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DEPENDENT FIRST ORDER THEORIES, CONTINUED

BY

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ABSTRACT

A dependent theory is a (first order complete theory) T which does not have the independence property. A major result here is: if we expand a model of T by the traces on it of sets definable in a bigger model then we preserve its being dependent. Another one justifies the cofinality restriction in the theorem (from a previous work) saying that pairwise perpendicular indiscernible sequences, can have arbitrary dual-cofinalities in some models containing them. We introduce "strongly dependent" and look at definable groups; and also at dividing, forking and relatives.

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Annotated Content

Recall: Dependent $T = T$ without the independence property.

0. Introduction

The work in [Sh:715] tries to deal with the investigation of (first order complete) theories T which have the dependence property, i.e., do not have the independence property.

If T is stable, we expand a model M of T by $p \restriction \varphi(\bar{x}, \bar{y})$, for $p \in \mathbf{S}^m(M)$. That is expanding M by the relation $R^{\varphi(\bar{x},\bar{y})}_{p,M} = \{\bar{a} \in {}^{\ell g(\bar{y})} M : \varphi(\bar{x}, \bar{a}) \in p\}$ is an inessential one, i.e., by a relation on M definable in M with parameters. This fails for unstable theories but in §1 we prove a weak relative: if T is a dependent theory then so is the expansion above, i.e., $\mathrm{Th}(M, R^{\varphi(\bar{x},\bar{y})}_{p,M})_{p,\varphi(\bar{x},\bar{y})}$.

In [Sh:715, §5] it is shown that for any model N of a dependent unstable T, we can find a κ-saturated model M extending N, such that the following set is quite arbitrary. Pairs of cofinalities of a cut in M, for some definable partial order in N (so not fulfilled in M), or even the set of pairs (κ_1, κ_2) of regular cardinals for which there is an indiscernible sequence $\langle a_\alpha : \alpha < \kappa_1\rangle^\frown\langle b_\beta : \beta < \kappa_2^*\rangle$ such that the (κ_1, κ_2)-cut is respected in M. That is, we cannot find an element in M which can be added after the a_α's but before the b_β's linearly ordered by a partial order some $\varphi(x, y; \bar{c})$. However, there were restrictions on the cofinalities being not too small. In §2 we show that, to a large extent, these restrictions are necessary.

The family of dependent theories is parallel to the family of stable theories. But actually a better balance than for stable of the "size" of the family of such theories and what we can tell about them is obtained by the family of superstable ones. In §3 a related family of strongly dependent theories, are defined. Now, every superstable T is strongly stable (defined as stable, strongly dependent), but the inverse fails (see also [Sh:839], [Sh:F660]). We then observe some basic properties. This is continued in [Sh:863].

In §4 we look at groups definable in models of dependent theories, and also in strongly dependent theories. In §5 we try to look systematically at a parallel to non-forking.

This work is continued in [Sh:876], [Sh:863], [Sh:886], [FiSh:E50], [CoSh:919], [Sh:F705], [Sh:877], [Sh:900] and [Sh:F906]. More specifically, on a parallel to uni-dimensionality for the theory of the real field see a hopefully forthcoming work of E. Firstenberg–S. Shelah [FiSh:E50]. For continuation of §2 see [Sh:950]. We try to investigate strongly dependent theories (see Section 3) in [Sh:863]. We should add to the history in [Sh:715] that Keisler [Ke87] connects dependent theories and measures on the set of definable subsets of a model. Also, [Sh:715, 3.2], is 5.2 of Baldwin–Benedikt [BlBn00]; we should also add Poizat [Po81] (and then [Sh:93, p. 202, 3] positively answering a question of Poizat). Poizat, dealing with the number of complete types in $\mathbf{S}(N)$ finitely satisfied in $M \prec N$, proves that the number is $\leq 2^{\|M\|}$ (when $|T| \leq \|M\|$) and asks whether it is $\leq (\mathrm{Ded}(\|M\|^{|T|}))$ so by [Sh:93], it is. In 5.26 we follow [Sh:93] proving that we can replace finitely satisfiable but does not split.

Note that Baisalov and Poizat [BaPo98] proved a theorem concerning an o-minimal T, which is a consequence of §1.

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NOTATION. As in [Sh:715] and, in addition

0.1. *Definition:* 1) For $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I \rangle$ an infinite indiscernible sequence, let $\mathrm{tp}'(\bar{\mathbf{b}}) = \langle\, \mathrm{tp}(\bar{b}_{t^n_0} \char`\^ \ldots \char`\^ \bar{b}_{t^n_{n-1}}, \emptyset, \mathfrak{C}) : n < \omega \rangle$ where $t^n_\ell <_I t^n_{\ell,k}$ for $\ell < k < n < \omega$; the choice of the t^n_ℓ's is immaterial.

2) Let "M is n-saturated" mean "M is $\aleph_0$-saturated" for $n < \omega$.

3) Let A/B mean $\mathrm{tp}(A, B)$, inside $\mathfrak{C}$ or $\mathfrak{C}^{\mathrm{eq}}$.

1. Expanding by making a type definable

What, in short, do we show here? We say that A is full over M, if every $p \in \mathbf{S}^{<\omega}(M)$ is realized in A, (Definition 1.5). We let $\mathfrak{B}_{A,M}$ be the expansion of M, for each $\varphi(\bar{x}, \bar{a}), \bar{a} \in {}^{\omega>}A$, by the following $\ell g(\bar{x})$-place relation: all realizations of $\varphi(-, \bar{a})$, i.e., by $\varphi(M, \bar{a})$ (see Definition 1.10(2)). We prove here that if A is full over M, then $\mathrm{Th}(\mathfrak{B}_{M,A})$ has elimination of quantifiers (see Claim 1.12(1), its proof depends only on 1.2, 1.7(2)). By this we prove that $\mathrm{Th}(\mathfrak{B}_{M,A})$ is dependent (in 1.13 depending on 1.19(4), 1.12(1), (5) only), so for this conclusion "A is full over M" is not needed.

1.1. CONTEXT: 1) T is a (first order complete) dependent theory in the language $\mathbb{L}(\tau_T)$.

2) $\mathfrak{C} = \mathfrak{C}_T$ is a monster model for T.

1.2. CLAIM: *Assume*

(a) *M a model*

(b) *D an ultrafilter on M, i.e. on the Boolean Algebra $\mathscr{P}(M)$.*

Then for any $\bar{c} \in {}^{\omega>}\mathfrak{C}$ and formula $\varphi(x, y, \bar{c})$ we have: if the set $\{a \in M : (\exists y \in M)(\mathfrak{C} \models \varphi[a, y, \bar{c}])\}$ belongs to D, then it belongs to $\mathrm{def}_2(D)$, *see definition below.*

1.3. *Definition:* 1) When D is an ultrafilter on a set $B \subseteq \mathfrak{C}$ let $\mathrm{def}_2(D) = \{A \in D$: some member of $\mathrm{def}_1(D)$ is included in $A\}$, where

$\mathrm{def}_1(D) = \{A \in D$: for some $\bar{c} \in {}^{\omega>}\mathfrak{C}$ and formula $\psi(x, \bar{c})$ the set $\psi(M, \bar{c}) = \{a \in M : \mathfrak{C} \models \psi(a, \bar{c})\}$ belongs to D and is equal to $A\}$.

2) Similarly, when D is an ultrafilter on ${}^m B, m < \omega$.

1.4. *Remark:* Note the following easy comments.

1) Of course, Claim 1.2 holds also for $\varphi = \varphi(\bar{x}, \bar{y}, \bar{c})$ when D an ultrafilter on ${}^m M$ and $m = \ell g(\bar{y})$ because, e.g. just work in $\mathfrak{C}^{\mathrm{eq}}$.
2) T is dependent if and only if $T^{\mathrm{eq}} = \mathrm{Th}(\mathfrak{C}^{\mathrm{eq}})$ is; this justifies the statement above (in part (1)); and $\mathrm{Th}(\mathfrak{C})$ is dependent if and only if $\mathrm{Th}(\mathfrak{C}, c)_{c \in C}$ is (for any $C \subseteq \mathfrak{C}$) and T dependent $\Rightarrow \mathrm{Th}(\mathfrak{C} \restriction \tau')$ is dependent when $\tau' \subseteq \tau_T$.
3) $\mathrm{def}_2\ (D)$ is a filter on A.
4) In the proof of 1.2 the hypothesis "T dependent" is used only for deducing that "$\varphi(x, y, \bar{c})$ is dependent" which is naturally defined.
5) Recall the following (which is used in the proof):
   (a) $\Delta \subseteq \mathbb{L}(\tau_T)$, means Δ is a set of objects of the form $\varphi(\bar{x})$, φ a (first order) formula from $\mathbb{L}(\tau_T), \bar{x}$ a sequence of variables with no repetitions including the free variables of φ, but changing the variables is allowed here, i.e., there is no difference between $\varphi(x)$ and $\varphi(y)$; we may write $\varphi(\bar{x}, \bar{y})$ instead of $\varphi(\bar{x}\hat{\ }\bar{y})$
   (b) $\mathrm{tp}_\Delta(\bar{a}, A) = \{\varphi(\bar{x}, \bar{b}) : \bar{x} = \langle x_\ell : \ell < \ell g(\bar{a})\rangle, \varphi(\bar{x}, \bar{y}) \in \Delta$ and $\mathfrak{C} \models \varphi[\bar{a}, \bar{b}]$ and $b \in {}^{\omega>}A\}$
   (c) $\langle \bar{b}_t : t \in I\rangle$ is Δ-indiscernible over B means that: I is a linear order and if $\varphi(\bar{x}_1, \ldots, \bar{x}_n, \bar{y}) \in \Delta, \ell g(\bar{x}_\ell) = \ell g(\bar{b}_t)$ for $\ell = 1, \ldots, n$ and $t \in I$ and $\bar{c} \in {}^{\ell g(\bar{y})}B$ then for any $s_1 <_I \cdots <_I s_n$ and $t_1 <_I \cdots <_I t_n$ we have $\mathfrak{C} \models$ "$\varphi[\bar{b}_{t_1}, \ldots, \bar{b}_{t_n}, \bar{c}] \equiv \varphi[\bar{b}_{s_1}, \ldots, \bar{b}_{s_n}, \bar{c}]$".
6) In the proof of Claim 1.2 we do not need to close Δ_1 to Δ_2, i.e., we can let $\Delta_2 = \Delta_1$ *provided that* we redefine $\mathrm{tp}_{\Delta_1}(a, A)$ as

$$\mathrm{tp}(a, A) \cap \{\varphi(a_0, \ldots, a_{m-1}, x, a_{m+1}, \ldots, a_n) : \varphi(x_0, \ldots, x_{m-1}, x_m, x_{m+1}, \ldots, x_{n-1}) \in \Delta\}$$

or, more specifically, in $(*)_1$ from $\boxtimes_1$, inside the proof of $\boxtimes_1$, we replace "a_ℓ realizes $\mathrm{tp}_{\Delta_2}(a_w, \ldots)$" by "$a_\ell$ realizes $\{\varphi(a_{\ell_0}, \ldots, a_{\ell_{m-1}}, x, a_{\omega+1}, \ldots, a_{\omega+n-1+m}, \bar{b}) : \varphi(x_0, \ldots, x_{n-1}, \bar{y}) \in \Delta_1$ and $\bar{b} \in {}^{\ell g(\bar{y})}B$, $m < n$, $\ell_0 < \cdots < \ell_{m-1} < \ell$ and $\mathfrak{C} \models \varphi[a_{\ell_0}, \ldots, a_{\ell_{m-1}}, a_\omega, a_{\omega+n+1}, \ldots, a_{\omega+n-1-m}, \bar{b})\}$".

Proof. We shall use "T is dependent" only in the last sentence of the proof toward contradiction. Assume that $\bar{c}, \varphi(x, y, \bar{c})$ form a counterexample.

