega_2\}$ is a sequence of ω_2 elements of A_ξ (more exactly, $\emptyset \Vdash^{P_\xi}$ "$\{f_i^n(\xi): i < \omega_2\} \subseteq A_\xi$"). (Set $f_i^n(\xi) = \emptyset$ if $\xi \notin \operatorname{Dom} f_i^n$.) So by (c) we have in V^{P_ξ} a regressive function $g_\xi^n: \omega_2 \to \omega_2$ and a closed unbounded $C_\xi^n \subseteq \omega_2$ (which we can assume is in V as P_ξ has the ω_2-C.C.). We find $f_i^{n+1} \geqq f_i^n$ such that for any $\xi \in \operatorname{Dom} f_i^n$, $f_i^{n+1} \restriction \xi \Vdash^{P_\xi} g_\xi^n(i) = \alpha_\xi^n(i)$ for some $\alpha_\xi^n(i) < i$. (f_i^{n+1} is a limit of ω steps.)

Let $\alpha_\xi^n(i) = 0$ for $\xi \notin \operatorname{Dom} f_i^n$. Let $f_i^\omega = \bigcup_{n<\omega} f_i^n$, $C_\xi = \bigcap_{n<\omega} C_\xi^n$. Fix $\{\xi_\alpha \mid \alpha < \omega_2\}$ an enumeration of $\bigcup_{i<\omega_2} \operatorname{Dom} f_i^\omega$ and $C = \Delta_{\alpha<\omega_2} C_{\xi_\alpha} = \{i < \omega_2: (\forall \alpha < i)\, i \in C_{\xi_\alpha}\}$. We can find a closed unbounded $E \subseteq C$ and a regressive $g: \omega_2 \to \omega_2$ such that if $cf(i) = cf(j) = \omega_1$, $i, j \in E$ and $g(i) = g(j)$, $i < j$ then:

(1) $\operatorname{Dom} f_i^\omega \cap \{\xi_\gamma: \gamma < i\} = \operatorname{Dom} f_j^\omega \cap \{\xi_\gamma: \gamma < j\}$,

(2) $\operatorname{Dom} f_i^\omega \subseteq \{\xi_\gamma: \gamma < j\}$,

(3) $\{\langle \gamma, n, \alpha_{\xi_\gamma}^n(i)\rangle: n < \omega, \gamma < i\} = \{\langle \gamma, n, \alpha_{\xi_\gamma}^n(j)\rangle: n < \omega, \gamma < j\}$.

Now we will show that in this case f_i^ω and f_j^ω are compatible, indeed that h, $h(\xi) = f_i^\omega(\xi) \cup f_j^\omega(\xi)$ for $\xi \in \operatorname{Dom} g_i^\omega \cup \operatorname{Dom} g_j^\omega$ is above them.

By induction on $\xi \leqq \delta$ we show $h \restriction \xi \geqq f_i^\omega \restriction \xi$, $f_j^\omega \restriction \xi$ (and $h \restriction \xi$ is well defined), for limit and for $\zeta + 1 = \xi$ if $\zeta \notin \operatorname{Dom} f_j^\omega$ it is immediate.

If $\zeta \in \operatorname{Dom} f_i^\omega \cap \operatorname{Dom} f_j^\omega$ then $\zeta = \xi_\gamma$ for some $\gamma < i$ (by (1) and (2)) and $\alpha_{\xi_\gamma}^n(i) = \alpha_{\xi_\gamma}^n(j)$ and $i, j \in C_{\xi_\gamma}$. By construction for each n, $h \restriction \zeta \Vdash f_i^n(\zeta)$, $f_j^n(\zeta)$ are compatible, so $h \restriction \zeta \Vdash$ "$f_i^\omega(\zeta)$, $f_j^\omega(\zeta)$ are compatible" (a common upper bound is $\bigcup_n (f_i^n(\zeta) \cup f_j^n(\zeta)))$ hence $h \restriction (\zeta + 1) \geqq f_i^\omega \restriction (\zeta + 1)$, $f_j^\omega \restriction (\zeta + 1)$.

