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Note

Axiom of choice and chromatic number: examples on the plane

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Abstract

In our previous paper (J. Combin. Theory Ser. A 103 (2) (2003) 387) we formulated a *conditional chromatic number theorem*, which described a setting in which the chromatic number of the plane takes on two different values depending upon the axioms for set theory. We also constructed an example of a distance graph on the real line R whose chromatic number depends upon the system of axioms we choose for set theory. Ideas developed there are extended in the present paper to construct a distance graph G_2 on the plane R^2, thus coming much closer to the setting of the chromatic number of the plane problem. The chromatic number of G_2 is 4 in the Zermelo–Fraenkel–Choice system of axioms, and is not countable (if it exists) in a consistent system of axioms with limited choice, studied by Solovay (Ann. Math. 92 Ser. 2 (1970) 1).

Keywords: Erdös problems and related topics to discrete geometry; Euclidean Ramsey theory; Graph theory; Chromatic number; Axiomatic set theory; Axiom of choice

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1. Question

Define a graph U^2 on the set of all points of the plane R^2 as its vertex set, with two points adjacent iff they are distance 1 apart. The graph U^2 ought to be called the *unit-distance plane*, and its chromatic number χ is called the *chromatic number of the plane*.[2] Finite subgraphs of U^2 are called *finite unit-distance plane graphs*.

In 1950, the 18-year old Edward Nelson posed the problem of finding χ (see the problem's history in [Soi1]). A number of relevant results were obtained under additional restrictions on monochromatic sets (see surveys in the fine problem monographs [KW,CFG], and also in [Soi2]). Amazingly though, the problem has withstood all assaults in the general case, leaving us with an embarrassingly wide range for χ being 4, 5, 6 or 7.

In their fundamental 1951 paper [EB], Erdös and de Bruijn have shown that the chromatic number of the plane is attained on some finite subgraph. This result has naturally channeled much of research in the direction of finite unit-distance graphs. One limitation of the Erdös–de-Bruijn result, however, has been that they used quite essentially the axiom of choice. What happens if we have no choice?

This question was addressed in our [SS]. We have formulated there a *conditional chromatic number theorem*, which specifically described a setting in which the chromatic number of the plane takes on two different values depending upon the axioms for set theory. We have also constructed an example of a distance graph on the real line R whose chromatic number depends upon the system of axioms we choose for set theory.

Ideas developed there are extended in the present paper to construct a distance graph G_2 on the plane R^2, thus coming much closer to the setting of the chromatic number of the plane problem. The chromatic number of G_2 is 4 in the Zermelo–Fraenkel–Choice system of axioms, and is not countable (if it, exists) in a consistent system of axioms with limited choice, studied by Solovay [Sol].

I. M. Gelfand once said that theories come and go, while examples live forever. We believe this example (and its analog G_3, presented here as well) may prove to be an important illumination and inspiration in this area of research.

2. Preliminaries

Let us recall basic set theoretic definitions and notations. In 1904, Zermelo [Z] formalized the axiom of choice that had previously been used informally.

Axiom of choice (AC): Every family Φ of nonempty sets has a choice function, i.e., there is a function f such that $f(S) \in S$ for every S from Φ.

Many results in mathematics really need just a countable version of choice.

Countable axiom of choice ($AC_{\aleph_0}$): Every countable family of nonempty sets has a choice function.

[2] The chromatic number $\chi(G)$ of a graph G is the smallest number of colors required for coloring the vertices, so that no two vertices of the same color are connected by an edge.

In 1942, Bernays [B] introduced the following axiom.

Principle of dependent choices (DC): If E is a binary relation on a nonempty set A, and for every $a \in A$ there exists $b \in A$ with aEb, then there is a sequence $a_1, a_2, \dots, a_n, \dots$ such that $a_n E a_{n+1}$ for every $n < \omega$.

AC implies DC (see [J, Theorem 8.2], for example), but not conversely. In turn, DC implies $AC_{\aleph_0}$, but not conversely. DC is a weak form of AC and is sufficient for the classical theory of Lebesgue measure. In fact, Solovay has contributed the following important observation in reply [Sol2] to Soifer's question "Do we need DC to develop the usual Lebesgue Measure Theory (in the absence of choice), or does the countable axiom of choice suffice?":

> I thought about this in the early 60's. The only theorem for which I needed DC was the Radon–Nikodym theorem. But I don't know that there isn't a clever way of getting by with just Countable Choice and proving Radon–Nikodym. I just noticed that the usual proof [found in Halmos] uses DC.