So

$\circledast_0$ (i) the set $A^* = \{a \in M$: for some $b \in M$ we have $\models \varphi[a, b, \bar{c}]\}$ belongs to D,

(ii) $A^* \notin \operatorname{def}_2(D)$, that is, no $A' \in \operatorname{def}_1(D)$ is included in A^*.

By the choice of A^* we can, for each $a \in A^*$, choose $b_a \in M$ such that $\models \varphi[a, b_a, \bar{c}]$. Let $D_1 = D$ and let D_2 be the following ultrafilter on ${}^2M : X \in D_2$ if and only if $X \subseteq {}^2M$ and for some $A \in D$ we have $\{(a, b_a) : a \in A \cap A^*\} \subseteq X$.

We can choose $\langle (a_{\omega+n}, b_{\omega+n}) : n < \omega \rangle$ from $\mathfrak{C}$ such that

$\circledast_1$ for $n_1 < n_2 < \omega$ the pair $(a_{\omega+n_1}, b_{\omega+n_1})$ realizes the type $\operatorname{Av}(M \cup \{a_{\omega+\ell}, b_{\omega+\ell} : \ell \in (n_1, n_2]\}, D_2)$.

It follows that $a_{\omega+n_1}$ realizes the type $\operatorname{Av}(M \cup \{a_{\omega+\ell}, b_{\omega+\ell} : \ell \in (n_1, n_2]\}, D_1)$ and

$\circledast_2$ for $n_1 < n_2$, the element $a_{\omega+n_1}$ realizes the type $\operatorname{Av}(M \cup \{a_{\omega+\ell} : \ell \in (n_1, n_2]\}, D)$;

$\circledast_3$ for $n_1 < n_2$ the triple $(a_{2n_1}, a_{2n_1+1}, b_{2n_1+1})$ realizes the type $\operatorname{Av}(M \cup \{a_{\omega+2\ell}, a_{\omega+2\ell+1}, b_{\omega+2\ell+1} : \ell \in (n_1, n_2]\}, D_3)$ for some ultrafilter D_3 on ${}^3M$, the set of triples of members of M.

(Why? We define $D_3 := \{X \subseteq {}^3M : \{a \in M : \{(b, c) \in M \times M : (a, b, c) \in X\} \in D_2\} \in D_1\}$.)

(We use mainly $\circledast_1$).

Now, clearly,

$\boxtimes_0$ $\langle (a_{\omega+n}, b_{\omega+n}) : n < \omega \rangle$ is an indiscernible sequence over M

$\boxtimes_1$ if $\Delta_1 \subseteq \mathbb{L}(\tau_T)$ is finite, then we can find $n(*) < \omega$ and finite $\Delta_2 \subseteq \mathbb{L}(T)$ such that

$(*)_1$ if $n_1 < \omega$ and $B \subseteq M$ is finite and for each $\ell < n_1$ the element $a_\ell \in M$ realizes the type $\operatorname{tp}_{\Delta_2}(a_\omega, \{a_0, \dots, a_{\ell-1}\} \cup \{a_{\omega+1}, \dots, a_{\omega+n(*)}\} \cup B)$, then $\langle a_\ell : \ell < n_1 \rangle \hat{\ } \langle a_{\omega+\ell} : \ell < \omega \rangle$ is a Δ_1-indiscernible sequence over B (and even Δ_2-indiscernible).

Note that this is close to [Sh:715, 1.16]; note that it follows from the result (that even for $n_1 = \omega$ this holds).

(Why does $\boxtimes_1$ hold? Let $n(*)$ be $\operatorname{arity}(\Delta_1)$, i.e., the maximal number of free variables of a formula from Δ_1, it is finite as Δ_1 is finite, so without loss of generality each $\varphi \in \Delta_1$ is $\varphi(\bar{x})$, $\mathrm{Rang}(\bar{x}) \subseteq \{x_\ell : \ell < n(*)\}$. Let Δ_2 be the closure of Δ_1 under identifying and permuting the variables and let $\Delta_{2,k}$ be defined as Δ_2 but we allow to add dummy variables from $\{x_0, \ldots, x_k\}$ to each formula (we can use below $\Delta_2 = \bigcup\{\Delta_{2,k} : k < \omega\}$). We have to prove that for this choice of $n(*)$ and Δ_2 the assertion $(*)_1$ holds.

So assume $n_1 < \omega$ and B, a_ℓ (for $\ell < n_1$) are as required in the assumption of $(*)_1$. Now we prove, by induction on $k \leq n_1$, that

- $(*)^1_k$ the sequences $\langle a_{\omega+\ell} : \ell < n_1 + n(*)\rangle$ and $\langle a_\ell : \ell < k\rangle\hat{\ }\langle a_{\omega+\ell} : \ell < n_1 + n(*) - k\rangle$ realize the same $\Delta_{2,n_1+n(*)}$-type over B which means that: if $m \leq n(*), \bar{d} \in {}^m B$ and $\varphi(\bar{y}_1, \bar{y}_2) \in \Delta_{2,n_1+n(*)+m}, \ell g(\bar{y}_1) = n_1 + n(*), \ell g(\bar{y}_2) = m$ then $\mathfrak{C} \models \varphi[\langle a_{\omega+\ell} : \ell < n_1 + n(*)\rangle, \bar{d}]$ if and only if $\mathfrak{C} \models \varphi[\langle a_\ell : \ell < k\rangle\hat{\ }\langle a_{\omega+\ell} : \ell < n_1 + n(*) - k\rangle, \bar{d}]$; note that we can allow $m \leq n(*)$.

For $k = 0$, the two expressions give the same sequence. Assume this holds for k and we shall prove it for $k+1$. First $\langle a_\ell : \ell < k+1\rangle\hat{\ }\langle a_{\omega+\ell} : \ell < n_1 + n(*) - (k+1)\rangle$ realize the same type as $\langle a_0, \ldots, a_k, a_{\omega+1}, \ldots, a_{\omega+n_1+n(*)-(k+1)}\rangle$, simply because $\langle a_{\omega+\ell} : \ell < \omega\rangle$ is an indiscernible sequence over M, by $\boxtimes_0$. Now by the assumption of $(*)_1$ we know that a_k, a_ω realizes the same Δ_2-type over $B \cup \{a_0, \ldots, a_{k-1}\} \cup \{a_{\omega+1}, \ldots, a_{\omega+n_1+n(*)-k}\}$.

As $n(*)$ is the arity of Δ_1 hence also of Δ_2 and from the definition of $\Delta_{2,n_1+n(*)}$ it follows that the sequence

$$\langle a_0, \ldots, a_{k-1}, a_k, a_{\omega+1}, \ldots, a_{\omega+n_1+n(*)-k-1}\rangle$$

realizes over B the same $\Delta_{2,n_1+n(*)}$-type as the sequence

$$\langle a_0, \ldots, a_{k-1}, a_\omega, a_{\omega+1}, \ldots, a_{\omega+n_1+n(*)-k-1}\rangle$$

but by the induction hypothesis on k the latter realizes over B the same $\Delta_{2,n_1+n(*)}$-type as the sequence $\langle a_\omega, a_{\omega+1}, \ldots, a_{\omega+n_1+n(*)-1}\rangle$, hence $(*)^1_{k+1}$ holds, so we have carried the induction on $k \leq n_1$. Now the desired conclusion follows from $(*)^1_m$ by $\boxtimes_0$ as each formula in Δ_1 and even Δ_2 has $\leq n(*)$ free variables.)

- $\boxtimes_2$ if $\Delta_1 \subseteq \mathbb{L}(\tau_T)$ is finite, then we can find $n(*) < \omega$ and finite $\Delta_2 \subseteq \mathbb{L}(\tau_T)$ such that
  - $(*)_2$ if $n_1 < \omega, B \subseteq M$ is finite and for each $\ell < n_1, a_{2\ell} \in M$ realizes

$$\mathrm{tp}_{\Delta_2}(a_\omega, \{a_{2m}, a_{2m+1}, b_{2m+1} : m < \ell\} \cup \{a_{\omega+\ell}, b_{\omega+\ell} : \ell = 1, \ldots, n(*)\} \cup B)$$

and $\langle a_{2\ell+1}, b_{2\ell+1}\rangle$ realizes

$$\mathrm{tp}_{\Delta_2}((a_\omega, b_\omega), \{a_{2m}, a_{2m+1}, b_{2m+1} : m < \ell\} \cup \{a_{2\ell}\} \cup \{a_{\omega+\ell}, b_{\omega+\ell} : \ell = 1, \ldots, n(*)\} \cup B)\}),$$

then

$$\langle (a_{2\ell}, a_{2\ell+1}, b_{2\ell+1}) : \ell < n_1\rangle\hat{\ }\langle a_{\omega+2\ell}, a_{\omega+2\ell+1}, b_{\omega+2\ell+1} : \ell < \omega\rangle)$$

is Δ_1-indiscernible over B (and even Δ_2-indiscernible).

(Why? The proof is similar to the proof of $\boxtimes_1$ mainly replacing the use of $\circledast_1$ by $\circledast_3$.)

$\boxtimes_3$ if $B \subseteq M$ is finite, $n^* < \omega$ and $\Delta \subseteq \mathbb{L}(\tau_T)$ is finite, then we can find $a \in M$ realizing the finite type $q = \mathrm{tp}_\Delta(a_\omega, B \cup \{a_{\omega+\ell}, b_{\omega+\ell} : \ell = 1, \ldots, n^*\})$ such that $\models \neg(\exists y \in M)\varphi(a, y, \bar{c})$.

(Why? The set $A := \{a \in M : a$ realizes q, equivalently satisfies the formula $\wedge q \in \mathrm{Av}(\mathfrak{C}, D)\}$ belongs to D because q is finite and the choice of $\langle a_{\omega+\ell}, b_{\omega+\ell} : \ell < \omega\rangle$; moreover, it belongs to $\mathrm{def}_1(D)$ by the definition of $\mathrm{def}_1(D)$ as $\wedge q$ is a formula. But $\mathrm{def}_1(D) \subseteq \mathrm{def}_2(D)$ hence $A \in \mathrm{def}_2(D)$.

So by the assumption towards a contradiction and choice of A^*, i.e., by $(*)_0$, we have $\neg(A \subseteq A^*)$ so there is $a \in A$ such that $a \notin A^*$ which means that $\neg(\exists y \in M)\varphi(a, y, \bar{c})$, so we are done.)

By the above and compactness (or use an ultrapower)

$\boxtimes_4$ there are $N, a_{2n}, a_{2n+1}, b_{2n+1}$ (for $n < \omega$) such that
- (a) N is $|T|^+$-saturated;
- (b) $a_{2n}, a_{2n+1}, b_{2n+1} \in N$;
- (c) $\langle a_n : n < \omega\rangle$ is an indiscernible sequence;
- (d) $\langle (a_{2n}, a_{2n+1}, b_{2n+1}) : n < \omega\rangle\hat{\ }\langle (a_{\omega+2n}, a_{\omega+2n+1}, b_{\omega+2n+1}) : n < \omega\rangle$ is an indiscernible sequence;
- (e) $\mathfrak{C} \models \varphi[a_{2n+1}, b_{2n+1}, \bar{c}]$;
- (f) for no $n < \omega$ and $b \in N$ do we have $\mathfrak{C} \models \varphi[a_{2n}, b, \bar{c}]$.