Claim 1.4. (1) If V satisfies $\diamondsuit_{\aleph_1}$ then V^P (from 1.1) satisfies $\diamondsuit_{\aleph_1}$ too.

(2) If $S \in V$ is a stationary subset of ω_1, it is stationary in V^P too.

(3) Every closed unbounded subset of ω_2 in V^P contains a closed unbounded subset from V.

Proof. (1), (2) This follows immediately by the $\aleph_1$-completeness of P.

(3) Easy as P satisfies the $\aleph_2$-chain condition.

Definition 1.1. $S_\beta^\alpha = \{\delta < \aleph_\alpha : f(\delta) = \aleph_\beta\}$.

Theorem 1.5. *Assume $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$. For $\delta \in S_1^2$ let η_δ be an unbounded sequence of ordinals below δ of order type ω_1, such that for every closed unbounded $C \subseteq \omega_2$, there is $\beta \in S_1^2$ such that for a stationary set of $\gamma < \beta$:*

(1) $\gamma \in C$,

(2) $[\gamma, \gamma^1) \cap \eta_\beta = \emptyset$ (where γ^1 is the successor of γ in C, and $[\gamma, \gamma^1)$ is the half-open interval).

Then there is a set of conditions P, $\aleph_1$-closed and satisfying the $\aleph_2$-C.C. such that in V^P the following hold: For every sequence of functions $f_\delta: \eta_\delta \to \omega_1$, $\delta \in S_1^2$ there *is F: $\omega_2 \to \omega_2$ such that F uniformize f_δ, i.e., for every $\delta \in S^2_1$ there is $\delta^1 < \delta$ such that $F \upharpoonright (\eta_\delta - \delta^1) = f_\delta \upharpoonright (\eta_\delta - \delta^1)$.*

REMARK. If $V = L$, S_δ is a stationary subset of δ for each δ, then we can find η_δ $(\delta \in S^2_1)$ as in the theorem, such that $S_\delta \cup \mathrm{Range}\,\eta_\delta$ contains a closed unbounded subset of δ.

PROOF. The forcing is done by the iteration. We describe at first the basic step of the iteration: Let $\bar{f} = \langle f_\delta : \in S^2_1 \rangle$ be a sequence of functions $f_\delta : \eta_\delta \to \omega_1$. $P_{\bar{f}}$ is the set of all countable functions p such that

(1) $\mathrm{Dom}\, p \subseteq S^2_1$,

(2) $p(\xi) < \xi$ for $\xi \in \mathrm{Dom}\, p$,

(3) If $\xi, \zeta \in \mathrm{Dom}\, p$ and $a \in \eta_\zeta$, $\max\{p(\xi), p(\zeta)\} < a$, then $f_\xi(a) = f_\zeta(a)$.

It is easy to see that $P_{\bar{f}}$ is $\aleph_1$-closed and satisfy $\aleph_2$-C.C. Now if G is V-generic over $P_{\bar{f}}$ then

$$F_{\bar{f}} = \bigcup_{\substack{p \in G \\ \xi \in \mathrm{Dom}\, p}} f_\xi \upharpoonright (\eta_\xi - p(\xi))$$

satisfies the claims of the theorem for $\bar{f}$.

Now we define by induction on $\alpha \leqq \omega_3$ sets of conditions P_α and $\bar{f}_\alpha$ names in P_α of sequences of functions $f_{\alpha,\delta} : \eta_\delta \to \omega_1$, $\delta \in S^2_1$ such that: P_0 is $P_{\bar{f}}$ for some $\bar{f}$ in V, the $\bar{f}_\alpha$ are chosen so that all possible $\bar{f}$ will appear in the $\bar{f}$ course of constructions. P_α is the set of all functions p such that:

(1) $\mathrm{Dom}\, p \subseteq \alpha$ is countable,

(2) for $\gamma \in \mathrm{Dom}\, p$, $p(\gamma)$ is a countable function (in V) $\mathrm{Dom}[p(\gamma)] \subseteq S^2_1$ and $p(\gamma)(\xi) < \xi$ for $\xi \in \mathrm{Dom}[p(\gamma)]$,

(3) for $\gamma \in \mathrm{Dom}\, p$, $p \upharpoonright \gamma \in P_\gamma$ and $p \upharpoonright \gamma \Vdash$ "$p(\gamma) \in P_{\bar{f}_\gamma}$".

