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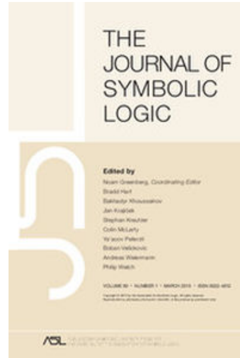


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# ON THE STANDARD PART OF NONSTANDARD MODELS OF SET THEORY

MENACHEM MAGIDOR, SAHARON SHELAH AND JONATHAN STAVI

**Abstract.** We characterize the ordinals $\alpha$ of uncountable cofinality such that $\alpha$ is the standard part of a nonstandard model of ZFC (or equivalently $KP$).

**§0. Introduction.** Let $\langle M, E\rangle$ be a model of ZF. The ordinals of $M$ have a largest initial segment isomorphic to some ordinal $\alpha$, which is usually called "the standard part of $M$". Without loss of generality we shall assume $\alpha \subseteq M$, and that for $\beta < \alpha$, $E|R^M(\beta) = \varepsilon|R^M(\beta)$, where $R^M(\beta)$ is the set of all elements of $M$ whose $M$ rank is less than $\beta$.

Which ordinals $\alpha$ can be the standard part of nonstandard models of ZF? It is easily verified that a necessary condition is that $\alpha$ is admissible. A well-known theorem of Friedman [2] implies that for countable $\alpha$'s the admissibility is also sufficient (provided there are enough countable standard models of ZF). Friedman's theorem can be generalized to other ordinals, and the proof imitated, using some decomposition of $\alpha$ into small sets. (See [4].) All these generalizations handle just $\alpha$'s of cofinality $\omega$.

In this paper we handle the problem for $\alpha$'s which are of cofinality larger than $\omega$, and we are able to give in this case a necessary and sufficient condition for an ordinal $\alpha$ to be the standard part of a nonstandard model of ZF (or equivalently $KP$). The condition involves the tree property. Let $\langle A, \in\rangle$ be a transitive set, $\alpha \in A$. An $\alpha$ tree in $A$ is a tree $\langle T, \prec\rangle$, $T \subseteq \alpha$, $\langle T, \prec\rangle \in A$ having $\alpha$ levels, such that each level has $A$ cardinality $< \alpha$ uniformly in the level (where uniformly in the level means that we have $g \in A$, $g\colon T \times \alpha \to \alpha$, such that for fixed $\gamma$, $g(x, \gamma)$ maps the $\gamma$th level of $T$ onto some ordinal $< \alpha$ in a one-to-one way) and such that the function mapping every element of $T$ to its level is in $A$. $\alpha$ has the tree property in $A$ if every $\alpha$ tree in $A$ has a branch of length $\alpha$, which is a member of $A$.

We say that $\alpha$ is a model of ZF if $\langle L_\alpha, \in\rangle \models$ ZF.

THEOREM. *For $\alpha$ such that* $\mathrm{cf}(\alpha) > \omega$, *the following are equivalent.*

(A) *$\alpha$ is the standard part of a nonstandard model of* ZF.

(B) *$\alpha$ is the standard part of a nonstandard model of $KP$.*

(C) *$\exists\gamma > \alpha$ such that $\gamma$ is a limit of ordinals which are models of* ZF, *and $\alpha$ has the tree property in $L_\gamma$.*

(D) *$\exists\gamma > \alpha$ such that $\gamma$ is a limit of admissible ordinals and $\alpha$ has the tree property in $L_\gamma$.*

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Note that if $\alpha$ is weakly compact in $L$, (D) is trivially satisfied by $\alpha$. Note that it is much more "difficult" for an ordinal of cofinality $>\omega$ to be a standard part than for an ordinal of cofinality $\omega$, since (as will become clear in §3) (D) implies that $\alpha$ is a model of ZF and actually of rather strong large cardinal axioms, whereas for $\alpha$ of cofinality $\omega$, $\alpha$ can be a successor or admissible ordinal.