(Why? By compactness it is enough to prove the following: for every $n_1 < \omega$ and finite $\Delta_1 \subseteq \mathbb{L}(\tau_T)$ to which φ belongs there are $a_{2n}, a_{2n+1}, b_{2n+1} \in M$ for $n < n_1$ such that clauses (a)–(f) hold when we restrict ourselves to $n < n_1$ and Δ_1-types and replace N by M. We first choose a finite $\Delta_2 \subseteq \mathbb{L}(\tau_T)$ as in $\boxtimes_2$, and then choose $(a_{2n}, a_{2n+1}, b_{2n+1})$ by induction on n such that the demand in

$(*)_2$ of $\boxtimes_2$ hold. Arriving to n, choose $a_{2n} \in M$ such that in addition, clause (f) holds, this is possible by $\boxtimes_3$, and then choose $(a_{2n+1}, b_{2n+1}) \in {}^2M$ recalling that (a_ω, b_ω) realizes $\mathrm{Av}(M \cup \{a_{\omega+n}, b_{\omega+n} : 1 \le n < \omega\}, D_2)$. So we are done proving $\boxtimes_4$.)

Next, by clause (d) of $\boxtimes_4$,

$\boxtimes_5$ there is an automorphism F of $\mathfrak{C}$ such that $n < \omega$ implies

$$F((a_{\omega+2n}, a_{\omega+2n+1}, b_{\omega+2n+1})) = (a_{2n}, a_{2n+1}, b_{2n+1}).$$

Hence we can find $b_{2n} \in \mathfrak{C}$ for $n < \omega$ such that $\langle (a_n, b_n) : n < \omega \rangle$ is an indiscernible sequence (over $\emptyset$, not necessarily over $\bar{c}$!) and as N is $|T|^+$-saturated, without loss of generality, $b_{2n} \in N$ for $n < \omega$. But $\mathfrak{C} \models \varphi[a_{2n+1}, b_{2n+1}, \bar{c}]$ for $n < \omega$ so as T is dependent for every large enough $n < \omega$, we have $\mathfrak{C} \models \varphi[a_{2n}, b_{2n}, \bar{c}]$. But as $b_{2n} \in N$ clearly $\{a_n, b_n : n < \omega\} \subseteq N$ hence $n < \omega \Rightarrow \mathfrak{C} \models \varphi[a_{2n}, b_{2n}, \bar{c}]$ contradicts clause (f) of $\boxtimes_4$. $\blacksquare_{1.2}$

Recall

1.5. *Definition:* For $A \subseteq C (\subseteq \mathfrak{C})$, we say that C is full over A when: for every $m < \omega$ and $p \in \mathbf{S}^m(A)$, there is $\bar{c} \in {}^mC$ which realizes p.

1.6. Observation: If

(a) D_1, D_2 are ultrafilters on ${}^mA$,
(b) $A \subseteq C$,
(c) C is full over A,
(d) $\mathrm{Av}(C, D_1) = \mathrm{Av}(C, D_2)$.

Then $\mathrm{def}_\ell(D_1) = \mathrm{def}_\ell(D_2)$ for $\ell = 1, 2$.

Proof. Easy. $\blacksquare$

1.7. Claim: 1) *Assume*

(a) $M \subseteq C$
(b) D_0 *is an ultrafilter on* ${}^{m_0}M$
(c) $\bar{b}_0$ *realizes* $\mathrm{Av}(C, D_0)$
(d) $\mathrm{tp}(\bar{b}_0 {}^\frown \bar{b}_1, C)$ *is f.s. in* M *and* $m_1 = \ell g(\bar{b}_1)$
(e) C *is full over* M.

Then for some ultrafilter D *on* ${}^{m_0+m_1}M$ *we have*

(α) $\mathrm{Av}(C, D) = \mathrm{tp}(\bar{b}_0 {}^\frown \bar{b}_1, C)$
(β) *the projection of* D *on* ${}^{m_0}M$ *is* D_0.

2) *Assume that clauses (a) and (e) of part (1) hold. Then for any* $\bar{c} \in {}^{\omega>}\mathfrak{C}$ *and formula* $\varphi(\bar{x}, y, \bar{z}) \in \mathbb{L}(\tau_T), \ell g(\bar{z}) = \ell g(\bar{c})$ *there are* $\psi(\bar{x}, \bar{z}')$ *and* $\bar{d}$ *of length* $\ell g(\bar{z}')$ *from* $\mathfrak{C}$*, (and even from* C*) such that* $\{\bar{a} \in M : (\exists y \in M)(\models \varphi[\bar{a}, y, \bar{c}])\} = \{\bar{a} \in M :\models \psi(\bar{a}, \bar{d})\}$.

Proof. 1) Let

$$\mathscr{E}_0 = \big\{\{\bar{a} \in {}^{m_0+m_1}M : \bar{a} \restriction m_0 \in X\} : X \in D_0\big\}$$

$$\mathscr{E}_1 = \big\{\{\bar{a} \in {}^{m_0+m_1}M : \mathfrak{C} \models \varphi[\bar{a}; \bar{c}]\} : \varphi(\bar{x}; \bar{y}) \in \mathbb{L}(\tau_T) \\ \ell g(\bar{x}) = m_0 + m_1, \ell g(\bar{y}) = \ell g(\bar{c}), \bar{c} \in {}^{\omega>}C \text{ and } \mathfrak{C} \models \varphi[\bar{b}_0{}^\frown\bar{b}_1; \bar{c}]\big\}.$$

Clearly, it suffices to prove that there is an ultrafilter on ${}^{m_0+m_1}M$ extending $\mathscr{E}_0 \cup \mathscr{E}_1$. For this it suffices to show that any finite subfamily of $\mathscr{E}_0 \cup \mathscr{E}_1$ has a non-empty intersection. But $\mathscr{E}_0$ is closed under finite intersections as D_0 is an ultrafilter on ${}^{m_0}M$ and $\mathscr{E}_1$ is closed under finite intersections as $\mathbb{L}(\tau_T)$ is closed under conjunctions, so it suffices to prove that $X_0 \cap X_1 \neq \emptyset$ when

(i) $X_0 = \{\bar{a} \in {}^{m_0+m_1}M : \bar{a} \restriction m_0 \in X\} \in \mathscr{E}_0$ for some $X \in D_0$

(ii) $X_1 = \{\bar{a} \in {}^{m_0+m_1}M : \mathfrak{C} \models \varphi[\bar{b}_0{}^\frown\bar{b}_1; \bar{c}]\} \in \mathscr{E}_1$, where $\varphi(\bar{x}, \bar{y})$ and $\bar{c}$ are as in the definition of $\mathscr{E}_1$.

As $\text{tp}(\bar{b}_0{}\hat{}\,\bar{b}_1, C)$ is finitely satisfiable in M (= assumption (d)), clearly there is an ultrafilter D'_1 on ${}^{m_0+m_1}M$ such that $\text{Av}(C, D'_1) = \text{tp}(\bar{b}_0{}\hat{}\,\bar{b}_1, C)$.

Let D'_0 be the projection of D'_1 to ${}^{m_0}M$, i.e., $\{Y \subseteq {}^{m_0}M : \{\bar{a} \in {}^{m_0+m_1}M : \bar{a} \restriction m_0 \in Y\} \in D'_1\}$. Clearly, D'_0 is an ultrafilter over ${}^{m_0}M$. We have $\mathfrak{C} \models \varphi[\bar{b}_0, \bar{b}_1; \bar{c}]$, so $X_1 \in D'_1$, hence $X'_0 = \{\bar{a} \restriction m_0 : \bar{a} \in X_1\} \in D'_0$; which implies that the set $X''_0 := \{\bar{a}_0 \in {}^{m_0}M$: for some $\bar{a}_1 \in {}^{m_1}M$ we have $\bar{a}_0{}\hat{}\,\bar{a}_1 \in X_1$, i.e., $\models \varphi[\bar{a}_0, \bar{a}_1; \bar{c}]\}$ belongs to D'_0.

By 1.2 (and 1.4(2)) it follows that X''_0 includes some $Y''_0 \in \text{def}_1(D'_0)$. Now $\text{Av}(C, D_0) = \text{tp}(\bar{b}_0, C) = \text{Av}(C, D'_0)$, because the first equality holds as by assumption (b) the sequence $\bar{b}_0$ realizes $\text{Av}(C, D_0)$ and second equality holds as $\bar{b}_0{}\hat{}\,\bar{b}_1$ realizes $\text{Av}(C, D'_1)$ and the choice of D'_0. But by assumption (e) every $p \in \mathbf{S}^{<\omega}(M)$ is realized by some sequence from C. Hence, by Observation 1.6 we have $\text{def}_2(D_0) = \text{def}_2(D'_0)$. But $Y''_0 \in \text{def}_1(D'_0)$ so $Y''_0 \in \text{def}_2(D_0)$ hence $Y''_0 \in D_0$. By the choice of Y''_0 we have $Y''_0 \subseteq X''_0 \subseteq {}^{m_0}M$ so by the previous sentence $X''_0 \in D_0$, but by clause (i) above also $X \in D_0$ hence $X \cap X''_0 \in D_0$, so we can find $\bar{a}_0 \in X \cap X''_0 \subseteq {}^{m_0}M$. By the definition of X''_0 there is $\bar{a}_1 \in {}^{m_1}M$ such that $\mathfrak{C} \models \varphi[\bar{a}_0, \bar{a}_1; \bar{c}]$. Now $\bar{a}_0{}\hat{}\,\bar{a}_1 \in X_1$, by the definition of X_1 from clause

(ii) and $\bar{a}_0\hat{\ }\bar{a}_1 \in X_0$, because $\bar{a}_0 \in X$ and X_0's definition from clause (i). So $\bar{a}_0\hat{\ }\bar{a}_1 \in X_0 \cap X_1$. Hence, $X_0 \cap X_1 \neq \emptyset$ and we are done.

2) Let $\varphi^*(\bar{x}, y, \bar{z}) \in \mathbb{L}(\tau_T)$ and $\bar{c}^* \in {}^{\ell g(\bar{z})}\mathfrak{C}$ and we should find $\psi(\bar{x}, \bar{z}'), \bar{d}$ as required. Assume that $\bar{c} \in {}^{\ell g(\bar{z})}C$ realizes $\mathrm{tp}(\bar{c}^*, M)$, for our purpose we may assume, without loss of generality, that $\bar{c}^* = \bar{c}$. For any formula $\psi(\bar{x}, \bar{z}') \in \mathbb{L}(\tau_T)$ and $\bar{d} \in {}^{\ell g(\bar{z}')}\mathfrak{C}$ let $Y_{\psi(\bar{x},\bar{d}),M} = \{\bar{a} \in {}^{\ell g(\bar{x})}M : \mathfrak{C} \models \psi[\bar{a}, \bar{d}]\}$ and let $X_{\varphi(\bar{x},y,\bar{c}),M} = \{\bar{a} \in {}^{\ell g(\bar{x})}M : \mathfrak{C} \models \varphi[\bar{a}, b, \bar{c}]$ for some $b \in M\}$.

Lastly, let $\mathscr{P} = \{Y_{\psi(\bar{x},\bar{d}),M} : \psi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_T), \bar{d} \in {}^{\ell g(\bar{z})}C$ and $Y_{\psi(\bar{x},\bar{d}),M} \subseteq X_{\varphi(\bar{x},y,\bar{c}),M}\}$. Clearly, $\mathscr{P}$ is closed under finite unions and is a family of subsets of M. Also if $X_{\varphi(\bar{x},y,\bar{c}),M}$ is equal to some member of $\mathscr{P}$ then we are done, so assume toward contradiction that this fails. So as $X_{\varphi(\bar{x},y,\bar{c})} \subseteq M$, there is an ultrafilter D on M such that $X_{\varphi(\bar{x},y,\bar{c}),M} \in D$ but D is disjoint to $\mathscr{P}$ which contradicts 1.2. $\blacksquare_{1.7}$

1.8. Conclusion: Assume

(a) $M \prec M_1$
(b) M_1 is $\|M\|^+$-saturated.

Then $\{A : A/M_1$ is f.s. in $M\}$ has amalgamation and JEP (the joint embedding property) by elementary maps from $\mathfrak{C}$ to $\mathfrak{C}$ which are the identity on M_1.