The order in P_α: $p \leqq q$ iff $\mathrm{Dom}\, p \subseteq \mathrm{Dom}\, q$ and for $\gamma \in \mathrm{Dom}\, p$, $p(\gamma) \subseteq q(\gamma)$.

LEMMA 1.6. *P_α is $\aleph_1$-closed.*

PROOF. Let $p_n \in P_\alpha$, $n < \omega$, $p_n \leqq p_{n+1}$. Define $\mathrm{Dom}\, p_\omega = \bigcup_{n<\omega} \mathrm{Dom}\, p_n$ for $\gamma \in \bigcup_{n<\omega} \mathrm{Dom}\, p_n$ from some n_0 onward $\gamma \in \mathrm{Dom}\, p_n$, define $p_\omega(\gamma) = \bigcup_{n_0 \leqq n} p_n(\gamma)$. p_ω is easily seen to be the supremum of the p_n.

LEMMA 1.7. *For $\alpha \leqq \omega_3$, $\gamma < \alpha$ and $\xi \in S^2_1$ the following set is dense in P_α: $\{p \in P : \gamma \in \mathrm{Dom}(p)$ and $\xi \in \mathrm{Dom}\, p(\xi)\}$.*

PROOF. Easy.

LEMMA 1.8. *Let $\beta \in S^2_1$ and $q_0 \in P_\alpha$; then there is a closed unbounded $D \subseteq \beta$,*

$D = \{d_\varepsilon \mid \varepsilon < \omega_1\}$ and an increasing sequence $q_0 \leqq q_\varepsilon$, $q_\varepsilon \in P_\alpha$, $\varepsilon < \omega_1$ such that for $\varepsilon < \omega_1$ the following hold:

(1) *For $\gamma \in \mathrm{Dom}\, q_\varepsilon$, $\mathrm{Dom}[q_\varepsilon(\gamma)] \cap \beta \subseteq d_\varepsilon$.*

(2) *For $\gamma \in \mathrm{Dom}\, q_\varepsilon$ and $\delta \in \mathrm{Dom}[q_\varepsilon(\gamma)]$ if $\delta > \beta$ then $\eta_\delta \cap \beta \subseteq d_\varepsilon$.*

(3) *For $\gamma \in \mathrm{Dom}\, q_\varepsilon$ and $\delta \in \mathrm{Dom}\, q_\varepsilon(\gamma)$ if $\delta > \beta$ then for some $q \in V$, $q_\eta \restriction \gamma \Vdash^{P_\gamma}$ "$f_{\gamma,\delta} \restriction (\eta_\delta \cap \beta) = q$".*

(4) *For $\gamma \in \mathrm{Dom}\, q_\varepsilon$, $q_\varepsilon \restriction \gamma \Vdash^{P_\gamma}$ "$f_{\gamma,\beta} \restriction d_\varepsilon = h$" for some $h \in V$.*

PROOF. By Lemma 1.6. At limit stages we take the unions and we obtain $q_{\varepsilon+1}$ and $c_{\varepsilon+1}$ from q_ε and c_ε through ω-steps.

The final Lemma is the essence of the proof.

LEMMA 1.9. *For $\alpha \leqq \omega_3$, P_α satisfy the $\aleph_2$-C.C.*

PROOF. Let $\{p_\delta: \delta < \omega_2\}$ be $\aleph_2$ conditions in P_α. $H(\aleph_3)$ is the collection of all sets whose cardinality is hereditarily $\leqq \aleph_3$. We construct an increasing and continuous chain of models M_l, $l < \omega_2$ such that:

(1) M_l is of cardinality $\aleph_1$,

(2) $M_{l+1} <_{L_{\aleph_1,\aleph_1}} (H(\aleph_3), \{p_\delta : \delta < \omega_2\}, P_\alpha, \Vdash)$.

