THE EXISTENCE OF LARGE ω_1-HOMOGENEOUS BUT NOT ω-HOMOGENEOUS PERMUTATION GROUPS IS CONSISTENT WITH ZFC+GCH

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Abstract

Denote by Perm(λ) the group of all permutations of a cardinal λ. A subgroup G of Perm(λ) is called *κ-homogeneous* if and only if for all $X, Y \in [\lambda]^\kappa$ there is a $g \in G$ with $g''X = Y$. We show that if either (i) $\diamondsuit^+$ holds and we add ω_1 Cohen reals to the ground model, or (ii) we add 2^{ω_1} Cohen reals to the ground model, then in the generic extension for each $\lambda \geqslant \omega_2$ there is an ω_1-homogeneous subgroup of Perm(λ) which is not ω-homogeneous.

1. *Introduction*

Denote by Perm(λ) the group of all permutations of a cardinal λ. The subgroups of Perm(λ) are called *permutation groups on* λ. We say that a permutation group G on λ is *κ-homogeneous* if and only if, for all $X, Y \in [\lambda]^\kappa$ with $|\lambda \setminus X| = |\lambda \setminus Y| = \lambda$, there is a $g \in G$ with $g''X = Y$. Given cardinals λ, κ and μ we write $H(\lambda, \kappa, \mu)$ to mean that every κ-homogeneous permutation group on λ is μ-homogeneous, as well. P. M. Neumann has raised the problem whether $\lambda > \kappa > \mu$ implies $H(\lambda, \kappa, \mu)$. He proved [3] that $H(\lambda, \kappa, \mu)$ holds for $\lambda > \kappa \geqslant \omega$ and $\mu < \omega$. P. Nyikos [4] and independently S. Shelah and S. Thomas [5] showed that $\neg H(2^\omega, \omega_1, \omega)$ is consistent with Martin's Axiom. Recently A. Hajnal [1] proved that $\square_{\omega_1}$ implies $\neg H(\omega_2, \omega_1, \omega)$. The aim of this paper is to construct models of ZFC in which $\neg H(\lambda, \omega_1, \omega)$ for each $\lambda \geqslant \omega_2$.

We shall use the standard notation of set theory, see [2]. For sets A and B let us denote by Fin(A, B) the poset whose underlying set consists of all functions mapping a finite subset of A into B and whose ordering is inclusion.

The $\diamondsuit^+$ principle asserts that *there is a sequence $\langle S_\alpha : \alpha < \omega_1 \rangle$ of countable sets such that for each $X \subset \omega_1$ we have a closed unbounded $C \subset \omega_1$ with $X \cap \nu \in S_\nu$ and $C \cap \nu \in S_\nu$ for each $\nu \in C$.*

THEOREM 1.1. *Assume that either* (i) *$\diamondsuit^+$ holds and $P =$* Fin$(\omega_1, 2)$, *or* (ii) $P =$ Fin$(2^{\omega_1}, 2)$. *Then*

$$V^P \models \text{“} \neg H(\lambda, \omega_1, \omega) \text{ for each } \lambda \geqslant \omega_1 \text{”}.$$

The proof of this theorem is based on the following observation. To formulate it we need some definitions. Let G be a permutation group on λ. Given X, $Y \subset \lambda$

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we say that X is (G, Y)-*large* if and only if for each $n \in \omega$ and $g_0, \dots, g_{n-1} \in G$ we have $|X \setminus \bigcup_{i<n} g_i'' Y| = |X|$; G is called κ-*inhomogeneous* if and only if there are $X, Y \in [\lambda]^\kappa$ such that X is (G, Y)-large.

OBSERVATION 1.2. *Assume that $V_0 \subset V_1$ are* ZFC *models and $\lambda \geqslant \omega_2$ is a cardinal in V_1. If*

(a) $V_0 \models$ "$|X| = \omega_1$" *iff* $V_1 \models$ "$|X| = \omega_1$" *for each* $X \in V_0$,
(b) $\forall X \in [\lambda]^{\omega_1} \cap V_1 \, \exists Y \in [\lambda]^{\omega_1} \cap V_0 \; X \subset Y$,
(c) $V_1 \models$ "*there is an ω_1-homogeneous, but ω-inhomogeneous permutation group G on ω_1 containing* $\mathrm{Perm}^{V_0}(\omega_1)$",

then $V_1 \models \neg H(\lambda, \omega_1, \omega)$.

Proof of the observation. We shall work in V_1. For each $g \in G$ define $g^* \in \mathrm{Perm}(\lambda)$ by taking $g^* \restriction \omega_1 = g$ and $g^*(\alpha) = \alpha$ for $\alpha \in \lambda \setminus \omega_1$. Let G^* be the subgroup of $\mathrm{Perm}(\lambda)$ generated by the set $\{g^* : g \in G\} \cup \mathrm{Perm}^{V_0}(\lambda)$. We claim that G^* witnesses $\neg H(\lambda, \omega_1, \omega)$. First we show that G^* is not ω-homogeneous. Pick $X, Y \in [\omega_1]^\omega$ such that X is (G, Y)-large. We shall show that X is (G^*, Y)-large. Assume on the contrary that $X \subset \bigcup_{i<n} g_i'' Y$, where $g_i \in G^*$. Then there must be an $i < n$ such that $X \cap g_i'' Y$ is (G, Y)-large. So we can assume that $n = 1$, that is, $X \subset g'' Y$ for some $g \in G^*$. Write $g = h_0 \circ (g_0)^* \circ h_1 \circ \dots \circ (g_{m-1})^* \circ h_m$, where $h_i \in \mathrm{Perm}^{V_0}(\lambda)$ and $g_i \in G$. We can assume that X and Y were chosen such that m is minimal. Define the function $d: Y \to m$ by the equation

$$d(y) = \max\{j : (h_j \circ (g_j)^* \circ \dots \circ h_m)(y) \in \omega_1\}.$$

Take $Y_j = d^{-1}\{j\}$ for $j \leqslant m$. For $y \in Y_j$ we have

$$(h_j \circ (g_j)^* \circ \dots \circ h_m)(y) = (h_j \circ h_{j+1} \circ \dots \circ h_m)(y).$$

Since $\mathrm{Perm}^{V_0}(\omega_1) \subset G$, for each $j \leqslant m$ there is an $f_j \in G$ such that

$$f_j(y) = (h_j \circ h_{j+1} \circ \dots \circ h_m)(y)$$

for each $y \in Y_j$. Putting $W_j = f_j'' Y_j$ and $Z_j = g_{j-1}'' W_j = (g_{j-1}^*)'' W_j$, this means that

$$X \subset \bigcup_{j \leqslant m} (h_0 \circ (g_0)^* \dots \circ h_{j-1})'' Z_j.$$

But $Z_j \subset (g_{j-1} \circ f_j)'' Y$, $g_{j-1} \circ f_j \in G$, so there is a $j \leqslant m$ such that, taking

$$X^* = X \cap (h_0 \circ (g_0)^* \dots \circ h_{j-1})'' Z_j,$$

we have that X^* is (G, Z_j)-large. But $X^* \subset (h_0 \circ (g_0)^* \dots \circ h_{j-1})'' Z_j$ and $j - 1 < m$, which contradicts the minimality of m. So G^* is ω-inhomogeneous.