The implications (A) $\rightarrow$ (B) and (C) $\rightarrow$ (D) are trivial, so we are left with proving (B) $\rightarrow$ (C) (which will be done in §1) and (D) $\rightarrow$ (A) (which will be done in §2).

Our notation and terminology are standard. When we relativize a set-theoretical notation to a particular model we use superscript. Thus $\lambda^{+M}$ is the successor cardinal of $\lambda$ is the sense of $M$. When $\langle M, E\rangle$ is a nonstandard model of ZFC and $x \in M$, we shall systematically confuse $x$ and $\{y | y\ E\ x\}$. Thus, for instance, if $f \in M$, $M \models f$ is a function from $x$ to $y$, we may consider $f$ to be a function from $\{z | z\ E\ x\}$ to $\{z | z\ E\ y\}$, etc. Similarly we shall use $<$ for the order on $M$ ordinals.

§**1.** In this section we prove that (B) of our main theorem implies (C). So let $\langle M, E\rangle$ be a nonstandard model of $KP$, whose standard part is $\alpha$ (cofinality $(\alpha) > \omega$). Without loss of generality we may assume $M \models V = L$, since $\langle L^M, E\rangle$ is easily verified to be a model of $KP$ and it has the same ordinals as $M$. As usual we assume $L_\alpha \subseteq M$.

Our first observation is

LEMMA 1. *There is no last $M$ cardinal in the standard part of $M$.*

PROOF. Assume (hoping for a contradiction) that $\lambda < \alpha$ is an $M$ cardinal, whereas every $\gamma$ with $\lambda < \gamma < \alpha$ is not an $M$ cardinal. Since $\lambda^{+M}$ (if it exists) cannot be the minimal nonstandard $M$ ordinal (There is no first nonstandard ordinal!), we can find a nonstandard $M$ ordinal $x$ such that

$$M \models \text{'}x \text{ has cardinality } \lambda\text{'}.$$

(If $\lambda^{+M}$ does not exist then every ordinal in the nonstandard part of $M$ has $M$ cardinality $\lambda$, and can be picked as $x$.) Hence let $f \in M$ satisfy

$$M \models f \text{ is a function from } \lambda \text{ onto } x.$$

We now distinguish two cases:

(I) The cofinality of $\lambda > \omega$, $\langle x, E\rangle$ is the direct limit of the structures $\langle f''\delta, E\rangle$ for $\delta < \lambda$. Since $\lambda$ is an $M$ cardinal, the order type (in $M$) of $\langle f''\delta, E\rangle$ is some ordinal less than $\lambda$. Hence it is really well founded since $\lambda$ is in the standard part of $M$. Therefore $\langle x, E\rangle$ is a direct limit of well-founded structures such that every countable subset of the directed system has an upper bound (using $\mathrm{cf}(\lambda) > \omega$). Therefore $\langle x, E\rangle$ is well founded, which contradicts the definition of $x$, as belonging to the nonstandard part of $M$.

(II) $\mathrm{cf}(\lambda) = \omega$. Since $\alpha \subseteq \{z | z\ E\ x\}$, $\alpha \subseteq f''\lambda$. Now $\mathrm{cf}(\lambda) = \omega$ and $\mathrm{cf}(\alpha) > \omega$; hence for some $\delta < \lambda, f''\delta \cap \alpha$ is cofinal in $\alpha$. $f''\delta$ has (in $M$) order type less than $\lambda$, hence it is really well founded. Therefore we can find in $f''\delta - \alpha$ a minimal element $y$. Let $z$ be the $M$ sup of $f''\delta \cap y$. Since $f''\delta \cap y \subseteq \alpha$ and $f''\delta \cap \alpha$ is cofinal in $\alpha$, $z$ is a supremum of the $M$ ordinals in the standard part of $M$; hence it is the minimal nonstandard $M$ ordinal, which is an obvious contradiction. □

Let $x$ be an $M$ ordinal which is nonstandard. (Note $\alpha \subseteq x$.) Let $p$ be a finite subset of $L_x^M$. We denote by $z(p, x)$ the set of elements of $L_x^M$, first order definable from $p \cup \alpha$ in $\langle L_x^M, E\rangle$.

