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Large continuum, oracles

Research Article

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Abstract: Our main theorem is about iterated forcing for making the continuum larger than $\aleph_2$. We present a generalization of [2] which deal with oracles for random, (also for other cases and generalities), by replacing $\aleph_1, \aleph_2$ by λ, λ^+ (starting with $\lambda = \lambda^{<\lambda} > \aleph_1$). Well, we demand absolute c.c.c. So we get, e.g. the continuum is λ^+ but we can get cov(meagre) $= \lambda$ and we give some applications. As in non-Cohen oracles [2], it is a "partial" countable support iteration but it is c.c.c.

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0. Introduction

Starting, e.g. with $\mathbf{V} \models$ G.C.H. and $\lambda = \lambda^{<\lambda} > \aleph_1$, we construct a forcing notion $\mathbb{P}$ of cardinality λ^+, by a partial of CS iteration but the result is a c.c.c. forcing.
The general iteration theorems (treated in section 1) seem generally suitable for constructing universes with $\mathrm{MA}_{<\lambda} + 2^{\aleph_0} = \lambda^+$, and taking more care, we should be able to get universes without $\mathrm{MA}_{<\lambda}$, see Discussion 0.1 below.
Our method is to immitate [2]; concerning the differences, some are inessential: using games not using diamonds in the framework itself, (inessential means that we could have in [2] immitate the choice here and vice versa).
An essential difference is that we deal here with large continuum – λ^+; we concentrate on the case we shall (in $\mathbf{V}^{\mathbb{P}}$) have $\mathrm{MA}_{<\lambda}$ but e.g. non(null) $= \lambda$ and $\mathfrak{b} = \lambda^+$ (or $\mathfrak{b} = \lambda$).
It seems to us that generally:

Thesis 0.1.
The iteration theorem here is enough to get results parallel to known results with $2^{\aleph_0} = \aleph_2$ replacing $\aleph_1, \aleph_2$ by λ, λ^+.

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To test this thesis we have asked Bartoszyński to suggest test problems for this method and he suggests:

Problem 0.1.
Prove the consistency of each of the

(A) $\aleph_1 < \lambda < 2^{\aleph_0}$ and the λ-Borel conjecture, i.e. $A \subseteq {}^\omega 2$ is of strong measure zero iff $|A| < \lambda$

(B) $\aleph_1 <$ non(null) $< 2^{\aleph_0}$, see Theorem 5.1

(C) $\aleph_1 < \mathfrak{b} = \lambda < 2^{\aleph_0}$ the dual λ-Borel conjecture (i.e. $A \subseteq {}^\omega 2$ is strongly meagre iff $|A| < \lambda$)

(D) $\aleph_1 < \mathfrak{b} = \lambda < 2^{\aleph_0}+$ the dual $2^{\aleph_0}$-Borel conjecture

(E) combine (A) and (C) and/or combine (A) and (D).

Parallely Steprans suggests:

Problem 0.2.
1) Is there a set $A \subseteq {}^\omega 2$ of cardinality $\aleph_2$ of p-Hausdorff measure > 0, but for every set of size $\aleph_2$ is null (for the Lebesgue measure)?
2) The (basic product) I think $\mathfrak{b} = \mathfrak{d} \vee \mathfrak{d} = 2^{\aleph_0}$ gives an answer, what about cov(meagre) $= \lambda < 2^{\aleph_0}$?

We shall deal with the iteration in section 1, give an application to a problem from [3] in section 2 (and 3, 4).
Lastly, in section 5 we deal with Bartoszyński's test problem (B), in fact, we get quite general such results.
It is natural to ask

Discussion 0.1.
1) In section 1, we may wonder if we can give "reasonable" sufficient condition for $\mathfrak{b} = \aleph_1$ or $\mathfrak{b} = \kappa < \lambda$? The answer is yes. It is natural to assume that we have in $\mathbf{V}$ a $<_{J^{bd}_\omega}$-increasing sequence $\bar{f} = \langle f_\alpha : \alpha < \kappa\rangle$ of functions from ${}^\omega\omega$ with no $<^*_{J^{bd}_\omega}$-upper bound and we would like to preserve this property of $\bar{f}$, i.e. in section 1 we

(a) restrict ourselves to $\mathbf{p} \in K^1_\lambda$ such that $\Vdash_{\mathbb{P}_\mathbf{p}}$ "$\bar{f}$ as above".

More formally redefine K^1_λ such that

(b) replace "$\mathbb{P}$ is absolute c.c.c." by "$\mathbb{P}$ is c.c.c., preserve $\bar{f}$ as above and if $\mathbb{Q}$ satisfies those two conditions then also $\mathbb{P} \times \mathbb{Q}$ satisfies those two conditions.

This has similar closure properties, that is, the proofs do not really change.
2) More generally consider K, a property of forcing notions such that:

(a) $\mathbb{P} \in K \Rightarrow \mathbb{P}$ is c.c.c.

(b) K is closed under $\lessdot$-increasing continuous unions

(c) K is closed under composition

(d) we replace in §1 "$\mathbf{p} \in K^1_\lambda$" by "$\mathbf{p} \in K$ has cardinality $< \lambda$"

(e) we replace in section 1, "$\mathbb{P}$ is absolutely c.c.c." by "$\mathbb{P} \in K$ and $\mathbb{R} \in K \Rightarrow \mathbb{P} \times \mathbb{R} \in K$".

3) What about using $\mathcal{P}(n)$-amalgamation of forcing notions? If we fix n this seems a natural way to get non-equality for many n-tuples of cardinal invariants; hopefully we shall return to this sometime.
4) What about forcing by the set of approximations $\mathbf{k}$? See Definition 1.5.

Definition 0.1.

1) We say a forcing notion $\mathbb{P}$ is absolutely c.c.c. when for every c.c.c. forcing notion $\mathbb{Q}$ we have $\Vdash_{\mathbb{Q}}$ "$\mathbb{P}$ is c.c.c."
2) We say $\mathbb{P}_2$ is absolutely c.c.c. over $\mathbb{P}_1$ when ($\mathbb{P}_1 \lessdot \mathbb{P}_2$ and) $\mathbb{P}_2/\mathbb{P}_1$ is absolutely c.c.c.
3) Let $\mathbb{P}_1 \subseteq_{\mathrm{ic}} \mathbb{P}_2$ means that $\mathbb{P}_1 \subseteq \mathbb{P}_2$ (as quasi orders) and if $p, q \in \mathbb{P}_1$ are incompatible in $\mathbb{P}_1$ then they are incompatible in $\mathbb{P}_2$ (the inverse holds too) .
The following tries to describe the iteration theorem, this may be more useful to the reader after having a first reading of section 1.
We treat λ as the vertical direction and λ^+ as the horizontal direction, the meaning will be clarified in section 2; our forcing is the increasing union of $\langle \mathbb{P}^{\mathbf{k}_\varepsilon} : \varepsilon < \lambda^+ \rangle$ where $\mathbf{k}_\varepsilon \in K_2$ (so $\mathbf{k}_\varepsilon$ gives an iteration $\langle \mathbb{P}_\alpha[\mathbf{k}_\varepsilon] : \alpha < \lambda \rangle$, i.e. a $\lessdot$-increasing continuous sequence of c.c.c. forcing notions) and for each such $\mathbf{k}_\varepsilon$ each iterand $\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}_\varepsilon]}$ is of cardinality $< \lambda$ and for each $\varepsilon < \lambda^+$ the forcing notion $\mathbb{P}^{\mathbf{k}_\varepsilon}$ is the union of increasing union of continuous sequence $\langle \mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}_\varepsilon]} : \alpha < \lambda \rangle$. So we can say that $\mathbb{P}^{\mathbf{k}_\varepsilon}$ is the limit of an FS iteration of length λ, each iterand of cardinality $< \lambda$ and for $\zeta \in (\varepsilon, \lambda^+), \mathbf{k}_\zeta$ gives a "fatter" iteration, which for "most" $\delta \in S (\subseteq \lambda)$, is a reasonable extension.

Question 0.1.

Can we get something interesting for the continuum $> \lambda^+$ and/or get cov(meagre) $< \lambda$? This certainly involves some losses! We intend to try elsewhere.

Definition 0.2.

1) For a set x let otrcl(x), the transitive closure over the ordinals of x, be the minimal set y such that $x \in y \wedge (\forall t \in y)(t \notin \text{ Ord } \rightarrow t \subseteq y)$.
2) For a set u of ordinals let $\mathcal{H}_{<\kappa}(u)$ be the set of x such that otrcl(x) is a subset of u of cardinality $< \kappa$.

Remark 0.1.

0) We use $\mathcal{H}_{<\kappa}(u)$ (in Definition 1.1) just for bookkeeping convenience.
1) It is natural to have Ord, the class of ordinals, a class of urelements.
2) If $\omega_1 \subseteq u$ for $\mathcal{H}_{<\aleph_1}(u)$ it makes no difference, but if $\omega_1 \nsubseteq u$ and $\beta = \min(\omega_1 \backslash u)$ then β is a countable subset of u but $\notin \mathcal{H}_{<\aleph_1}(u)$. Also we use $\mathcal{H}_{<\aleph_0}(u)$ where $\omega \subseteq u$, so there are no problems.

1. The iteration theorem

If we use the construction for $\lambda = \aleph_1$, the version we get is closer to, but not the same as [2] with the forcing being locally Cohen.
Here there are "atomic" forcings used below coming from three sources:

(a) the forcing given by the winning strategies $\mathbf{s}_\delta$(see below), i.e. the quotient

(b) forcing notions intended to generate $\mathrm{MA}_{<\lambda}$
[see Claim 1.6; we are given $\mathbf{k}_1 \in K_f^2$, an approximation of size λ, see Definition 1.5, and a $\mathbb{P}_{\mathbf{k}_1}$-name $\underset{\sim}{\mathbb{Q}}$ of a c.c.c. forcing and sequence $\langle \underset{\sim}{\mathcal{I}}_i : i < i(*) \rangle$ of $< \lambda$ dense subsets of $\mathbb{Q}$. We would like to find $\mathbf{k}_2 \in K_2$ satisfying $\mathbf{k}_1 \leq_{K_f^2} \mathbf{k}_2$ such that $\Vdash_{\mathbb{P}_{\mathbf{k}_2}}$ "there is a directed $G \subseteq \underset{\sim}{\mathbb{Q}}$ not disjoint to any $\underset{\sim}{\mathcal{I}}_i (i < i(*))$". We <u>do not</u> use composition, only $\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}_2]} = \mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}_1]} * \underset{\sim}{\mathbb{Q}}$ for some $\alpha \in E_{\mathbf{k}_1} \cap E_{\mathbf{k}_2}$]]

(c) given $\mathbf{k}_1 \in K_f^2$, and $\underset{\sim}{\mathbb{Q}}$ which is a $\mathbb{P}_{\mathbf{k}_1}$-name of a suitable c.c.c. forcing of cardinality λ can we find $\mathbf{k}_2$ such that $\mathbf{k}_1 \leq_{K_f^2} \mathbf{k}_2$ and in $\mathbf{V}$ we have $\Vdash_{\mathbb{P}[\mathbf{k}_2]}$ "there is a subset of $\underset{\sim}{\mathbb{Q}}$ generic over $\mathbf{V}[\underset{\sim}{G} \cap \mathbb{P}_{\mathbf{k}_1}]$".

Let us describe the roles of some of the definitions. We shall construct (in the main case) a forcing notion of cardinality λ^+ by approximations $\mathbf{k} \in K_f^2$ of size (= cardinality) λ, see Definition 1.5, which are constructed by approximations $\mathbf{p} \in K_1$ of cardinality $< \lambda$, see Definition 1.1.
Now $\mathbf{p} \in K_1$ is essentially a forcing notion of cardinality $< \lambda$, i.e. $\mathbb{P}_{\mathbf{p}} = (P_{\mathbf{p}}, \leq_{\mathbf{p}})$, and we add the set $u = u_{\mathbf{p}}$ to help the bookkeeping, so (in the main case) $u_{\mathbf{p}} \in [\lambda^+]^{<\lambda}$. For the bookkeeping we let $P_{\mathbf{p}} \subseteq \mathcal{H}_{<\aleph_1}(u_{\mathbf{p}})$, see Definition 0.2(2).
More specifically $\mathbf{k}$ (from Definition 1.5) is mainly a $\lessdot$-increasing continuous sequence $\bar{\mathbf{p}} = \langle \mathbf{p}_\alpha : \alpha \in E_{\mathbf{k}} \rangle = \langle \mathbf{p}_\alpha[\mathbf{k}] : \alpha \in E_{\mathbf{k}} \rangle$, where $E_{\mathbf{k}}$ is a club of λ. Hence $\mathbf{k}$ represents the forcing notion $\mathbb{P}_{\mathbf{k}} = \cup\{(P_{\mathbf{p}_\alpha}, \leq_{\mathbf{p}_\alpha}) : \alpha < \lambda\}$; the union of a $\lessdot$-increasing continuous sequence of forcing notions $\mathbb{P}_{\mathbf{p}_\alpha} = \mathbb{P}[\mathbf{p}_\alpha] = (P_{\mathbf{p}_\alpha}, \leq_{\mathbf{p}_\alpha})$, so we can look at $\mathbb{P}_{\mathbf{k}}$ as a FS-iteration.

But then we would like to construct say an "immediate successor" $\mathbf{k}^+$ of $\mathbf{k}$, so in particular $\mathbb{P}_{\mathbf{k}} \lessdot \mathbb{P}_{\mathbf{k}^+}$, e.g. taking care of (b) above so $\underset{\sim}{\mathbb{Q}}$ is a $\mathbb{P}_{\mathbf{k}}$-name and even a $\mathbb{P}_{\min(E_{\mathbf{k}})}$-name of a c.c.c. forcing notion. Toward this we choose $\mathbf{p}_\alpha^{\mathbf{k}^+} = p_\alpha[\mathbf{k}^+]$ by induction on $\alpha \in E_{\mathbf{k}}$. So it makes sense to demand $\mathbf{p}_\alpha \le_{K_1} \mathbf{p}_\alpha[\mathbf{k}^+]$, which naturally implies that $u[\mathbf{p}_\alpha] \subseteq u[\mathbf{p}_\alpha^{\mathbf{k}^+}], \mathbb{P}_{\mathbf{p}_\alpha} \lessdot \mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}^+]}$. So as $\mathbf{p}_\alpha[\mathbf{k}^+]$ for $\alpha \in E_{\mathbf{k}}$ is $\le_{K_1}$-increasing continuous, the main case is when $\beta = \min(E_{\mathbf{k}} \backslash (\alpha+1)$, can we choose $\mathbf{p}_\beta[\mathbf{k}^+]$?

Let us try to draw the picture:

$$\begin{tikzcd}
\mathbb{P}_{\mathbf{p}_\beta[\mathbf{k}]} \arrow[r, dashed] & ? \\
\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}]} \arrow[u] \arrow[r, "\lessdot"] & \mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}^+]} \arrow[u]
\end{tikzcd}$$

So we have three forcing notions, $\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}]}, \mathbb{P}_{\mathbf{p}_\beta[\mathbf{k}]}, \mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}^+]}$, where the second and third are $\lessdot$-extensions of the first. The main problem is the c.c.c. As in the main case we like to have $\mathrm{MA}_{<\lambda}$, there is no restriction on $\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}^+]}/\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}]}$, so it is natural to demand "$\mathbb{P}_{\mathbf{p}_\beta[\mathbf{k}]}/\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}]}$ is absolutely c.c.c. for $\alpha < \beta$ from $E_{\mathbf{k}}$" (recall $\mathbf{p}_\alpha[\mathbf{k}]$ is demanded to be $<^+_{K_1}$-increasing).
How do we amalgamate? There are two natural ways which say that "we leave $\mathbb{P}_{\mathbf{p}_\beta[\mathbf{k}]}/\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}]}$ as it is".

First way: We decide that $\mathbb{P}_{\mathbf{p}_\beta[\mathbf{k}^+]}$ is $\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}]} * ((\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}^+]}/\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}]}) \times (\mathbb{P}_{\mathbf{p}_\beta[\mathbf{k}]}/\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}]}))$.
[This is the "do nothing" case, the lazy man strategy, which in glorified fashion we may say: do nothing when in doubt. Note that $\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}^+]}/\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}]}$ and $\mathbb{P}_{\mathbf{p}_\beta[\mathbf{k}]}/\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}]}$ are $\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}]}$-names of forcing notions.]

Second way: $\mathbb{P}_{\mathbf{p}_\beta[\mathbf{k}]}/\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}]}$ is defined in some way, e.g. is a random real forcing in the universe $\mathbf{V}[\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}]}]$ and we decide that $\mathbb{P}_{\mathbf{p}_\beta[\mathbf{k}^+]}/\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}^+]}$ is defined in the same way: the random real forcing in the universe $\mathbf{V}[\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}^+]}]$; this is expressed by the strategy $\mathbf{s}_\alpha$.
[That is: retain the same definition of the forcing in the α-th place, so in some sense we again do nothing novel.]

Context 1.1.

Let $\lambda = cf(\lambda) > \aleph_1$ or just[1] $\lambda = cf(\lambda) \ge \aleph_1$.

Below, $\le^+_{K_1}$ is used in defining $\mathbf{k} \in K^2_f$ as consisting also of $\le^+_{K_1}$-increasing continuous sequence $\langle \mathbf{p}_\alpha : \alpha \in E \subseteq \lambda \rangle$ (so increasing vertically).

Definition 1.1.