If $X, Y \in [\lambda]^{\omega_1}$ first pick $X_0, Y_0 \in [\lambda]^{\omega_1} \cap V_0$ with $X \subset X_0$ and $Y \subset Y_0$ such that $|X_0 \setminus X| = |Y_0 \setminus Y| = \omega_1$. Fix $f, h \in \mathrm{Perm}^{V_0}(\lambda)$ with $f'' X_0 = \omega_1$ and $h'' Y_0 = \omega_1$. Since G is ω_1-homogeneous, we have a $g \in G$ with $g''(f'' X) = h'' Y$. Then $(h^{-1} \circ g \circ f)'' X = Y$ and $h^{-1} \circ g \circ f \in G^*$.

By this observation the following theorem yields Theorem 1.1.

THEOREM 1.3. *If either* (i) $\diamondsuit^+$ *holds and* $P = \mathrm{Fin}(\omega_1, 2)$, *or* (ii) $P = \mathrm{Fin}(2^{\omega_1}, 2)$, *then* $V^P \models$ "*there is an* ω_1*-homogeneous, but* ω*-inhomogeneous permutation group G on* ω_1 *with* $G \supset \mathrm{Perm}^V(\omega_1)$".

2. *Proof of Theorem* 1.3

Since the proof of Theorem 1.3 is quite long and technical, we first sketch the main ideas.

In both cases we shall define an iterated forcing system with finite support

$$\langle P_\nu : -1 \leqslant \nu \leqslant \kappa,\ Q_\nu : -1 \leqslant \nu < \kappa \rangle$$

and an increasing sequence of permutation groups $\langle G_\nu : \nu < \kappa \rangle$, $G_\nu \prec \mathrm{Perm}^{V^{P_\nu}}(\omega_1)$, simultaneously, where $\kappa = \omega_1$ in case (i) and $\kappa = 2^{\omega_1}$ in case (ii).

Take $G_0 = \mathrm{Perm}^V(\omega_1)$ and $P_0 = Q_{-1} = \mathrm{Fin}(\omega, 2)$. Denote by r the Cohen real given by the V-generic filter over P_0. By standard density arguments it is easy to see that ω is (G_0, r)-large. We try to carry out the inductive construction of the sequence $\langle G_\nu : \nu < \kappa \rangle$ in such a way that

(a) for each $\nu < \kappa$ we pick X_ν, $Y_\nu \in ([\omega_1]^{\omega_1})^{V^{P_\nu}}$, then we construct a permutation $g_\nu \in V^{P_{\nu+1}}$ with $g_\nu'' X_\nu = Y_\nu$ and take $G_{\nu+1}$ as the subgroup of $\mathrm{Perm}^{V^{P_{\nu+1}}}(\omega_1)$ generated by $G_\nu \cup \{g_\nu\}$,

(b) ω is (G_ν, r)-large,

and we hope that the sequence $\langle G_\nu : \nu < \kappa \rangle$ will give us a required permutation group. In case (ii) we use a book-keeping function to ensure that every pair $X, Y \in ([\omega_1]^{\omega_1})^{V^{P_{2^{\omega_1}}}}$ with $|\omega_1 \setminus X| = |\omega_1 \setminus Y| = \omega_1$ will be chosen as X_ν, Y_ν in some step. Then $G = \bigcup_{\nu < 2^{\omega_1}} G_\nu$ will be ω_1-homogeneous. So the question is whether we can preserve (b) during the induction. In case (i) we can pick only ω_1-many pairs $\langle X, Y \rangle$, so we cannot expect that $G' = \bigcup_{\nu < \omega_1} G_\nu$ will be ω_1-homogeneous. But in this case we use the $\diamondsuit^+$ principle to pick the sets X_ν, Y_ν for $\nu < \omega_1$ in such a way that, if we consider the elements of G' as the family of countable approximations of our required group, and if we take

$$G = \{g \in \mathrm{Perm}^{V^{P_{\omega_1}}}(\omega_1) : \forall \nu < \omega_1\, \exists g' \in G'\, g \restriction \nu = g' \restriction \nu\},$$

then G will be ω_1-homogeneous. Obviously, ω is (G, r)-large provided that ω is (G', r)-large.

So the question is whether one can preserve (*b*). Unfortunately we cannot prove a preservation theorm for the (G, r)-largeness of ω, but we shall introduce the notion of goodness of a pair (G, r) which can be preserved during a suitable iteration and it will be clear from the definition that the goodness of (G, r) implies that ω is (G, r)-large.

After this introduction let us see the details.

For $g \in \mathrm{Perm}(\omega_1)$ and $r \subset \omega_1$ we shall write, by an abuse of notation, $g(r)$ for $g''r$. Given sets X and Y let us denote by $\mathrm{Bij}_p(X, Y)$ the set of all bijections between subsets of X and Y.

If a_0, a_1, b_0 and b_1 are subsets of ω_1 with $a_0 \cap a_1 = b_0 \cap b_1 = \varnothing$ take

$$Q_{a_0, a_1, b_0, b_1} = \{g_0 \cup g_1 : g_i \in \mathrm{Bij}_p(a_i, b_i) \text{ and } |g_i| < \omega \text{ for } i \in 2\}$$

and $\mathcal{Q}_{a_0, a_1, b_0, b_1} = \langle Q_{a_0, a_1, b_0, b_1}, \supseteq \rangle$.

DEFINITION 2.1. Let $\mathscr{F} \subset \mathrm{Bij}_p(\omega_1, \omega_1)$ and $r \subset \omega$ be given.

1. A sequence $\mathscr{X} = \langle x_\nu : \nu < \mu \rangle \subset [\omega_1]^{<\omega}$ is called $(\mathscr{F}, r)$-*large* if and only if for each $n \in \omega$ and $h_0, \ldots, h_{n-1} \in \mathscr{F}$ there exists a $\nu < \mu$ such that
$$x_\nu \cap \bigcup \{h_i(r) : i < n\} = \emptyset.$$
2. We say that the pair $(\mathscr{F}, r)$ is *good* if and only if every uncountable sequence $\langle y_\nu : \nu < \omega_1 \rangle$ of pairwise disjoint, finite subsets of ω_1 has a countable initial segment $\langle y_\nu : \nu < \mu \rangle$ which is $(\mathscr{F}, r)$-large.

Case (i): $\diamondsuit^+$ *holds and* $P = \mathrm{Fin}(\omega_1, 2)$. We shall define an iterated forcing system
$$\langle P_\nu : -1 \leqslant \nu \leqslant \omega_1, Q_\nu : -1 \leqslant \nu \leqslant \omega_1 \rangle$$
with finite support, such that $P_0 = Q_{-1} = \mathrm{Fin}(\omega, 2)$ and for each $\nu < \omega_1$ either $Q_\nu = \mathrm{Fin}(\omega, 2)$ or
$$V^{P_\nu} \models \text{“} Q_\nu = \mathscr{Q}_{A_0^\nu, A_1^\nu, B_0^\nu, B_1^\nu} \textit{ for some infinite, countable subsets } A_0^\nu, A_1^\nu, B_0^\nu, B_1^\nu \textit{ of } \omega_1 \text{”}.$$
Since $V^{P_\nu} \models$ “*the completions of* Q_ν *and* $\mathrm{Fin}(\omega, 2)$ *are isomorphic*”, we have $V^P = V^{P_{\omega_1}}$, and so we may construct our desired permutation group in $V^{P_{\omega_1}}$. For $\nu \leqslant \omega_1$ let $\mathscr{G}_\nu$ be the P_ν-generic filter. Take $g = \cup \mathscr{G}_0 : \omega \to 2$ and $r = g^{-1}\{1\}$.