LEMMA 2. *For $p$ and $x$ as above, $\langle z(p, x), E\rangle$ is well founded.*

PROOF. Fix $p$ and $x$. For $\beta < \alpha$ let $z_\beta$ be the set of all elements of $L_x^M$ which are first order definable in $\langle L_x, E\rangle$ from $p \cup \beta$. Note that $z = \bigcup_{\beta<\alpha} z_\beta$, and that $z_\beta$ is essentially a member of $M$ (since $\beta \cup \{\beta\} \in M$). Hence, in $M$, $\langle z_\beta, E\rangle$ is isomorphic to some $\langle L_y^M, E\rangle$ for some $M$ ordinal $y$ such that $M \models |y| \leq \beta$. (One can use here the result by Boolos [1], where there exists a first order sentence $\sigma$, such that it follows from $KP$ that a transitive set satisfies $\sigma$ iff it is of the form $\langle L_\gamma, G\rangle$. Hence an elementary substructure of $L_x^M$, which is in $M$, is isomorphic in $M$ to a structure of the form $L_y^M$.)

Now use Lemma 1. $\beta$ is in the standard part of $M$, hence $|\beta|$ is not the last $M$ cardinal there. Therefore $|\beta|^{+M} < \alpha$, but since $y < |\beta|^{+M}$, we get that $y$ is in the standard part of $M$, hence $y$, and consequently $\langle L_y^M, E\rangle$, is well founded. Therefore $z_\beta$ is well founded. We proved that $\langle z(p, x), E\rangle$ is an increasing union of well-founded structures, where the union has cofinality $= \mathrm{cf}(\alpha) > \omega$. We get that $\langle z, E\rangle$ is well founded. □

By Lemma 2 $z(p, x)$ is isomorphic to a unique structure of the form $\langle L_\delta, \in\rangle$. (Again one can use Boolos [1].) For some $\delta \leq \alpha$, let $\pi(p, x)$ be the unique isomorphism of $z(p, x)$ onto $L_\delta$, and denote the unique $\delta$ by $\delta(p, x)$. Note that if $p$ contains some nonstandard element of $L_x^M$ then $\delta(p, x) > \alpha$. We shall confine our attention to such $p$, so from now on when we from $z(p, x)$, $\pi(p, x)$, $\delta(p, x)$ we assume that $p$ contains a nonstandard element of $L_x^M$.

LEMMA 3. *Let $p$, $x$, $z(p, x)$, $\delta(p, x)$ be as above. Then there exist $q \supseteq p$ and an elementary embedding $j\colon L_{\delta(p,x)} \to L_{\delta(q,x)}$ such that the first ordinal moved by $j$ is $\alpha$. (Note that $p \subseteq q$ implies $\delta(p, x) \leq \delta(q, x)$.)*

PROOF. $\langle z(p, x), E\rangle$ is well founded. By our assumption $z(p, x)$ contains a nonstandard ordinal $t$. Hence let $t$ be the minimal nonstandard $M$ ordinal in $z(p, x)$. Let $t'$ be a nonstandard ordinal such that $t' < t$. Let $q \mathrel{\bar{=}} p \cup \{t'\}$. Clearly $\langle z(p, x), E\rangle$ is an elementary substructure of $\langle z(q, x), E\rangle$ (since both are elementary substructures of $\langle L_x^M, E\rangle$). Since $\pi(p, x)$ is an isomorphism of $\langle z(p, x), E\rangle$ onto $\langle L_{\delta(p,x)}, \in\rangle$ and $\pi(q, x)$ is an isomorphism of $\langle z(q, x), E\rangle$ onto $\langle L_{\delta(q,x)}, \in\rangle$, $j = \pi(q, x) \circ \pi(p, x)^{-1}$ is an elementary embedding of $\langle L_{\delta(p,x)}, \in\rangle$ into $\langle L_{\delta(q,x)}, \in\rangle$. $z(p, x)$ and $z(q, x)$ both contain $\alpha$, hence $\pi(p, x)$ and $\pi(q, x)$ are both the identity on $\alpha$. Hence $j$ is the identity on $\alpha$. By definition of $t$, $\pi(p, x)(t) = \alpha$, but since $t' < t$, $t' \in q$, $\alpha \leq \pi(q, x)(t') < \pi(q, x)(t)$. Hence