1) Let K_1 be the class of $\mathbf{p}$ such that:

(a) $\mathbf{p} = (u, P, \le) = (u_{\mathbf{p}}, P_{\mathbf{p}}, \le_{\mathbf{p}}) = (u_{\mathbf{p}}, \mathbb{P}_{\mathbf{p}})$

(b) $\omega \subseteq u \subseteq \mathrm{Ord}$,

(c) P is a set $\subseteq \mathcal{H}_{<\aleph_1}(u)$,

(d) $\le$ is a quasi-order on P,

satisfying

(e) the pair $(P, \le)$ which we denote also by $\mathbb{P} = \mathbb{P}_{\mathbf{p}}$ is a c.c.c. forcing notion.

1A) We may write $u[\mathbf{p}], P[\mathbf{p}], \mathbb{P}[\mathbf{p}]$.
2) $\le_{K_1}$ is the following two-place relation on $K_1 : \mathbf{p} \le_{K_1} \mathbf{q}$ iff $u_{\mathbf{p}} \subseteq u_{\mathbf{q}}$ and $\mathbb{P}_{\mathbf{p}} \lessdot \mathbb{P}_{\mathbf{q}}$ and $\mathbb{P}_{\mathbf{q}} \cap \mathcal{H}_{<\aleph_1}(u_{\mathbf{p}}) = \mathbb{P}_{\mathbf{p}}$; moreover, just for transparency $q \le_{\mathbb{P}[\mathbf{q}]} p \in \mathbb{P}_{\mathbf{p}} \Rightarrow q \in \mathbb{P}_{\mathbf{p}}$.
3) $\le^+_{K_1}$ is the following two-place relation on $K_1 : \mathbf{p} \le^+_{K_1} \mathbf{q}$ iff $\mathbf{p} \le_{K_1} \mathbf{q}$ and $\mathbb{P}_{\mathbf{q}}/\mathbb{P}_{\mathbf{p}}$ is absolutely c.c.c., see Definition 0.1(1).
4) K^1_λ is the family of $\mathbf{p} \in K_1$ such that $u_{\mathbf{p}} \subseteq \lambda^+$ and $|u_{\mathbf{p}}| < \lambda$.

[1] *if $\lambda = \aleph_1$, we can change the definitions of $\mathbf{k} \in K_2$, instead $\langle \mathbb{P}_\alpha[\mathbf{k}] : \alpha < \lambda \rangle$ is $\lessdot$-increasing, we carry with us large enough family of dense subsets, e.g. coming from some countable N.*

5) We say $\mathbf{p}$ is the exact limit of $\langle \mathbf{p}_\alpha : \alpha \in v\rangle, v \subseteq$ Ord, in symbols $\mathbf{p} = \cup\{\mathbf{p}_\alpha : \alpha \in v\}$ when $u_{\mathbf{p}} = \cup\{u_{\mathbf{p}_\alpha} : \alpha \in v\}$ or the union $\mathbb{P}_{\mathbf{p}} = \cup\{\mathbb{P}_{\mathbf{p}_\alpha} : \alpha \in v\}$ and $\alpha \in v \Rightarrow \mathbf{p}_\alpha \le_{K_1} \mathbf{p}$; hence $\mathbf{p} \in K_1$.
6) We say $\mathbf{p}$ is just a limit of $\langle \mathbf{p}_\alpha : \alpha \in v\rangle$ when $u_{\mathbf{p}}$ is $\cup\{u_{\mathbf{p}_\alpha} : \alpha \in v\}, \mathbb{P}_{\mathbf{p}} \supseteq \cup\{\mathbb{P}_{\mathbf{p}_\alpha} : \alpha \in v\}$ and $\alpha \in v \Rightarrow \mathbf{p}_\alpha \le_{K_1} \mathbf{p}$.
7) We say $\bar{\mathbf{p}} = \langle \mathbf{p}_\alpha : \alpha < \alpha^*\rangle$ is $\le_{K_1}$-increasing continuous [strictly $\le_{K_1}$-increasing continuous] when it is $\le_{K_1}$-increasing and for every limit $\alpha < \alpha^*, \mathbf{p}_\alpha$ is a limit of $\bar{\mathbf{p}} \restriction \alpha$ [is the exact limit of $\bar{\mathbf{p}} \restriction \alpha$], respectively.

Observation 1.1.

1) $\le_{K_1}$ is a partial order on K_1.
2) $\le^+_{K_1} \subseteq \le_{K_1}$ is a partial order on K_1.
3) If $\bar{\mathbf{p}} = \langle \mathbf{p}_\alpha : \alpha < \delta\rangle$ is a $\le_{K_1}$-increasing sequence and $\cup\{\mathbb{P}_{\mathbf{p}_\alpha} : \alpha < \delta\}$ satisfies the c.c.c. and $\delta < \lambda$ then some $\mathbf{p} \in K_1$ is the union $\cup\{\mathbf{p}_\alpha : \alpha < \delta\}$ of $\bar{\mathbf{p}}$, i.e. $\cup\bar{\mathbf{p}} \in K_1$ and $\alpha < \delta \Rightarrow \mathbf{p}_\alpha \le_{K_1} \mathbf{p}$; this determines $\mathbf{p}$ uniquely and $\mathbf{p}$ is the exact union of $\bar{\mathbf{p}}$.
4) If $\bar{\mathbf{p}} = \langle \mathbf{p}_\alpha : \alpha < \delta\rangle$ is $\le_{K_1}$-increasing and $\mathrm{cf}(\delta) = \aleph_1$ implies $\{\alpha < \delta : \mathbf{p}_\alpha$ the exact limit of $\bar{\mathbf{p}} \restriction \alpha$ or just $\bigcup\limits_{\beta<\alpha} \mathbb{P}_{\mathbf{p}_\beta} \lessdot \mathbb{P}_{\mathbf{p}_\alpha}\}$ is a stationary subset of δ then $\cup\bar{\mathbf{p}} \in K_1$ is a $\le_{K_1}$-upper bound of $\bar{\mathbf{p}}$ and is the exact limit of $\bar{\mathbf{p}}$.
5) If in part (4), $\bar{\mathbf{p}}$ is also $\le^+_{K_1}$-increasing then $\alpha < \delta \Rightarrow \mathbf{p}_\alpha \le^+_{K_1} \mathbf{p}$.

Proof. Should be clear, e.g. in part (5) recall that c.c.c. forcing preserve stationarity of subsets of δ. $\square$

We now define the partial order $\le^*_{K_1}$; it will be used in describing $\mathbf{k}_1 <_{K_2} \mathbf{k}_2$, i.e. demanding $(\mathbf{p}^{\mathbf{k}_1}_\alpha, \mathbf{p}^{\mathbf{k}_2}_\alpha) \le^*_{K_1} (\mathbf{p}^{\mathbf{k}_1}_{\alpha+1}, \mathbf{p}^{\mathbf{k}_2}_{\alpha+1})$ for many $\alpha < \lambda$.

Definition 1.2.

1) Let $\le^*_{K_1}$ be the following two-place relation on the family of pairs $\{(\mathbf{p}, \mathbf{q}) : \mathbf{p} \le_{K_1} \mathbf{q}\}$. We let $(\mathbf{p}_1, \mathbf{q}_1) \le^*_{K_1} (\mathbf{p}_2, \mathbf{q}_2)$ iff

(a) $\mathbf{p}_1 \le^+_{K_1} \mathbf{p}_2$

(b) $\mathbf{q}_1 \le^+_{K_1} \mathbf{q}_2$

(c) $\Vdash_{\mathbb{P}[\mathbf{p}_2]}$ "$\mathbb{P}_{\mathbf{q}_1}/(\underset{\sim}{G}_{\mathbb{P}[\mathbf{p}_2]} \cap \mathbb{P}_{\mathbf{p}_1}) \lessdot \mathbb{P}_{\mathbf{q}_2}/\underset{\sim}{G}_{\mathbb{P}[\mathbf{p}_2]}$"

(d) $u_{\mathbf{p}_2} \cap u_{\mathbf{q}_1} = u_{\mathbf{p}_1}$

2) Let $\le'_{K_1}$ be the following two-place relation on the family $\{(\mathbf{p}, \mathbf{q}) : \mathbf{p} \le_{K_1} \mathbf{q}\}$ of pairs. We let $(\mathbf{p}_1, \mathbf{q}_1) \le'_{K_1} (\mathbf{p}_2, \mathbf{q}_2)$ iff clauses (a),(b),(d) from part (1) above and

(c)′ if $p_1 \in \mathbb{P}_{\mathbf{p}_1}, q_1 \in \mathbb{P}_{\mathbf{q}_1}$ and $p_1 \Vdash_{\mathbb{P}_{\mathbf{p}_1}}$ "$q_1 \in \mathbb{P}_{\mathbf{p}_2}/\underset{\sim}{G}_{\mathbb{P}_{\mathbf{p}_1}}$" then $p_1 \Vdash_{\mathbb{P}_{\mathbf{p}_2}}$ "$q_1 \in \mathbb{P}_{\mathbf{q}_2}/\underset{\sim}{G}_{\mathbb{P}_{\mathbf{p}_2}}$".

3) Assume $\mathbf{p}_\ell \in K_1$ for $\ell = 0, 1, 2$ and $\mathbf{p}_0 \le_{K_1} \mathbf{p}_1$ and $\mathbf{p}_0 \le_{K_1} \mathbf{p}_2$ and $u_{\mathbf{p}_1} \cap u_{\mathbf{p}_2} = u_{\mathbf{p}_0}$. We define the amalgamation $\mathbf{p} = \mathbf{p}_3 = \mathbf{p}_1 \times_{\mathbf{p}_0} \mathbf{p}_2$ or $\mathbf{p}_3 = \mathbf{p}_1 \times \mathbf{p}_2/\mathbf{p}_0$ as the triple $(u_{\mathbf{p}}, P_{\mathbf{p}} \le_{\mathbf{p}})$ as follows[2]:

(a) $u_{\mathbf{p}} = u_{\mathbf{p}_1} \cup u_{\mathbf{p}_2}$

(b) $P_{\mathbf{p}} = P_{\mathbf{p}_1} \cup P_{\mathbf{p}_2} \cup \{(p_1, p_2) : p_1 \in P_{\mathbf{p}_1} \backslash P_{\mathbf{p}_0}, p_2 \in P_{\mathbf{p}_2} \backslash P_{\mathbf{p}_0}$ and for some $p \in P_{\mathbf{p}_0}$ we have $p \Vdash_{P[\mathbf{p}_0]}$ "$p_\ell \in \mathbb{P}_{\mathbf{p}_\ell}/\mathbb{P}_{\mathbf{p}_0}$" for $\ell = 1, 2\}$

(c) $\le_{\mathbf{p}}$ is defined naturally as $\le_{\mathbf{p}_1} \cup \le_{\mathbf{p}_2} \cup \{((p_1, p_2), (q_1, q_2)) : (p_1, p_2), (q_1, q_2) \in P_{\mathbf{p}}$ and $p_1 \le_{\mathbf{p}_1} q_1$ and $p_2 \le_{\mathbf{p}_2} q_2\} \cup \{(p'_\ell, (p_1, p_2)) : p'_\ell \in P_{\mathbf{p}_\ell}, (p_1, p_2) \in P_{\mathbf{p}}$ and $p'_\ell \le_{\mathbf{p}_1} p_\ell$ for some $\ell \in \{1, 2\}\}$.

Remark 1.1.

Why not use u instead $\mathcal{H}_{<\aleph_1}(u)$? Not a real difference but, e.g. there may not be enough elements in a union of two.

[2] *If in clause (b) of Definition 1.2(3) we would like to avoid "$p_\ell \in \mathbb{P}_{p_\ell} \backslash \mathbb{P}_{\mathbf{p}_0}$" we may replace (p_1, p_2) by $(p_1, p_2, u_{\mathbf{p}_1} \cup u_{\mathbf{p}_2})$ when $\mathbf{p}_1 \ne \mathbf{p}_1 \wedge \mathbf{p}_0 \ne \mathbf{p}_2$ equivalently $\mathbf{p}_0 \ne \mathbf{p}_1 \wedge \mathbf{p}_0 \ne \mathbf{p}_2$.*

Observation 1.2.

1) $\leq^*_{K_1}, \leq'_{K_1}$ are partial orders on their domains.
2) $(\mathbf{p}_1, \mathbf{q}_1) \leq^*_{K_1} (\mathbf{p}_1, \mathbf{q}_1)$ implies $(\mathbf{p}_1, \mathbf{q}_1) \leq'_{K_1} (\mathbf{p}_2, \mathbf{q}_2)$.

For the "successor case vertically and horizontally" we shall use

Claim 1.1.

Assume that $\mathbf{p}_1 \leq^+_{K_1} \mathbf{p}_2$ and $\mathbf{p}_1 \leq_{K_1} \mathbf{q}_1$ and $u_{\mathbf{p}_2} \cap u_{\mathbf{q}_1} = u_{\mathbf{p}_1}$ <u>then</u> $\mathbf{q}_2 \in K_1$ and $(\mathbf{p}_1, \mathbf{q}_1) \leq^*_{K_1} (\mathbf{p}_2, \mathbf{q}_2)$ when we define $\mathbf{q}_2 = \mathbf{q}_1 \times_{\mathbf{p}_1} \mathbf{p}_2$ as in Definition 1.2(3).

Proof. Straight. □

The following claim will be applied to a pair of vertically increasing continuous sequences, one laying horizontally above the other.

Claim 1.2.

Assume $\varepsilon(*) < \lambda$ and

- (a) *$\langle \mathbf{p}^\ell_\varepsilon : \varepsilon \leq \varepsilon(*) \rangle$ is $\leq^+_{K_1}$-increasing continuous for $\ell = 1, 2$*
- (b) *$(\mathbf{p}^1_\varepsilon, \mathbf{p}^2_\varepsilon) \leq'_{K_1} (\mathbf{p}^1_{\varepsilon+1}, \mathbf{p}^2_{\varepsilon+1})$ for $\varepsilon < \varepsilon(*)$.*

Then

- (α) *$\mathbf{p}^1_{\varepsilon(*)} \leq_{K_1} \mathbf{p}^2_{\varepsilon(*)}$*
- (β) *for $\varepsilon < \zeta \leq \varepsilon(*)$ we have $(\mathbf{p}^1_\varepsilon, \mathbf{p}^2_\varepsilon) \leq'_{K_1} (\mathbf{p}^1_\zeta, \mathbf{p}^2_\zeta)$.*

Proof. Easy. □

For the "successor case horizontally, limit case vertically when the relevant game, i.e. the relevant winning strategy is not active" we shall use

Claim 1.3.

Assume $\varepsilon(*) < \lambda$ is a limit ordinal and

- (a) *$\langle \mathbf{p}_\varepsilon : \varepsilon \leq \varepsilon(*) \rangle$ is $\leq^+_{K_1}$-increasing, and $\langle \mathbf{q}_\varepsilon : \varepsilon < \varepsilon(*) \rangle$ is $\leq^+_{K_1}$-increasing*
- (b) *$\mathbf{p}_\varepsilon \leq_{K_1} \mathbf{q}_\varepsilon$ for $\varepsilon < \varepsilon(*)$*
- (c) *if $\varepsilon < \zeta < \varepsilon(*)$ then $(\mathbf{p}_\varepsilon, \mathbf{q}_\varepsilon) \leq'_{K_1} (\mathbf{p}_\zeta, \mathbf{q}_\zeta)$*
- (d) *if $\zeta < \varepsilon(*)$ is a limit ordinal then $\Vdash_{\mathbb{P}_{[\mathbf{p}_\zeta]}}$ "$\mathbb{P}_{\mathbf{q}_\zeta} / \underset{\sim}{G}_{\mathbb{P}[\mathbf{p}_\zeta]} = \cup \{\mathbb{P}_{\mathbf{q}_\varepsilon} / (\underset{\sim}{G}_{\mathbb{P}_{[\mathbf{p}_\zeta]}} \cap \mathbb{P}_{\mathbf{p}_\varepsilon}) : \varepsilon < \zeta\}$".*

Then we can choose $\mathbf{q}_{\varepsilon(*)}$ such that

- (α) *$\mathbf{p}_{\varepsilon(*)} \leq_{K_1} \mathbf{q}_{\varepsilon(*)}$*
- (β) *$(\mathbf{p}_\varepsilon, \mathbf{q}_\varepsilon) \leq'_{K_1} (\mathbf{p}_{\varepsilon(*)}, \mathbf{q}_{\varepsilon(*)})$ for every $\varepsilon < \varepsilon(*)$*
- (γ) *clause (d) holds also for $\zeta = \varepsilon(*)$.*

Remark 1.2.

We can replace $\leq'_{K_1}$ by $\leq^*_{K_1}$ in (c) and (β).

Proof. Should be clear. □

The game defined below is the non-FS ingredient; (in the main application below, $\gamma = \lambda$), it is in the horizontal direction; it lasts $\gamma \le \lambda$ steps but will be used in $\le_{K^2_f}$-increasing subsequences of $\langle \mathbf{k}_i : i < \lambda^+ \rangle$.

Definition 1.3.