Fix a large enough regular cardinal κ and let $\langle N_\nu : \nu < \omega_1 \rangle$ be a sequence of countable, elementary submodels of $\mathscr{H}_\kappa = \langle H_\kappa, \in, \prec \rangle$, where H_κ consists of the sets whose transitive closure has cardinality less than κ and $\prec$ is a well-ordering of H_κ, with the property that $\langle N_\mu : \mu < \nu \rangle \in N_\nu$ for each $\nu < \omega_1$. Fix a $\diamondsuit^+$-sequence
$$\mathscr{S} = \langle S_\alpha : \alpha < \omega_1 \rangle \in N_0.$$
If $\nu \leqslant \mu < \omega_1$ and $\mathscr{G}$ is a P_ν-generic filter, then take
$$N_\mu[\mathscr{G}] = \{K_{\mathscr{G}}(x) : x \in N_\mu\}.$$
A P_{ω_1}-name $\tilde{\chi}$ of the characteristic function of a subset of ω_1 is called a *canonical* ν-*name* if and only if $\nu \leqslant \omega_1$ and there exist maximal antichains $D_\xi^{\tilde{\chi}} \subset P_{\omega_1}$ and functions $h_\xi^{\tilde{\chi}} : D_\xi^{\tilde{\chi}} \to 2$ for $\xi \in \nu$ such that
$$\tilde{\chi} = \{\langle p, \widehat{\langle \xi, i \rangle} \rangle : i = h_\xi^{\tilde{\chi}}(p), p \in D_\xi^{\tilde{\chi}}, \xi \in \nu\};$$
$\tilde{\chi}$ is called a *nice* ν-*name* if and only if $\tilde{\chi}$ is a canonical ν-name and $\bigcup_{\xi < \nu} D_\xi^{\tilde{\chi}} \subset P_{\omega\nu}$. We shall usually identify sets and their characteristic functions. To simplify our notation we shall write $p \Vdash$ “$\rho \in \tilde{\chi}$” for $p \Vdash$ “$\tilde{\chi}(\rho) = 1$”.

If $\tilde{A}$ is a nice ν-name and $\mathscr{G}$ is a $P_{\omega\nu}$-generic filter, then we write
$$\tilde{A}[\mathscr{G}] = \{\xi \in \nu : \exists p \in \mathscr{G}\, p \Vdash \xi \in \tilde{A}\}.$$
Given a P_{w_1}-name $\tilde{A}$ and $\nu < \omega_1$ take
$$\tilde{A} \restriction {}^*\nu = \{\langle p, x \rangle \in \tilde{A} : p \in P_{\omega\nu}\}.$$
A triple $\langle \tilde{A}, \tilde{B}, C \rangle$ is called ν-*remarkable* if and only if $C \subset \nu$ is a club, $\tilde{A} \restriction {}^*\mu$ and $\tilde{B} \restriction {}^*\mu$ are nice μ-names and $\langle \tilde{A} \restriction {}^*\mu, \tilde{B} \restriction {}^*\mu, C \cap \mu \rangle \in N_\mu$ for each $\mu \in C \cup \{\nu\}$.

Take
$$\mathrm{Rem}(N_\nu) = \{\langle \tilde{A}, \tilde{B}, C \rangle : \langle \tilde{A}, \tilde{B}, C \rangle \text{ is } \nu\text{-remarkable}\}.$$
First we construct the family $\mathscr{F}'$ of ‘countable approximations’ of elements of our required permutation group.

By induction on ν we shall define the posets $Q_{\omega\nu+i}$ for $i<\omega$, functions $f_{\langle\tilde{A},\tilde{B},C\rangle}$ for $\langle\tilde{A},\tilde{B},C\rangle\in\operatorname{Rem}(N_\nu)$ and families $\mathscr{F}_\nu\subset\operatorname{Bij}_p(\omega_1,\omega_1)$ in $V^{P_{\omega_1}}$ such that

(a) $\mathscr{F}_0=\operatorname{Bij}_p(\omega_1,\omega_1)\cap V$,
(b) $\langle\mathscr{F}_\mu:\mu<\nu\rangle\in N_\nu[\mathscr{G}_{\omega\nu}]$,
(c) $\mathscr{F}_\nu=\bigcup_{\mu<\nu}\mathscr{F}_\mu\cup\{f_{\langle\tilde{A},\tilde{B},C\rangle}:\langle\tilde{A},\tilde{B},C\rangle\in\operatorname{Rem}(N_\nu)\}$,
(d) $\operatorname{dom}f_{\langle\tilde{A},\tilde{B},C\rangle}=\operatorname{ran}f_{\langle\tilde{A},\tilde{B},C\rangle}=\gamma_{\langle\tilde{A},\tilde{B},C\rangle}$ for some $\gamma_{\langle\tilde{A},\tilde{B},C\rangle}\in C\cup\{\nu\}$,
(e) if $\langle\tilde{A},\tilde{B},C\rangle\in\operatorname{Rem}(N_\nu)$, $\mu\in C$, then

$$f_{\langle\tilde{A}\restriction^*\mu,\tilde{B}\restriction^*\mu,C\cap\mu\rangle}=f_{\langle\tilde{A},\tilde{B},C\rangle}\restriction\gamma_{\langle\tilde{A}\restriction^*\mu,\tilde{B}\restriction^*\mu,C\cap\mu\rangle}.$$

Assume that $\langle\mathscr{F}_\mu:\mu<\nu\rangle$ is constructed. From now on we shall work in $V[\mathscr{G}_{\omega\nu}]$. Put $\mathscr{F}_{<\nu}=\bigcup_{\mu<\nu}\mathscr{F}_\mu$. For a triple $\langle\tilde{A},\tilde{B},C\rangle\in\operatorname{Rem}(N_\nu)$ take

$$f^*_{\langle\tilde{A},\tilde{B},C\rangle}=\bigcup_{\mu\in C}f_{\langle\tilde{A}\restriction^*\mu,\tilde{B}\restriction^*\mu,C\cap\mu\rangle}$$

and

$$\gamma^*_{\langle\tilde{A},\tilde{B},C\rangle}=\sup_{\mu\in C}\gamma_{\langle\tilde{A}\restriction^*\mu,\tilde{B}\restriction^*\mu,C\cap\mu\rangle}.$$

Clearly $f^*_{\langle\tilde{A},\tilde{B},C\rangle}\in N_\nu[\mathscr{G}_{\omega\nu}]$.