Proof. The joint embedding property is trivial. For the amalgamation, by compactness, we should consider finite sequence $\bar{a}_0, \bar{a}_1, \bar{a}_2$ such that $\mathrm{tp}(\bar{a}_0\hat{\ }\bar{a}_\ell, M_1)$ is f.s. in M for $\ell = 1, 2$ and we should find sequences $\bar{b}_0, \bar{b}_1, \bar{b}_2$ such that $\ell g(\bar{b}_\ell) = \ell g(\bar{a}_\ell)$ for $\ell = 0, 1, 2$ and $\mathrm{tp}(\bar{a}_0\hat{\ }\bar{a}_\ell, M_1) = \mathrm{tp}(\bar{b}_0\hat{\ }\bar{b}_\ell, M_1)$ for $\ell = 1, 2$ and $\mathrm{tp}(\bar{b}_0\hat{\ }\bar{b}_1\hat{\ }\bar{b}_2, M_1)$ is f.s. in M.

Let $m_\ell = \ell g(a_\ell)$, let D_0 be an ultrafilter on ${}^{m_0}M$ such that $\mathrm{tp}(\bar{a}_0, M_1) = \mathrm{Av}(M_1, D_0)$. By 1.7(1) for $\ell \in \{1, 2\}$ there is an ultrafilter D_ℓ on ${}^{m_0+m_\ell}M$ such that

$(*)_1$ $\mathrm{tp}(\bar{a}_0\hat{\ }\bar{a}_\ell, M_1)$ is $\mathrm{Av}(M_1, D_\ell)$;
$(*)_2$ the projection of D_ℓ on ${}^{m_0}M$ is D_0.

Let $m = m_0 + m_1 + m_2$ and let D'_1 be the filter on ${}^mM$ consisting of $\{Y \subseteq {}^mM$: for some $X \in D_1$ for every $\bar{a} \in {}^mM$ we have $\bar{a} \restriction (m_0 + m_1) \in X \Rightarrow \bar{a} \in Y\}$. Let D'_2 be the filter on ${}^mM$ consisting of $\{Y \subseteq {}^mM$: for some $X \in D_2$ for every $\bar{a} \in {}^mM$ we have $(\bar{a} \restriction m_0)\hat{\ }(\bar{a} \restriction [m_0 + m_1, m)) \in X \Rightarrow \bar{a} \in Y\}$. Easily, $Y_1 \in D'_1$ and $Y_2 \in D'_2 \Rightarrow Y_1 \cap Y_2 \neq \emptyset$ because D_1, D_2 has the same projection on ${}^{m_0}M$.

Hence, we can find an ultrafilter D^* on ${}^{m_0+m_1+m_2}M$ which extends $D'_1 \cup D'_2$. Hence, if $\bar b_0{}^\frown \bar b_1{}^\frown \bar b_2$ realizes $\mathrm{Av}(M_1, D^*)$, then $\bar b_0{}^\frown \bar b_\ell$ realizes $\mathrm{tp}(\bar a_0{}^\frown \bar a_\ell, M_1)$ for $\ell = 1, 2$. This completes the proof. $\blacksquare_{1.8}$

1.9. Discussion: Next we shall deduce the promised results. If M^+ is an expansion of a model $M \prec \mathfrak{C}$ by the restriction of relations definable in $\mathfrak{C}$ (with parameters), then $\mathrm{Th}(M^+)$ is still dependent. Moreover, if we do this for close enough family of such relations then $\mathrm{Th}(M^+)$ has elimination of quantifiers. Toward formulating this result we define several extensions of T.

1.10. *Definition:* Let $M \prec \mathfrak{C}$, $A \subseteq \mathfrak{C}$ and for simplicity τ_T has predicate symbols only.

1) We define a universal first order theory $T_{M,A}$ as follows
   (a) the vocabulary is $\tau_{M,A} = \{P_{\varphi(\bar x,\bar a)} : \varphi \in \mathbb{L}(\tau_T)$ and $\bar a \in {}^{\ell g(\bar y)}A\} \cup \{c_a : a \in M\}$ with
      (i) c_a an individual constant
      (ii) $P_{\varphi(\bar x,\bar a)}$ being a predicate with arity $\ell g(\bar x)$; but we identify $P_{R(\bar x)}$ with R (where $\bar x = \langle x_\ell : \ell < \mathrm{arity}(R)\rangle$) so $\tau_T \subseteq \tau_{M,A}$)
   (b) $T_{M,A}$ is the set of universal (first order) sentences satisfied in $\mathfrak{B}_{M,M,A}$, see part (2).
2) Assume $M \subseteq C \prec \mathfrak{C}$ and $\mathrm{tp}(C, M \cup A)$ is f.s. in M (e.g., $C = M$). We define $\mathfrak{B} = \mathfrak{B}_{C,M,A}$ as the $\tau_{M,A}$-model with universe C such that $P^{\mathfrak{B}}_{\varphi(\bar x,\bar a)} = \{\bar b \in {}^{\ell g(\bar x)}C : \mathfrak{C} \models \varphi[\bar b, \bar a]\}$ for $\varphi(\bar x, \bar y) \in \mathbb{L}(\tau_T), \bar a \in {}^{\ell g(\bar y)}(A)$ and such that $c^{\mathfrak{B}}_a = a$ for $a \in M$. If $C = M$ we may omit C.
3) A model $\mathfrak{B}$ of $T_{M,A}$ is called quasi-standard if $c^{\mathfrak{B}}_a = a$ for $a \in M$.

3A) A model $\mathfrak{B}$ of $T_{M,A}$ is called standard if it is $\mathfrak{B}_{C,M,A}$ for some $C, M \subseteq C \subseteq \mathfrak{C}$ satisfying $\mathrm{tp}(C, M \cup A)$ is finitely satisfiable in M.

4) Let $T^*_{M,A}$ be the model completion of $T_{M,A}$ (well defined only if it exists!)

1.11. Observation: 1) If $M \subseteq C$ and $\mathrm{tp}(C, M \cup A)$ is finitely satisfiable in M, then $\mathfrak{B}_{C,M,A}$ is a model of $T_{M,A}$.

2) If $\mathfrak{B}$ is a model of $T_{M,A}$, then $\mathfrak{B}$ is isomorphic to the standard model $\mathfrak{B} = \mathfrak{B}_{C,M,A}$ of $T_{M,A}$ for some C.

3) Moreover, if $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$ are models of $T_{M,A}$ and $\mathfrak{B}_1$ is standard, then $\mathfrak{B}_2$ is (quasi standard and is) isomorphic over $\mathfrak{B}_1$ to some standard $\mathfrak{B}'_2$ satisfying $\mathfrak{B}_1 \subseteq \mathfrak{B}'_2$.

4) If $A_1 \subseteq A_2, M \subseteq C$ and $\mathrm{tp}(C, M \cup A_2)$ is f.s. in M, then $\mathfrak{B}_{C,M,A_1}$ is a reduct of $\mathfrak{B}_{C,M,A_2}$.
5) If $M \subseteq C_1 \subseteq C_2$ and $\mathrm{tp}(C_2, M \cup A)$ is f.s. in M, then $\mathfrak{B}_{C_1,M,A}$ is a submodel of $\mathfrak{B}_{C_2,M,A}$ (and $\mathrm{tp}(C_1, M \cup A)$ is finitely satisfiable in M, hence $\mathfrak{B}_{C_1,M,A}$ is well-defined).

Proof. Easy. ∎

1.12. Claim: *Assume A is full over M.*

1) *$\mathfrak{B}_{M,M,A}$ is a model of $T_{M,A}$ with elimination of quantifiers; in fact, every subset of ${}^m(\mathfrak{B}_{M,M,A})$, i.e., of ${}^m|M|$ definable in $\mathfrak{B}_{M,M,A}$ by some first order formula with parameters, is definable by an atomic formula $R(x_0, \ldots, x_{m-1})$ in this model.*
2) *If $\mathrm{tp}(C, A)$ is f.s. in M, then we can find M^+ such that*
   (a) $M \cup C \subseteq M^+ \prec \mathfrak{C}$
   (b) $\mathrm{tp}(M^+, A)$ *is f.s. in* M
   (c) $\mathfrak{B}_{M^+,M,A}$ *is an elementary extension of* $\mathfrak{B}_{M,M,A}$.
3) *$T_{M,A}$ has amalgamation and JEP.*
4) *$\mathrm{Th}(\mathfrak{B}_{M,M,A})$ is the model completion of $T_{M,A}$ so is equal to $T^*_{M,A}$ (which is well-defined).*
5) *$T^*_{M,A}$ is a dependent (complete first order) theory.*

Proof. 1) By Claim 1.7(2), Definition 1.10(1) and A being full over M.

2) E.g., use an ultrapower $\mathfrak{C}^\kappa/D$ of $\mathfrak{C}$ with $\kappa \geq |T| + |C| + |A|, D$ a regular filter on κ and let $\mathbf{j}$ be the canonical embedding of $\mathfrak{C}$ into $\mathfrak{C}^\kappa/D$. So we can find $f : C \to M^\kappa/D$ such that $f \cup (\mathbf{j} \restriction A)$ is an elementary mapping, i.e., a $(\mathfrak{C}, \mathfrak{C}^\kappa/D)$-elementary embedding, now it should be clear.

3) The JEP is trivial because of the individual constants $c_a (a \in M)$. The amalgamation property holds by 1.8 as we can replace M_1 there by any set full over M.

4) By parts (1),(2),(3) we have already proved.

5) As $\mathfrak{B}_{M,M,A}$ is a model of it and reflects.

That is, assume $\psi(x, \bar{y})$ is a formula with the independence property in $T^*_{M,A}$. Then, by part (1), without loss of generality, ψ is an atomic relation hence for some formula $\varphi(x, \bar{y}, \bar{z}) \in \mathbb{L}(\tau_T)$ and $\bar{c} \in {}^{\ell g(\bar{z})}A$, for every $a, \bar{b}$ from $M, \mathfrak{C} \models \varphi[a, \bar{b}, \bar{c}]$ if and only if $\mathfrak{B}_{M,M,A} \models \psi(a, \bar{b})$.

By the choice of $\psi(x,\bar{y})$, for every $n<\omega$ there are $\bar{a}^n_\ell \in {}^{\ell g(\bar{y})}(\mathfrak{B}_{M,M,A}) = {}^{\ell g(\bar{y})}(M)$ for $\ell<\omega$ and $b_w \in \mathfrak{B}_{M,M,A}$, i.e., $b^n_w \in M$ for $w \subseteq \{0,\dots,n-1\}$ such that, for every $w \subseteq \{0,\dots,n-1\}$ and $\ell<n$, we have $\mathfrak{B}_{M,M,A} \models \psi[b^n_w, a^n_\ell]^{\text{if}(\ell\in w)}$. Hence, $\mathfrak{C} \models \varphi[b^n_w,\bar{a}^n_\ell,\bar{c}]^{\text{if}(\ell\in w)}$. So $\varphi(x;\bar{y},\bar{z})$ has the independence property in T. $\blacksquare_{1.12}$

1.13. Conclusion: Assume $M \prec \mathfrak{C}$ and $A \subseteq \mathfrak{C}$. Then $\mathrm{Th}(\mathfrak{B}_{M,M,A})$ is a dependent (complete first order) theory.

Proof. By 1.11(4) and 1.12(5) it is the reduct of a dependent (complete first order) theory. More fully, let A_1 be full over M such that $A \subseteq A_1$ and let $\kappa = |A_1|+|T|$. Clearly if $\mathrm{Th}(\mathfrak{B}_{M,MA_1})$ is dependent, then so is $\mathrm{Th}(\mathfrak{B}_{M,M,A}) = \mathrm{Th}(\mathfrak{B}')$. By 1.12(5) we are done. $\blacksquare_{1.13}$

1.14. *Definition:* 1) For any model $\mathfrak{B}$ (not necessarily of T) and $A \subseteq \mathfrak{B}$ let $\mathbb{B}^m[A,\mathfrak{B}]$ be the family of subsets of ${}^mA$ of the form $\{\bar{a} \in {}^mA : \varphi(\bar{x},\bar{a}) \in p\}$ for some $p \in \mathbf{S}^m(A,\mathfrak{B})$.