For $\delta < \lambda$ and $\gamma \le \lambda$ let $\Game_{\delta,\gamma}$ be the following game between the player INC (incomplete) and COM (complete).
A play last γ moves. In the β-th move a pair $(\mathbf{p}_\beta, \mathbf{q}_\beta)$ is chosen such that $\mathbf{p}_\beta \le^+_{K_1} \mathbf{q}_\beta$ and $\beta(1) < \beta \Rightarrow (\mathbf{p}_{\beta(1)} \le_{K_1} \mathbf{p}_\beta) \wedge (\mathbf{q}_{\beta(1)} \le_{K_1} \mathbf{q}_\beta) \wedge (u_{\mathbf{p}_\beta} \cap u_{\mathbf{q}_{\beta(1)}} = u_{\mathbf{p}_{\beta(1)}})$ and $u_{\mathbf{p}_\beta} \cap \lambda = \delta$ and $u_{\mathbf{q}_\beta} \cap \lambda = u_{\mathbf{q}_0} \cap \lambda \supseteq \delta + 1$.
In the β-th move <u>first</u> INC chooses $(\mathbf{p}_\beta, u_\beta)$ such that $\mathbf{p}_\beta$ satisfies the requirements and u_β satisfies the requirements on $u_{\mathbf{q}_\beta}$ (i.e. $\cup\{u_{\mathbf{q}_\alpha} : \alpha < \beta\} \cup u_{\mathbf{p}_\beta} \subseteq u_\beta \in [\lambda^+]^{<\lambda}$ and $u_\beta \cap \lambda = u_{\mathbf{q}_0} \cap \lambda$) and say $u_\beta \backslash u_{\mathbf{p}_\beta} \backslash \cup \{u_{\mathbf{q}_\gamma} : \gamma < \beta\}$ has cardinality $\ge |\delta|$ (if λ is weakly inaccessible we may be interested in asking more).
<u>Second</u>, COM chooses $\mathbf{q}_\beta$ as required such that $u_\beta \subseteq u[\mathbf{q}_\beta]$.
A player which has no legal moves loses the play, and arriving to the γ-th move, COM wins.

Remark 1.3.

It is not problematic for COM to have a winning strategy. But having "interesting" winning strategies is the crux of the matter. More specifically, any application of this section is by choosing such strategies.
Such examples are the

- (a) <u>lazy man strategy</u>: preserve $\mathbb{P}_{\mathbf{q}_\beta} = \mathbb{P}_{\mathbf{q}_0} \times_{\mathbb{P}_{\mathbf{p}_0}} \mathbb{P}_{\mathbf{p}_\beta}$ recalling Claim 1.1
- (b) it is never too late to become lazy, i.e. arriving to $(\mathbf{p}_{\beta(*)}, \mathbf{q}_{\beta(*)})$ the COM player may decide that $\beta \ge \beta(*) \Rightarrow \mathbb{P}_{\mathbf{q}_\beta} = \mathbb{P}_{q_{\beta(*)}} \times_{\mathbb{P}_{\mathbf{p}_{\beta(*)}}} \mathbb{P}_{\mathbf{p}_\beta}$
- (c) definable forcing strategy, i.e. preserve "$\mathbb{P}_{\mathbf{q}_\beta}/\mathbb{P}_{\mathbf{p}_\beta}$ is a definable c.c.c. forcing (in $\mathbf{V}^{\mathbb{P}[\mathbf{p}_\beta]}$)".

Definition 1.4.

We say f is λ-appropriate if

- (a) $f \in {}^\lambda(\lambda + 1)$
- (b) $\alpha < \lambda \wedge f(\alpha) < \lambda \Rightarrow (\exists \beta)[f(\alpha) = \beta + 1]$
- (c) if $\varepsilon < \lambda^+, \langle u_\alpha : \alpha < \lambda \rangle$ is an increasing continuous sequence of subsets of ε of cardinality $< \lambda$ with union ε then $\{\delta < \lambda\text{: } \mathrm{otp}(u_\delta) < f(\delta)\}$ is a stationary subset of λ.

Convention 1.1.

Below f is λ-appropriate function.

We arrive to defining the set of approximations of size λ (in the main application f_* is constantly λ); we shall later connect it to the oracle version (also see the introduction).

Definition 1.5.

For f_* a λ-appropriate function let $K^2_{f_*}$ be the family of $\mathbf{k}$ such that:

- (a) $\mathbf{k} = \langle E, \bar{\mathbf{p}}, S, \bar{\mathbf{s}}, \bar{\mathbf{g}}, f \rangle$
- (b) E is a club of λ
- (c) $\bar{\mathbf{p}} = \langle \mathbf{p}_\alpha : \alpha \in E \rangle$
- (d) $\mathbf{p}_\alpha \in K^1_\lambda$
- (e) $\mathbf{p}_\alpha \le_{K_1} \mathbf{p}_\beta$ for $\alpha < \beta$ from E
- (f) if $\delta \in$ acc(E) then $\mathbf{p}_\delta = \cup\{\mathbf{p}_\alpha : \alpha \in E \cap \delta\}$
- (g) $S \subseteq \lambda$ is a stationary set of limit ordinals

(h) if $\delta \in S \cap E$ (hence a limit ordinal) then $\delta + 1 \in E$

(i) $\bar{\mathbf{s}} = \langle \mathbf{s}_\delta : \delta \in E \cap S \rangle$

(j) $\mathbf{s}_\delta$ is a winning strategy for the player COM in $\partial_{\delta, f(\delta)}$, see Remark 1.4(1)

(k) $\bar{\mathbf{g}} = \langle \mathbf{g}_\delta : \delta \in S \cap E \rangle$

(l) • $\mathbf{g}_\delta$ is an initial segment of a play of $\partial_{\delta, f_*(\delta)}$ in which the COM player uses the strategy $\mathbf{s}_\delta$

• if its length is $< f_*(\delta)$ then $\mathbf{g}_\delta$ has a last move

• $(\mathbf{p}_\delta, \mathbf{p}_{\delta+1})$ is the pair chosen in the last move, call it $\mathrm{mv}(\mathbf{g}_\delta)$

• let $S_0 = \{\delta \in S \cap E : \mathbf{g}_\delta$ has length $< f_*(\delta)\}$ and $S^1 = S \cap E \backslash S_0$

(m) if $\alpha < \beta$ are from E then $\mathbf{p}_\alpha \leq^+_{K_1} \mathbf{p}_\beta$, so in particular $\mathbb{P}_\beta / \mathbb{P}_\alpha$ is absolutely c.c.c. that is if $\mathbb{P} \lessdot \mathbb{P}'$ and $\mathbb{P}'$ is c.c.c. then $\mathbb{P}' *_{\mathbb{P}_\alpha} \mathbb{P}_\beta$ is c.c.c.; this strengthens clause (e)

(n) $f \in {}^\lambda\lambda$

(o) if $\delta \in S \cap E$ then $f(\delta) + 1$ is the length of $\mathbf{g}_\delta$

(p) for every $\delta \in E$, if $f_*(\delta) < \lambda$ then $f(\delta) \leq \mathrm{otp}(u_{\mathbf{p}_\delta})$.

Remark 1.4.

1) Concerning clause (j), recall (using the notation of Definition 1.3) that during a play the player INC chooses $\mathbf{p}_\varepsilon$ and COM chooses $\mathbf{q}_\varepsilon, \varepsilon \leq f(\delta)$ and recalling clause (o) we see that $(\mathbf{p}_{f(\delta)}, \mathbf{q}_{f(\delta)})$ there stands for $(\mathbf{p}_\delta, \mathbf{p}_{\delta+1})$ here. You may wonder from where does the $(\mathbf{p}_\varepsilon, \mathbf{q}_\varepsilon)$ for $\varepsilon < f(\delta)$ comes from; the answer is that you should think of $\mathbf{k}$ as a stage in an increasing sequence of approximations of length $f(\delta)$ and $(\mathbf{p}_\varepsilon, \mathbf{q}_\varepsilon)$ comes from the δ-place in the ε-approximation. This is cheating a bit - the sequence of approximations has length $< \lambda^+$, but as on a club of λ this reflects to length $< \lambda$, all is O.K.
2) Below we define the partial order $\leq_{K_2}$ (or $\leq_{K^2_{f_*}}$) on the set $K^2_{f_*}$, recall our goal is to choose an $\leq_{K_2}$-increasing sequence $\langle \mathbf{k}_\varepsilon : \varepsilon < \lambda^+ \rangle$ and our final forcing will be $\cup \{\mathbb{P}_{\mathbf{k}_\varepsilon} : \varepsilon < \lambda^+\}$.
3) Why clause (d) in Definition 1.6(2) below? It is used in the proof of the limit existence Claim 1.5. This is because the club $E_{\mathbf{k}}$ may decrease (when increasing $\mathbf{k}$).
Note that we use $\leq^*_{K^1_f}$ "economically". We cannot in general demand (in Definition 1.6(2) below) that for $\alpha < \beta$ from $E_{\mathbf{k}_2} \backslash \alpha(*)$ we have $(\mathbf{p}^{\mathbf{k}_1}_\alpha, \mathbf{p}^{\mathbf{k}_1}_\beta) \leq^*_{K_1} (\mathbf{p}^{\mathbf{k}_2}_\alpha, \mathbf{p}^{\mathbf{k}_2}_\beta)$ as the strategies $\mathbf{s}_\delta$ may defeat this. How will it still help? Assume $\langle \mathbf{k}_\varepsilon : \varepsilon < \varepsilon(*) \rangle$ is increasing, $\varepsilon(*) < \lambda$ for simplicity and $\gamma \in \cap \{E_{\mathbf{k}_\varepsilon} : \varepsilon < \varepsilon(*)\} \cap \bigcap \{S_{\mathbf{k}_\varepsilon} : \varepsilon < \varepsilon(*)\} \backslash \cup \{\alpha(\mathbf{k}_\varepsilon, \mathbf{k}_\zeta) : \varepsilon < \zeta < \varepsilon(*)\}$ and $\gamma_\varepsilon = \mathrm{Min}(E_{\mathbf{k}_\varepsilon} \backslash (\gamma + 1))$ for $\varepsilon < \varepsilon(*)$. We shall have $\langle \gamma_\varepsilon : \varepsilon < \varepsilon(*) \rangle$ is increasing; there may be $\delta \in (\gamma_\varepsilon, \gamma_{\varepsilon+1})$ where $\mathbf{s}_\delta$ was active between $\mathbf{k}_\varepsilon$ and $\mathbf{k}_{\varepsilon+1}$ so it contributes to $\mathbb{P}^{\mathbf{k}_{\varepsilon+1}}_{\gamma_{\varepsilon+1}} / \mathbb{P}^{\mathbf{k}_\varepsilon}_{\gamma_\varepsilon}$.
4) If we omit the restriction $u \in [\lambda^+]^{<\lambda}$ and allow $f : \lambda \to \delta^* + 1$, replace the club E by an end segment, we can deal with sequences of length $\delta^* < \lambda^+$.
In the direct order in Definition 1.6(3) we have $\alpha(*) = 0$. Using e.g. a stationary non-reflecting $S \subseteq S^{\delta^*}_\lambda$ we can often allow $\alpha(*) \neq 0$.
5) Is the "$\mathbf{s}_\delta$ a winning strategy" in addition for telling us what to do, crucial? The point is preservation of c.c.c. in limit of cofinality $\aleph_1$.
6) If we use $f_* \in {}^\lambda(\lambda + 1)$ constantly λ, we do not need $f_{\mathbf{k}}$ so we can omit clauses (n),(o),(p) of Definition 1.5 and (c), and part of another in Definition 1.6.
6A) Alternatively we can omit clause (o) in Definition 1.5 but demand "$\prod\limits_{\alpha < \lambda} f(\alpha) / \mathcal{D}$ is λ^+-directed", fixing a normal filter $\mathcal{D}$ on λ (and demand $S_{\mathbf{k}} \in \mathcal{D}^+$).
7) The "omitting type" argument here comes from using the strategies.

Definition 1.6.

1) In Definition 1.5, let $E = E_{\mathbf{k}}, \bar{\mathbf{p}} = \bar{\mathbf{p}}_{\mathbf{k}}, \mathbf{p}_\alpha = \mathbf{p}^{\mathbf{k}}_\alpha = \mathbf{p}_\alpha[\mathbf{k}], \mathbb{P}_\alpha = \mathbb{P}^{\mathbf{k}}_\alpha = \mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}]}, S = S_{\mathbf{k}}, S_{[\ell]} = S_{\mathbf{k},\ell}$ for $\ell = 0, 1$, etc. and we let $\mathbb{P}_{\mathbf{k}} = \cup \{\mathbb{P}^{\mathbf{k}}_\alpha : \alpha \in E_{\mathbf{k}}\}$ and $u_{\mathbf{k}} = u[\mathbf{k}] = \cup \{u_{\mathbf{p}^{\mathbf{k}}_\alpha} : \alpha \in E_{\mathbf{k}}\}$.
2) We define a two-place relation $\leq_{K^2_f}$ on $K^2_f : \mathbf{k}_1 \leq_{K^2_f} \mathbf{k}_2$ iff (both are from K^2_f and) for some $\alpha(*) < \lambda$ (and $\alpha(\mathbf{k}_1, \mathbf{k}_2)$ is the first such $\alpha(*)$) we have:

(a) $E_{\mathbf{k}_2} \backslash E_{\mathbf{k}_1}$ is bounded in λ, moreover $\subseteq \alpha(*)$

(b) for $\alpha \in E_{\mathbf{k}_2} \backslash \alpha(*)$ we have $\mathbf{p}^{\mathbf{k}_1}_\alpha \leq_{K_1} \mathbf{p}^{\mathbf{k}_2}_\alpha$

(c) if $\alpha \in E_{\mathbf{k}_2} \backslash \alpha(*)$ then $f_{\mathbf{k}_1}(\alpha) \leq f_{\mathbf{k}_2}(\alpha)$

(d) if $\gamma_0 < \gamma_1 \leq \gamma_2 < \lambda$, $\gamma_0 \in E_{\mathbf{k}_2} \backslash (\alpha(*) \cup S_{\mathbf{k}_1})$, $\gamma_1 = \min(E_{\mathbf{k}_1} \backslash (\gamma_0 + 1))$ and $\gamma_2 = \min(E_{\mathbf{k}_2} \backslash (\gamma_0 + 1))$, then $(\mathbf{p}^{\mathbf{k}_1}_{\gamma_0}, \mathbf{p}^{\mathbf{k}_2}_{\gamma_0}) \leq_{K_1} (\mathbf{p}^{\mathbf{k}_1}_{\gamma_1}, \mathbf{p}^{\mathbf{k}_2}_{\gamma_2})$, see Definition 1.2(2)

(e) if $\delta \in S_{\mathbf{k}_1} \cap E_{\mathbf{k}_2} \backslash \alpha(*)$ then $\delta \in S_{\mathbf{k}_2} \cap E_{\mathbf{k}_2} \backslash \alpha(*)$; but note that if $f_{\mathbf{k}_1}(\delta) \geq f(\delta)$ we put δ into $S_{\mathbf{k}_2}$ just for notational convenience

(f) if $\delta \in S_{\mathbf{k}_1} \cap E_{\mathbf{k}_2} \backslash \alpha(*)$ then $\mathbf{s}^{\mathbf{k}_2}_\delta = \mathbf{s}^{\mathbf{k}_1}_\delta$ and $\mathbf{g}^{\mathbf{k}_1}_\delta$ is an initial segment of $\mathbf{g}^{\mathbf{k}_2}_\delta$

(g) if $\mathbf{k}_1 \neq \mathbf{k}_2$ then $u[\mathbf{k}_1] \neq u[\mathbf{k}_2]$

(h) if $\alpha < \beta$ are from $E_{\mathbf{k}_2} \backslash \alpha(*)$ then $(\mathbf{p}^{\mathbf{k}_1}_\alpha, \mathbf{p}^{\mathbf{k}_2}_\alpha) \leq'_{K_1} (\mathbf{p}^{\mathbf{k}_1}_\beta, \mathbf{p}^{\mathbf{k}_2}_\beta)$, see Definition 1.2(2), i.e. if $p \in \mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}_1]}$, $q \in \mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}_2]}$ and $p \Vdash_{\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}_1]}}$ "$q \in \mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}_2]} / \mathop{G}\limits_{\sim}{}_{\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}_1]}}$" then $p \Vdash_{\mathbb{P}_{\mathbf{p}_\beta[\mathbf{k}_1]}}$ "$q \in \mathbb{P}_{\mathbf{p}_\beta[\mathbf{k}_2]} / \mathop{G}\limits_{\sim}{}_{\mathbb{P}_{\mathbf{p}_\beta[\mathbf{k}_1]}}$".

3) We define a two-place relation $\leq^{\rm dir}_{K^2_f}$ on K^2_f as follows: $\mathbf{k}_1 \leq^{\rm dir}_{K^2_f} \mathbf{k}_2$ iff

(a) $\mathbf{k}_1 \leq_{K^2_f} \mathbf{k}_2$

(b) $E_{\mathbf{k}_2} \subseteq E_{\mathbf{k}_1}$; no real harm here if we add $\mathbf{k}_1 \neq \mathbf{k}_2 \Rightarrow E_{\mathbf{k}_2} \subseteq \operatorname{acc}(E_{\mathbf{k}_1})$

(c) $\alpha(\mathbf{k}_1, \mathbf{k}_2) = \operatorname{Min}(E_{\mathbf{k}_2})$.

4) We write $K^2_\lambda, \leq_{K^2_\lambda}, \leq^{\rm dir}_{K^2_\lambda}$ or just $K_2, \leq_{K_2}, <^{\rm dir}_{K_2}$ for $K^2_f, \leq_{K^2_f}, \leq^{\rm dir}_{K^2_f}$ when f is constantly λ.

Remark 1.5.

1) In [2] we may increase S as well as here but we may replace clause (e) by

(e)′ $\delta \in S_{\mathbf{k}_1} \cap E_{\mathbf{k}_2} \backslash \alpha(*)$ iff $f_{\mathbf{k}_2}(\delta) < f(\delta) \wedge \delta \in S_{\mathbf{k}_2} \cap E_{\mathbf{k}_2} \backslash \alpha(*)$.