Let

$$\mathscr{F}^*_\nu=\{f^*_{\langle\tilde{A},\tilde{B},C\rangle}:\langle\tilde{A},\tilde{B},C\rangle\in\operatorname{Rem}(N_\nu)\}.$$

Then $N_\nu[\mathscr{G}_{\omega\nu}]\in N_{\nu+1}[\mathscr{G}_{\omega\nu}]$ implies that $\mathscr{F}^*_\nu\in N_{\nu+1}[\mathscr{G}_{\omega\nu}]$ and $\mathscr{F}^*_\nu$ is countable in $N_{\nu+1}[\mathscr{G}_{\omega\nu}]$. So we can fix an enumeration

$$\mathscr{F}^*_\nu=\{f^*_{\langle\tilde{A}^\nu_j,\tilde{B}^\nu_j,C^\nu_j\rangle}:j<\omega\}$$

in $N_{\nu+1}[\mathscr{G}_{\omega\nu}]$. To simplify our notation we shall write

$$f^*_{\nu,j}=f^*_{\langle\tilde{A}^\nu_j,\tilde{B}^\nu_j,C^\nu_j\rangle}.$$

We define the posets $Q_{\omega\nu+j}$ and the functions

$$f_{\langle\tilde{A}^\nu_j,\tilde{B}^\nu_j,C^\nu_j\rangle}\supset f^*_{\langle\tilde{A}^\nu_j,\tilde{B}^\nu_j,C^\nu_j\rangle}$$

by induction on j. Assume that we have done it for $i<j$. Take

$$\mathscr{F}^j_\nu=\mathscr{F}_{<\nu}\cup\mathscr{F}^*_\nu\cup\{f_{\langle\tilde{A}^\nu_i,\tilde{B}^\nu_i,C^\nu_i\rangle}:i<j\}.$$

Put

$$I_{\nu,j}=[\gamma^*_{\langle\tilde{A}^\nu_j,\tilde{B}^\nu_j,C^\nu_j\rangle},\nu).$$

Take $\xi=\omega\nu+j$. Let $A^\xi_0=I_{\nu,j}\cap\tilde{A}^\nu_j[\mathscr{G}_{\omega\nu}]$, $A^\xi_1=I_{\nu,j}\setminus\tilde{A}^\nu_j[\mathscr{G}_{\omega\nu}]$, $B^\xi_0=I_{\nu,j}\cap\tilde{B}^\nu_j[\mathscr{G}_{\omega\nu}]$, $B^\xi_1=I_{\nu,j}\setminus\tilde{B}^\nu_j[\mathscr{G}_{\omega\nu}]$ and $\mathscr{Q}^\xi=\mathscr{Q}_{A^\xi_0,A^\xi_1,B^\xi_0,B^\xi_1}$.

If the sets A^ξ_0,A^ξ_1,B^ξ_0 and B^ξ_1 are all $(\mathscr{F}^j_\nu,r)$-large, then put $Q_\xi=\mathscr{Q}^\xi$. Let $\mathscr{G}^\xi$ be the Q_ξ-generic filter over V^{P_ξ}. Take

$$f'_{\langle\tilde{A}^\nu_j,\tilde{B}^\nu_j,C^\nu_j\rangle}=\bigcup\mathscr{G}^\xi$$

and let

$$f_{\langle\tilde{A}^\nu_j,\tilde{B}^\nu_j,C^\nu_j\rangle}=f^*_{\langle\tilde{A}^\nu_j,\tilde{B}^\nu_j,C^\nu_j\rangle}\cup f'_{\langle\tilde{A}^\nu_j,\tilde{B}^\nu_j,C^\nu_j\rangle}$$

and so $\gamma_{\langle\tilde{A}^\nu_j,\tilde{B}^\nu_j,C^\nu_j\rangle}=\nu$.

Otherwise, if at least one of the sets A^ξ_0,A^ξ_1,B^ξ_0 and B^ξ_1 is not $(\mathscr{F}^j_\nu,r)$-large, put

$$Q_{\omega\nu+j}=\operatorname{Fin}(\omega,2),\quad f'_{\langle\tilde{A}^\nu_j,\tilde{B}^\nu_j,C^\nu_j\rangle}=\varnothing\quad\text{and}\quad f_{\langle\tilde{A}^\nu_j,\tilde{B}^\nu_j,C^\nu_j\rangle}=f^*_{\langle\tilde{A}^\nu_j,\tilde{B}^\nu_j,C^\nu_j\rangle}.$$

The inductive construction is complete.

Take $\mathscr{F} = \bigcup_{\nu\in\omega_1} \mathscr{F}_\nu$. Let $\mathscr{F}'$ be the closure of $\mathscr{F}$ under composition and inverse image.

LEMMA 2.2. *The pair* $(\mathscr{F}', r)$ *is a good pair.*

Since its proof is quite long we postpone it for a while.

Having the family $\mathscr{F}'$ of countable approximations we are ready to construct G. *A pair* $\langle \tilde{A}, \tilde{B}\rangle$ is called *nice* if and only if both $\tilde{A}$ and $\tilde{B}$ are nice ω_1-names of uncountable, co-uncountable subsets of ω_1. Using the $\diamondsuit^+$ principle, for each nice pair $\langle \tilde{A}, \tilde{B}\rangle$ choose a club set $C = C_{\tilde{A},\tilde{B}} \subset \omega_1$ such that $\tilde{A}\lceil{}^*\nu$ and $\tilde{B}\lceil{}^*\nu$ are nice ν-names and $\langle \tilde{A}\lceil{}^*\nu, \tilde{B}\lceil{}^*\nu, C\cap\nu\rangle \in N_\nu$ for all $\nu\in C$, and put

$$f_{\tilde{A},\tilde{B}} = \bigcup_{\nu\in C} f_{\langle \tilde{A}\lceil{}^*\nu, \tilde{B}\lceil{}^*\nu, C\cap\nu\rangle}.$$

Let G be the subgroup of $\mathrm{Perm}(\omega_1)$ generated by the set

$$\{f_{\tilde{A},\tilde{B}} : \langle \tilde{A}, \tilde{B}\rangle \text{ is a nice pair with } \operatorname{dom} f_{\tilde{A},\tilde{B}} = \omega_1\}.$$

We claim that G satisfies the requirements of the theorem

Since cofinally many countable initial segments of the elements of G are in $\mathscr{F}'$, Lemma 2.2 implies that (G, r) is a good pair. Applying this fact for ω_1 as an uncountable sequence of pairwise disjoint finite subsets of ω_1 we obtain that there is a countable ordinal α which is (G, r)-large. This means that G is ω-inhomogeneous.