2) If $\mathfrak{B} \prec \mathfrak{C}$ we may omit $\mathfrak{B}$.

Remark: If $\mathfrak{B} = \mathfrak{C}$ (or just if $\mathfrak{B}$ is $|A|^+$-saturated), then $\mathbb{B}^m[A,\mathfrak{B}] = \{\{\bar{a} : \mathfrak{B} \models \varphi[\bar{b},\bar{a}]\} : \varphi(\bar{x},\bar{y}) \in \mathbb{L}(\tau_{\mathfrak{B}})$ and $\bar{b} \in {}^{\ell g(\bar{y})}\mathfrak{B}\}$.

1.15. Question: Assume $M \subseteq A \subseteq \mathfrak{C}$ and $\mathfrak{B}$ a standard model of $T_{M,A}$ and $N = \mathfrak{B} \restriction \tau_T$. Then do we have

$(*)_{T,T_{M,A}}$ for any ultrafilter D_0 on $\mathbb{B}[N,N]$, the number of ultrafilters D_1 on $\mathbb{B}[N,\mathfrak{B}]$ extending it is at most $2^{|T|+|A|}$?

1.16. *Remark:* 1) For complete (first order theories) $T \subseteq T_1$, the condition $(*)_{T,T_1}$ of 1.15 has affinity to conditions like "any model of T has <1 or $\le \aleph_0$ or $<\|M\|$ expansions to a model of T_1". What is the syntactical characterization?

2) When is $\mathfrak{B}_{N,M,A}$ a model of $T^*_{M,A}$? Assume $T^*_{M,A}$ has elimination of quantifiers does the following condition implies it, i.e., implies $\mathfrak{B}_{N,M,A} \models T^*_{M,A}$?

$\boxdot_{N,M,A}$ every formula over $N \cup A$ which does not fork over N is realized in N.

1.17. Discussion: 1) Note that in the proof 1.2 we use "T is dependent" just to deduce that the formula $\varphi(x, y, \bar{z})$ is dependent, i.e., for some $n = n_{\varphi(x,y,\bar{z})}$

$\circledast$ $\mathfrak{C} \models \neg(\exists x_0 y_0, \dots, x_{n-1} y_{n-1}) \bigwedge\limits_{w \subseteq n} (\exists \bar{z}) \bigwedge\limits_{\ell < n} \varphi(x_\ell, y_\ell, \bar{z})^{\text{if}(\ell \in w)}$.

In the proof we can use finite Δ_1, Δ_2 large enough for $\varphi(x, y, \bar{c})$, i.e., such that for a suitable n:

$\circledast_2$ $\Delta_1 = \Big\{ (\exists \bar{z}) \Big(\bigwedge\limits_{\ell < n} \varphi(x_0, y_0, \dots, x_{n-1}, y_{n-1}, \bar{z})^{\text{if}(\ell \in w)} \Big) : w \subseteq n \Big\}$.

In particular we need

$\circledast_3$ there is Δ_1-indiscernible sequence $\langle (a_\ell, b_\ell) : \ell < 2n \rangle$ and $\bar{c}'$ such that $\mathfrak{C} \models \varphi[a_\ell, b_\ell, \bar{c}']$ if and only if ℓ is odd

$\circledast_4$ $\Delta_2 = \{(\exists y_0, y_2, \dots, y_{2n-2})(\wedge q(x_0, y_0, \dots, x_{2n-1}, y_{2n-1}) : q$ is a complete Δ_1-type of a Δ_1-indiscernible sequence of pairs of length $2n\}$, hence,

$\circledast_5$ there is no Δ_2 indiscernible sequence

$$\langle (a_{2\ell}, a_{2\ell+1}, b_{2\ell+1}) : \ell < n \rangle \hat{\ } \langle (a_{\omega+2\ell}, a_{\omega+2\ell+1}, b_{\omega+2\ell+1}) : \ell < n \rangle$$

such that $\mathfrak{C} \models \varphi[a_{2\ell+1}, b_{2\ell+1}, \bar{c}]$ for $\ell < n$ and $\{\bar{a}_{2\ell}, a_{2\ell}, b_{2\ell+1} : \ell < n\} \subseteq M$ and for each $\ell < n$ for no $b' \in M$ do we have $\models \varphi[a_{2\ell}, b', \bar{c}]$.

2) So, looking at the proof and 1.7(2)

$\circledast_6$ there is a finite set $\Delta = \Delta^*_\varphi$ of formulas of the form $\psi(x, \bar{z})$ computable from $\varphi(x, y, \bar{z})$ (and n_φ) such that:

(a) if $M, \bar{c}, D$ are as in 1.2, then for some $\bar{c}'$ the set $\psi(M, \bar{c}')$ belongs to D and is included in $\{a \in M$: for no $b \in M$ do we have $\models \varphi[a, b, \bar{c}]\}$

(b) $\{a \in M : (\exists b \in M)(\varphi(a, b, \bar{c})\}$ is a finite union of sets from $\{\psi(M, \bar{C}) : \bar{c}' \in {}^{\bar{z}'}\mathfrak{C}$ and $\psi(x, \bar{z}') \in \Delta\}$.

If in $\circledast_6(b)$ there is a bound n on the size of the set not depending on $(M, \bar{c})$, let $\Delta^*_\varphi = \{\psi_\ell(\bar{x}, \bar{z}_\ell) : \ell < n_*\}$ and let $\psi^*(\bar{x}, \bar{z}) = \bigwedge_{\ell > n} z^n = z^\ell \rightarrow \psi_\ell(x, \bar{z}_\ell)$ so in $\circledast_6$, without loss of generality, $\Delta^*_\varphi = \{\psi^*_\varphi(\bar{x}, \bar{z}^*)\}$.

3) We elaborate; we know that if $\mathbf{I} = \{a \in {}^m M$: there is $b \in M$ such that $\mathfrak{C} \models \varphi[a, b, \bar{c}]\}$ where $\varphi = \varphi(x, y, \bar{z}) \in \mathbb{L}(\tau_T), \bar{c} \in \mathfrak{C}, M \prec \mathfrak{C}$, then for some $\psi(x, \bar{z}') \in \mathbb{L}(\tau_T)$ and $\bar{c}' \in {}^{\ell g(\bar{z}')}\mathfrak{C}$ we have $\mathbf{I} = \psi(M, \bar{c}')$. Can we characterize ψ? Yes, but not so well. Toward proving this, first let $n(*)$

be minimal, such that there are no $a_\ell, b_\ell, (\ell < n(*), \bar{c}_\eta, (\eta \in {}^{\eta(*)}2)$ from $\mathfrak{C}$ such that $M \models \varphi(a_\ell, b_\ell, \bar{c}_\eta)$ if and only if $\eta(\ell) = 1$.

Let $\psi_n(x_0, y_0, \ldots, x_{n(*)-1}{}^\frown y_{n(*)-1}) = (\exists \bar{z}) \bigwedge_{\ell<n(*)} \varphi(x_\ell, y_\ell, \bar{z})^{\eta(\ell)}$ for $\eta \in {}^{n(*)}2$ and $\Delta_1 = \{\psi_\eta(\bar{x}_0, \bar{y}_0, \ldots, \bar{x}_{n(*)}, \bar{y}_{n(*)-1})\}$. Let Δ_2 be the closure of Δ_1 under permuting the variables.

Let $\Delta_{3,k}$ be the set of formulas of the form

$$\vartheta(y_{2k(*)};$$
$$x_0, y_0, x_1, y_1, \ldots, x_{2k-1}, y_{2k-1} - y_{2k};$$
$$x_{2k+1}, y_{2k+1}, \ldots, x_{2n(*)-2}, x_{2n(*)-1}, y_{2n(*)-1}) = (\exists y_{2k+2}) \ldots (\exists y_{2n(*)_2})\psi^*$$

where ψ^* is a conjunction or formula from Δ_2 and their negation.

Now ψ belongs to $\Delta_{3,k}$ for some $k < n(*)$. (In fact, we could be somewhat more specific).

Why? We work with $\bigcup \Delta_{3,\ell}$ choose $a_{2\ell}, a_{2\ell+1}, k_{2\ell+1}$ as in the proof for it. Then we choose $b_{2\ell+1} \in M$ by induction on $\ell < n(*)$ such that $\langle (a_\ell, b_\ell) : \ell < 2n(*) \rangle$ is Δ_1-indiscernible. So for every $\eta \in {}^{(*)}2$ we have

$$(\exists \bar{z}) \bigwedge_{\ell<n(*)} \varphi(a_\ell, b_\ell, \bar{z})^{\eta(\ell)}.$$

2. More on indiscernible sequences

2.1. Context: 1) T is a (first order complete) dependent theory.

2) $\mathfrak{C}$ is the monster model of T.

This section is complimentary to [Sh:715, §5] so recall the definition.

2.2. *Definition:* Let $\bar{\mathbf{a}}^\ell = \langle \bar{a}^\ell_t : t \in I_\ell \rangle$ be an indiscernible sequence which is endless (i.e., I_ℓ having no last element) for $\ell = 1, 2$.

1) We say that $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are **perpendicular** when:

(∗) if $\bar{b}^\ell_n$ realizes $\mathrm{Av}(\{\bar{b}^k_m$: we have $m < n$ and $k \in \{1,2\}$ or we have $m = n$ and $k < \ell\} \cup \bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2, \bar{\mathbf{a}}^\ell)$ for $\ell = 1, 2$, then $\bar{\mathbf{b}}^1, \bar{\mathbf{b}}^2$ are mutually indiscernible (i.e., each is indiscernible over the set of elements appearing in the other) where $\bar{\mathbf{b}}^\ell = \langle \bar{b}^\ell_n : n < \omega \rangle$ for $\ell = 1, 2$.

We define "Δ-perpendicular" in the obvious way.

2) We say $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are equivalent and write $\approx$ if for every $A \subseteq \mathfrak{C}$ we have $\mathrm{Av}(A, \bar{\mathbf{a}}^1) = \mathrm{Av}(A, \bar{\mathbf{a}}^2)$.

3) If $\bar{\mathbf{a}}^1 \subseteq A$, let dual-cf$(\bar{\mathbf{a}}^1, A) = $ Min$\{|B| : B \subseteq A$ and no $\bar{c} \in {}^{\omega>}A$ realizes Av$(B, \bar{\mathbf{a}}^1)\}$; we usually apply this when $A = M$.

2.3. Claim: *Assume*

(α) $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I_0 \rangle$ *is an infinite indiscernible sequence over* A.

(β) $B \subseteq \mathfrak{C}$.

Then we can find I_1, J *and* $\bar{b}_t$ *for* $t \in I_1 \backslash I_0$ *such that:*

(a) $I_0 \subseteq I_1, I_1 \backslash I_0 \subseteq J \subseteq I_1$ *and* $|I_1 \backslash I_0| \leq |J| \leq |B| + |T|$

(b) $\bar{\mathbf{b}}' = \langle \bar{b}_t : t \in I_1 \rangle$ *is an indiscernible sequence over* A

(c) *if* I_2 *is a* J*-free extension of* I_1 *(see below) and* $\bar{b}_t$ *for* $t \in I_2 \backslash I_1$ *are such that* $\bar{\mathbf{b}}'' = \langle \bar{b}_t : t \in I_2 \rangle$ *is an indiscernible sequence over* A*, then*
  - $\circledast$ *if* $n < \omega, \bar{s}, \bar{t} \in {}^n(I_2)$ *and* $\bar{s} \sim_J \bar{t}$ *(see below), then* $\bar{b}_{\bar{s}}, \bar{b}_{\bar{t}}$ *realize the same type over* $A \cup B$ *where* $\bar{b}_{\langle t_\ell : \ell < n \rangle} = \bar{b}_{t_0} {}^\frown \bar{b}_{t_1} {}^\frown \ldots {}^\frown \bar{b}_{t_{n-1}}$.