If we do this, is it a great loss? No! This can still be done here by choosing $\mathbf{s}_\delta$ such that as long as INC chooses u_β of certain form (e.g. $u_\beta \backslash u^{\mathbf{p}_\beta} = \{\delta\}$) the player COM chooses $\mathbf{q}_\beta = \mathbf{p}_\beta$. We can allow in Definition 1.6(2) to extend S but a priori start with $\langle S_\varepsilon : \varepsilon < \lambda^+ \rangle$ such that $S_\varepsilon \subseteq \lambda$ and $S_\varepsilon \backslash S_\zeta$ is bounded in λ when $\varepsilon < \zeta < \lambda$ and demand $S_{\mathbf{k}_\epsilon} = S_\epsilon$.
2) We can weaken clause (e) of Definition 1.6(2) to

(e)″ if $\delta \in S_{\mathbf{k}_1} \cap E_{\mathbf{k}_2} \backslash \alpha(*)$ and $f_{\mathbf{k}_2}(\delta) < f(\delta)$ then $\delta \in S_{\mathbf{k}_2}$.

But then we have to change accordingly, e.g. Definition 1.6(c),(f), Definition 1.7(c).
3) We can define $\mathbf{k}_1 \leq_{K^2_f} \mathbf{k}_2$ demanding $(S_{\mathbf{k}_1}, \bar{\mathbf{s}}_{\mathbf{k}_1}) = (S_{\mathbf{k}_2}, \bar{\mathbf{s}}_{\mathbf{k}_2})$ but replace everywhere "$\delta \in S_\mathbf{k} \cap E_\mathbf{k}$" by "$\delta \in S_\mathbf{k} \cap E_\mathbf{k} \wedge f_\mathbf{k}(\delta) \leq f(\delta)$" so omit clause (e) of Definition 1.6.

Observation 1.3.

1) $\leq_{K^2_f}$ is a partial order on K^2_f.
2) $\leq^{\rm dir}_{K^2_f} \subseteq \leq_{K^2_f}$ is a partial order on K^2_f.
3) If $\mathbf{k}_1 \leq_{K^2_f} \mathbf{k}_2$ then $\mathbb{P}_{\mathbf{k}_1} \lessdot \mathbb{P}_{\mathbf{k}_2}$.
4) If $\langle \mathbf{k}_\varepsilon : \varepsilon < \lambda^+ \rangle$ is $<_{K^2_f}$-increasing and $\mathbb{P} = \cup\{\mathbb{P}_{\mathbf{k}_\varepsilon} : \varepsilon < \lambda^+\}$ then

(a) $\mathbb{P}$ is a c.c.c. forcing notion of cardinality $\leq \lambda^+$

(b) $\mathbb{P}_{\mathbf{k}_\varepsilon} \lessdot \mathbb{P}$ for $\varepsilon < \lambda^+$.

Definition 1.7.

1) Assume $\bar{\mathbf{k}} = \langle \mathbf{k}_\varepsilon : \varepsilon < \varepsilon(*) \rangle$ is $\leq_{K^2_f}$-increasing with $\varepsilon(*)$ a limit ordinal $< \lambda$. We say $\mathbf{k}$ is a limit of $\bar{\mathbf{k}}$ when $\varepsilon < \varepsilon(*) \Rightarrow \mathbf{k}_\varepsilon \leq_{K^2_f} \mathbf{k} \in K^2_f$ and for some $\alpha(*)$

(a) $\alpha(*) = \cup\{\alpha(\mathbf{k}_\varepsilon, \mathbf{k}_\zeta) : \varepsilon < \zeta < \varepsilon(*)\}$

(b) $E_\mathbf{k} \backslash \alpha(*) \subseteq \cap\{E_{\mathbf{k}_\varepsilon} \backslash \alpha(*) : \varepsilon < \varepsilon(*)\}$

(c) $S_\mathbf{k} = (\cup\{S_{\mathbf{k}_\varepsilon} : \varepsilon < \varepsilon(*)\}) \cap (\cap\{E_{\mathbf{k}_\varepsilon} : \varepsilon < \varepsilon(*)\}) \backslash \alpha(*)$

(d) if $\delta \in S_\mathbf{k}$ then $\mathbf{g}^{\mathbf{k}_\varepsilon}_\delta$ is an initial segment of $\mathbf{g}^\mathbf{k}_\delta$ for every $\varepsilon < \varepsilon(*)$

(e) $f_\mathbf{k}(\delta) = \cup\{f_{\mathbf{k}_\varepsilon}(\delta) : \varepsilon < \varepsilon(*)\} + 1$ for $\delta \in S_\mathbf{k}$.

2) Assume $\bar{\mathbf{k}} = \langle \mathbf{k}_\varepsilon : \varepsilon < \lambda \rangle$ is $\leq_{K^2_f}$-increasing continuous. We say $\mathbf{k}$ is a limit of $\bar{\mathbf{k}}$ when $\varepsilon < \lambda \Rightarrow \mathbf{k}_\varepsilon \leq \mathbf{k} \in K^2_f$ and for some $\bar{\alpha}$

(a) $\bar{\alpha} = \langle \alpha_\varepsilon : \varepsilon < \lambda \rangle$ is increasing continuous, $\lambda > \alpha_\varepsilon \in \cap\{E_{\mathbf{k}_\zeta} : \zeta < \varepsilon\} \backslash \cup \{\alpha(\mathbf{k}_{\zeta_1}, \mathbf{k}_{\zeta_2}) : \zeta_1 < \zeta_2 < 1 + \varepsilon\}$

(b) $E_\mathbf{k} = \{\alpha_\varepsilon : \varepsilon < \lambda\} \cup \{\alpha_\varepsilon + 1 : \varepsilon < \lambda$ and $\varepsilon \in S\}$ and $\mathbf{p}^\mathbf{k}_{\alpha_\varepsilon} = \mathbf{p}^{\mathbf{k}_\varepsilon}_{\alpha_\varepsilon}$, $\mathbf{p}^\mathbf{k}_{\alpha_\varepsilon} = \mathbf{p}^{\mathbf{k}_\varepsilon + 1}_{\alpha_\varepsilon + 1}$

(c) $S_\mathbf{k} = \{\alpha_\varepsilon : \alpha_\varepsilon \in \mathbf{S}_{\mathbf{k}_\zeta}$ for every $\zeta < \varepsilon$ large enough$\}$

(d) if $\delta = \alpha_\varepsilon \in S_{\mathbf{k}_\varepsilon}$ then $\mathbf{g}^\mathbf{k}_\delta = \mathbf{g}^{\mathbf{k}_\varepsilon}_\delta$

(e) if $\alpha < \delta$ and $\zeta =$ Min$\{\varepsilon : \alpha \leq \alpha_{\varepsilon+1}\}$ then $f_\mathbf{k}(\alpha) = f_{\mathbf{k}_\zeta}(\alpha)$.

3) We say that $\langle \mathbf{k}_\varepsilon : \varepsilon < \varepsilon(*) \rangle$ is $\leq_{K^2_f}$-increasing continuous when :

(a) $\mathbf{k}_\varepsilon \leq_{K^2_f} \mathbf{k}_\zeta$ for $\varepsilon < \zeta < \varepsilon(*)$

(b) $\mathbf{k}_\varepsilon$ is a limit of $\langle \mathbf{k}_{\xi(\zeta)} : \zeta <$ cf$(\varepsilon)\rangle$ for some increasing continuous sequence $\langle \xi(\zeta) : \zeta <$ cf$(\varepsilon)\rangle$ of ordinals with limit ε, for every limit $\varepsilon < \varepsilon(*)$, by part (1) or part (3).

Definition 1.8.

1) In part (1) of Definition 1.7, we say "a direct limit" when in addition

(α) the sequences are $\leq^{\rm dir}_{K^2_f}$-increasing

(β) in clause (b) we have equality

(γ) $\mathbf{p}^\mathbf{k}_{\min(E_\mathbf{k})}$ is the exact union of $\langle \mathbf{p}^\mathbf{k}_{\min(E_{\mathbf{k}_\varepsilon})} : \varepsilon < \varepsilon(*) \rangle$

(δ) if $\gamma \in E_\mathbf{k}$, $\xi < \varepsilon(*)$, $\gamma \notin S^1_{\mathbf{k}_\xi}$ and $\langle \gamma_\varepsilon : \varepsilon \in [\xi, \varepsilon(*)] \rangle$ is defined by $\gamma_\xi = \gamma$, $\gamma_\varepsilon =$ min$(E_{\mathbf{k}_\varepsilon} \backslash (\gamma + 1))$ when $\xi < \varepsilon \leq \varepsilon(*)$, so $\langle \gamma_\varepsilon : \varepsilon \in [\xi, \varepsilon(*)] \rangle$ is an $\leq$-increasing continuous sequence of ordinals, then $\mathbf{p}^{\mathbf{k}_\xi}_{\gamma_{\varepsilon(*)}} / \mathbf{p}^{\mathbf{k}_\xi}_\gamma = \cup\{\mathbf{p}^\mathbf{k}_{\gamma_\varepsilon} / \mathbf{p}^\mathbf{k}_\gamma : \varepsilon \in [\xi, \varepsilon(*))\}$ with the obvious meaning.

2) In part (2) of Definition 1.7 we say a "direct limit" when in addition

(α) the sequence is $\leq^{\rm dir}_{K^2_f}$

(β) α_ε is minimal under the restriction.

3) We say that $\bar{\mathbf{k}} = \langle \mathbf{k}_\varepsilon : \varepsilon < \varepsilon(*) \rangle$ is $\leq^{\rm dir}_{K^2_f}$-increasing continuous or directly increasing continuous when :

(a) $\mathbf{k}_\varepsilon \leq^{\rm dir}_{K^2_f} \mathbf{k}_\zeta$ for $\varepsilon \leq \zeta < \varepsilon(*)$

(b) if $\varepsilon < \varepsilon(*)$ is a limit ordinal then $\mathbf{k}_\varepsilon$ is a (really the) direct limit of $\bar{\mathbf{k}} \restriction \varepsilon$.

Claim 1.4.

If $\mathbf{k}_1 \leq_{K^2_f} \mathbf{k}_2$ then for some $\mathbf{k}'_2$ we have

(a) $\mathbf{k}_1 \leq^{\rm dir}_{K^2_f} \mathbf{k}'_2$

(b) $\mathbf{k}_2 \leq_{K^2_f} \mathbf{k}'_2 \leq_{K^2_f} \mathbf{k}_2$

(c) $\mathbf{k}_2, \mathbf{k}'_2$ *are almost equal - the only differences being* $E_{\mathbf{k}'_2} = E_{\mathbf{k}_2} \backslash \min(E_{\mathbf{k}'_2}), S_{\mathbf{k}'_2} \subseteq S_{\mathbf{k}_2}$, *etc.*

Claim 1.5.

The limit existence claim *1) If* $\varepsilon(*) < \lambda$ *is a limit ordinal and* $\bar{\mathbf{k}} = \langle \mathbf{k}_\varepsilon : \varepsilon < \varepsilon(*) \rangle$ *is a [directly] increasing continuous then* $\bar{\mathbf{k}}$ *has a [direct] limit.*
2) Similarly for $\varepsilon(*) = \lambda$.

Proof. It is enough to prove the direct version.
1) We define $\mathbf{k} = \mathbf{k}_{\varepsilon(*)}$ as in the definition, we have no freedom left.
The main points concern the c.c.c. and the absolute c.c.c., $\leq'_{K^0_1}, \leq_{K_1}$ demands. We prove the relevant demands by induction on $\beta \in E_{\mathbf{k}_{\varepsilon(*)}}$.

Case 1: $\beta = \min(E_{\mathbf{k}_{\varepsilon(*)}})$.
First note that $\langle \mathbf{p}^\varepsilon_{\min(E_{\mathbf{k}_\varepsilon})} : \varepsilon \leq \varepsilon(*) \rangle$ is increasing continuous (in K^1_λ) moreover $\langle \mathbb{P}[\mathbf{p}^{\mathbf{k}_\varepsilon}_{\min(E_{\mathbf{k}_\varepsilon})}] : \varepsilon \leq \varepsilon(*) \rangle$ is increasing continuous, see clause (γ) of Definition 1.8(1). As each $\mathbb{P}[\mathbf{p}_{\min(E_{\mathbf{k}_\varepsilon})}]$ is c.c.c. if $\varepsilon < \varepsilon(*)$, we know that this holds for $\varepsilon = \varepsilon(*)$, too.

Case 2: $\beta = \delta + 1, \delta \in S^1_{\mathbf{k}} \cap E_{\mathbf{k}}$.
Since $\mathbf{s}^{\mathbf{k}}_\delta$ is a winning strategy in the game $\partial_{\delta, f(\delta)}$ we have $\mathbf{p}^{\mathbf{k}_{\varepsilon(*)}}_\delta \leq^+_{K_1} \mathbf{p}^{\mathbf{k}_{\varepsilon(*)}}_\beta$. But what if the play is over? Recall that in Definition 1.4, $f(\delta) = \lambda$ or $f(\delta)$ is successor and $\langle f_{\mathbf{k}_\varepsilon}(\delta) : \varepsilon < \varepsilon(*) \rangle$ is (strictly) increasing, so this never happens; it may happen when we try to choose $\mathbf{k}'$ such that $\mathbf{k} <_{K^2_f} \mathbf{k}'$, see Claim 1.6.
We also have to show: if $\alpha \in \beta \cap E_{\mathbf{k}}$ then $\mathbb{P}[\mathbf{p}^{\mathbf{k}}_\beta]/\mathbb{P}[\mathbf{p}^{\mathbf{k}}_\alpha]$ is absolutely c.c.c. First, if $\alpha = \delta$ this holds by Definition 1.1(3) of $\leq^+_{K_1}$ and the demand $\mathbf{p}_\beta \leq^+_{K_1} \mathbf{q}_\beta$ in Definition 1.3 (and clause (ℓ) of Definition 1.5). Second, if $\alpha < \delta$, it is enough to show that $\mathbb{P}[\mathbf{p}^{\mathbf{k}}_\beta]/\mathbb{P}[\mathbf{p}^{\mathbf{k}}_\delta]$ and $\mathbb{P}[\mathbf{p}^{\mathbf{k}}_\delta]/\mathbb{P}[\mathbf{p}^{\mathbf{k}}_\alpha]$ are absolutely c.c.c., but the first holds by the previous sentence, the second by the induction hypothesis. In particular, when $\varepsilon < \varepsilon(*) \Rightarrow \mathbb{P}^{\mathbf{k}_\varepsilon}_\beta \lessdot \mathbb{P}^{\mathbf{k}}_\beta$.

Case 3: For some $\gamma, \gamma = \max(E_{\mathbf{k}} \cap \beta), \gamma \notin S^1_{\mathbf{k}}$.
As $\gamma \notin S_{\mathbf{k}}$ there is $\xi < \varepsilon(*)$ such that $\gamma \notin S^1_{\mathbf{k}_\xi}$ let $\gamma_\xi = \gamma$ and for $\varepsilon \in (\xi, \varepsilon(*)]$ we define $\gamma_\varepsilon =: \min(E_{\mathbf{k}_\varepsilon} \backslash (\beta + 1))$. Now as $\bar{\mathbf{k}}$ is directly increasing continuous we have

⊛ (a) $\langle \gamma_\varepsilon : \varepsilon \in [\xi, \varepsilon(*)] \rangle$ is increasing continuous

(b) $\gamma_\xi = \gamma$

(c) $\gamma_{\varepsilon(*)} = \beta$

(d) $\langle \mathbf{p}^{\mathbf{k}_\varepsilon}_{\gamma_\varepsilon} : \varepsilon \in [\xi, \varepsilon(*)] \rangle$ is increasing continuous.

So by Claim 1.3 we are done, the main point is that clause (d) there holds by clause (d) of the definition of $\leq_{K^2_f}$ in Definition 1.6(2).

Case 4: $\beta = \sup(E_{\mathbf{k}} \cap \beta)$.
It follows by the induction hypothesis and Observation 1.1(3) as $\langle \mathbf{p}^{\mathbf{k}}_\gamma : \gamma \in E_k \cap \beta \rangle$ is $\leq^+_{K_1}$-increasing continuous with union $\mathbf{p}^{\mathbf{k}}_\beta$; of course we use clause (h) of Definition 1.6, so Definition 1.2(2),(5) applies.
2) Similarly. $\square$

The following is an atomic step toward having $\mathrm{MA}_{<\lambda}$.

Claim 1.6.

Assume

(a) $\mathbf{k}_1 \in K^2_f$

(b) $\alpha(*) \in E_{\mathbf{k}_1}$

(c) $\underset{\sim}{\mathbb{Q}}$ *is a* $\mathbb{P}[\mathbf{p}^{\mathbf{k}_1}_{\alpha(*)}]$*-name of a c.c.c. forcing (hence* $\Vdash_{\mathbb{P}_{\mathbf{k}_1}}$ *"*$\underset{\sim}{\mathbb{Q}}$ *is a c.c.c. forcing")* (*d*) $u_* \subseteq \lambda^+$ *is disjoint to* $u[\mathbf{k}_1] = \cup\{u_{\mathbf{p}_\alpha[\mathbf{k}_1]} : \alpha \in E_{\mathbf{k}}\}$ *and of cardinality* $< \lambda$ *but* $\geq |\underset{\sim}{\mathbb{Q}}|$.