$C = \{M_l \cap \omega_2 = c_l : l < \omega_2\}$ is closed unbounded in ω_2. By the assumption of the theorem there is $\beta \in S^2_1 \cap C$ such that $S_\beta = \{\gamma < \beta : \gamma \in C \,\&\, [\gamma, \gamma^1) \cap \eta_\beta = \emptyset\}$ is stationary. (γ^1 is the successor of γ in C.)

By Lemma 1.8 for q_0, P_α, P_β we have a closed unbounded $D \subseteq \beta$ and an increasing sequence $q_\xi \in P_\alpha$, $\xi < \omega_1$ with the properties listed there and $q_\xi \in M_\beta$. Hence we can find $c_\gamma \in S_\beta \cap D \cap C^\xi$ as $c_\gamma \in D$, $c_\gamma = d_\varepsilon$ for some $\varepsilon < \omega_1$. Look at q_ε and $M_{\gamma+1}$, for $\gamma \in \mathrm{Dom}(q_\varepsilon) \cap M_{\gamma+1}$. Use (1)–(4) of Lemma 1.8 to describe the behaviour of q_ε below β by a $L_{\aleph_1,\aleph_1}$ sentence with parameters in $M_{\gamma+1}$. Take the conjunction of these descriptions and the sentence which says that some extensions of an element in $\{p_\delta: \delta < \omega_2\}$ satisfies the conjunction. Now as $M_{\gamma+1}$ is an elementary submodel for some $\beta' < c_{\gamma+1}$ and $q'_\varepsilon \geqq p_{\beta'}$, q'_ε, $p_{\beta'} \in M_{\gamma+1}$, q'_ε satisfy that $L_{\aleph_1,\aleph_1}$ sentence described above. Now we claim that q'_ε and q_ε are compatible and hence that p_β and $p_{\beta'}$ are compatible. Indeed, we prove by induction on ξ that $p_\beta \restriction \xi \vee p_{\beta'} \restriction \xi$ is a condition, for limit ξ or $\xi = \zeta + 1$ such that $\zeta \in (\alpha - \mathrm{Dom}\, p_{\beta'}) \cup (\alpha - \mathrm{Dom}\, p_\beta)$ it is immediate. If $\xi = \zeta + 1$ and $\zeta \in \mathrm{Dom}\, p_\beta \cap \mathrm{Dom}\, p_{\beta'}$ then $\zeta \in M_{\gamma+1}$ and so the behaviour of $q_\varepsilon(\zeta)$ below β was described and hence $q'_\varepsilon(\zeta)$ has the same behaviour below β'. Using also the facts that $\mathrm{Dom}\, q'_\varepsilon(\gamma) \subseteq c_{\gamma+1}$ and that $[c_\gamma, c_{\gamma+1}) \cap \eta_\beta \subseteq d_\varepsilon$ it follows that $q_\varepsilon \restriction \zeta \vee q' \restriction \zeta \Vdash$ "$(q_\varepsilon(\zeta) \cup q'_\varepsilon(\zeta))$ is a condition".

We can similarly prove:

THEOREM 1.10. *Assume $2^{\aleph_0}=\aleph_1$, $2^{\aleph_1}=\aleph_2$, and for each $\delta \in S^2_1$, S_δ is a stationary subset of δ.*

Then there is a set of conditions P, $\aleph_1$-closed and satisfying the $\aleph_2$-chain condition such that in V^P the following holds: for every sequence of functions $f_\delta: \delta \to \omega_1$ $(\delta \in S^2_1)$ there is $F: \omega_2 \to \omega_1$ such that for every $\delta \in S^2_1$, $S_\delta \cup \{i < \delta : F(i) = f_\delta(i)\}$ contains a closed unbounded subset of δ.

The proof of Theorem 1.10 gives more than the theorem stated. We give here an attempt for generalizing.