2.4. *Definition:* 1) For linear orders J, I_1, I_2 we say that I_2 is a J-free extension of I_1 when: $J \subseteq I_1 \subseteq I_2$ and
  - $\circledast$ if $t \in I_2 \backslash I_1$ and $s \in J$, then for some $t' \in I_1$ we have $I_2 \models s < t' < t$ or $I_2 \models t < t' < s$.

2) For linear orders J, I_1, I_2 we say that I_2 is a strong J-free extension of I_1 when $J \subseteq I_1 \subseteq I_2$ and:
  - $\circledast$ if $t \in I_2 \backslash I_1$, then for some $s_1, s_2 \in I_1$ we have $s_1 <_{I_2} t <_{I_2} s_2$ and $[s_1, s_2]_{I_1} \cap J = \emptyset$.

3) For linear orders $J \subseteq I$ and $\bar{s}, t \in {}^nI$, let $\bar{s} \sim_J \bar{t}$ mean that $(s_\ell <_I s_k) \equiv (t_\ell <_I t_k)$ and $(s_\ell <_I r) \equiv (t_\ell <_I r)$ and $(r <_I s_\ell \equiv r <_I t_\ell)$ whenever $\ell, k < n, r \in J)$. Similarly, for $\bar{s}, \bar{t} \in {}^\alpha I$.

2.5. *Remark:* In 2.3 why do we need "J-free"? Let $M = (\mathbb{R}, <, Q^M), Q^M = \mathbb{Q}, B = \{0\}, A = \emptyset, I_0$ the irrationals, $b_t = t$ for $t \in I_0$.

Proof. We try to choose by induction on $\zeta < \lambda^+$ where $\lambda = |T| + |B|$ a sequence $\bar{\mathbf{b}}^\zeta = \langle \bar{b}_t : t \in J_\zeta \rangle$ and together with $\bar{\mathbf{b}}^{\zeta+1}$ we choose $n_\zeta, \bar{s}_\zeta, \bar{t}_\zeta, J'_\zeta, \varphi_\zeta, \bar{c}_\zeta, \bar{d}_\zeta$ such that

(a) J_ζ is a linear order, increasing continuous with ζ;

(b) $J_0 = I_0$ (so $\bar{\mathbf{b}}^0 = \bar{\mathbf{b}}$), $J_{\varepsilon+1} \backslash J_\varepsilon$ is finite so $|J_\varepsilon \backslash I_0| < |\varepsilon|^+ + \aleph_0$;

(c) $\bar{\mathbf{b}}^\zeta$ is an indiscernible sequence over A;

(d) $J'_\zeta \subseteq J_\zeta, J_\zeta = I_0 \cup J'_\zeta, J'_\zeta$ is increasing continuous with ζ and $|J'_\zeta| < |\zeta|^+ + \aleph_0$;

(e) if $\zeta = \varepsilon + 1$, then

$$n_\varepsilon < \omega, \bar{s}_\varepsilon \in {}^{n_\varepsilon}(J'_\zeta), \bar{t}_\varepsilon \in {}^{n_\varepsilon}(J'_\zeta), \varphi_\varepsilon = \varphi_\varepsilon(\bar{x}_0, \dots, \bar{x}_{n_\varepsilon - 1}, \bar{c}_\varepsilon, \bar{d}_\varepsilon), \bar{c}_\varepsilon \subseteq B, \bar{d}_\varepsilon \subseteq A \text{ and } J'_\zeta = J'_\varepsilon \cup (\bar{s}_\varepsilon \hat{\ } \bar{t}_\varepsilon);$$

(f) $\bar{s}_\varepsilon \sim_{J'_\varepsilon} \bar{t}_\varepsilon \wedge \models \varphi_\varepsilon[\bar{b}_{\bar{s}_\varepsilon}, \bar{c}_\varepsilon, \bar{d}_\varepsilon] \wedge \neg\varphi_\varepsilon[\bar{b}_{\bar{t}_\varepsilon}, \bar{c}_\varepsilon, \bar{d}_\varepsilon]$;

(g) $J_{\zeta+1}$ is a J'_ζ-free extension of J_ζ.

If we succeed, for some unbounded $w \subseteq \lambda^+$ and $n_*, \varphi_\xi, \bar{c}^*$ and u for every $\varepsilon \in w$ we have $n_\varepsilon = n_*, \varphi_\varepsilon = \varphi_*, \bar{c}_\varepsilon = \bar{c}^*$ and $u = \{\ell < n_* : s_{\varepsilon,\ell} \in J'_\zeta\}$. Now let $J^* = \bigcup\{J'_\zeta : \zeta < \lambda^+\}$, so every $J' \subseteq J^*$ of cardinality $\le \lambda$ is included in J'_ζ for some $\zeta < \lambda^+$ and we get contradiction to clause (b) of [Sh:715, 3.2], hence we fail, i.e., we cannot choose for some ζ. But we can choose $\bar{\mathbf{b}}_\zeta = \langle b_t : t \in J_\zeta \rangle$, if $\zeta = 0$ by clause (b) and if ζ is a limit ordinal by clause (a). So $\zeta = \varepsilon + 1$, we have chosen $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in J_\zeta \rangle$ but we cannot choose $J_{\zeta+1}, \bar{\mathbf{b}}^{\zeta+1}, n_\zeta, \bar{s}_\zeta, \bar{t}_\zeta, J'_\zeta, \varphi_\zeta, \bar{c}_\zeta, \bar{d}_\zeta$ as required. Then $\bar{\mathbf{b}}^\varepsilon$ is as required. $\blacksquare_{2.3}$

The aim of 2.6 and 2.9 below is to show a complement of [Sh:715, §5]; that is, in the case of small cofinality, what occurs in one cut is the "same" as what occurs in others.

2.6. Claim: *Assume*

(a) $\mu \ge |T|$;

(b) I_ℓ *for* $\ell < 4$ *are pairwise disjoint linear orders;*

(c) $I_\ell = \cup_{\beta<\mu^+} I^\beta_\ell, I^\beta_\ell$ *(strictly) increasing with* β *and* $|I^\beta_\ell| \le \mu$ *for* $\ell < 4$;

(d) $\ell \in \{0, 2\} \Rightarrow I^\beta_\ell$ *an end segment of* I_ℓ;

(e) $\ell \in \{1, 3\} \Rightarrow I^\beta_\ell$ *is an initial segment of* I_ℓ;

(f) $I = I_0 + I_1 + I_2 + I_3$ *and* $I^\beta = I^\beta_0 + I^\beta_1 + I^\beta_2 + I^\beta_3$;

(g) $\langle \bar{b}_t : t \in I \rangle$ *is an indiscernible sequence.*

Then we can find a limit ordinal $\beta(*) < \mu^+$ *and* $\langle \bar{b}^*_t : t \in I \rangle$ *such that:*

(A) $\bar{b}^*_t = \bar{b}_t$ *if* $t \in I \backslash I^{\beta(*)}$;

(B)$_1$ $\langle \bar{b}^*_t : t \in I \backslash I^{\beta(*)}_0 \backslash I^{\beta(*)}_1 \rangle$ *is an indiscernible sequence;*

(B)$_2$ $\langle \bar{b}^*_t : t \in I \backslash I^{\beta(*)}_2 \backslash I^{\beta(*)}_3 \rangle$ *is an indiscernible sequence;*

(C)$_1$ $\mathrm{tp}_*(\langle \bar{b}^*_t : t \in I^{\beta(*)}_0 \cup I^{\beta(*)}_1 \rangle, \cup \{\bar{b}^*_t : t \in (I \backslash I^\beta) \cup I^{\beta(*)}_2 \cup I^{\beta(*)}_3 \cup (I^{\beta(*)+\omega}_0 \backslash I^{\beta(*)}_0) \cup (I^{\beta(*)+\omega}_1 \backslash I^{\beta(*)}_1)\})$
$\vdash \mathrm{tp}_*(\langle \bar{b}^*_t : t \in I^{\beta(*)}_0 \cup I^{\beta(*)}_1 \rangle, \cup \{\bar{b}^*_t : t \in (I \backslash I^{\beta(*)}) \cup I^{\beta(*)}_2 \cup I^{\beta(*)}_3\})$
for any $\beta \in [\beta(*) + \omega, \mu^+)$;

(C)$_2$ $\mathrm{tp}_*(\langle \bar b^*_t : t \in I^{\beta(*)}_2 \cup I^{\beta(*)}_3\rangle, \cup\{\bar b^*_t : t \in (I\backslash I^\beta) \cup I^{\beta(*)}_0 \cup I^{\beta(*)}_1$
$\cup\, (I^{\beta(*)+\omega}_2 \backslash I^{\beta(*)}_2) \cup (I^{\beta(*)+\omega}_3 \backslash I^{\beta(*)}_3)\}) \vdash$
$\mathrm{tp}(\langle b^*_t : t \in I^{\beta(*)}_2 \cup I^{\beta(*)}_3\rangle, \cup\{\bar b^*_t : t \in (I\backslash I^{\beta(*)}) \cup I^{\beta(*)}_0 \cup I^{\beta(*)}_1\})$
for any $\beta \in [\beta(*) + \omega, \mu^+)$*;*

(D)$_1$ $\langle \bar b^*_t : t \in I_0\backslash I^{\beta(*)}_0\rangle$ *is an indiscernible sequence over* $\cup\{\bar b^*_t : t \in I^{\beta(*)}_0 \cup I_1 \cup I_2 \cup I_3\}$*;*

(D)$_2$ $\langle \bar b^*_t : t \in (I_1\backslash I^{\beta(*)}_1) + (I_2\backslash I^{\beta(*)}_2)\rangle$ *is an indiscernible sequence over* $\cup\{\bar b^*_t : t \in I_0 \cup I^{\beta(*)}_1 \cup I^{\beta(*)}_2 \cup I_3\}$*;*

(D)$_3$ $\langle \bar b^*_t : t \in I_3\backslash I^{\beta(*)}_3\rangle$ *is an indiscernible sequence over* $\cup\{\bar b^*_t : t \in I_0 \cup I_1 \cup I_2 \cup I^{\beta(*)}_3\}$*.*

2.7. *Remark:* What occurs if T is stable (or just $\bar{\mathbf{b}}$ is)? We get something like $\{\bar b^*_t : t \in I^{\beta(*)}_0 \cup I^{\beta(*)}_1\} = \{\bar b^*_t : t \in I^{\beta(*)}_2 \cup I^{\beta(*)}_3\}$.

Proof. For simplicity assume $I^0_\ell = \emptyset$.