We will make use of the following axiom:

(LM). Every set of real numbers is Lebesgue measurable.

As always, ZF stands for the Zermelo–Fraenkel system of axioms for sets, and ZFC for Zermelo–Fraenkel with the addition of the axiom of choice.

Assuming the existence of an inaccessible cardinal, Solovay constructed in 1964 (and published in 1970) a model that proved the following consistency result [Sol1].

Solovay theorem. *The system of axioms* ZF + DC + LM *is consistent*.

As Jech [J] observes, in the Solovay model, every set of reals differs from a Borel set by a set of measure zero.

3. Example on the plane

Let Q denote the set of all rational numbers, so that Q^2 is the "rational plane". Let Z denote the set of all integers. We define a graph G_2 as follows: the set R^2 of points on the plane serves as the vertex set, and the set of edges is the union of the four sets $\{(s,t) : s,t \in R; s - t - \varepsilon \in Q^2\}$ for $\varepsilon = (\sqrt{2}, 0), \varepsilon = (0, \sqrt{2}), \varepsilon = (\sqrt{2}, \sqrt{2})$, and $\varepsilon = (-\sqrt{2}, \sqrt{2})$, respectively.

Claim 1. *In* ZFC *the chromatic number of* G_2 *is equal to* 4.

Proof. Let $S = \{(q_1 + n_1\sqrt{2}, q_2 + n_2\sqrt{2}) : q_i \in Q, n_i \in Z\}$. We define an equivalence relation E on R as follows: $sEt \Leftrightarrow s - t \in S$.

Let Y be a set of representatives for E. For $t \in R^2$ let $y(t) \in Y$ be such that $tEy(t)$. We define a 4-coloring $c(t)$ as follows: $c(t) = (l_1, l_2)$, $l_i \in \{0, 1\}$ iff there is a pair $(n_1, n_2) \in Z^2$ such that $t - y(t) - 2\sqrt{2}(n_1, n_2) - \sqrt{2}(l_1, l_2) \in Q^2$. □

Without AC the chromatic situation changes dramatically.

Claim 2. *In* ZF + $AC_{\aleph_0}$ + LM *the chromatic number of the graph G_2 cannot be equal to any positive integer n nor even to $\aleph_0$.*

The proof of Claim 2 immediately follows from the first of the following two statements:

Statement 1: If $A_1, \dots, A_n, \dots$ are measurable subsets of R^2 and $\bigcup_{n<\omega} A_n = [0,1)^2$, then at least one set A_n contains two adjacent vertices of the graph G_2.

Statement 2: If $A \subseteq [0,1)^2$ and A contains no pair of adjacent vertices of G_2, then A is null (of Lebesgue measure zero).

Proof. We start with the proof of statement 2. Assume to the contrary that $A \subseteq [0,1)^2$ contains no pair of adjacent vertices of G_2, yet A has positive measure. Then there is a rectangle I, with a side parallel to the coordinate axis x of length, say, a, such that

$$\frac{\mu(A \cap I)}{\mu(I)} > \frac{9}{10}. \tag{1}$$

Choose $q \in Q$ such that $\sqrt{2} < q < \sqrt{2} + \frac{a}{10}$. Define a translate B of A as follows:

$$B = A - (q - \sqrt{2}, 0) = \{(x - q + \sqrt{2}, y) : (x, y) \in A\}.$$

Then

$$\frac{\mu(B \cap I)}{\mu(I)} > \frac{8}{10}. \tag{2}$$

Inequalities (1) and (2) imply that there is $v = (x, y) \in I \cap A \cap B$. Since $(x, y) \in B$, we have $w = (x + (q - \sqrt{2}), y) \in A$. So, we have v, $w \in A$ and $v - w - (\sqrt{2}, 0) = (-q, 0) \in Q^2$. Thus, $\{v, w\}$ is an edge of the graph G with both endpoints in A, which is the desired contradiction.