DEFINITION 1.2. (1) A five-tuple $B = \langle \kappa, \lambda, \chi, A, R\rangle$ will be called a proper basis if

(a) $\lambda > \chi$ are infinite regular cardinals,

(b) $A = \langle A_i : i < \lambda\rangle$, A_i sets,

(c) R is a five-place relation on $\bigcup_{i<\lambda} A_i \cup \kappa \cup \lambda$;

(2) B is good if whenever for each $\alpha < \lambda$, a function h_α is given, $|\mathrm{Dom}\, h_\alpha| < \chi$, $\mathrm{Dom}\, h_\alpha \subseteq \kappa$, $\mathrm{Range}\, h_\alpha \subseteq A_\alpha$, *then* some $\alpha < \beta$ are B-compatible, which means that $(\forall \gamma < \kappa)[\gamma \in \mathrm{Dom}\, h_\alpha \cap \mathrm{Dom}\, h_\beta \to R(\gamma, \alpha, \beta, h_\alpha(\gamma), h_\beta(\gamma))]$;

(3) a partial order P is B-good, if whenever sequence $\langle p_\alpha : \alpha < \lambda\rangle$ is given, $p_\alpha \in P$, we can find h_α $(\alpha < \lambda)$ as in (2) such that whenever $\alpha, \beta < \lambda$ are B-compatible p_α, p_β are compatible in P.

THEOREM 1.11. *Suppose B is a good proper basis, $\mu < \chi$ is regular, $\kappa \geqq \lambda$ is regular, $\kappa = 2^{<\kappa}$.*

Then there is a set of forcing conditions P, χ-closed, satisfying the λ-chain condition, $|P| = \kappa$ such that V^P satisfy:

$(*)$ If Q is a set of forcing conditions of cardinality $< \kappa$, $D_i \subseteq Q$ $(i < i_0 < \kappa)$ are dense subsets of P, then there is a filter $G \subseteq Q$ which intersect each D_i, provided that

(a) if $p, q \in Q$ are compatible then they have a least upper bound $p \cup q \in Q$,

(b) if p_i $(i < i_0 < \kappa)$ is an increasing sequence of elements of Q, then $\{p_i : i < i_0\}$ has an upper bound,

(c) Q is B-good.

REMARK. (1) In fact P satisfies (b) and (c) of $(*)$. Also the parallel of Claim 1.4 holds.

(2) Usually every Q of cardinality $< \lambda$ is B-good, so in V^P, λ is the successor of χ, and 2^λ is κ.

(3) We can amalgamate Theorem 1.11 with Theorem 1.5.

(4) The main case of Theorem 1.11 is $\lambda = \aleph_2$, $\chi = \aleph_1$, $\mu = \aleph_0$.

§2. Applications

In this section we assume $2^{\aleph_1} > \aleph_2$ and $(*)$ of Section 1 holds, or an appropriate strengthening.

Claim 2.1. Suppose $\boldsymbol{S} = \boldsymbol{S}_1 \cup \boldsymbol{S}_2$ is a family of $< 2^{\aleph_1}$ sets, each of cardinality $\aleph_1$, the intersection of any two is countable, and $\boldsymbol{S}_1 \cap \boldsymbol{S}_2 = \emptyset$, and

(a) $\boldsymbol{S}$ is a subset of $\{\{\eta \restriction \alpha : \alpha < \omega_1\} : \eta \in {}^{(\omega_1)}2\}$ or

(b) $\boldsymbol{S} \subseteq \{A_\alpha : \alpha < 2^{\aleph_1}, \ cf(\alpha) = \omega_1\}$, where $\cup\{\beta : \beta \in A_\alpha\} = \alpha$ and A_α has order-type ω_1 (but here we need Theorem 1.11).

Then for some set S

(1) $A \in \boldsymbol{S}_1$ implies $A - S$ is countable,

(2) $A \in \boldsymbol{S}_2$ implies $A \cap S$ is countable.