We choose by induction on $n < \omega$ an ordinal $\beta(n)$ and $\langle \bar b^n_t : t \in I\rangle$ such that:

(α) $\beta(n) < \mu^+, \beta(0) = 0, \beta(n) + \omega \le \beta(n + 1)$;

(β) $\bar b^n_t = \bar b_t$ if $t \in I\backslash I^{\beta(n)}$ or if $n = 0$;

$(\gamma)_1$ $\langle \bar b^n_t : t \in I\backslash I^{\beta(n)}_0 \backslash I^{\beta(n)}_1\rangle$ realizes the same type as $\langle \bar b_t : t \in I\backslash I^{\beta(n)}_0 \backslash I^{\beta(n)}_1\rangle$;

$(\gamma)_2$ $\langle \bar b^n_t : t \in I\backslash I^{\beta(n)}_2 \backslash I^{\beta(n)}_3\rangle$ realizes the same type as $\langle \bar b_t : t \in I\backslash I^{\beta(n)}_2 \backslash I^{\beta(n)}_3\rangle$;

$(\delta)_1$ if n is even, then:

(1) $\bar b^{n+1}_t = \bar b^n_t$ for $t \in I\backslash I^{\beta(n)}_2 \backslash I^{\beta(n)}_3$;

(2) if $\beta(n+1) < \beta < \mu^+$ then the type which $\langle \bar b^{n+1}_t : t \in I^{\beta(n)}_2 \cup I^{\beta(n)}_3\rangle$ realizes over $\cup\{\bar b^n_t : t \in (I_0\backslash I^\beta_0) \cup I^{\beta(n+1)}_0 \cup (I_1\backslash I^\beta_1) \cup I^{\beta(n+1)}_1 \cup (I_2\backslash I^\beta_2) \cup (I^{\beta(n+1)}_2 \backslash I^{\beta(n)}_2) \cup (I_3\backslash I^\beta_3) \cup (I^{\beta(n+1)}_3 \backslash I^{\beta(n)}_3)\}$ has a unique extension over $\cup\{\bar b^n_t : t \in I\backslash I^{\beta(n)}_2 \backslash I^{\beta(n)}_3\}$;

(3) $\bar b^{n+1}_t = b^n_t$ if $t \in I^{\beta(k)}_2 \cup I^{\beta(k)}_3, k < n$

$(\delta)_2$ if n is odd like (δ_1) inverting the roles of $(I_0, I_1), (I_2, I_3)$;

(ε) $\langle \bar b^n_t : t \in I\rangle$ satisfies clauses $(D)_1, (D)_2, (D)_3$ of the claim with $\beta(n)$ instead of $\beta(*)$.

The induction step is as in the proof of 2.3 (though we use the finite character for the middle clause (2) of clauses $(\delta)_1, (\delta)_2$).

Alternatively, letting n be even we try to choose $\beta_n(\varepsilon), \bar{\mathbf{b}}^{n,\varepsilon} = \langle \bar b^{n,\varepsilon}_t : t \in I^{\beta(n)}_2 + I^{\beta(n)}_3\rangle$ by induction on $\varepsilon \le \mu^+$ such that:

$\odot$ (a) $\beta_n(\varepsilon) < \mu^+$;

(b) $\beta_n(0) = \beta(n)$;

(c) $\beta_n(\varepsilon)$ is increasing and continuous;

(d) $\zeta < \varepsilon \Rightarrow \ \mathrm{tp}(\bar{\mathbf{b}}^{n,\varepsilon}, \cup\{b^n_t : t \in (I \backslash I^{\varepsilon}_{\beta_n(\varepsilon)}) \cup I_{\beta_n(\zeta)}\}) \vdash$ $\mathrm{tp}(\bar{\mathbf{b}}^{n,\zeta}, \cup\{\bar{b}^n_t : t \in (I \backslash I_{\beta_n(\varepsilon)}) \cup I_{\beta_n(\zeta)}\})$;

(e) if $\varepsilon = \zeta + 1$, then $(\delta)_1(2)$ fails if we let

$$\bar{b}^{n+1}_t = \begin{cases} b^n_t & \text{if } t \in I \backslash I^{\beta(n)}_2 \backslash I^{\beta(n)}_3 \\ \bar{b}^{n,\zeta}_t & \text{if } t \in I^{\beta(n)}_2 \cup I^{\beta(n)}_3 \end{cases}.$$

If we succeed to carry the induction, by [Sh:715], for some ε, the sequences $\langle \bar{b}^n_t : t \in I^{\beta_n(\varepsilon)}_0 \rangle, \langle \bar{b}^n_t : t \in I^{\beta_n(\varepsilon)}_1 + I^{\beta_n(\varepsilon)}_2 \rangle, \langle \bar{b}^n_t : t \in I^{\beta_n(\varepsilon)}_3 \rangle$ are mutually indiscernible over $\bigcup\{\bar{b}^{n,\mu^+}_t : t \in I^{\beta(n)}_2 + I^{\beta(n)}_3\} \cup \{b^n_t : t \in (I \backslash I_{\beta_n(\varepsilon)})\}$ (because $\langle \bar{b}_t : t \in I_0 \backslash I^{\beta_n(\varepsilon)}_0 \rangle, \langle \bar{b}_t : t \in (I_1 \backslash I_{\beta_n(\varepsilon)}) + I_2 \backslash I^{\beta_n(\varepsilon)}_2 \rangle, \langle \bar{b}_t : t \in I_3 \backslash I^{\beta_n(\varepsilon)}_3 \rangle$ are mutually indiscernible, recalling (β).

This contradicts (e). So we cannot complete the induction. We certainly succeed for $\varepsilon = 0$, and there is no problem for limit $\varepsilon \leq \mu^+$. So for some $\varepsilon = \zeta + 1$ we have success for ζ and cannot choose for ε. We define $\bar{b}^{n+1}_i$ as in (e) of $\odot$ above, and choose $\beta(n+1) \in [\beta_n(\varepsilon), \mu^+)$ such that clauses (ε) holds.

Let $\beta(*) = \bigcup\{\beta(n) : n < \omega\} < \mu^+, \bar{b}^*_t$ is $\bar{b}^n_t$ for every n large enough (exists by clause (β) if $t \in I \backslash I^{\beta(*)}$ and by $(\delta)_\ell$ (1) and (3) if $t \in I^{\beta(*)}$). Clearly, we are done. $\blacksquare_{2.6}$

2.8. Claim: *Assume*

(a) $I, I^\beta, I_\ell, I^\beta_\ell$ *for* $\ell < 4, \beta < \mu^+$ *are as in the assumption of claim 2.6;*

(b) $\beta(*)$ *and* $\langle \bar{b}^*_t : t \in I \rangle$ *are as in the conclusion of claim 2.6;*

(c) $J^+ = J^+_0 + J^+_1 + J^+_2 + J^+_3 + J^+_4$ *linear orders;*

(d) $J = J_0 + J_1 + J_2 + J_3 + J_4$ *linear orders;*

(e) $J_1 = J^+_1 + I^{\beta(*)}_0 + I^{\beta(*)}_1$ *and* $J_3 = I^{\beta(*)}_2 + I^{\beta(*)}_3$;

(f) $J_0 \subseteq J^+_0$ *and* $I_0 \backslash I^{\beta(*)}_0 \subseteq J^+_0$;

(g) $J_2 \subseteq J^+_2$ *and* $(I_1 \backslash I^{\beta(*)}_1) + (I_2 \backslash I^{\beta(*)}_2) \subseteq J_2$;

(h) $J_4 \subseteq J^+_4$ *and* $(I_3 \backslash I^{\beta(*)}_3) \subseteq J^+_4$;

(i) $\langle \bar{b}^*_t : t \in J^+ \rangle$ *is an indiscernible sequence.*

1) If J'_0, J'_2, J'_4 *are infinite initial segments of* J_0, J_2, J_4 *respectively, then*

(α) $\mathrm{tp}(\langle \bar{b}^*_t : t \in J_3 \rangle, \cup\{\bar{b}_s : s \in J'_0 \cup J_1 \cup J'_2 \cup J'_4) \vdash \mathrm{tp}(\langle \bar{b}^*_t : t \in J_3 \rangle, \cup\{\bar{b}^*_s : s \in J_0 \cup J_1 \cup J_2 \cup J_4\})$

(β) *like* (α) *interchanging* J_3, J_1.

2) If J_0 has no first element, $J'_0 \subseteq J_0$ is unbounded from below, $J'_2 \subseteq J_2$ is infinite and J_4 has no last element and $J'_4 \subseteq J_4$ is unbounded from above, then the conclusions of (1) holds

(α) $\mathrm{tp}(\langle \bar b^*_t : t \in J_3\rangle, \bigcup\{\bar b_s : s \in J'_0 \cup J_1 \cup J'_2 \cup J'_4) \vdash \mathrm{tp}(\langle \bar b^*_t : t \in J_3\rangle, \cup\{\bar b^*_s : s \in J_0 \cup J_1 \cup J_2 \cup J_4\})$

(β) $\mathrm{tp}(\langle \bar b^*_t : t \in J_1\rangle, \bigcup\{\bar b_s : s \in J'_0 \cup J'_2 \cup J_3 \cup J'_4) \vdash \mathrm{tp}(\langle \bar b^*_t : t \in J_1\rangle, \cup\{\bar b^*_s : s \in J_0 \cup J_2 \cup J_3 \cup J_4\})$.

3) If J^*_0, J^*_2, J^*_4 has neither first element nor last element and J'_0, J'_2, J'_4 are subsets of J_0, J_2, J_4 respectively unbounded from below and J''_0, J''_2, J''_4 are subsets of J_0, J_2, J_4 respectively unbounded from above, then the conclusion of part (1) holds.

Proof. The result follows by the local character of $\vdash$ and by the indiscernibility demands in 2.6, i.e., clauses $(D)_1, (D)_2, (D)_3$. $\blacksquare_{2.8}$

2.9. CONCLUSION: 1) If $\mu \geq \kappa \geq |T|$, then for some linear order J^* of cardinality κ we have

$\boxtimes_{\bar{\mathbf{b}}^*, J^*}$ Assume

(a) $J = J_0 + J_1 + J_2 + J_3 + J_4$;

(b) the cofinalities of J_0, J_2, J_4 and their inverse are $\leq \mu$ but are infinite;

(c) $J_1 \cong J^*$ and $J_3 \cong J^*$ (hence J_1, J_3 have cardinality $\leq \kappa$);

(d) $\langle \bar b_t : t \in J \backslash J_3\rangle$ is an indiscernible sequence (of m-tuples);

(e) M is a μ^+-saturated model;

(f) $\bigcup\{\bar b_t : t \in J\backslash J_3\} \subseteq M$.

Then we can find $\bar b_t \in {}^m M$ for $t \in J_3$ such that $\langle \bar b_t : t \in J\backslash J_1\rangle$ is an indiscernible sequence.

2) If we allow J^* to depend on $\mathrm{tp}'(\bar{\mathbf{b}}^*)$, see Definition 0.1(1), then we can use J^* of the form $\delta^* + \delta, \delta < \kappa^+$ (δ^* – the inverse of δ).

Proof. Let $\bar{\mathbf{b}}^*$ be an infinite indiscernible sequence.

Let J_0, J_2, J_4 be disjoint linear orders as in (b). Apply 2.6 with I_1, I_3 isomorphic to $(\mu^+, <)$ and I_0, I_2 isomorphic to $(\mu^+, >)$, say $I_\ell = \{t^\ell_\alpha : \alpha < \mu^+\}$ with t^ℓ_α increasing with α if $\ell \in \{1,3\}$ and decreasing with α if $\ell \in \{0,2\}$, we get $\bar{\mathbf{b}}^* = \langle b^*_t : t \in \sum_{\ell<4} I_\ell\rangle, \beta(*)$ as in 2.6 with $\mathrm{tp}'(\bar{\mathbf{b}}^* \restriction I_0) = \mathrm{tp}'(\bar{\mathbf{b}}^{\circledast})$, see Definition 0.1. Let $J^+_0 = J_0 + (I_0 \backslash I_0^{\beta(*)})$, $J^+_1 = J_1 = I_0^{\beta(*)} + I_1^{\beta(*)}$,

$J_2^+ = J_2 + (I_1 \backslash I_1^{\beta(*)}) + (I_2 \backslash I_2^{\beta(*)})$, $J_3^+ = I_2^{\beta(*)} + I_3^{\beta(*)} + J_3$ and $J_4^+ = J_4 + (I_3 \backslash I_3^{\beta(*)})$ and $J^+ = J_0^+ + J_1^+ + J_2^+ + J_3^+ + J_4^+$. All J_ℓ are infinite linear orders, choose $J^* = J_1$, clearly $J_3 \cong J^*$. Now

(∗) $\langle \bar{b}_t^* : t \in J \backslash J_3 \rangle$ is an indiscernible sequence and

(∗∗) if $M \supseteq \bigcup \{\bar{b}_t^* : t \in J \backslash J_3\}$ is μ^+-saturated then we can find $\bar{b}_t' \in {}^m M$ for $t \in J_3$ such that

$$\langle \bar{b}_t^* : t \in J_0 \rangle \hat{} \langle \bar{b}_t' : t \in J_3 \rangle \hat{} \langle \bar{b}_t^* : t \in J_4 \rangle$$

is an indiscernible sequence.