Next we show that G is ω_1-homogeneous. Pick a nice pair $(\tilde{A}, \tilde{B})$. Let $C = C_{\tilde{A},\tilde{B}}$ and consider the function

$$f = f_{\tilde{A},\tilde{B}} = \bigcup_{\nu\in C} f_{\langle \tilde{A}\lceil{}^*\nu, \tilde{B}\lceil{}^*\nu, C\cap\nu\rangle}.$$

If $\operatorname{dom}(f) = \omega_1$ then $f\in G$ and $f''A = B$, so we are done. Assume that $\operatorname{dom}(f) = \mu < \omega_1$. Then $\mu\in C$. But the pair $(\mathscr{F}', r)$ is good, so there is a $\nu > \mu$, $\nu\in C$, such that the sets $(\nu\setminus\mu)\cap A$, $(\nu\setminus\mu)\setminus A$, $(\nu\setminus\mu)\cap B$ and $(\nu\setminus\mu)\setminus B$ are all $(\mathscr{F}', r)$-large. So we get

$$\operatorname{dom}(f'_{\langle \tilde{A}\lceil{}^*\nu, \tilde{B}\lceil{}^*\nu, C\cap\nu\rangle}) = \nu\setminus\mu.$$

Thus $\operatorname{dom}(f) \supset \nu$ which gives a contradiction, so G is ω_1-homogeneous.

Finally we present the postponed proof of Lemma 2.2.

Proof of Lemma 2.2. First we introduce some notion.

A condition $p\in P_{\omega_1}$ is called *determined* if and only if, for each $\xi < \omega_1$ if $p(\xi) \neq 1_{Q_\xi}$, then $p\lceil\xi \Vdash$ "$Q_\xi = \mathrm{Fin}(\omega, 2)$" or $p\lceil\xi \Vdash$ "$Q_\xi = \mathscr{Q}^\xi$" and there is a function $F_{p,\xi}$ in the ground model such that $p(\xi) = \hat{F}_{p,\xi}$. Put

$$\mathrm{Det}(P_{\omega_1}) = \{p\in P_{\omega_1} : p \text{ is determined}\}.$$

The proof of the following sublemma is straightforward.

SUBLEMMA 2.2.1. *We have* $\mathrm{Det}(P_{\omega_1})$ *is dense in* P_{ω_1}.

Consider a condition $p \in \mathrm{Det}(P_{\omega_1})$. For each $\langle \tilde{A}^\nu_j, \tilde{B}^\nu_j, C^\nu_j \rangle \in \mathrm{Rem}(N_\nu)$ take

$$f^{[p]}_{\nu,j} = \bigcup \{p(\omega\mu + i) : p \restriction (\omega\mu + i) \models \text{“} Q_{\omega\mu+i} = \mathcal{Q}^{\omega\mu+i} \text{”}, \mu \in C^\nu_j \cup \{\nu\} \text{ and } \langle \tilde{A}^\mu_i, \tilde{B}^\mu_i, C^\mu_i \rangle = \langle \tilde{A}^\nu_j \restriction {}^*\mu, \tilde{B}^\nu_j \restriction {}^*\mu, C^\nu_j \cap \mu \rangle\}.$$

The finite function $f^{[p]}_{\nu,j}$ gathers all the information contained in p about $f_{\langle \tilde{A}^\nu_j, \tilde{B}^\nu_j, C^\nu_j \rangle}$.

An *$\mathcal{F}$-term* t is a sequence $\langle h_0, h_1, \ldots, h_{n-1} \rangle$ where $h_i = f_i$ or $h_i = f_i^{-1}$ for some $f_i \in \mathcal{F}$. We also use t to denote the function $h_0 \circ \ldots \circ h_{n-1}$. We say that t' is a *subterm of* t if $t' = \langle h_{i_0}, \ldots, h_{i_l} \rangle$, where $0 \leqslant i_0 < \ldots < i_l \leqslant n-1$.

Given an $\mathcal{F}$-term $t = \langle h_0, h_1, \ldots, h_{n-1} \rangle$ and a condition $p \in \mathrm{Det}(P_{\omega_1})$ take $t^{[p]} = h'_0 \circ \ldots \circ h'_{n-1}$, where

$$h'_i = \begin{cases} h_i & \text{if } h_i \in \mathcal{F}_0 \, (= \mathrm{Bij}_p(\omega_1, \omega_1) \cap V), \\ f^{[p]}_{\nu,j} & \text{if } h_i = f_{\langle \tilde{A}^\nu_j, \tilde{B}^\nu_j, C^\nu_j \rangle} \text{ for some } \langle \tilde{A}^\nu_j, \tilde{B}^\nu_j, C^\nu_j \rangle \in \mathrm{Rem}(N_\nu), \\ (f^{[p]}_{\nu,j})^{-1} & \text{if } h_i = (f_{\langle \tilde{A}^\nu_j, \tilde{B}^\nu_j, C^\nu_j \rangle})^{-1} \text{ for some } \langle \tilde{A}^\nu_j, \tilde{B}^\nu_j, C^\nu_j \rangle \in \mathrm{Rem}(N_\nu). \end{cases}$$

If T is a set of $\mathcal{F}$-terms let $T^{[p]} = \{t^{[p]} : t \in T\}$.

Given $p \in \mathrm{Det}(P_{\omega_1})$, $\xi \in \omega_1$ and a function $F \in \mathrm{Bij}_p(\omega_1, \omega_1)$, if $p \restriction \xi \Vdash \text{“} p(\xi) \cup F \in Q_\xi \text{”}$ then write $p \wedge_\xi F$ for the condition $q \in \mathrm{Det}(P_{\omega_1})$ defined by the formula

$$q(\zeta) = \begin{cases} p(\zeta) \cup F & \text{if } \zeta = \xi, \\ p(\zeta) & \text{otherwise.} \end{cases}$$

If $p \restriction \xi \in \mathcal{G}_\xi$, then let

$$t^{[p]}[\mathcal{G}_\xi] = \bigcup \{t^{[p \wedge q]} : q \in \mathcal{G}_\xi] \quad \text{and} \quad T^{[p]}[\mathcal{G}_\xi] = \{t^{[p]}[\mathcal{G}_\xi] : t \in T\}.$$

If $a \subset \omega_1$ then, by an abuse of notation, we shall write $t(a)$ for $t''a$ and $T(a)$ for $\bigcup_{t \in T} t(a)$.

Assume now on the contrary that
$p' \Vdash \text{“}Z = \{\tilde{z}_\alpha : \alpha < \omega_1\}$ *is a sequence of pairwise disjoint finite subsets of* ω_1, $\{\tilde{U}_\beta : \beta < \omega_1\}$ *is a sequence of finite sets of* $\mathcal{F}$*-terms such that* $\tilde{z}_\alpha \cap \tilde{U}_\beta(r) \neq \emptyset$ *for each* $\alpha < \beta < \omega_1$”.

For each $\alpha < \omega_1$ choose a condition $p_\alpha \in \mathrm{Det}(P_{\omega_1})$, a set $y_\alpha \in [\omega_1]^{<\omega}$ and a finite set of $\mathcal{F}$-terms T_α such that $p_\alpha \Vdash$ “$\tilde{z}_\alpha = \hat{y}_\alpha$ and $\tilde{U}_\alpha = \hat{T}_\alpha$”. We can assume that every T_α is closed under subterms.

SUBLEMMA 2.2.2. *There are* $\alpha < \beta < \omega_1$ *and a condition* $p \in \mathrm{Det}(P_{\omega_1})$ *such that* $p \leqslant p_\alpha, p_\beta$ *and* $p \Vdash \text{“} T^{[p]}_\beta(r) \cap y_\alpha = \emptyset \text{”}$.