$$j(\alpha) = \pi(q, x) \circ \pi(p, x)^{-1}(\alpha) = \pi(q, x)(t) > \alpha. \quad \square$$

LEMMA 4. *$\alpha$ is a model of* ZF.

PROOF. Take any $p$ containing a nonstandard element of $M$. (Hence $\delta(p, x) > \alpha$.) By Lemma 3 $\langle L_{\delta(p,x)}, \in\rangle$ can be elementary embedded by an embedding $j$ satisfying $j(\gamma) = \gamma$ for $\gamma < \alpha$, $j(\alpha) > \alpha$. It follows that $\langle L_\alpha, \in\rangle$ is an elementary substructure of $\langle L_{j(\alpha)}, \in\rangle$. Standard argument now implies that $\langle L_\alpha, \in\rangle$ is a model of ZF. (The argument for the replacement axiom is that a definition of function in $L_\alpha$ on a set $X \in L_\alpha$ yields the same function in $L_{j(\alpha)}$, because $j$ is elementary and $j$ is the identity on $L_\alpha$. Hence in $L_{j(\alpha)}$ there exists a set, namely $L_\alpha$, containing the range of $f$. The same must hold in $L_\alpha$ by $j$ being elementary.) But

$$L_\alpha \models \text{there is no last cardinal;}$$

otherwise we get a contradiction to Lemma 1. Hence, again a standard argument, using basic properties of the $L_\alpha$ hierarchy, can be invoked to get that $\langle L_\alpha, \in\rangle$ is also a model of the power set axiom. (Recall that any subsets of $L_\beta$ in $L_\alpha$ are constructed before the next $L_\alpha$ cardinal.) Thus $\langle L_\alpha, \in\rangle \models$ ZF. □

If $x$ and $p$ are as above, using Lemma 4 we get

$$\langle z(p, x), E\rangle \models \pi(p, x)^{-1}(\alpha) \text{ is a model of ZF}',$$

but using the fact that $\pi(p, x)^{-1}(\alpha)$ is a nonstandard $M$ ordinal, and that $\langle z(p, x), E\rangle$ is an elementary substructure of $\langle L_x^M, E\rangle$, we get

COROLLARY 5. *Some nonstandard ordinal in $M$, $y$, satisfies $\langle L_y^M, E\rangle \models$ ZF.*

The following lemma is rather commonly known.

LEMMA 6. *Let $j$ be an elementary embedding of $\langle L_\delta, \in\rangle$ into $\langle L_\rho, \in\rangle$ where $\delta > \alpha$ and such that the first ordinal moved by $j$ is $\alpha$. Then every $\alpha$ tree in $L_\delta$ has a branch of length $\alpha$ in $L_\rho$.*

PROOF. Let $\langle T, \prec\rangle$ be an $\alpha$ tree in $L_\delta$. $j(\langle T, \prec\rangle)$ is clearly a $j(\alpha)$ tree in $L_\rho$. Using the fact that $j$ is the identity on $\alpha$, one can show that the tree made up of the first $\alpha$ levels of $j(\langle T, \prec\rangle)$ is exactly $\langle T, \prec\rangle$. Use $T \subseteq \alpha$, and the fact that the first $\alpha$ levels have uniform cardinality $<\alpha$, hence in $j(T)$ the $\beta$th level (for $\beta < \alpha$) contains no new members. Let $z$ be any member of $j(T)$ of level $\alpha$. (It exists since $j(\alpha) > \alpha$.) $\{y \mid y \prec z\}$ is a branch in $j(T)$ of length $\alpha$, but the first $\alpha$ levels of $j(T)$ are $T$, hence we get a branch of length $\alpha$ in $T$ which lies in $L_\rho$. (A structure of the form $\langle L_\rho, \in\rangle$ always satisfies $\Delta_0$ separation!) □