Then we can find $\mathbf{k}_2$ *such that*

(*α*) $\mathbf{k}_1 \leq^{dir}_{K^2_f} \mathbf{k}_2 \in K^2_f$

(*β*) $E_{\mathbf{k}_2} = E_{\mathbf{k}_1} \backslash \alpha(*)$

(*γ*) $u^{\mathbf{k}_2}_\alpha = u^{\mathbf{k}_1}_\alpha \cup u_*$ *for* $\alpha \in E_{\mathbf{k}_2} \cap S^1_{\mathbf{k}_1}$

(*δ*) $\mathbb{P}_{\mathbf{p}_{\alpha(*)}[\mathbf{k}_2]}$ *is isomorphic to* $\mathbb{P}_{\mathbf{p}_{\alpha(*)}[\mathbf{k}_1]} * \underset{\sim}{\mathbb{Q}}$ *over* $\mathbb{P}_{\mathbf{p}_{\alpha(*)}[\mathbf{k}_1]}$

(*ε*) $S_{\mathbf{k}_2} = S_{\mathbf{k}_1}$ *and* $\bar{\mathbf{s}}_{\mathbf{k}_2} = \bar{\mathbf{s}}_{\mathbf{k}_1} \restriction G_{\mathbf{k}_2}$

(*ζ*) $f_{\mathbf{k}_2} = f_{\mathbf{k}_1} + 1$

(*η*) *if* $\Vdash_{\mathbb{P}_{\mathbf{k}_1} * \mathbb{Q}}$ *"*$\underset{\sim}{\rho} \in {}^\omega 2$ *but* $\underset{\sim}{\rho} \notin \mathbf{V}[\underset{\sim}{G}_{\mathbb{P}_{\mathbf{k}_1}}]$*" then* $\Vdash_{\mathbb{P}_{\mathbf{k}_2}}$ *"*$\underset{\sim}{\rho} \in {}^\omega 2$ *but* $\underset{\sim}{\rho} \notin \mathbf{V}[\underset{\sim}{G}_{\mathbb{P}_{\mathbf{k}_1}}]$*" provided that the strategies preserve this which they do under the criterion here.*

Proof. We choose $\mathbf{p}^{\mathbf{k}_2}_\alpha$ by induction on $\alpha \in E_{\mathbf{k}_1} \backslash \alpha(*)$, keeping all relevant demands (in particular $u_{\mathbf{p}_\alpha[\mathbf{k}_2]} \cap u[\mathbf{k}_1] = u_{\mathbf{p}_\alpha[\mathbf{k}_1]}$).

Case 1: $\alpha = \alpha(*)$.
As only the isomorphism type of $\underset{\sim}{\mathbb{Q}}$ is important, without loss of generality $\Vdash_{\mathbb{P}[\mathbf{p}^{\mathbf{k}_1}_{\alpha(*)}]}$ "every member of $\underset{\sim}{\mathbb{Q}}$ belongs to u_*".
So we can interpret the set of elements of $\mathbb{P}_{\mathbf{p}_{\alpha(*)}[\mathbf{k}_1]} * \underset{\sim}{\mathbb{Q}}$ such that it is $\subseteq \mathcal{H}_{<\aleph_1}(u_{\mathbf{p}_{\alpha(*)}[\mathbf{k}_1]} \cup u_*)$.
Now $\mathbb{P}_{\mathbf{p}_{\alpha(*)}[\mathbf{k}_1]} \lessdot \mathbb{P}_{\mathbf{p}_{\alpha(*)}[\mathbf{k}_2]}$ by the classical claims on composition of forcing notions.

Case 2: $\alpha = \delta + 1, \delta \in S_{\mathbf{k}_1} \cap E_{\mathbf{k}_1} \backslash \alpha(*)$.
The case split to two subcases.

Subcase 2A: The play $\mathbf{g}^{\mathbf{k}_1}_\delta$ is not over, i.e. $f(\delta)$ is larger than the length of the play so far.
In this case do as in case 2 in the proof of Claim 1.5, just use $\mathbf{s}_\delta$.

Subcase 2B: The play $\mathbf{g}^{\mathbf{k}_1}_\delta$ is over.
In this case let $\mathbb{P}^{\mathbf{k}_2}_{\delta+1} = \mathbb{P}^{\mathbf{k}_1}_{\delta+1} *_{\mathbb{P}^{\mathbf{k}_1}_\delta} \mathbb{P}^{\mathbf{k}_2}_\delta$, in fact, $\mathbf{p}^{\mathbf{k}_2}_{\delta+1} = \mathbf{p}^{\mathbf{k}_1}_{\delta+1} *_{\mathbf{p}^{\mathbf{k}_1}_\delta} \mathbf{p}^{\mathbf{k}_2}_\delta$ (and choose $u_{\mathbf{p}_{\delta+1}[\mathbf{k}_2]}$ appropriately). Now possible and $(\mathbf{p}^{\mathbf{k}_1}_\delta, \mathbf{p}^{\mathbf{k}_2}_\delta) <'_{K_1} (\mathbf{p}^{\mathbf{k}_1}_{\delta+1}, \mathbf{p}^{\mathbf{k}_2}_{\delta+1})$ by Claim 1.1.

Case 3: For some $\gamma, \gamma = \max(E_{\mathbf{k}} \cap \beta) \geq \alpha(*)$ and $\gamma \notin S_{\mathbf{k}}$.
Act as in Subcase 2B of the proof of Claim 1.5

Case 4: $\beta = \sup(E_{\mathbf{k}} \cap \beta)$.
As in Case 4 in the proof of Claim 1.5. □

2. $\mathfrak{p} = \mathfrak{t}$ does not decide the existence of a peculiar cut

We deal here with a problem raised in [3], toward this we quote from there. Recall (Definition [3, 1.10]).

Definition 2.1.
Let κ_1, κ_2 be infinite regular cardinals. A (κ_1, κ_2)-peculiar cut in ${}^\omega\omega$ is a pair $(\langle f_i : i < \kappa_1\rangle, \langle f^\alpha : \alpha < \kappa_2\rangle)$ of sequences of functions in ${}^\omega\omega$ such that:

(*α*) $(\forall i < j < \kappa_1)(f_j <_{J^{\rm bd}_\omega} f_i)$,

(*β*) $(\forall \alpha < \beta < \kappa_2)(f^\alpha <_{J^{\rm bd}_\omega} f^\beta)$,

(*γ*) $(\forall i < \kappa_1)(\forall \alpha < \kappa_2)(f^\alpha <_{J^{\rm bd}_\omega} f_i)$,

(δ) if $f : \omega \to \omega$ is such that $(\forall i < \kappa_1)(f \le_{J^{\rm bd}_\omega} f_i)$, then $f \le_{J^{\rm bd}_\omega} f^\alpha$ for some $\alpha < \kappa_2$,

(ε) if $f : \omega \to \omega$ is such that $(\forall \alpha < \kappa_2)(f^\alpha \le_{J^{\rm bd}_\omega} f)$, then $f_i \le_{J^{\rm bd}_\omega} f$ for some $i < \kappa_1$.

Recall that if $\mathfrak{p} < \mathfrak{t}$ then for some regular $\kappa < \mathfrak{p}$ there is a $(\kappa, \mathfrak{p})$-peculiar cut, ([3, 1.12]). Also $\mathfrak{p} = \aleph_1 \Rightarrow \mathfrak{t} = \mathfrak{p}$ by the classic theorem of Rothenberg and $\mathrm{MA}_{\aleph_1} + \mathfrak{p} = \aleph_2 \Rightarrow \mathfrak{t} = \aleph_2$ by [3, 2.3].

Recall (from [3]) that

Claim 2.1.

1) If there is a (κ_1, κ_2)-peculiar then recall from there that the motivation of looking at (κ_1, κ_2)-peculiar type is understanding the case $\mathfrak{p} > \mathfrak{t}$.
1A) In particular, if $\mathfrak{p} < \mathfrak{t}$ then there is a (κ_1, κ_2)-peculiar type for some (regular) κ_1, κ_2 satisfying $\kappa_1 < \kappa_2 = \mathfrak{t}$, see [3], $\mathfrak{t} \le \mathfrak{p} \le \max\{\kappa_1, \kappa_2\}$.
2) There is a (κ_1, κ_2)-peculiar cut iff there is a (κ_2, κ_1)-peculiar cut.

Proof. 1A) See [3].
2) Trivial. □

Observation 2.1.

If $(\bar\eta^{up}, \bar\eta^{dn})$ is a peculiar $(\kappa_{up}, \kappa_{dn})$-cut and if $A \subseteq \omega$ is infinite, $\eta \in {}^\omega\omega$ then:

(a) $\eta <_{J^{bd}_A} \eta^{up}_\alpha$ for every $\alpha < \kappa_{up}$ iff $\eta <_{J^{bd}_A} \eta^{dn}_\beta$ for every large enough $\beta < \kappa_{dn}$

(b) $\neg(\eta^{up}_\alpha <_{J^{bd}_A} \eta)$ for every $\alpha < \kappa_{up}$ iff $\neg(\eta^{dn}_\beta <_{J^{bd}_A} \eta)$ for every large enough $\beta < \kappa_{dn}$.

Proof. Clause (a): The implication $\Leftarrow$ is trivial as $\beta < \kappa_{\rm dn} \wedge \alpha < \kappa_{\rm up} \Rightarrow \eta^{\rm dn}_\beta <_{J^{\rm bd}_\omega} \eta^{\rm up}_\alpha$. So assume the leftside. We define $\eta' \in {}^\omega\omega$ by: $\eta'(n)$ is $\eta(n)$ if $n \in A$ and is 0 if $n \in \omega \backslash A$. Clearly $\eta' <_{J^{\rm bd}_\omega} \eta^{\rm up}_\alpha$ for every $\alpha < \kappa_{\rm up}$ hence by clause (δ) of Definition 2.1 we have $\eta' \le_{J^{\rm bd}_\omega} \eta^{\rm dn}_\beta$ for some large enough $\beta < \kappa_{\rm dn}$ hence $\eta = \eta' \restriction A \le_{J^{\rm bd}_A} \eta^{\rm dn}_{\beta+1} <_{J^{\rm bd}_A} \eta^{\rm dn}_\beta$ for every large enough $\beta < \kappa_{\rm dn}$.

Clause (b): Again the direction $\Leftarrow$ is obvious. For the other direction define $\eta' \in {}^\omega\omega$ by $\eta'(n)$ is $\eta(n)$ if $n \in A$ and is $\eta^{\rm up}_0(n)$ if $n \in \omega \backslash A$. So clearly $\alpha < \kappa_{\rm up} \Rightarrow \neg(\eta^{\rm up}_\alpha <_{J^{\rm bd}_\omega} \eta')$ hence $\alpha < \kappa_{\rm up} \Rightarrow \neg(\eta^{\rm up}_\alpha \le_{J^{\rm bd}_\omega} \eta)$ hence by clause (ε) of Definition 2.1 for some $\beta < \kappa_{\rm dn}$ we have $\neg(\eta^{\rm dn}_\beta <_{J^{\rm bd}_\omega} \eta')$. As $\eta^{\rm dn}_\beta <_{J^{\rm bd}_\omega} \eta^{\rm up}_0$, necessarily $\neg(\eta^{\rm dn}_\beta <_{J^{\rm bd}_A} \eta')$ but $\gamma \in [\beta, \kappa_{\rm dn}) \Rightarrow \eta^{\rm dn}_\beta \le_{J^{\rm bd}_A} \eta^{\rm dn}_\gamma$ hence $\gamma \in [\beta, \kappa_{\rm dn}) \Rightarrow \neg(\eta^{\rm dn}_\gamma <_{J^{\rm bd}_A} \eta') \Rightarrow \neg(\eta^{\rm dn}_\gamma <_{J^{\rm bd}_A} \eta)$, as required. □

We need the following from [3, 2.1]:

Claim 2.2.

Assume that $\kappa_1 \le \kappa_2$ are infinite regular cardinals, and there exists a (κ_1, κ_2)-peculiar cut in ${}^\omega\omega$.
Then for some σ-centered forcing notion $\mathbb{Q}$ of cardinality κ_1 and a sequence $\langle I_\alpha : \alpha < \kappa_2 \rangle$ of open dense subsets of $\mathbb{Q}$, there is no directed $G \subseteq \mathbb{Q}$ such that $(\forall \alpha < \kappa_2)(G \cap I_\alpha \ne \emptyset)$. Hence MA_{κ_2} fails.

Theorem 2.1.

Assume $\lambda = \mathrm{cf}(\lambda) = \lambda^{<\lambda} > \aleph_2, \lambda > \kappa = \mathrm{cf}(\kappa) \ge \aleph_1$ and $2^\lambda = \lambda^+$ and $(\forall \mu < \lambda)(\mu^{\aleph_0} < \lambda)$.
For some forcing $\mathbb{P}^*$ of cardinality λ^+ not adding new members to ${}^\lambda\mathbf{V}$ and $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}^*$ of a c.c.c. forcing we have $\Vdash_{\mathbb{P}^* * \underset{\sim}{\mathbb{Q}}^*}$ "$2^{\aleph_0} = \lambda^+$ and $\mathfrak{p} = \lambda$ and $\mathrm{MA}_{<\lambda}$ and there is a pair $(\bar f_1, \bar f^1)$ which is a peculiar (κ, λ)-cut".

Remark 2.1.

1) The proof of Theorem 2.1 is done in section 4 and broken into a series of Definitions and Claims, in particular we specify some of the free choices in the general iteration theorem.
2) In Choice 4.1(1), is $\mathrm{cf}(\delta) > \aleph_0$ necessary?
3) What if $\lambda = \aleph_2$? The problem is Claim 3.1(2). To eliminate this we may, instead quoting Claim 3.1(2), start by forcing $\bar\eta = \langle \eta_\alpha : \alpha < \omega_1 \rangle$ in $\mathbb{P}_{\aleph_0}$ and change some points.

Complementary to Theorem 2.1 is

Observation 2.2.

Assume $\lambda = \operatorname{cf}(\lambda) > \aleph_1$ and $\mu = \operatorname{cf}(\mu) = \mu^{<\lambda} > \lambda$ then for some c.c.c. forcing notion $\mathbb{P}$ of cardinality μ we have: $\Vdash_{\mathbb{P}}$ "$2^{\aleph_0} = \mu, \mathfrak{p} = \lambda$ and for no regular $\kappa < \lambda$ is there a peculiar (κ, λ)-cut so $\mathfrak{t} = \lambda$".

Proof. We choose $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \le \mu, \beta < \mu \rangle$ such that:

(*a*) $\bar{\mathbb{Q}}$ is an FS-iteration

(*b*) $\underset{\sim}{\mathbb{Q}}_\beta$ is a σ-centered forcing notion of cardinality $< \lambda$

(*c*) if $\alpha < \mu, \underset{\sim}{\mathbb{Q}}$ is a $\mathbb{P}_\alpha$-name of a σ-centered forcing notion of cardinality $< \lambda$ then for some $\beta \in [\alpha, \mu)$ we have $\underset{\sim}{\mathbb{Q}}_\beta = \underset{\sim}{\mathbb{Q}}$

(*d*) $\mathbb{Q}_0$ is adding λ Cohens, $\langle \underset{\sim}{r}_\varepsilon : \varepsilon < \lambda \rangle$ will witness $\mathfrak{p} \le \lambda$.

Clearly in $\mathbf{V}^{\mathbb{P}_\lambda}$ we have $2^{\aleph_0} = \lambda$, also every σ-centered forcing notion of cardinality $< \mu$, is from $\mathbf{V}^{\mathbb{P}_\alpha}$ for some $\alpha < \mu$, so as μ is regular we have

$(*)$ MA for σ-centered forcing notions of cardinality $\le \lambda$ or just $< \mu$ dense sets

Hence by Claim 2.2 there is no peculiar (κ_1, κ_2)-cut when $\aleph_1 \le \kappa_1 < \kappa_2 = \lambda$ (even $\kappa_1 < \kappa_2 < \mu, \kappa_1 < \lambda < \mu$). $\square$

3. Some specific forcing

Definition 3.1.

Let $\bar{\eta} =: \langle \eta_\alpha : \alpha < \alpha^* \rangle$ be a sequence of members of ${}^\omega\omega$ which is $<_{J^{\mathrm{bd}}_\omega}$-increasing or just $\le_{J^{\mathrm{bd}}_\omega}$-directed. We define the set $\mathcal{F}_{\bar{\eta}}$ and the forcing notion $\mathbb{Q} = \mathbb{Q}_{\bar{\eta}}$ and a generic real $\underset{\sim}{\nu}$ for $\mathbb{Q} = \mathbb{Q}_{\bar{\eta}}$ as follows:

(*a*) $\mathcal{F}_{\bar{\eta}} = \{\nu \in {}^\omega(\omega+1)$: if $\alpha < \ell g(\bar{\eta})$ then $\eta_\alpha <_{J^{\mathrm{bd}}_\omega} \nu\}$, here $\bar{\eta}$ is not[3] necessarily $<_{J^{\mathrm{bd}}_\omega}$-directed

(*b*) $\mathbb{Q}$ has the set of elements consisting of all triples $p = (\rho, \alpha, g) = (\rho^p, \alpha^p, g^p)$ (and $\alpha(p) = \alpha^p$) such that

- (α) $\rho \in {}^{\omega>}\omega$,
- (β) $\alpha < \ell g(\bar{\eta})$,
- (γ) $g \in \mathcal{F}_{\bar{\eta}}$, and
- (δ) if $n \in [\ell g(\rho), \omega)$ then $\eta_\alpha(n) \le g(n)$;

(*c*) $\le_{\mathbb{Q}}$ is defined by: $p \le_{\mathbb{Q}} q$ iff (both are elements of $\mathbb{Q}$ and)

- (α) $\rho^p \trianglelefteq \rho^q$,
- (β) $\alpha^p \le \alpha^q, \eta_{\alpha^p} \le_{J^{\mathrm{bd}}_\omega} \eta_{\alpha^q}$,
- (γ) $g^q \le g^p$,
- (δ) if $n \in [\ell g(\rho^q), \omega)$ then $\eta_{\alpha(p)}(n) \le \eta_{\alpha(q)}(n)$,
- (ε) if $n \in [\ell g(\rho^p), \ell g(\rho^q))$ then $\eta_{\alpha(p)}(n) \le \rho^q(n) \le g^p(n)$.