Proof. Let P be the family of countable sets $p = p_1 \cup p_2$, where p_l is a set of pairs (A, B), $A \in \boldsymbol{S}_l$, $B \subseteq A$, B countable, such that $D_1(p)$, $D_2(p)$ are disjoint, where $D_l(p) = \cup\{A - B : (A, B) \in p_l\}$.

$\boldsymbol{P}$ is ordered, of course, by inclusion. Condition (a) is obvious.

The rest is left to the reader.

Claim 2.2. In Claim 2.1, without (a) or (b) the conclusion may fail (see Luzin [1]).

Definition 2.2. A subset $A \subseteq {}^{(\omega_1)}2$ will be called "of the first category" if $A = \bigcup_{i<\omega_1} B_i$, B_i closed nowhere-dense subset of ${}^{(\omega_1)}2$ in the topology generated by countable intersections of open sets in the Tychonov topology.

Claim 2.3. The union of $\alpha < 2^{\aleph_1}$ subsets of ${}^{(\omega_1)}2$ which are of the first category, is of the first category.

Proof. So let $\alpha < 2^{\aleph_1}$, $B = \bigcup_{i<\alpha} B_i$, B_i of the first category, so w.l.o.g. each B_i is closed, nowhere-dense. Notice that the union of countably many nowhere-dense sets is nowhere-dense. A condition p is a countable set of atomic conditions which are of one of the two forms:

$B_j \subset X_i$ $(j < \alpha, i < \omega_1)$,

$A \cap X_i = \emptyset$ (A a basic clopen subset of ${}^{(\omega_1)}2$, $i < \omega_1$) such that $\{B_j \subset X_i, A \cap X_i = \emptyset\} \subseteq p \Rightarrow B_j \cap A = \emptyset$.

Theorem 2.4. *The monadic theory of ω_2 (as an ordered set) is decidable (using Theorem 1.11).*

Notation. $S^\alpha_\beta = \{i < \aleph_\alpha : cf(i) = \aleph_\beta\}$, and for a set S of ordinals $F(S) = \{\alpha < \sup S : S \cap \alpha$ is a stationary subset of $\alpha\}$.

PROOF. By [3] it suffices to prove the following three assertions:
$(*)$ If $A \subset S_0^2$, $F(A) = B \cup C$, B, C disjoint and, necessarily, $B, C \subseteq S_1^2$, *then* for some disjoint sets B_1, C_1, $A = B_1 \cup C_1$ and $F(B_1) = B$, $F(C_1) = C$.

For each $\delta \in F(A)$ choose an increasing and continuous sequence of ordinals η_δ of length ω_1 with limit δ, and let its range be A_δ. By Claim 2.1 there is a set $S \subseteq \omega_2$ such that $\delta \in B \Rightarrow |A_\delta \cap S| \leqq \aleph_0$ and $\delta \in C$, $|A_\delta - S| \leqq \aleph_0$. Now we choose $B_1 = A - S$, $C_1 = A \cap S$.
$(**)$ If $A \subseteq S_0^2$ is stationary then for some disjoint B_1, C_1, $A = B_1 \cup C_1$ and $F(B_1) = F(C_1) = F(A)$.

Unfortunately, we do not see how to prove this from Theorem 1.1 $(*)$, but V^P satisfied it. Because for some $\alpha, A \in V^{P_\alpha}$, and we can assume A_α from §1 is just the addition of a new subset of ω_2, i.e., the conditions are countable functions f, $\operatorname{Dom} f \subseteq \omega_2$, $\operatorname{Range} f \subseteq \{0, 1\}$. Remember that as our forcings are $\aleph_1$-complete, stationary subsets of ω_1 remain stationary.

Similarly we can prove:
$(***)$ If $A \subseteq S_0^2$ is stationary, then for some stationary $B \subseteq A$, $F(B) = 0$, and also $A - B$ is stationary.

This time we choose for each $\delta \in S_1^2$ a closed unbounded set $A_\delta \subseteq \delta$ of order type ω_1. The conditions will be countable sets whose elements have the form $\alpha \in X$, $A_\delta \cap X \subseteq \beta$ (for $\beta < \delta$, $\delta \in S_1^2$).