LEMMA 7. *Let $j$ be an elementary embedding of $\langle L_\delta, \in\rangle$ into itself such that $\alpha$ is the first ordinal moved by $j$. Then* (C) *holds for $\alpha$.*

PROOF. Since $\langle L_\alpha, \in\rangle$ is a model of ZFC, $\langle L_{j^n(\alpha)}, \in\rangle$ is a model of ZFC, where $j^n(\alpha)$ is the $n$th iterate of $j$. Also the restriction of $j$ is an elementary embedding of $\langle L_{j^n(\alpha)}, \in\rangle$ into $\langle L_{j^{n+1}(\alpha)}, \in\rangle$ for every $n < \omega$.

By Lemma 6, every $\alpha$ tree which lies in $L_{j^n(\alpha)}$ has a branch of length $\alpha$ in $L_{j^{n+1}(\alpha)}$. Define $\gamma = \sup_{n<\omega} j^n(\alpha)$; then $\alpha$ has the tree property in $L_\gamma$ because every $\alpha$ tree in $\gamma$ is an $\alpha$ tree in some $L_{j^n(\alpha)}$, hence it has a branch in $L_{j^{n+1}(\alpha)}$ which in included in $L_\gamma$. Since $\gamma$ is a limit of ordinals which are models of ZFC, $\gamma$ is a witness to (C). □

PROOF OF (B) → (C). By Corollary 5 pick a nonstandard $x \in M$ such that $\langle L_x^M, E\rangle$ is a model of ZFC. Define a sequence of finite subsets of $L_x^M$, $p_n$, by induction on $n < \omega$.

$p_0 = \{t\}$ where $t$ is any nonstandard element of $L_x^M$. $p_{n+1}$ is any finite subset of $L_x^M$, $p_n \subseteq p_{n+1}$ such that $p_{n+1}$ satisfies Lemma 3, with respect to $x$ and $p_n$. Hence we get an elementary embedding $j_n$: $\langle L_{\delta(p_n, x)}, \in\rangle \to \langle L_{\delta(p_{n+1}, x)}, \in\rangle$ such that $j_n$ is the identity on $\alpha$ and $j_n(\alpha) > \alpha$.

Without loss of generality we assume $\delta(p_{n+1}, x) > \delta(p_n, x)$ for all $n < \omega$. Otherwise $\delta(p_{n+1}, x) = \delta(p_n, x) = \delta$ and we get a $\delta$ and $j$ for which Lemma 7 holds, and we have that (C) holds for $\alpha$. Therefore we are finished. So assume that for all $n < \omega$, $\delta(p_n, x) < \delta(p_{n+1}, x)$.

Remember that $\langle z(p_n, x), E\rangle$ is an elementary submodel of $\langle L_x^M, E\rangle$, hence $\langle z(p_n, x), E\rangle$, and consequently, $\langle L_{\delta(p_n,x)}, \in\rangle$, are models of ZF.

So if we define $\gamma = \sup_{n<\omega} \delta(p_n, x)$ we get that $\gamma$ is a limit of ordinals which are models of ZF. Now if $T$ is an $\alpha$ tree in $L_\gamma$, $T$ is an $\alpha$ tree in $L_{\delta(p_n,x)}$ for some $n < \omega$. Hence we can use $j_n$ and Lemma 6 to get that $T$ has a branch of length $\alpha$ in $L_{\delta(p_{n+1},x)} \subseteq L_\gamma$. Thus we verified that $\gamma$ is a witness that $\alpha$ satisfies (C). □

§**2.** In this section we assume that for some $\gamma$ which is a limit of admissible ordinals, $\alpha$ has the tree property in $L_\gamma$. We shall prove that $\alpha$ is the standard part of a nonstandard model of ZF.