Proof. Without loss of generality we can assume that $\{\mathrm{supp}(p_\alpha) : \alpha < \omega_1\}$ forms a Δ-system with kernel d, such that $-1 \in d$ and $p_\alpha \restriction d = q$ for each $\alpha < \omega_1$. Then $q(-1) \in \mathrm{Fin}(\omega, 2)$. Take $D = \mathrm{dom}(q(-1))$.

Consider a $t \in T_\omega$. If $\alpha_0 < \alpha_1 < \omega$ then $u = (t^{[p_{\alpha_0} \cup p_{\alpha_1} \cup p_\omega]})^{-1}$ is a $1-1$ function, and so $u(y_{\alpha_0}) \cap u(y_{\alpha_1}) = \emptyset$. Thus

$$|\{\alpha < \omega : (t^{[p_\alpha \cup p_\omega]})^{-1}(y_\alpha) \cap D \neq \emptyset\}| \leqslant |D|.$$

But T_ω is finite, so there is an $\alpha < \omega$ such that $T^{[p_\alpha \cup p_\omega]}_\omega(D) \cap y_\alpha = \emptyset$. Take $E = (T^{[p_\alpha \cup p_\omega]}_\omega)^{-1} y_\alpha$. Then $D \cap E = \emptyset$. Put $p = p_\alpha \wedge p_\omega \wedge q'$, where q' is defined by the equations $\mathrm{supp}(q') = \{-1\}$ and $g'(-1) = \{\langle m, 0 \rangle : m \in E\}$. Then p is determined and $p \Vdash \text{“} T^{[p]}_\omega(r) \cap y_\alpha = \emptyset \text{”}$, because $(T^{[p]}_\omega)^{-1}(y_\alpha) \subset E$ and $p \Vdash \text{“} r \cap E = \emptyset \text{”}$.

Take $T = (T_\beta)^{-1} \overset{\text{def}}{=} \{t^{-1} : t \in T_\beta\}$ and $y = y_\alpha$. Then

$$p \Vdash \text{“} T^{[p]}(y) \cap r = \emptyset \text{ but } T(y) \cap r \neq \emptyset \text{”}.$$

We shall find $p^* \leqslant p$ such that $p^* \Vdash$ “$T^{[p^*]}(y) \cap r = \emptyset$” and p^* ‘fully decides’ $T(y)$, that is, $p^* \Vdash$ “$T(y) = T^{[p^*]}(y)$” contradicting $p \Vdash$ “$T(y) \cap r \neq \emptyset$”.

DEFINITION 2.3. Let $p^* \in \mathrm{Det}(P_{\omega_1})$, s be an $\mathscr{F}$-term, S be a set of $\mathscr{F}$-terms, $\rho \in \omega_1$, $z \subset \omega_1$, $\xi \in \omega_1$, $s = \langle h_0, h_1, \ldots, h_{n-1}\rangle$.

1. We say that p^* *fully decides* $s(\rho)$ if and only if there is a sequence $\langle \rho_0, \ldots, \rho_m\rangle$ such that $\rho_0 = \rho$ and $p^* \Vdash$ “$h_i(\hat{\rho}_i) = \hat{\rho}_{i+1}$” for $i < m$ and either $m = n$ or $m < n$ and $p^* \Vdash$ “$\hat{\rho}_m \notin \mathrm{dom}(h_m)$”.
2. We also say that p^* *fully decides* $S(z)$ if and only if it fully decides $s(\rho)$ whenever $s \in S$ and $\rho \in z$.
3. Take $\mathrm{dcd}(p^*, S, z) = \{\langle s, \rho\rangle \in S \times z : p^* \text{ fully decides } s(\rho)\}$.
4. If $p^* \lceil \xi \in \mathscr{G}_\xi$ then put $\mathrm{dcd}(p^*, S, z)[\mathscr{G}_\xi] = \bigcup \{\mathrm{dcd}(p^* \wedge q, S, z) : q \in \mathscr{G}_\xi\}$.

If $q \in \mathrm{Det}(P_{\omega_1})$, $y \in [\omega_1]^{<\omega}$ and S is a set of $\mathscr{F}$-terms we shall define the set of undecided evaluations as follows: let $\mathrm{Und}(q, S, y)$ be the set of triples $\langle x, \langle \nu, j\rangle, e\rangle$ for which there exist $s \in S$, $s = \langle h_0, \ldots, h_l\rangle$, and $\eta \in y$ satisfying (a) to (d) below:

(a) $x \in \omega_1$, $e \in \{-1, +1\}$ and $h_l = (f_{\langle \tilde{A}^\nu_j, \tilde{B}^\nu_j, C^\nu_j\rangle})^e$,
(b) taking $u = \langle h_0, \ldots, h_{l-1}\rangle$ we have $u^{[q]}(\eta) = x$,
(c) $x \notin \mathrm{dom} f^{[q]}_{\langle \nu, j\rangle}$ if $e = 1$,
(d) $x \notin \mathrm{ran} f^{[q]}_{\langle \nu, j\rangle}$ if $e = -1$.

For $\langle \mu, i\rangle \in \omega_1 \times \omega$ define $\mathrm{Und}^+(q, T, y, \mu, i)$, the set of undecided evaluations which will be decided in step μ, as follows: it contains a triple $\langle x, \langle \nu, j\rangle, e\rangle \in \mathrm{Und}(q, T, y)$ if and only if (A) to (C) below are satisfied:

(A) $\langle \tilde{A}^\mu_i, \tilde{B}^\mu_i, C^\mu_i\rangle = \langle \tilde{A}^\nu_j \lceil {}^*\mu, \tilde{B}^\nu_j \lceil {}^*\mu, C^\nu_j \cap \mu\rangle$,
(B) $q \Vdash$ “$x \in \mathrm{dom} f'_{\langle \tilde{A}^\mu_i, \tilde{B}^\mu_i, C^\mu_i\rangle}$” provided $e = 1$,
(C) $q \Vdash$ “$x \in \mathrm{ran} f'_{\langle \tilde{A}^\mu_i, \tilde{B}^\mu_i, C^\mu_i\rangle}$” provided $e = -1$.

Let $\mathrm{Und}^-(q, T, y, \mu, i)$, the family of undecided evaluations which remain undecided in step μ, be the set of triples $\langle x, \langle \nu, j\rangle, e\rangle \in \mathrm{Und}(q, T, y, \mu, i)$ with $\langle \tilde{A}^\mu_i, \tilde{B}^\mu_i, C^\mu_i\rangle \neq \langle \tilde{A}^\nu_j \lceil {}^*\mu, \tilde{B}^\nu_j \lceil {}^*\mu, C^\nu_j \cap \mu\rangle$ or with $q \Vdash$ “$f'_{\langle \tilde{A}^\mu_i, \tilde{B}^\mu_i, C^\mu_i\rangle} = \emptyset$”.