(*d*) For $\mathcal{F} \subseteq \mathcal{F}_{\bar{\eta}}$ which is downward directed (by $<_{J^{\mathrm{bd}}_\omega}$) we define $\mathbb{Q}_{\bar{\eta},\mathcal{F}}$ as $\mathbb{Q}_{\bar{\eta}} \restriction \{p \in \mathbb{Q}_{\bar{\eta}} : g^p \in \mathcal{F}\}$

(*e*) $\underset{\sim}{\nu} = \underset{\sim}{\nu}_{\mathbb{Q}} = \underset{\sim}{\nu}_{\mathbb{Q}_{\bar{\eta}}} = \cup\{\rho^p : p \in \underset{\sim}{G}_{\mathbb{Q}_{\bar{\eta}}}\}$.

[3] *the main case if that $\bar{\eta}$ is $\aleph_2$-directed; if $\bar{\eta}$ is $\le_{J^{\mathrm{bd}}_\omega}$-increasing, we can in clause (c)(β) omit $\eta_{\alpha(p)} \le_{J^{\mathrm{bd}}_\omega} \eta_{\alpha(q)}$*

Claim 3.1.
1) If $\bar{\eta} \in {}^{\gamma}({}^{\omega}\omega)$ then $\mathcal{F}_{\bar{\eta}}$ is downward directed, in fact if $g_1, g_2 \in \mathcal{F}_{\bar{\eta}}$ then $g = \min\{g_1, g_2\} \in \mathcal{F}_{\bar{\eta}}$, i.e., $g(n) = \min\{g_1(n), g_2(n)\}$ for $n < \omega$. Also "$f \in \mathcal{F}_{\bar{\eta}}$" is absolute.
[But possibly for every $\nu \in {}^{\omega}(\omega+1)$ we have: $\nu \in \mathcal{F}_{\bar{\eta}} \Leftrightarrow (\forall^* n)\nu(n) = \omega$].
2) If $\bar{\eta} \in {}^{\delta}({}^{\omega}\omega)$ is $<_{J^{bd}_{\omega}}$-increasing and $\mathrm{cf}(\delta) > \aleph_1$ then $\mathbb{Q}_{\bar{\eta}}$ is c.c.c.
3) Moreover any set of $\aleph_1$ members of $\mathbb{Q}_{\bar{\eta}}$ is included in the union of countably many directed subsets of $\mathbb{Q}_{\bar{\eta}}$.
4) Assume $\langle \mathbb{P}_\varepsilon : \varepsilon \leq \zeta\rangle$ is a $\lessdot$-increasing sequence of c.c.c. forcing notions, $\bar{\eta} = \langle \eta_\alpha : \alpha < \delta\rangle$ is a $\mathbb{P}_0$-name of a $<_{J^{bd}_{\omega}}$-increasing sequence of members of ${}^{\omega}\omega$ and $\mathrm{cf}(\delta) > \aleph_1$. For $\varepsilon \leq \zeta$ let $\underset{\sim}{\mathbb{Q}}_\varepsilon$ be the $\mathbb{P}_\varepsilon$-name of the forcing notion $\mathbb{Q}_{\bar{\eta}}$ as defined in $\mathbf{V}^{\mathbb{P}_\varepsilon}$. Then $\Vdash_{\mathbb{P}_\zeta}$ "$\underset{\sim}{\mathbb{Q}}_\varepsilon$ is $\subseteq$-increasing and $\leq_{ic}$-increasing for $\varepsilon \leq \zeta$ and it is c.c.c. and $\mathrm{cf}(\zeta) > \aleph_0 \Rightarrow \underset{\sim}{\mathbb{Q}}_\zeta = \cup\{\underset{\sim}{\mathbb{Q}}_\varepsilon : \varepsilon < \zeta\}$ is c.c.c."
5) Let $\bar{\eta} \in {}^{\delta}({}^{\omega}\omega)$ be as in part (2).

(a) If $\mathcal{F} \subseteq \mathcal{F}_{\bar{\eta}}$ is downward directed (by $\leq_{J^{bd}_{\omega}}$) then $\mathbb{Q}_{\bar{\eta},\mathcal{F}}$ is absolutely c.c.c.

(b) If $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_{\bar{\eta}}$ are downward directed then $\mathbb{Q}_{\bar{\eta},\mathcal{F}_1} \leq_{ic} \mathbb{Q}_{\bar{\eta},\mathcal{F}_2}$.

6)

(a) $\Vdash_{\mathbb{Q}_{\bar{\eta}}}$ "$\underset{\sim}{\nu} \in {}^{\omega}\omega$ and $\mathbf{V}[\underset{\sim}{G}] = \mathbf{V}[\underset{\sim}{\nu}]$"

(b) $p \Vdash_{\mathbb{Q}_{\bar{\eta}}}$ "$\rho^p \triangleleft \underset{\sim}{\nu}$ and $n \in [\ell g(\rho), \omega) \Rightarrow \eta_{\alpha(p)}(n) \leq \underset{\sim}{\nu}(n) \leq g^p(n)$"

(c) $\Vdash_{\mathbb{Q}_{\bar{\eta}}}$ "$p \in G$ iff $\rho \triangleleft \underset{\sim}{\nu} \wedge (\forall n)(\ell g(\rho) \leq n < \omega \Rightarrow \eta_\alpha(n) \leq \underset{\sim}{\nu}(n) \leq g^p(n))$"

(d) $\Vdash_{\mathbb{Q}_{\bar{\eta}}}$ "$\underset{\sim}{\nu} \in \mathcal{F}_{\bar{\eta}}$, i.e. $\underset{\sim}{\nu}(n) \in \mathcal{F}^{\mathbf{V}[\mathbb{Q}_{\bar{\eta}}]}$"

(e) $\Vdash_{\mathbb{Q}_{\bar{\eta}}}$ "for every $f \in ({}^{\omega}\omega)^{\mathbf{V}}$ we have $f \in \mathcal{F}_{\bar{\eta}}$ iff $f \in \mathcal{F}^{\mathbf{V}}_{\bar{\eta}}$ iff $\underset{\sim}{\nu} \leq_{J^{bd}_{\omega}} f$".

Proof. 1) Trivial.
2) Assume $p_\varepsilon \in \mathbb{Q}_{\bar{\eta}}$ for $\varepsilon < \omega_1$. So $\{\alpha(p_\varepsilon) : \varepsilon < \omega_1\}$ is a set of $\leq \aleph_1$ ordinals $< \delta$. But $\mathrm{cf}(\delta) > \aleph_1$ hence there is $\alpha(*) < \delta$ such that $\varepsilon < \omega_1 \Rightarrow \alpha(p_\varepsilon) < \alpha(*)$. For each ε let $n_\varepsilon =$ Min$\{n$: for every $k \in [n, \omega)$ we have $\eta_{\alpha(p_\varepsilon)}(k) \leq \eta_{\alpha(*)}(k) \leq g^{p_\varepsilon}(k)\}$. It is well defined because $\eta_{\alpha(p_\varepsilon)} <_{J^{bd}_{\omega}} \eta_{\alpha(*)} <_{J^{bd}_{\omega}} g^{p_\varepsilon}$ recalling $\alpha(p_\varepsilon) < \alpha(*)$ and $g^{p_\varepsilon} \in \mathcal{F}_{\bar{\eta}}$.
So clearly for some $\mathrm{x} = (\rho^*, n^*, \eta^*, \nu^*)$ the following set is uncountable

$$\begin{aligned}\mathcal{U} = \mathcal{U}_{\mathrm{x}} = \{\varepsilon < \omega_1 :\ & \rho^{p_\varepsilon} = \rho^* \text{ and } n_\varepsilon = n^* \text{ and } \eta_{\alpha(p_\varepsilon)} \restriction n^* = \eta^* \\ & \text{and } g^{p_\varepsilon} \restriction n^* = \nu^*\}.\end{aligned}$$

Let

$$\begin{aligned}\mathbb{Q}' = \mathbb{Q}'_{\mathrm{x}} =: \{p \in \mathbb{Q}_{\bar{\eta}} :\ & \ell g(\rho^p) \geq \ell g(\rho^*), \rho^p \restriction \ell g(\rho^*) = \rho^* \text{ and } \rho^p \restriction [\ell g(\rho^*), \ell g(\rho^p)) \subseteq \eta_{\alpha(*)} \\ & \text{and } \alpha(p) < \alpha(*), \text{ and } \eta_{\alpha(p)} \restriction n^* = \eta^* \text{ and } g^p \restriction n^* = \nu^* \\ & \text{and } n \in [n^*, \omega) \Rightarrow \eta_{\alpha(p)}(n) \leq \eta_{\alpha(*)}(n) \leq g^p(n)\}.\end{aligned}$$

Clearly

$\circledast_1$ $\{p_\varepsilon : \varepsilon \in \mathcal{U}\} \subseteq \mathbb{Q}'$

$\circledast_2$ $\mathbb{Q}' \subseteq \mathbb{Q}_{\bar{\eta}}$ is directed.

So we are done.
3) The proof of part (2) proves this.
4),5) First we can check clause (b) of part (5) by the definitions of $\mathbb{Q}_{\bar{\eta},\mathcal{F}}, \mathbb{Q}_{\bar{\eta}}$. Second, concerning "$\mathbb{Q}_{\bar{\eta},\mathcal{F}}$ is absolutely c.c.c." (i.e. clause (a) of part (5)) note that if $\mathbb{P}$ is c.c.c., $G \subseteq \mathbb{P}$ is generic over $\mathbf{V}$ then $\mathbb{Q}^{\mathbf{V}}_{\bar{\eta},\mathcal{F}} = \mathbb{Q}^{\mathbf{V}[G]}_{\bar{\eta},\mathcal{F}}$ and $\mathbb{Q}^{\mathbf{V}}_{\bar{\eta},\mathcal{F}} \leq_{\mathrm{ic}} \mathbb{Q}^{\mathbf{V}}_{\bar{\eta}} \leq_{\mathrm{ic}} \mathbb{Q}^{\mathbf{V}[G]}_{\bar{\eta}}$ by clause (b) and the last one is c.c.c. (as $\mathbf{V}[G] \models$ "$\mathrm{cf}(\ell g(\bar{\eta})) > \aleph_1$"). Hence $\mathbb{Q}^{\mathbf{V}}_{\bar{\eta},\mathcal{F}}$ is c.c.c. even in $\mathbf{V}[G]$ as required. Turning to part (4), letting $\mathcal{F}_\varepsilon = (\mathcal{F}_{\bar{\eta}})^{\mathbf{V}[\mathbb{P}_\varepsilon]}$, clearly $\Vdash_{\mathbb{P}_{\varepsilon_2}}$ "$\underset{\sim}{\mathbb{Q}}_{\varepsilon_1} = \underset{\sim}{\mathbb{Q}}_{\bar{\eta},\mathcal{F}_{\varepsilon_1}}$" for $\varepsilon_1 < \varepsilon_2 < \zeta$. Now about the c.c.c., as $\mathbb{P}_\varepsilon$ is c.c.c., it preserves "$\mathrm{cf}(\delta) > \aleph_1$", so the proof of part (2) works.
6) Easy, too. $\square$

Definition 3.2.

Assume $\bar{A} = \langle A_\alpha : \alpha < \alpha^* \rangle$ is a $\subseteq^*$-decreasing sequence of members of $[\omega]^{\aleph_0}$. We define the forcing notion $\mathbb{Q}_{\bar{A}}$ and the generic real $\underset{\sim}{w}$ by:

(A) $p \in \mathbb{Q}_{\bar{A}}$ iff

(a) $p = (w, n, A_\alpha) = (w_p, n_p, A_{\alpha(p)})$,

(b) $w \subseteq \omega$ is finite,

(c) $\alpha < \alpha^*$ and $n < \omega$,

(B) $p \leq_{\mathbb{Q}_{\bar{A}}} q$ iff

(a) $w_p \subseteq w_q \subseteq w_p \cup (A_{\alpha(p)} \backslash n_p)$

(b) $n_p \leq n_q$

(c) $A_{\alpha(p)} \backslash n_p \supseteq A_{\alpha(q)} \backslash n_q$

(C) $\underset{\sim}{w} = \cup\{w_p : p \in \underset{\sim}{G}_{\mathbb{Q}_{\bar{A}}}\}$.

Claim 3.2.

Let $\bar{A}$ be as in Definition 3.2.
1) $\mathbb{Q}_{\bar{A}}$ is a c.c.c. and even σ-centered forcing notion.
2) $\Vdash_{\mathbb{Q}_{\bar{A}}}$ "$\underset{\sim}{w} \in [\omega]^{\aleph_0}$ is $\subseteq^* A_\alpha$ for each $\alpha < \alpha^*$" and $\mathbf{V}[\underset{\sim}{G}] = \mathbf{V}[\underset{\sim}{w}]$.
3) Moreover, for every $p \in \mathbb{Q}_{\bar{A}}$ we have $\Vdash$ "$p \in \underset{\sim}{G}$ iff $w_p \subseteq \underset{\sim}{w} \subseteq (A_{\alpha(p)} \backslash n_p) \cup w_p$".

Proof. Easy. $\square$

Claim 3.3.

Assume $\bar{\eta} \in {}^{\delta}({}^{\omega}\omega)$ is $\leq_{J^{bd}_\omega}$-increasing.
1) If $\mathcal{F} \subseteq \mathcal{F}_{\bar{\eta}}$ is downward cofinal in $(\mathcal{F}_{\bar{\eta}}, <_{J^{bd}_\omega})$, i.e. $(\forall \nu \in \mathcal{F}_{\bar{\eta}})(\exists \rho \in \mathcal{F})(\rho <_{J^{bd}_\omega} \nu)$ and $\mathcal{U} \subseteq \delta$ is unbounded then $\mathbb{Q}_{\bar{\eta} \restriction \mathcal{U}, \mathcal{F}} = \{p \in \mathbb{Q}_{\bar{\eta}} : \alpha^p \in \mathcal{U}$ and $g^p \in \mathcal{F}\}$ is (not only $\subseteq \mathbb{Q}_{\bar{\eta}}$ but also is) a dense subset of $\mathbb{Q}_{\bar{\eta}}$.
2) If $\mathrm{cf}(\delta) > \aleph_0$ and $\mathbb{R}$ is Cohen forcing then $\Vdash_{\mathbb{R}}$ "$\mathbb{Q}^{\mathbf{V}}_{\bar{\eta}}$ is dense in $\mathbb{Q}^{\mathbf{V}[\underset{\sim}{G}]}_{\bar{\eta}}$".

Remark 3.1.

1) We can replace "$\eta_\alpha \leq_{J^{bd}_\omega} \rho$" by "$\rho$ belongs to the F_σ-set $\mathbf{B}_\alpha$", where $\mathbf{B}_\alpha$ denotes a Borel set from the ground model, i.e. its definition.
2) Used in Definition/Claim 4.2.

Proof. 1) Check.
2) See next claim. $\square$

Claim 3.4.

Let $\bar{\eta} = \langle \eta_\gamma : \gamma < \delta \rangle$ is $\leq_{J^{bd}_\omega}$-increasing in ${}^{\omega}\omega$.
1) If $\mathbb{P}$ is a forcing notion of cardinality $< \mathrm{cf}(\delta)$ then $\Vdash_{\mathbb{P}}$ "$\mathbb{Q}^{\mathbf{V}}_{\bar{\eta}}$ is dense in $\mathbb{Q}^{\mathbf{V}[\underset{\sim}{G}_\eta]}$".
2) A sufficient condition for the conclusion of part (1) is:

$\odot^{cf(\delta)}_{\mathbb{P}}$ *for every $X \in [\mathbb{P}]^{cf(\delta)}$ there is $Y \in [\mathbb{P}]^{<cf(\delta)}$ such that $(\forall p \in X)(\exists q \in Y)(p \leq q)$.*

2A) We can weaken the condition to: if $X \in [\mathbb{P}]^{\mathrm{cf}(\delta)}$ then for some $q \in \mathbb{P}$, $\mathrm{cf}(\delta) \leq |\{p \in X : p \leq_{\mathbb{P}} q\}|$.
3) If $\langle A_\alpha : \alpha < \delta^* \rangle$ is $\subseteq^*$-decreasing sequence of infinite subsets of ω and $\mathrm{cf}(\delta^*) \neq \mathrm{cf}(\delta)$ then $\odot^{cf(\delta)}_{\mathbb{Q}_{\bar{A}}}$ holds.

Proof. 1) By part (2).
2) Let $\mathcal{U} \subseteq \delta$ be unbounded of order type $\mathrm{cf}(\delta)$. Assume $p \in \mathbb{P}$ and $\underset{\sim}{\nu}$ satisfies $p \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\nu} \in \mathcal{F}^{\mathbf{V}[\underset{\sim}{G}]}_{\bar{\eta}}$". So for every $\gamma \in \mathcal{U}$ we have $p \Vdash_{\mathbb{P}}$ "$\eta_\gamma <_{J^{bd}_\omega} \underset{\sim}{\nu} \in {}^{\omega}\omega$", hence there is a pair (p_γ, n_γ) such that:

(∗) (a) $p \le_{\mathbb{P}} p_\gamma$

(b) $n_\gamma < \omega$

(c) $p_\varepsilon \Vdash_{\mathbb{P}}$ "$(\forall n)(n_\gamma \le n < \omega \Rightarrow \eta_\varepsilon(n) < \underset{\sim}{\nu}(n))$.