Clearly $L_\gamma \models \alpha$ is regular, because from a cofinal sequence in $\alpha$ of order type $< \alpha$ one could easily define an $\alpha$ tree with no branches of length $\alpha$. In particular, we get $L_\gamma \models \alpha$ is a cardinal.

LEMMA 8. *$L_\gamma \models \alpha$ is a limit cardinal.*

PROOF. Assume $\alpha = (\delta^+)^{L_\gamma}$. Using the fact that $L_\gamma \models \alpha$ is a cardinal, one can reproduce the proof of the existence of Souslin tree on successor cardinals in $L$ [3, pp. 292–295] in $L_\gamma$ to get an $\alpha$ tree such that it has no branches of length $\alpha$ in $L_\alpha$. (The definitions and the proofs of the combinatorial principles $\diamondsuit$ and $\square$ pass verbatim to $\alpha$ and $\delta$, respectively.) □

It follows from Lemma 8 that $\langle L_\alpha, \in\rangle$ is a model of ZF.

LEMMA 9. *Let $\alpha \leq \delta < \gamma$ such that $L_\gamma \models |\delta| = \alpha$. Then there exist $\rho$, $\delta \leq \rho < \gamma$, $\alpha < \rho$, $L_\gamma \models |\rho| = \alpha$ and an elementary embedding $j\colon \langle L_\delta, \in\rangle \to \langle L_\rho, \in\rangle$ such that $j(\mu) = \mu$ for $\mu < \alpha$ and $j(\alpha) > \alpha$ (if $\alpha < \delta$).*

PROOF. The proof will be easily recognized by anybody familiar with weakly compact cardinals. Consider the Boolean Algebra $B$ of all subsets of $\alpha$, definable (with parameters) in $\langle L_\delta, \in\rangle$. Using the tree property of $\alpha$ and the fact that $\alpha$ is a limit cardinal in $L_\gamma$, one can find a nonprincipal ultrafilter $U$ in $B$ with $U \in L_\gamma$ such that $L_\gamma \models U$ is $\alpha$ complete. Namely if $\langle A_\eta | \eta < \mu\rangle \subseteq U$ $\langle A_\eta | \eta < \mu\rangle \in L_\gamma$ ($\mu < \alpha$) then $\bigcap_{\beta<\mu} A_\rho \neq \emptyset$. (See Silver [5] for this type of argument.) One gets $U$ by enumerating the elements of the Boolean Algebra $B$ in a sequence of length $\alpha$. (It is possible since $L_\gamma \models |L| = \alpha$) $\langle B_\eta | \eta < \alpha\rangle$, and one considers the tree of all functions $f\colon \mu \to 2$ ($\mu < \alpha$), $f \in L_\alpha$, such that $\bigcap_{f(\eta)=1} B_\eta \cap \bigcap_{f(\eta)=0} (\alpha - B_\eta)$ has order type $\alpha$. Using the fact that $\alpha$ is a regular limit cardinal in $L_\gamma$ one can get that this tree is essentially an $\alpha$ tree in $L_\gamma$, hence has a branch. Such a branch is a function $f\colon \alpha \to 2$ and one can easily verify that $U = \{A_\mu | f(\mu) = 1\}$ is the required ultrafilter.

Using $U$ one can consider the "ultrapower" of $\langle L_\delta, \in\rangle$ reduced by $U$, i.e., consider all functions from $\alpha$ into $L_\delta$, definable over $L_\delta$ (using parameters) where we identify $f$ and $g$ if $A = \{\rho | f(\rho) = g(\rho)\} \in U$. (Note that if $f$ and $g$ are definable then $A$ is definable.) Let $\langle M, E\rangle$ be the resulting structure. As usual we get an elementary embedding $j'\colon \langle L_\delta, \in\rangle \to \langle M, E\rangle$.