If $q \lceil \xi \in \mathscr{G}_\xi$, $\xi \leqslant \omega\mu + i$, then take

$$\mathrm{Und}(q, T, y)[\mathscr{G}_\xi] = \bigcup \{\mathrm{Und}(q \wedge q', T, y) : q' \in \mathscr{G}_\xi\},$$

$$\mathrm{Und}^+(q, T, y, \mu, i)[\mathscr{G}_\xi] = \bigcup \{\mathrm{Und}^+(q \wedge q', T, y, \mu, i) : q' \in \mathscr{G}_\xi\}$$

and

$$\mathrm{Und}^-(q, T, y, \mu, i)[\mathscr{G}_\xi] = \bigcup \{\mathrm{Und}^-(q \wedge q', T, y, \mu, i) : q' \in \mathscr{G}_\xi\}.$$

SUBLEMMA 2.3.1. *There is a condition $p^* \leqslant p$ which fully decides $T(Y)$ and $p^* \Vdash$ “$T^{[p^*]}(y) \cap r = \emptyset$”.*

Proof. We shall define a finite, decreasing sequence of determined conditions, $p_0, p_1, \ldots, p_{l^*} = p^*$ satisfying (I) and (II) below for each $l \leqslant l^*$:

(I) $p_l \Vdash$ “$T^{[p_l]}(y) \cap r = \emptyset$”;
(II) $|\mathrm{dcd}(p_{l-1}, T, y)| < |\mathrm{dcd}(p_l, T, y)|$.

Put $p_0 = p$ and assume that p_l is constructed.

Choose the minimal $\langle \mu, i\rangle$ in the lexicographical order of $\omega_1 \times \omega$ with the property (*) below:

(*) there is a $p' \in P_{\omega\mu+i}, p' \leqslant p_l \restriction (\omega\mu+i)$, such that taking $p'_l = p' \wedge p_l$ we have $\mathrm{Und}^+(p'_l, T, y, \mu, i) \neq \varnothing$.

To simplify our notation take $\xi = \omega\mu + i$. Let $\mathscr{G}_l$ be a P_ξ-generic filter with $p' \in \mathscr{G}_l$. By the minimality of $\langle \mu, i\rangle$ we have $T^{[p_l]}[\mathscr{G}_l](y) = T^{[p_l]}(y)$ and so

$$(\dagger) \qquad V[\mathscr{G}_l] \models \text{“} T^{[p_l]}[\mathscr{G}_l](y) \cap r = \varnothing \text{”}.$$

Clearly

$$\mathrm{Und}(p'_l, T, y)[\mathscr{G}_l] = \mathrm{Und}^+(p'_l, T, y, \mu, i)[\mathscr{G}_l] \cup \mathrm{Und}^-(p'_l, T, y, \mu, i)[\mathscr{G}_l].$$

The condition p_{l+1} will be constructed by a finite induction. Working in $V[\mathscr{G}_l]$ we shall define a natural number o_l and determined conditions $p_{l,0}, p_{l,1}, \ldots, p_{l,o_l}$ such that for all $k \leqslant o_l$ (i) to (vi) below are satisfied:

(i) $p_{l,k} \restriction \xi \in \mathscr{G}_l$,
(ii) $p_{l,k} \leqslant p_{l,k-1}$,
(iii) $p_{l,k} \restriction [\xi+1, \omega_1) = p_l \restriction [\xi+1, \omega_1)$,
(iv) $V[\mathscr{G}_l] \models \text{“}T^{[p_{l,k}]}[\mathscr{G}_l](y) \cap r = \varnothing\text{”}$,
(v) $|\mathrm{dcd}(p_{l,k-1}, T, y)[\mathscr{G}_l]| < |\mathrm{dcd}(p_{l,k}, T, y)[\mathscr{G}_l]|$,
(vi) $\mathrm{Und}^+(p_{l,o_l}, T, y, \mu, i)[\mathscr{G}_l] = \varnothing$.

Let $p_{l,0} = p_l$. Then (iv) holds by (†). Assume that $p_{l,k}$ is chosen. If

$$\mathrm{Und}^+(p_{l,k}, T, y, \mu, i)[\mathscr{G}_l] = \varnothing$$

then put $o_l = k$. We remark that (v) and $|\mathrm{dcd}(p_{l,k+1}, T, y)[\mathscr{G}_l]| \leqslant |T \times y|$ imply that $o_l \leqslant |T \times y|$. Suppose now that (vi) fails for k and pick a witness

$$\langle x_k, \langle v_k, j_k\rangle, e_k\rangle \in \mathrm{Und}^+(p_{l,k}, T, y, \mu, i)[\mathscr{G}_l].$$

Let $T_k = T^{[p_{l,k}]}[\mathscr{G}_l]$, choose a

$$z_k \in \begin{cases} \tilde{A}^\mu_i[\mathscr{G}_l] \setminus (T_k^{-1}(r \cup y) \cup T^k(r \cup y)) & \text{if } e_k = 1, \\ \tilde{B}^\mu_i[\mathscr{G}_l] \setminus (T_k^{-1}(r \cup y) \cup T^k(r \cup y)) & \text{if } e_k = -1 \end{cases}$$

and take

$$F_k = \begin{cases} \{\langle x_k, z_k\rangle\} & \text{if } e_k = 1, \\ \{\langle z_k, x_k\rangle\} & \text{if } e_k = -1. \end{cases}$$

We remark that z_k is chosen from a non-empty set, because both $\tilde{A}^\mu_i[\mathscr{G}_l]$ and $\tilde{B}^\mu_i[\mathscr{G}_l]$ are $(\mathscr{F}^{\,i}_\mu, r)$-large.

Choose $p^*_{l,k} \in \mathscr{G}_l$ such that

$$p^*_{l,k} \models \text{“}p_{l,k} \wedge_\xi F_k \in P_{w_1} \text{ and } \langle x_k, \langle v_k, j_k\rangle, e_k\rangle \in \mathrm{Und}^+(p_{l,k}, T, y, \mu, i)[\mathscr{G}_l]\text{”}$$

and put $p_{l,k+1} = (p^*_{l,k} \wedge p_{l,k}) \wedge_\xi F_k$. We need to check only (iv). Working in $V[\mathscr{G}_l]$ assume on the contrary that there are $\alpha \in y$ and $t \in T$, $t = \langle h_0, \ldots, h_{m-1}\rangle$, such that $t^{[p_{l,k+1}]}[\mathscr{G}_l](\alpha) \in r$. We can assume further that t is of minimal length. Write $\alpha_0 = \alpha$ and $\alpha_{s+1} = h_s(\alpha_s)$ for $s < m$. Since (iv) holds for k, it follows that $t^{[p_{l,k}]}[\mathscr{G}_l](\alpha) \notin r$. Suppose first that $e_k = 1$. Then there must be an $s < m$ such that

$$\alpha_s = x_k \quad \text{and} \quad h_s = f_{\langle \tilde{A}^\eta_l, \tilde{B}^\eta_l, C^\eta_l\rangle},$$

or

$$\alpha_s = z_k \quad \text{and} \quad h_s = (f_{\langle \tilde{A}^\eta_l, \tilde{B}^\eta_l, C^\eta_l \rangle})^{-1},$$

where

$$\langle \tilde{A}^\eta_l \restriction {}^*\mu, \tilde{B}^\eta_l \restriction {}^*\mu, C^\eta_l \cap \mu \rangle = \langle \tilde{A}^\mu_l, \tilde{B}^\mu_l, C^\mu_l \rangle.$$