We apply the assumption to the set $X = \{p_\varepsilon : \gamma \in \mathcal{U}\}$ and get $Y \in [\mathbb{P}]^{<\mathrm{cf}(\delta)}$ as there. So for every $\gamma \in \mathcal{U}$ there is q_γ such that $p_\gamma \le_{\mathbb{P}} q_\gamma \in Y$. As $|Y \times \omega| = |Y| + \aleph_0 <$ cf$(\delta) = |\mathcal{U}|$ there is a pair $(q_*, n_*) \in Y \times \omega$ such that $\mathcal{U}' \subseteq \delta$ is unbounded where $\mathcal{U}' := \{\gamma \in \mathcal{U} : q_\gamma = q_*$ and $n_\gamma = n_*\}$. Lastly, define $\nu_* \in {}^\omega(\omega+1)$ by $\nu_*(n)$ is 0 if $n < n_*$ is $\cup\{\eta_\alpha(n)+1 : \alpha \in \mathcal{U}'\}$ when $n \ge n_*$.
Clearly

⊛ (a) $\nu_* \in {}^\omega(\omega+1)$

(b) $\gamma \in \mathcal{U}' \Rightarrow \eta_\alpha \restriction [n_*, \omega) < \nu_* \restriction [n_*, \omega)$

(c) if $\gamma < \delta$ then $\eta_\alpha <_{J^{\mathrm{bd}}_\omega} \nu_*$

(d) $\nu_* \in \mathcal{F}^{\mathbf{V}}_{\bar\eta}$

(e) $p \le q_*$

(f) $q_* \Vdash_{\mathbb{P}}$ "$\nu_* \le \underset{\sim}{\nu}$".

So we are done.
2A) Similarly.
3) If cf$(\delta^*) <$ cf(δ) let $\mathcal{U} \subseteq \delta^*$ be unbounded of order type cf(δ^*) and $\mathbb{Q}'_{\bar A} = \{p \in \mathbb{Q}_{\bar A} : \alpha^p \in \mathcal{U}\}$, it is dense in $\mathbb{Q}_{\bar A}$ and has cardinality $\le \aleph_0 +$ cf$(\delta^*) <$ cf(δ), so we are done.
If cf$(\delta^*) >$ cf(δ) and $X \in [\mathbb{P}]^{\mathrm{cf}(\delta)}$, let $\alpha(*) = \sup\{\alpha^p : p \in X\}$ and $Y = \{p \in \mathbb{Q}_{\bar A} : \alpha^p = \alpha(*)\}$.
The rest should be clear. □

4. Proof of Theorem 2.1

Choice 4.1.

1) $S \subseteq \{\delta < \lambda : cf(\delta) > \aleph_0\}$ stationary.
2) $\bar\eta$ is as in Definition/Claim 4.1 below, so possibly a preliminary forcing of cardinality κ we have such $\bar\eta$.

Definition/Claim 4.1.

1) Assume $\kappa =$ cf$(\kappa) \in [\aleph_2, \lambda)$ and $\bar\eta = \langle \eta_\alpha : \alpha < \kappa\rangle$ is an $<_{J^{\mathrm{bd}}_\omega}$-increasing sequence in ${}^\omega\omega$ and $\delta \in \lambda \backslash \omega_1$ a limit ordinal and $\gamma \le \lambda$. Then the following $\mathbf{s} = \mathbf{s}_{\delta,\gamma}$ is a winning strategy of COM in the game $\partial_{\delta,<\gamma}$: COM just preserves:

⊗ (a) if for every $\zeta < \varepsilon$ we have $(\alpha) + (\beta)$ then we have $(*)$ where

(α) $\mathbb{P}_{\mathbf{q}_\zeta} = \mathbb{P}_{\mathbf{p}_\zeta} * \underset{\sim}{\mathbb{Q}}_{\bar\eta}$ where $\underset{\sim}{\mathbb{Q}}_{\bar\eta}$ is from Definition 3.1 and in $\mathbf{V}^{\mathbb{P}[\mathbf{p}_\zeta]}$, i.e. is a $\mathbb{P}_{\mathbf{p}_\zeta}$-name

(β) $\mathbb{P}_{\mathbf{p}_\zeta} * \underset{\sim}{\mathbb{Q}}_{\bar\eta} \lessdot \mathbb{P}_{\mathbf{p}_\varepsilon} * \underset{\sim}{\mathbb{Q}}_{\bar\eta}$

($*$) $\mathbb{P}_{\mathbf{q}_\varepsilon} = \mathbb{P}_{\mathbf{p}_\varepsilon} * \underset{\sim}{\mathbb{Q}}_{\bar\eta}$, so we have to interpret $\mathbb{P}_{\mathbf{q}_\varepsilon}$ such that its set of elements is $\subseteq \mathcal{H}_{<\aleph_1}(u^{\mathbf{q}_\varepsilon})$ which is easy, i.e. it is $\mathbb{P}_{\mathbf{p}_\varepsilon} \cup \{(p, \underset{\sim}{r})$: $p \in \mathbb{P}_{\mathbf{p}_\varepsilon}$ and $\underset{\sim}{r}$ is a canonical $\mathbb{P}_{\mathbf{p}_\varepsilon}$-name of a member of $\underset{\sim}{\mathbb{Q}}_{\bar\eta}$ (i.e. use $\aleph_0$ maximal antichains, etc.)$\}$

(b) if in (a) clause (α) holds but (β) fail then

(α) the set of elements of $\mathbb{P}_{\mathbf{q}_\varepsilon}$ is $\mathbb{P}_{\mathbf{p}_\varepsilon} \cup \{(p, \underset{\sim}{r})$: for some $\zeta < \varepsilon$ and $(p', \underset{\sim}{r}) \in \mathbb{P}_{\mathbf{q}_\zeta}$ we have $\mathbb{P}_{\mathbf{p}_\varepsilon} \models$ "$p' \le p$"$\}$

(β) the order is defined naturally

(c) if in (a), clause (α) fail, let ζ be minimal such that it fails, and then

(α) the set of elements of $\mathbb{P}_{\mathbf{q}_\varepsilon}$ is $\mathbb{P}_{\mathbf{p}_\varepsilon} \cup \{(p, \underset{\sim}{r})$: for some $\xi < \zeta$ and p' we have $(p', \underset{\sim}{r}) \in \mathbb{P}_{\mathbf{q}_\zeta}$ and $\mathbb{P}_{\mathbf{p}_\varepsilon} \models "p' \leq p"\}$

(β) the order is natural.

Remark 4.1.
In Definition/Claim 4.1 we can combine clauses (b) and (c).

Proof. By Claim 3.1 this is easy, see in particular Claim 3.1(4). $\square$

Technically it is more convenient to use the (essentially equivalent) variant.

Definition/Claim 4.2.
1) We replace $\mathbb{P}_{\mathbf{q}_\zeta} = \mathbb{P}_{\mathbf{p}_\zeta} * \mathbb{Q}_{\bar{\eta}}$ by $\mathbb{P}_{\mathbf{q}_\zeta} = \mathbb{P}_{\mathbf{p}_\zeta} * \mathbb{Q}_{\bar{\eta}, \mathcal{F}_\zeta}$ where

$$\begin{aligned} \mathcal{F}_\zeta = \{\nu :\ & \text{for some } \varepsilon \leq \zeta, \nu \in \mathcal{F}_{\bar{\eta}}^{\mathbf{V}[\mathbb{P}[\mathbf{p}_\varepsilon]]} \text{ but} \\ & \text{for no } \xi < \varepsilon \text{ and } \nu_1 \in \mathcal{F}_{\bar{\eta}}^{\mathbf{V}[\mathbb{P}[\mathbf{p}_\xi]]} \text{ do we have} \\ & \nu_1 \leq_{J_\omega^{\mathrm{bd}}} \nu\}. \end{aligned}$$

2) No change by Claim 3.3s(1).

Remark 4.2.
In Definition/Claim 4.1 we can use $\underset{\sim}{\bar{\eta}} = \langle \underset{\sim}{\eta}_\alpha : \alpha < \kappa \rangle$ say a $\mathbb{P}_{\mathbf{k}_0}$-name, but then for the game $\partial_{\delta, f(\delta)}$ we better assume $\delta \in E_{\mathbf{k}_0}$ and $\underset{\sim}{\bar{\eta}}$ is a $\mathbb{P}[\mathbf{p}_\delta^{\mathbf{k}}]$-name.

Definition/Claim 4.3.
1) Let $\mathbf{k}_* \in K_\lambda^2$ and $\underset{\sim}{\nu}_\alpha$ ($\alpha < \lambda$) be chosen as follows:

(a) $E_{\mathbf{k}_*} = \lambda$ and $u[\mathbf{p}_\alpha^{\mathbf{k}_*}] = \omega_1 + \alpha$ hence $u[\mathbf{k}_*] = \lambda$

(b) $\mathbb{P}_\alpha^{\mathbf{k}_*}$ is $\lessdot$-increasing continuous

(c) $\mathbb{P}_{\alpha+1}^{\mathbf{k}_*} = \mathbb{P}_\alpha^{\mathbf{k}_*} * \underset{\sim}{\mathbb{Q}}_{\bar{\eta}}$ and $\underset{\sim}{\nu}_\delta$ is the generic (for this copy) of $\underset{\sim}{\mathbb{Q}}_{\bar{\eta}}$ where $\bar{\eta}$ is from In Definition/Claim 4.1

(d) $S_{\mathbf{k}_*} = S$ (a stationary subset of λ), $\delta \in S \Rightarrow \operatorname{cf}(\delta) > \aleph_0$

(e) for each $\delta \in S_{\mathbf{k}_*}, \mathbf{s}_\delta^{\mathbf{k}_*} = \mathbf{s}_{\delta, \lambda}$ is from In Definition/Claim 4.1 or better Definition/Claim 4.2

(f) $\mathbf{g}_\delta^{\mathbf{k}_*}$ is $\langle (\mathbf{p}_\delta^{\mathbf{k}_*}, \mathbf{p}_{\delta+1}^{\mathbf{k}_*}) \rangle$, $\operatorname{mv}(\mathbf{g}_\delta^{\mathbf{k}_*}) = 0$, only one move was done.

2) If $\mathbf{k}_* \leq_{K_2} \mathbf{k}$ then $\Vdash_{\mathbb{P}_\mathbf{k}}$ "the pair $(\langle \underset{\sim}{\nu}_\alpha : \alpha < \lambda \rangle, \langle \eta_i : i < \kappa \rangle)$ is a (λ, κ)-peculiar cut".

Proof. Clear (by In Definition/Claim 4.1). $\square$

Definition 4.4.
Let $\mathbb{P}^*$ be the following forcing notion:

(A) the members are $\mathbf{k}$ such that

(a) $\mathbf{k}_* \leq_{K_2} \mathbf{k} \in K_\lambda^2$

(b) $u[\mathbf{k}] = \cup\{u[\mathbf{p}_\alpha^{\mathbf{k}}] : \alpha \in E_\mathbf{k}\}$ is an ordinal $< \lambda^+$ (but of course $\geq \lambda$)

(c) $S_\mathbf{k} = S_{\mathbf{k}_*}$ and $\mathbf{s}_\delta^{\mathbf{k}} = \mathbf{s}_\delta^{\mathbf{k}_*}$ for $\delta \in S_\mathbf{k}$

(B) the order: $\leq_{K_\lambda^2}$.

Definition 4.5.
We define the $\mathbb{P}^*$-name $\underset{\sim}{\mathbb{Q}}^*$ as

$$\cup\{\mathbb{P}^{\mathbf{k}}_\lambda : \mathbf{k} \in \underset{\sim}{\mathbb{G}}_{\mathbb{P}^*}\} = \cup\{\mathbb{P}_{\mathbf{p}}[\mathbf{p}^{\mathbf{k}}_\alpha] : \alpha \in E_{\mathbf{k}} \text{ and } \mathbf{k} \in \underset{\sim}{G}_{\mathbb{P}^*}\}.$$

Claim 4.1.
1) $\mathbb{P}^*$ has cardinality λ^+.
2) $\mathbb{P}^*$ is strategically $(\lambda+1)$-complete hence add no new member to ${}^\lambda\mathbf{V}$.
3) $\Vdash_{\mathbb{P}^*}$ "$\underset{\sim}{\mathbb{Q}}^*$ is c.c.c. of cardinality $\leq \lambda^+$".
4) $\mathbb{P}^* * \underset{\sim}{\mathbb{Q}}^*$ is a forcing notion of cardinality λ^+ neither collapsing any cardinal nor changing cofinalities.
5) If $\mathbf{k} \in \mathbb{P}^*$ then $\mathbf{k} \Vdash_{\mathbb{P}^*}$ "$\mathbb{P}_{\mathbf{k}} \lessdot \underset{\sim}{\mathbb{Q}}^*$" hence $\Vdash_{\mathbb{P}^*}$ "$\mathbb{P}_{\mathbf{k}_*} \lessdot \underset{\sim}{\mathbb{Q}}^*$".

Proof. 1) Trivial.
2) By Claim 1.5.
3) $\underset{\sim}{G}_{\mathbb{P}^*}$ is $(< \lambda^+)$-directed.
4),5) Should be clear. $\square$

Claim 4.2.
If $\mathbf{k} \in \mathbb{P}^*$ and $G \subseteq \mathbb{P}_{\mathbf{k}}$ is generic over $\mathbf{V}$ then

(a) $\langle \underset{\sim}{\nu}_\alpha[G \cap \mathbb{P}_{\mathbf{k}_*}] : \alpha < \lambda\rangle$ is $<_{J^{bd}_\omega}$-decreasing and $i < \kappa \Rightarrow \eta_i <_{J^{bd}_\omega} \underset{\sim}{\nu}_\alpha[G \cap \mathbb{P}_{\mathbf{k}_*}]$, (this concerns $\mathbb{P}_{\mathbf{k}_*}$ only)

(b) if $\rho \in ({}^\omega\omega)^{\mathbf{V}[G]}$ and $i < \kappa \Rightarrow \eta_i <_{J^{bd}_\omega} \rho$ then for every $\alpha < \lambda$ large enough we have $\underset{\sim}{\nu}_\alpha[G] <_{J^{bd}_\omega} \rho$

(c) if $\rho \in ({}^\omega\omega)^{\mathbf{V}[G]}$ and $i < \kappa \Rightarrow \eta_i \not\leq_{J^{bd}_\omega} \rho$ then for every $\alpha < \lambda$ large enough we have $\underset{\sim}{\nu}_\alpha[G] \not\leq_{J^{bd}_\omega} \rho$.

Proof. Should be clear. $\square$

Claim 4.3.
1) If $\mathbf{k} \in \mathbb{P}^*$ and $\underset{\sim}{\mathbb{Q}}$ is a $\mathbb{P}_{\mathbf{k}}$-name of a c.c.c. forcing of cardinality $< \lambda$ and $\alpha \in E_{\mathbf{k}}$ and $\underset{\sim}{\mathbb{Q}}$ is a $\mathbb{P}[\mathbf{p}^{\mathbf{k}}_\alpha]$-name then for some $\mathbf{k}_1$ we have:

(a) $\mathbf{k} \leq_{K_1} \mathbf{k}_1 \in \mathbb{P}^*$

(b) $\Vdash_{\mathbb{P}_{\mathbf{k}_1}}$ "there is a subset of $\underset{\sim}{\mathbb{Q}}$ generic over $\mathbf{V}[\underset{\sim}{G}_{\mathbb{P}_{\mathbf{k}_1}} \cap \mathbb{P}[\mathbf{p}^{\mathbf{k}}_\alpha]]$".

2) In (1) if $\Vdash_{\mathbb{P}[\mathbf{p}^{\mathbf{k}}_\alpha] * \underset{\sim}{\mathbb{Q}}}$ "there is $\rho \in {}^\omega 2$ not in $\mathbf{V}[\underset{\sim}{G}_{\mathbb{P}_{\mathbf{k}}}]$" then $\Vdash_{\mathbb{P}_{\mathbf{k}_1}}$ "there is $\rho \in {}^\omega 2$ not in $\mathbf{V}[\underset{\sim}{G}_{\mathbb{P}_{\mathbf{k}}}]$".

Proof. 1) By Claim 1.6.
2) By part (1) and clause (η) of Claim 1.6. $\square$

Proof of Theorem 2.1. We force by $\mathbb{P}^* * \underset{\sim}{\mathbb{Q}}^*$ where $\mathbb{P}^*$ is defined in Definition 4.4 and the $\mathbb{P}^*$-name $\underset{\sim}{\mathbb{Q}}^*$ is defined in Definition 4.5. By Claim 4.1(4) we know that no cardinal is collapsed and no cofinality is changed. We know that $\Vdash_{\mathbb{P}^* * \underset{\sim}{\mathbb{Q}}^*}$ "$2^{\aleph_0} \leq \lambda^+$" because $|\mathbb{P}^*| = \lambda^+$ and $\Vdash_{\mathbb{P}^*}$ "$\underset{\sim}{\mathbb{Q}}^*$ has cardinality $\leq \lambda^+$", so $\mathbb{P}^* * \underset{\sim}{\mathbb{Q}}^*$ has cardinality λ^+, see Claim 4.1(3),(4).
Also $\Vdash_{\mathbb{P}^* * \underset{\sim}{\mathbb{Q}}}$ "$2^{\aleph_0} \geq \lambda^+$" as by Claim 4.1(2) it suffices to prove: for every $\mathbf{k}_1 \in \mathbb{P}^*$ there is $\mathbf{k}_2 \in \mathbb{P}^*$ such that $\mathbf{k}_1 \leq_{K_2} \mathbf{k}_2$ and forcing by $\mathbb{P}_{\mathbf{k}_2}/\mathbb{P}_{\mathbf{k}_1}$ adds a real, which holds by Claim 4.3(2).
Lastly, we have to prove that $(\langle \eta_i : i < \kappa\rangle, \langle \underset{\sim}{\nu}_\alpha : \alpha < \lambda\rangle)$ is a peculiar cut. In Definition 2.1 clauses $(\alpha), (\beta), (\gamma)$ holds by the choice of $\mathbf{k}_*$. As for clauses $(\delta), (\varepsilon)$ to check this it suffices to prove that for every $f \in {}^\omega\omega$ they hold, so it is suffice to check it in any sub-universe to which $(\bar{\eta}, \bar{\nu}), f$ belong. Hence by claim 4.1(1) it suffices to check it in $\mathbf{V}^{\mathbb{P}_{\mathbf{k}}}$ for any $\mathbf{k} \in \mathbb{P}^*$. But this holds by Definition/Claim 4.3(2). $\square$

5. Quite general applications

Theorem 5.1.