The proof of the statement 1 is now obvious. Since $\bigcup_{n<\omega} A_n = [0,1)^2$ and Lebesgue measure is a countably additive function in $AC_{\aleph_0}$, there is a positive integer n such that A_n is a nonnull set. By statement 2, A_n contains a pair of adjacent vertices of G_2 as required. □

4. Another example on the plane

We can define a graph G_3 slightly differently from G_2: the set R^2 of points on the plane serves as the vertex set, and the set of edges is the union of the *two* sets $\{(s, t) : s, t \in R; s - t - \varepsilon \in Q^2\}$ for $\varepsilon = (\sqrt{2}, 0)$ and $\varepsilon = (0, \sqrt{2})$, respectively.

Claim 1. *In* ZFC *the chromatic number of* G_2 *is equal to* 2.

Proof. Let $S = \{(q_1 + n_1\sqrt{2}, q_2 + n_2\sqrt{2}) : q_i \in Q, n_i \in Z\}$. We define an equivalence relation E on R as follows: $sEt \Leftrightarrow s - t \in S$

Let Y be a set of representatives for E. For $t \in R^2$ let $y(t) \in Y$ be such that $tEY(t)$. We define a 2-coloring $c(t)$ as follows: $c(t) = (\varepsilon_1 + \varepsilon_2)_{\text{mod } 2}$ iff there is a pair $(\varepsilon_1, \varepsilon_2) \in Z^2$ such that $t - y(t) - \sqrt{2}(\varepsilon_1, \varepsilon_2) \in Q^2$. $\square$

Claim 2. *In* $\mathrm{ZF} + \mathrm{AC}_{\aleph_0} + \mathrm{LM}$ *the chromatic number of the graph* G_2 *cannot be equal to any positive integer* n *nor even to* $\aleph_0$.

The proof closely repeats the one presented for G_2 in Section 3.

5. Remark

1. One may wonder what is so special about $\sqrt{2}$ in our constructions. Well, $\sqrt{2}$ is the oldest known irrational number: a proof of its irrationality, apparently, comes from the Pythagoras School. Our reasoning and results would not change if we were to replace $\sqrt{2}$ everywhere with another irrational number.
2. Constructions presented here can be generalized to produce examples of distance G with all points of n-dimensional Euclidean space R^n as their vertex sets, and whose chromatic number $\chi(G)$ depends upon the system of axioms we choose for set theory.
3. See [Soi3] for more results and history related to this problem and early Ramsey Theory.

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References

[B] P. Bernays, A system of axiomatic set theory III, J. Symbolic Logic 7 (1942) 65–89.
[EB] P. Erdös, N.G. de Bruijn, A colour problem for infinite graphs and a problem in the theory of relations, Indag. Math. 13 (1951) 371–373.
[CFG] H.T. Croft, K.J. Falconer, R.K. Guy, Unsolved Problems in Geometry, Springer, New York, 1991.
[KW] V. Klee, S. Wagon, Old and New Unsolved Problems in Plane Geometry and Number Theory, The Mathematical Association of America, 1991.
[J] T.J. Jech, The Axiom of Choice, North-Holland, Amsterdam, 1973.

[SS] S. Shelah, A. Soifer, Axiom of choice and chromatic number of the plane, J. Combin. Theory. Ser. A 103 (2) (2003) 391–397.

[Soi1] A. Soifer, Chromatic number of the plane & its relatives, Part I: the problem & its history, Geombinatorics XII (3) (2003) 131–148.

[Soi2] A. Soifer, Chromatic number of the plane & its relatives, Part II: results & further problems, Geombinatorics XII(4) (2003).

[Soi3] A. Soifer, *Mathematical Coloring Book*, CEME, Colorado Springs, CO (to appear).

[Sol1] R.M. Solovay, A model of set theory in which every set of reals is Lebesgue measurable, Ann. Math. 92 Ser. 2 (1970) 1–56.

[Sol2] R.M. Solovay, e-mail to A. Soifer of April 11, 2003.

[Z] E. Zermelo, Beweis dass jede Menge wohlgeordnet werden kann, Math. Ann. 59 (1904) 514–516.