$\langle M, E\rangle \in L_\gamma$, and since $L_\gamma \models U$ is $\alpha$ complete, we get, as usual, that $L_\gamma \models \langle M, E\rangle$ is well founded. Note $L_\gamma \models |M| = \alpha$, since the cardinality of the $L_\gamma$ definable functions is $\alpha$. Since $\gamma$ is a limit of admissible ordinals, $\langle M, E\rangle$ is isomorphic to a structure of the form $\langle L_\rho, \in\rangle$ by a collapsing map $\pi$. $j = \pi \circ j'$ is the required $j$ since one can easily verify, using the $\alpha$ completeness of $U$, that $j(\mu) = \mu$ for $\mu < \alpha$ and $j(\alpha) >\mu\ \alpha$. $L_\gamma \models |\rho| = \alpha$ since $L_\gamma \models |M| = \alpha$. □

PROOF OF (A) → (D). By induction we construct $\alpha_n$ and $j_n$ ($\alpha_0 = \alpha$, $\alpha_n > \alpha$ for $n > 0$) such that:

(a) $\alpha_n$ is a model of ZFC;

(b) $L_\gamma \models |\alpha_n| = \alpha$;

(c) $j_n$ is an elementary embedding of $\langle L_{\alpha_n}, \in\rangle$ into $\langle L_{\alpha_{n+1}}, \in\rangle$, where $\alpha$ is the first ordinal moved by $j_n$ if $n > 0$.

$\alpha_0$ is $\alpha$. $\alpha_{n+1}$ and $j_n$ are defined from $\alpha_n$. Using Lemma 9 $\alpha_{n+1}$ is clearly a model of ZF by $j_n$ being an elementary embedding. Therefore we get a directed system

$$\langle L_{\alpha_0}, \in\rangle \xrightarrow{j_0} \langle L_{\alpha_1}, \in\rangle \xrightarrow{j_1} \langle L_{\alpha_2}, \in\rangle \to \dots .$$

Let $\langle M, E\rangle$ be the directed limit of this system. Note the elements of $M$ are equivalence classes of pairs of the form $\langle x, n\rangle$ where $x \in L_{\alpha_n}$, under an appropriate equivalence relation $\sim$. Denote this equivalence class by $[\langle x, n\rangle]$.

$\langle M, E\rangle$ is a model of ZFC since it is the directed limit of an elementary chain of models of ZFC. The standard part of $M$ contains $\alpha$ since all of the embeddings $j_n$ are the identity on $\alpha$, so for every $n$, $\mu < \alpha$, $\langle \mu, n\rangle \sim \langle \mu, n+1\rangle$ and the equivalence classes of the pairs $\langle \mu, n\rangle$ form na initial segment of the ordinals of $M$, isomorphic to $\alpha$. The standard part of $M$ cannot be greater than $\alpha$, because if $[\langle \rho, n\rangle]$ is the $\alpha$th ordinal of $M$, we must have $\rho \ge \alpha$, but then

$$\alpha < j_n(\alpha) \le j_n(\rho) \quad \text{and} \quad [\langle \alpha, n+1\rangle] < [\langle \rho, n\rangle].$$

But $[\langle \alpha, n+1\rangle]$ has a copy of $\alpha$ below it (namely $[\langle \mu, n+1\rangle]$ for $\mu < \alpha$). Hence $[\langle \rho, n\rangle]$ cannot be the $\alpha$th ordinal of $\langle M, E\rangle$ and we have proved that $\alpha$ is exactly the standard part of $\langle M, E\rangle$. □

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HEBREW UNIVERSITY
JERUSALEM, ISRAEL

BAR ILAN UNIVERSITY
RAMAT GAN, ISRAEL