Assume $\lambda = \mathrm{cf}(\lambda) = \lambda^{<\lambda} > \aleph_2$ and $2^\lambda = \lambda^+$ and $(\forall \mu < \lambda)(\mu^{\aleph_0} < \lambda)$. Then for some forcing $\mathbb{P}^*$ of cardinality λ^+ not adding new members to ${}^\lambda\mathbf{V}$ and $\mathbb{P}^*$-name $\underset{\sim}{\mathbb{Q}}^*$ of a c.c.c. forcing it is forced, i.e. $\Vdash_{\mathbb{P}^* * \underset{\sim}{\mathbb{Q}}^*}$ that $2^{\aleph_0} = \lambda^+$ and

(a) $\mathfrak{p} = \lambda$ and $\mathrm{MA}_{<\lambda}$

(b) for every regular $\kappa \in (\aleph_1, \lambda)$ there is a (κ, λ)-peculiar cut $(\langle \eta^\kappa_i : i < \kappa\rangle, \langle \nu^\kappa_\alpha : \alpha < \lambda\rangle)$ hence $\mathfrak{p} = \mathfrak{t} = \lambda$

(c) if $\mathbb{Q}$ is a (definition of a) Suslin c.c.c. forcing notion defined by $\bar{\varphi}$ possibly with a real parameter from $\mathbf{V}$, then we can find a sequence $\langle \nu_{\mathbb{Q},\underset{\sim}{\eta},\alpha} : \alpha < \lambda\rangle$ which is positive for $(\mathbb{Q}, \underset{\sim}{\eta})$, see [1], e.g. $\mathrm{non}(\mathrm{null}) = \lambda$

(d) in particular $\mathfrak{b} = \mathfrak{d} = \lambda$.

Remark 5.1.

0) In clause (c) we can let $\mathbb{Q}$ be a c.c.c nep forcing (see [1]), with $\mathfrak{B}, \mathfrak{C}$ of cardinality $\leq \lambda$ and $\underset{\sim}{\eta}$ is a $\mathbb{Q}$-name of a real (i.e. member of ${}^\omega 2$).
1) Concerning Theorem 5.1 as remarked earlier in Remark 1.5(1), if we like to deal with Suslin forcing defined with a real parameter from $\mathbf{V}^{\mathbb{P}^* * \underset{\sim}{\mathbb{Q}}^+}$ and similarly for $\mathfrak{B}, \mathfrak{C}$ we in a sense have to change/create new strategies. We could start with $\langle S_\alpha : \alpha < \lambda^+\rangle$ such that $S_\alpha \subseteq \lambda, \alpha < \beta \Rightarrow |S_\alpha \backslash S_\beta| < \lambda$ and $S_{\alpha+1} \backslash S_\alpha$ is a stationary subset of λ. But we can code this in the strategies, do nothing till you know the definition of the forcing.
2) We may like to strengthen Theorem 5.1 by demanding

(c) for some $\mathbb{Q}$ as in clause (c) of Theorem 5.1, $\mathrm{MA}_{\mathbb{Q}}$ holds or even for a dense set of $\mathbf{k}_1 \in \mathbb{P}^*$, see below, there is $\mathbf{k}_2 \in \mathbb{P}^*$ such that $\mathbf{k}_1 \leq_{K_2} \mathbf{k}_2$ and $\mathbb{P}_{\mathbf{k}_2}/\mathbb{P}_{\mathbf{k}_1}$ is $\mathbb{Q}^{\mathbf{V}[\mathbb{P}_{\mathbf{k}_1}]}$.

For this we have to restrict the family of $\mathbb{Q}$'s in clause (c) such that those two families are orthogonal, i.e. commute. Note, however, that for Suslin c.c.c forcing this is rare, see [1].
3) This solves the second Bartoszynski test problem, i.e. (B) of Problem 0.1.
4) So $(\bar{\varphi}, \mathbb{Q}, \nu, \underset{\sim}{\eta})$ in clause (c) of Theorem 5.1 satisfies

(a) $\nu \in {}^\omega 2$

(b) $\bar{\varphi} = (\varphi_0, \varphi_1, \varphi_2), \Sigma_1$ formulas with the real parameter ν

(c) $\mathbb{Q}$ is the forcing notion defined by:

- set of elements $\{\rho \in {}^\omega 2 : \varphi_0[\rho]\}$
- quasi order $\leq_{\mathbb{Q}} = \{(\rho_1, \rho_2) : \rho_1, \rho_2 \in {}^\omega 2, \varphi_1(\rho_1, \rho_2)\}$
- incompatibility in $\mathbb{Q}$ is defined by φ_3

(d) $\underset{\sim}{\eta}$ is a $\mathbb{Q}$-name of a real, i.e. $\langle p_{n,k} : k \leq \omega\rangle$ a (absolute) maximal antichain of $\mathbb{Q}, \mathbf{t}_k = \langle \mathbf{t}_{n,k} : k < \omega\rangle, \mathbf{t}_{k,n}$ a truth value.

Proof. The proof is like the proof of Theorem 2.1 so essentially broken to a series of definitions and Claims. □

Claim/Choice 5.1.

Without loss of generality there is a sequence $\langle S_\alpha : \alpha < \lambda^+\rangle$ such that:

(a) $S_\alpha \subseteq S^\lambda_{\aleph_0}$ is stationary

(b) if $\alpha < \beta$ then $S_\alpha \backslash S_\beta$ is bounded (in λ)

(c) $\diamondsuit_{S_{\alpha+1} \backslash S_\alpha}$ and $\diamondsuit_{S^\lambda_{\aleph_0} \backslash \cup\{S_\alpha : \alpha < \lambda^+\}}$.

Proof. E.g. by a preliminary forcing. □

Definition 5.1.
Let $\mathbb{P}^*$ be the following forcing notion:

(A) The members are $\mathbf{k}$ such that

(a) $\mathbf{k} \in K^2_\lambda$

(b) $u[\mathbf{k}] = \cup\{u[\mathbf{p}^{\mathbf{k}}_\alpha] : \alpha \in E_{\mathbf{k}}\}$ is an ordinal $< \lambda^+$ (but of course $\geq \lambda$)

(c) $S_{\mathbf{k}} \in \{S_\alpha : \alpha < \lambda^+\}$.

(B) The order: $\leq_{K^2_\lambda}$.

Definition 5.2.
We define the $\mathbb{P}^*$-name $\underset{\sim}{\mathbb{Q}}^*$ as

$$\cup\{\mathbb{P}^{\mathbf{k}}_\lambda : \mathbf{k} \in \underset{\sim}{\mathbb{G}}_{\mathbb{P}^*}\} = \cup\{\mathbb{P}[\mathbf{p}^{\mathbf{k}}_\alpha] : \alpha \in E_{\mathbf{k}} \text{ and } \mathbf{k} \in \underset{\sim}{G}_{\mathbb{P}^*}\}.$$

Claim 5.2.
As in Claim 4.1:
1) $\mathbb{P}^*$ has cardinality λ^+.
2) $\mathbb{P}^*$ is strategically $(\lambda+1)$-complete hence add no new member to ${}^\lambda\mathbf{V}$.
3) $\Vdash_{\mathbb{P}^*}$ "$\underset{\sim}{\mathbb{Q}}^*$ is c.c.c. of cardinality $\leq \lambda^+$".
4) $\mathbb{P}^* * \tilde{\mathbb{Q}}^*$ is a forcing notion of cardinality λ^+ neither collapsing any cardinal nor changing cofinalities.
5) If $\mathbf{k} \tilde{\in} \mathbb{P}^*$ then $\mathbf{k} \Vdash_{\mathbb{P}^*}$ "$\mathbb{P}_{\mathbf{k}} \lessdot \underset{\sim}{\mathbb{Q}}^*$" hence $\Vdash_{\mathbb{P}^*}$ "$\mathbb{P}_{\mathbf{k}_*} \lessdot \underset{\sim}{\mathbb{Q}}^*$".

Proof. 1) Trivial.
2) By Claim 1.5.
3) $\underset{\sim}{G}_{\mathbb{P}^*}$ is $(< \lambda^+)$-directed.
4),5) Should be clear. $\square$

Claim 5.3.
Assume

(A) (a) $\mathbf{k} \in \mathbb{P}^*$

(b) $S_{\mathbf{k}} = S_\alpha, \alpha < \lambda^+$

(c) *$\underset{\sim}{\nu}$ is a $\mathbb{P}^{\mathbf{k}}_\varepsilon$-name of a member of ${}^\omega 2, \varepsilon < \kappa$*

(d) *$\underset{\sim}{\mathbb{Q}}$ is a $\mathbb{P}_{\mathbf{k}_1}$-name of a c.c.c. Suslin forcing and $\underset{\sim}{\eta}$ a $\underset{\sim}{\mathbb{Q}}$-name both definable from $\underset{\sim}{\nu}$.*

Then there is $\mathbf{k}_2$ such that

(B) (a) $\mathbf{k}_1 \leq \mathbf{k}_2$

(b) $S_{\mathbf{k}_2} = S_{\alpha+1}$

(c) *if $\varepsilon \in S_{\alpha+1} \backslash S_\alpha$ then $\mathbb{P}^{\mathbf{k}_2}_{\varepsilon+1} = \mathbb{P}^{\mathbf{k}_2}_\varepsilon * \underset{\sim}{\mathbb{Q}}$ and $\underset{\sim}{\eta}_\varepsilon$ is the copy of $\underset{\sim}{\eta}$*

(d) *if $\varepsilon \in S_{\alpha+1} \backslash S_\varepsilon$ then the strategy $\mathbf{st}_\varepsilon$ is as in In Definition/Claim 4.1, using $\underset{\sim}{\mathbb{Q}}$ instead of $\underset{\sim}{\mathbb{Q}}_{\bar{\eta}}$.*

Proof. Straight. $\square$

Claim 5.4.
Like Claim 4.3:
1) If $\mathbf{k} \in \mathbb{P}^*$ and $\underset{\sim}{\mathbb{Q}}$ is a $\mathbb{P}_{\mathbf{k}}$-name of a c.c.c. forcing of cardinality $< \lambda$ and $\alpha \in E_{\mathbf{k}}$ and $\underset{\sim}{\mathbb{Q}}$ is a $\mathbb{P}[\mathbf{p}^{\mathbf{k}}_\alpha]$-name then for some $\mathbf{k}_1$ we have:

(a) $\mathbf{k} \le_{K_2} \mathbf{k}_1 \in \mathbb{P}^*$

(b) $\Vdash_{\mathbb{P}_{\mathbf{k}_1}}$ *"there is a subset of $\underset{\sim}{\mathbb{Q}}$ generic over $\mathbf{V}[G_{\mathbb{P}_{\mathbf{k}_1}} \cap \mathbb{P}[\mathbf{p}^{\mathbf{k}}_\alpha]]$.*

2) In (1) if $\Vdash_{\mathbb{P}_{\mathbf{k}} * \underset{\sim}{\mathbb{Q}}}$ "there is $\rho \in {}^\omega 2$ not in $\mathbf{V}[\underset{\sim}{G}_{\mathbb{P}_{\mathbf{k}}}]$" then $\Vdash_{\mathbb{P}_{\mathbf{k}_1}}$ "there is $\rho \in {}^\omega 2$ not in $\mathbf{V}[\underset{\sim}{G}_{\mathbb{P}_{\mathbf{k}}}]$".

Proof. 1) By Claim 1.6.
2) By part (1) and clause (η) of Claim 1.6. □

Proof of Theorem 5.1. We force by $\mathbb{P}^* * \underset{\sim}{\mathbb{Q}}^*$ where $\mathbb{P}^*$ is defined in Definition 5.1 and the $\mathbb{P}^*$-name $\underset{\sim}{\mathbb{Q}}$ is defined in Definition 5.2. By Claim 5.2(4) we know that no cardinal is collapsed and no cofinality is changed. We know that $\Vdash_{\mathbb{P}^* * \underset{\sim}{\mathbb{Q}}^*}$ "$2^{\aleph_0} \le \lambda^+$" because $|\mathbb{P}^*| = \lambda^+$ and $\Vdash_{\mathbb{P}^*}$ "$\underset{\sim}{\mathbb{Q}}^*$ has cardinality $\le \lambda^+$", so $\mathbb{P}^* * \underset{\sim}{\mathbb{Q}}^*$ has cardinality λ^+, see Claim 5.2(3),(4).
Also $\Vdash_{\mathbb{P}^* * \underset{\sim}{\mathbb{Q}}}$ "$2^{\aleph_0} \ge \lambda^+$" as by Claim 4.1(2) it suffices to prove: for every $\mathbf{k}_1 \in \mathbb{P}^*$ there is $\mathbf{k}_2 \in \mathbb{P}^*$ such that $\mathbf{k}_1 \le_{K_2} \mathbf{k}_2$ and forcing by $\mathbb{P}_{\mathbf{k}_2}/\mathbb{P}_{\mathbf{k}_1}$ add a real, which holds by Claim 5.4(2). Similarly $\Vdash_{\mathbb{P} * \underset{\sim}{\mathbb{Q}}^*}$ "$\mathrm{MA}_{<\lambda}$" even for $< \lambda$ dense subsets by Claim 5.4(1) we have proved clause (a) of Theorem 5.1.
Clause (b) of Theorem 5.1 is proved as in the proof of Theorem 2.1, $\mathbf{k}_*$ is above $\mathbf{k}_0$.
As for clause (c) we are given $\mathbf{k}_0$ and $\underset{\sim}{\mathbb{Q}}, \underset{\sim}{\nu}, \underset{\sim}{\eta}$ such that $\underset{\sim}{\nu}$ is a $(\mathbb{P}^* * \underset{\sim}{\mathbb{Q}}^*)$-name of a real and $\underset{\sim}{\mathbb{Q}}$ is a Suslin c.c.c. forcing definable (say by $\bar{\varphi}_0$) from the real $\underset{\sim}{\nu}$ and $\underset{\sim}{\eta}$ a $(\mathbb{P}^* * \underset{\sim}{\mathbb{Q}}^*)$-name of $\underset{\sim}{\mathbb{Q}}$-name for $\underset{\sim}{\mathbb{Q}}$ of a real defined by $\aleph_0$ maximal antichain of $\underset{\sim}{\mathbb{Q}}$, absolutely of course.
As $\Vdash_{\mathbb{P}^*}$ "$\underset{\sim}{\mathbb{Q}}^*$ satisfies the c.c.c.", for some $\mathbf{k}_1 \in \mathbb{P}^*$ above $\mathbf{k}_0$ and $\mathbb{P}_{\mathbf{k}_1}$-name $\underset{\sim}{\nu}'$ of a member of ${}^{\lambda\ge}2$ and $\underset{\sim}{\eta}'$ is a $\mathbb{P}_{\mathbf{k}_1}$-name in $\underset{\sim}{\mathbb{Q}}_{\bar{\varphi},\nu'}$ we have $\mathbf{k}_1 \Vdash_{\mathbb{P}^*}$ "$\underset{\sim}{\nu} = \underset{\sim}{\nu}' \wedge \underset{\sim}{\eta} = \underset{\sim}{\eta}'$".
As $\mathbb{P}_{\mathbf{k}_1}$ satisfies the c.c.c. for some $\varepsilon < \lambda, (\mathbf{k}_1, \varepsilon, \underset{\sim}{\nu}', \underset{\sim}{\mathbb{Q}}_{\nu'}, \underset{\sim}{\eta}')$ satisfies the assumptions on $(\mathbf{k}, \varepsilon, \underset{\sim}{\nu}', \underset{\sim}{eta}')$ is as in Claim 5.3 so there is $\mathbf{k}_2$ and $\langle \underset{\sim}{\eta}_\alpha : \alpha \in S_{\alpha+1} \backslash S_\alpha \rangle$ as there. So $\mathbf{k}_0 \le \mathbf{k}_1 \le \mathbf{k}_2$ and

$(*)$ if $\mathbf{k}_2 \le \mathbf{k}_3$ then for a club of $\zeta < \lambda, \underset{\sim}{\nu}'$ is a $\mathbb{P}^{\mathbf{k}_3}_\zeta$-name and $\underset{\sim}{\eta}_\zeta$ is $(\underset{\sim}{\mathbb{Q}}_{\bar{\varphi},\bar{\nu}'}, \underset{\sim}{\eta})$-generic over $\mathbf{V}^{\mathbb{P}_\zeta[\mathbf{k}_3]}$.

This is clearly enough, so clause (c) of Theorem 5.1 holds. For clause (d) of Theorem 5.1, first Random real forcing is a Suslin c.c.c. forcing so non(null) $\le \lambda$ follows from clause (c) and non(nul) $\ge \lambda$ follows from clause (a).
Lastly, $\mathfrak{b} \ge \lambda$ by $\mathrm{MA}_{<\lambda}$ and we know $\mathfrak{d} \ge \mathfrak{b}$. As dominating real forcing = Hechler forcing is a c.c.c. Suslin forcing so by clause (c) we have $\mathfrak{d} \le \lambda$, together $\mathfrak{d} = \mathfrak{b} = \lambda$, i.e. clause (d) holds. □

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