Pick the minimal s. Then $\alpha_s \in T^{[p_{l,k}]}[\mathcal{G}_l](y)$, because T is closed under subterms. But $z_k \notin T^{[p_{l,k}]}[\mathcal{G}_l](y)$, so $\alpha_s = x_k$ and $\alpha_{s+1} = z_k$. But t is of minimal length and T is closed under subterms, so we have that there are no $s' > s$ with $\alpha_{s'} = x_k$ and $h_{s'} = f_{\langle \tilde{A}^\eta_l, \tilde{B}^\eta_l, C^\eta_l \rangle}$, or with $\alpha_{s'} = z_k$ and $h_{s'} = (f_{\langle \tilde{A}^\eta_l, \tilde{B}^\eta_l, C^\eta_l \rangle})^{-1}$. But this means that $z_k = \alpha_{s+1} \in (T^{[p_{l,k}]}[\mathcal{G}_l])^{-1}(r)$, which contradicts the choice of z_k. If $e_k = -1$, then a similar argument works. Condition (v) obviously holds by the construction.

So we have p_{l,o_l}. Choose a condition $p^*_l \in \mathcal{G}_l$ such that

$$p^*_l \leqslant p_{l,o_l} \restriction \xi, p^*_l \Vdash \text{“} t^{[p_{l,o_l}]}[\mathcal{G}_l](y) \cap r = \varnothing \text{”}$$

and taking $p_{l+1} = p^*_l \wedge p_{l,o_l}$ we have

$$p^*_l \Vdash \text{“} t^{[p_{l,o_l}]}[\mathcal{G}_l](y) = t^{[p_{l+1}]}(y) \text{”}.$$

Since $t^{[p_{l,o_l}]}[\mathcal{G}_l](y) = t^{[p_{l+1}]}[\mathcal{G}_l](y)$, it follows that p_{l+1} satisfies (I); (II) is clear from the construction.

If there are no more $\langle \mu, i \rangle$ with property (*), then we finish the construction of the decreasing sequence $p_0, p_1, \ldots, p_{l^*}$. This must happen after at most $|T \times y|$ steps, because (II) holds and $|\mathrm{dcd}(p_l, T, y)| \leqslant |T \times y|$. Let $p^* = p_{l^*}$. Since there are no more $\langle \mu, i \rangle$ satisfying (*), it follows that p^* fully decides $T(y)$ and $p^* \Vdash \text{“} T^{[p^*]}(y) \cap r = \varnothing \text{”}$, and so p^* satisfies the requirements of the sublemma.

We continue the proof of Lemma 2.2. Let $\mathcal{G}_{\omega_1}$ be any P_{ω_1}-generic filter with $p^* \in \mathcal{G}$. Then in $V[\mathcal{G}_{\omega_1}]$ we have $T^{[p^*]}(y) = T(y)$, because p^* fully decides $T(y)$, and so $T(y) \cap r = \varnothing$. But $p \in \mathcal{G}_{\omega_1}$, and $p \Vdash \text{“} T(y) \cap r \neq \varnothing \text{”}$. Contradiction, the lemma is proved.

We return to the proof of Theorem 1.3.

Case (ii): $P = \mathrm{Fin}(2^{\omega_1}, 2)$. Since the proof in this case is simpler than in Case (i) and it does not require any new ideas, we shall sketch it only and leave the details to the readers.

Consider the iterated forcing system $\langle P_\nu : \nu \leqslant 2^{\omega_1}, Q_\nu : \nu < 2^{\omega_1} \rangle$ with finite support, where $Q_0 = \mathrm{Fin}(\omega, 2)$ and $Q_\nu = \mathrm{Fin}(\omega_1, \omega)$ for $1 \leqslant \nu < 2^{\omega_1}$. Since P is isomorphic to a dense subset of $P_{2^{\omega_1}}$, we must show that $V^{P_{2^{\omega_1}}}$ contains a suitable permutation group.

Let r be the Cohen real in V^{Q_0} given by the Q_0-generic filter. We shall define, by induction on μ, permutation groups G_μ on ω_1 for $1 \leqslant \mu \leqslant 2^{\omega_1}$ such that

(a) $G_\nu \subset G_\mu$ for $\nu < \mu$,
(b) $V^{P_\mu} \models$ “the pair (G_μ, r) is good”.

Case 1: $\mu = 1$. Take $G_0 = \mathrm{Perm}^V(\omega_1)$. Standard density arguments show that G_0 satisfies (b).

Case 2: *μ is a limit*. Take $G_\mu = \bigcup_{\nu < \mu} G_\nu$. A suitable modification of the proof of Lemma 2.2 shows that G_μ satisfies (b).

Case 3: $\mu = \nu+1$. We shall work in V^{P_ν}. Pick sets $X_\nu, Y_\nu \subset \omega_1$ with

$$|X_\nu| = |\omega_1 \setminus X_\nu| = |Y_\nu| = |\omega_1 \setminus Y_\nu| = \omega_1.$$

Since (G_ν, r) is a good pair, we can fix partitions $\{X^0_{\nu,\alpha} : \alpha < \omega_1\}$, $\{Y^0_{\nu,\alpha} : \alpha < \omega_1\}$, $\{X^1_{\nu,\alpha} : \alpha < \omega_1\}$ and $\{Y^1_{\nu,\alpha} : \alpha < \omega_1\}$ of $X_\nu, Y_\nu, \omega_1 \setminus X_\nu$ and $\omega_1 \setminus Y_\nu$, respectively, such that each of the $X^i_{\nu,\alpha}$ and $Y^i_{\nu,\alpha}$ are countable and (G_ν, r)-large. Let

$$R_\nu = \{f \in \mathrm{Bij}_p(\omega_1, \omega_1) : |f| < \omega, f'' X^i_{\nu,\alpha} \subset Y^i_{\nu,\alpha} \text{ for each } \alpha < \omega_1 \text{ and } i \in 2\}.$$

Take $\mathscr{R}_\nu = \langle R_\nu, \supseteq \rangle$. Then $\mathscr{R}_\nu$ is isomorphic to Q_ν, so the Q_ν-generic filter over V^{P_ν} gives us an $\mathscr{R}_\nu$-generic filter $\mathscr{G}_\nu$ over V^{P_ν}. Take $g_\nu = \bigcup \mathscr{G}_\nu$ and let G_μ be the subgroup of $\mathrm{Perm}^{V^{P_\mu}}(\omega_1)$ generated by $G_\nu \cup \{g_\nu\}$. The proof of that (b) remains true is a suitable modification of the proof of Lemma 2.2 and it is left to the readers.

So $G = G_{2^{\omega_1}}$ is ω-inhomogeneous, because the pair (G, r) is good. Using a book-keeping function we can ensure that $\{(X_\nu, Y_\nu) : \nu < 2^{\omega_1}\}$ enumerates all the pairs (X, Y) of uncountable, co-uncountable subsets of ω_1 in $V^{P_{2^{\omega_1}}}$. Since $g''_\nu X_\nu = Y_\nu$, it follows that G is ω_1-homogeneous.

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