Now we give an application (where we get a similar universe, i.e., $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} > \aleph_2$, $\diamondsuit_{\aleph_1}$ hold).

CLAIM 2.5. There is an abelian group $G = \bigcup_{i<\omega_2} G_i$, $|G_i| \leqq \aleph_1$, G_i is free, and G/G_i is $\aleph_2$-free iff $cf(i) \neq \aleph_1$, hence G is not free, but G is still a Whitehead group.

PROOF. Let G^1 be the abelian group generated freely by $\{\zeta_i^\alpha : \alpha < \omega_2, i < \omega_1\} \cup \{y^\alpha : \alpha < \omega_2\}$. For each limit $\delta < \omega_1$ and $\alpha < \omega_2$, choose an increasing ω-sequence ν_δ^α of ordinals whose limit is δ. We get G^1 from G by adding to G^1, for each δ, $\alpha \in S_1^2$, $\delta < \omega_1$, δ limit

$$\left(\zeta_\delta^\alpha - \sum_{l \leqq n} 2^l \zeta^\alpha_{\nu_\delta^\alpha(l)} - \sum_{l \leqq n} 2^l y^{\eta_\alpha(\delta\omega + l)} \right) \Big/ 2^n.$$

CLAIM 2.6. If $\boldsymbol{S} = \boldsymbol{S}_1 \cup \boldsymbol{S}_2$ is a family of sets, each of cardinality $\aleph_1$, and $A_l \in \boldsymbol{S}_l \Rightarrow |A_1 \cap A_2| < \aleph_1$ *then* there is a set A^* such that

$$\boldsymbol{S}_1 = \{A \in \boldsymbol{S} : |A \cap A^*| \leqq \aleph_0\}.$$

PROOF. Left to the reader.

Added in proof, March 1978. Here are some historical and mathematical remarks.

(1) Baumgartner has proved the consistency of a quite similar statement (see comparison below) but has not published it, as Laver had previously a similar thing which he also did not pursue. His conditions on P are

(a)′ if $p, q \in P$ are compatible, then they have a least upper bound,

(b)′ any increasing ω-chain has an upper bound,

(c)′ $P = \bigcup_{i<\omega_1} P_i$, every two members of P_i are compatible.

Notice (c)′ is stronger, and (b)′ is weaker; however, the last difference is not essential (see next remark).

(2) If P satisfies (b)′ let

$$Q = \{A : A \text{ a countable directed subset of } P\},$$

$$A \leqq_Q B =^{\text{def}} (\forall a \in A)\ (\exists b \in B)\ (a \leqq b).$$

Now P has a natural embedding to $Q(a \mapsto \{a\})$ preserving compatibility, if P satisfies (a) [(c)], then Q satisfies (a) [(c)], and Q satisfies (b).

(3) In $(*)$ we can omit (a) if we replace (c) by (c)″, which is as (c) when we replace the conclusion by "then P_α, P_β have a least upper bound".

The advantage of this is that all the paper does not change, but the forcing of Theorem 1.1 satisfies this version of $(*)$.

(4) If P is a partial order satisfying (c) of $(*)$, and C.H. holds, then $Q = \{h : h$ a countable function from P to ω_1, $f^{-1}(\alpha)$ a directed set for $\alpha \in \text{Dom}\, h\}$ ordered by inclusion, satisfies (a), (b), (c). Hence if $(*)$ holds, P is the union of $\aleph_1$ directed sets. So if P is a Boolean algebra $P - \{0\}$ is the union of $\aleph_1$ filters. This, and the next remark is proved in theorem 4.13 in S. Shelah, *Simple unstable theories*, preprint.

(5) If C.H. holds, P a partial order satisfying the countable chain condition, then it satisfies (c) from $(*)$.

(6) The conclusion of Theorems 1.5 and 2.4 are consistent with G.C.H. Also a related theorem (essentially that we can deal with any stationary subset of ω_2 with no stationary initial segment) will appear in S. Shelah, *Remarks on λ-collectionwise Hausdorf spaces*, Topology Proc., accepted.

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