(Why? Choose $J_0' \subseteq J_0$ unbounded from below of cardinality $\mathrm{cf}(J_0, >_{J_0})$ which is $\leq \mu$ but $\geq \aleph_0$, and similarly $J_2' \subseteq J_2, J_4' \subseteq J_4$ and choose $J_0'' \subseteq J_0$ unbounded from above of cardinality $\mathrm{cf}(J_0)$ which is $\leq \mu$ and similarly $J_2'' \subseteq J_2, J_4'' \subseteq J_4$ (all O.K. by clause (b) of the assumption).

Now $p = \mathrm{tp}(\langle b_t^* : t \in J_3 \rangle, \bigcup \{\bar{b}_s : s \in J_0' \cup J_0'' \cup J_2' \cup J_2'' \cup J_4' \cup J_4'\})$ is a type of cardinality $\leq |T| + |J_0'| + |J_0''| + |J_2'| + |J_2''| + |J_4'| + |J_4''| \leq \mu$ hence is realized by some sequence $\langle \bar{b}_t' : t \in J_3 \rangle$ from M.

By Claim 2.8 the desired conclusion in (∗∗) holds.)

So we have gotten the desired conclusion for any $\langle J_\ell : \ell \leq 4 \rangle$ and indiscernible sequence, $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in J \backslash J_5 \rangle$ as long as $\mathrm{tp}'(\bar{\mathbf{b}}) = \mathrm{tp}'(\bar{\mathbf{b}}^*)$ and the order type of J_1, J_3 is as required for $\bar{\mathbf{b}}^*$. This is enough for part (2), we are left with (1).

Note that by the proof of 2.3, the set of $\beta(*)$ as required contains $E \cap \{\delta < \mu^+ : \mathrm{cf}(\delta) = \aleph_0\}$ for some club E (in fact even contains E). So if $\mu \geq 2^{|T|}$, as $\{\mathrm{tp}'(\bar{\mathbf{b}}) : \bar{\mathbf{b}}$ an infinite indiscernible sequence$\}$ has cardinality $\leq 2^{|T|}$ we are done.

Otherwise, choose J^* a linear order of cardinality μ isomorphic to its inverse, to $J^* \times \omega$ and to $J^* \times (\gamma + 1)$ ordered lexicographically for every $\gamma \leq \mu$ hence for every $\gamma < \mu^+$, (e.g. note if J^{**} is dense with no first and last element and saturated, or special, of cardinality $> \mu$, then $J^{**} \times \omega$ satisfies this and use the L.S. argument). So we can in 2.6 and hence in 2.8, use $I_\ell (\ell < 4)$, such that $I_\ell^{\beta+1} \cong J^*$ for $\beta < \mu^+, \ell < 4$. So $I_0^{\beta(*)} + I_1^{\beta(*)} \cong J^* \cong I_2^{\beta(*)} + I_3^{\beta(*)}$. $\blacksquare_{2.9}$

2.10. Conclusion: In 2.9:

(A) we can choose $J^* = \mu^* + \mu$ i.e. $\{0\} \times (\mu, >) + \{1\} \times (\mu, <)$;

(B) if J is a linear order ($\neq \emptyset$) of cardinality $\leq \mu$, we can use $J^* = (\mu^* + \mu) \times J$ ordered lexicographically;

(C) we can change the conclusion of 2.9 to make it symmetrical between J_3 and J_1;

(D) we use only clause $(E)_2$ of 2.6, or we could use only clause $(E)_1$.

Proof. (A),(B) combine the proofs of 2.3 and 2.6 trying to contradict each formula, by bookkeeping trying for it enough times. $\blacksquare_{2.10}$

We may look at it differently, part (2) is close in formulation to be a complement to [Sh:715, §5].

2.11. Conclusion: 1) Assume

(a) $J = I \times J^*$ lexicographically, J^*, μ are as in 2.9, I infinite;

(b) $\langle \bar{b}_t : t \in J \rangle$ an indiscernible sequence, $\ell g(\bar{b}_t) = m$ or just $\ell g(\bar{b}_t) < \mu^+$;

(c) for $s \in I$ let $\bar{c}_s$ be $\langle \bar{b}_t : t \in \{s\} \times J^* \rangle$, more exactly the concatanation of the sequences in $\bar{b}_t$ for $t \in \{s\} \times J^*$.

Then

(α) $\langle \bar{c}_s : s \in I \rangle$ is an infinite indiscernible sequence

(β) if $s_0 <_I \cdots <_I s_7$ then there is $\bar{c}$ realizing $\mathrm{tp}(\bar{c}_{s_2}, \bigcup\{\bar{c}_{s_\ell} : \ell \leq 7, \ell \neq 2\})$ such that $\mathrm{tp}(\bar{c}, \bigcup\{\bar{c}_{s_\ell} : \ell \leq 7, \ell \neq 2\}) \vdash \mathrm{tp}(\bar{c}_{s_2}, \bigcup\{\bar{c}_s : s_0 \leq_I s \leq_I s_1$ or $s_3 \leq_I s \leq_I s_4$ or $s_6 \leq_I s \leq_I s_7\})$

(γ) similarly inverting the order (i.e. interchanging the roles of s_2, s_5 in clause (β)).

2) Assume the sequence $\langle \bar{c}_s : s \in I \rangle$ from part (1) satisfies $M \supseteq \bigcup\{\bar{c}_s : s \in I\}$ and $(I_1, I_2), (I_3, I_4)$ are Dedekind cuts of I, each of $I_1, (I_2)^*, I_3, (I_4)^*$ is non-empty of cofinality $\leq \mu$. Let $I^+ \supseteq I, t_2, t_5 \in I_1^+$ realize the cuts $(I_1, I_2), (I_3, I_4)$, respectively, and $\bar{c}_t$ for $t \in I^+ \backslash I$ are such that $\langle \bar{c}_t : t \in I^+ \rangle$ is indiscernible (then for notational simplicity), then

$\boxdot$ there is a sequence in M realizing $\mathrm{tp}(\bar{c}_{t_2}, \bigcup\{\bar{c}_s : s \in I\})$ if and only if there is a sequence in M realizing $\mathrm{tp}(\bar{c}_{t_5}, \bigcup\{\bar{c}_s : s \in I\})$.

Concluding Remark: There is a gap between [Sh:715, 5.11] and the results in §2, some light is thrown by

2.12. Claim: *In* [Sh:715, 5.11]*; we can omit the demand* $\mathrm{cf}(\mathrm{Dom}(\bar{\mathbf{a}}^\zeta)) \geq \kappa_1$ *(= clause (f) there) if we add* $\zeta < \zeta^* \Rightarrow (\theta^1_\zeta)^+ = \lambda$.

Proof. By the omitting type argument. $\blacksquare$

2.13. QUESTION: Assume:

(a) $\langle (N_i, M_i) : i \leq \kappa \rangle$ is $\prec$-increasing (as pairs), M_{i+1}, N_{i+1} are λ_i^+-saturated, $\|N_i\| \leq \lambda_i, \langle \lambda_i : i < \kappa \rangle$ increasing, $\kappa < \lambda_0$;

(b) $p(\bar{x})$ is a partial type over $N_0 \cup M_\kappa$ of cardinality $\leq \lambda_0$.

1) Does $p(\bar{x})$ have a λ_0^+-isolated extension?

2) Does this help to clarify DOP?

3) Does this help to clarify "if any M is a benign set" (see [BBSh:815]).

2.14. CLAIM: *Assume*

(a) *M is λ^+-saturated;*

(b) *$p(\bar{x})$ is a type of cardinality $\leq \kappa, \ell g(\bar{x}) \leq \kappa$;*

(c) *$\mathrm{Dom}(p) \subseteq A \cup M, |A| \leq \kappa \leq \lambda$;*

(d) *$B \subseteq M, |B| \leq \lambda$.*

Then there is a type $q(\bar{x})$ over $A \cup M$ of cardinality $< \kappa$ and $r(\bar{x}) \in \mathbf{S}^{\ell g(\bar{x})}(A \cup B)$ such that

$$p(\bar{x}) \subseteq q(\bar{x}) \quad q(\bar{x}) \vdash r(\bar{x})$$

Remark: This defines a natural quasi order (type definable) is it directed?

3. Strongly dependent theories

3.1. CONTEXT: T complete first order, $\mathfrak{C}$ a monster model of T.

3.2. *Definition:* 1) T is strongly[1] dependent (we may omit the 1) if : there are no $\bar{\varphi} = \langle \varphi_n(\bar{x}, \bar{y}_n) : n < \omega \rangle$ and $\langle \bar{a}_\alpha^n : n < \omega, \alpha < \lambda \rangle$ such that

(*) for every $\eta \in {}^\omega\lambda$ the set $p_\eta = \{\varphi_n(\bar{x}, \bar{a}_\alpha^n)^{\mathrm{if}(\eta(n)=\alpha)} : \alpha < \lambda\}$ is consistent; so $\ell g(\bar{a}_\alpha^n) = \ell g(\bar{y}_n)$.

2) T is strongly stable if it is stable and strongly dependent.

3) $\kappa_{\mathrm{ict}}(T)$ is the first κ such that there is no $\bar{\varphi} = \langle \varphi_\alpha(\bar{x}, \bar{y}_\alpha) : \alpha < \kappa \rangle$ satisfying the parallel of part (1), in this case we say that $\bar{\varphi}$ witnesses $\kappa < \kappa_{\mathrm{ict}}(T)$ and let $m(\bar{\varphi}) = \ell g(\bar{x})$.

3.3. CLAIM: *1) If T is superstable, then T is strongly dependent.*

2) If T is strongly dependent, then T is dependent.

3) There are stable T which are not strongly dependent.

4) There are stable not superstable T which are strongly dependent.

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5) There are unstable strongly dependent theories.

6) The theory of real closed fields is strongly dependent; moreover every o-minimal (complete first-order) T is strongly dependent.

7) If T is stable, then $\kappa_{\rm ict}(T) \leq \kappa(T)$.

8) If T is dependent, then we may add, in 3.2(1)

(∗∗) *for each $n < \omega$ for some k_n any k_n of the formulas $\{\varphi_n(\bar{x}, \bar{a}^n_\alpha) : \alpha < \lambda\}$ are contradictory.*

Proof. 1), 2), 7) and 8) are easy.

3) E.g., $T = \text{Th}({}^\omega\omega, E^1_n)_{n<\omega}$ where $\eta E^1_n \nu \Leftrightarrow \eta(n) = \nu(n)$ and use $\varphi_n(x, y_n) = x E^1_n y_n$ for $n < \omega$.

4) E.g., $T = \text{Th}({}^\omega\omega, E^2_n)_{n<\omega}$ where $(\eta E_n \nu) \equiv (\eta \restriction n = \nu \restriction n)$.

5) E.g., $T =$ Th$(\mathbb{Q}, <)$, the theory of dense linear orders with no first and no last element.

6) For simplicity we use $\bar{x} = \langle x \rangle$, (justified in [Sh:863, Observation,1.7](1)). Assume $\langle \varphi_n(x, \bar{y}_n) : n < \omega \rangle$ and $\langle \bar{a}^n_\alpha : \alpha < \lambda \rangle$ are as in Definition 3.2. Clearly we can replace $\varphi_n(x, \bar{y}_n), \bar{a}^n_\alpha$ by $\varphi'(x, \bar{y}'_n), \bar{b}^n_\alpha$ when $\bar{y}_n \trianglelefteq \bar{y}'_n, \bar{a}^n_\alpha \trianglelefteq \bar{b}^n_\alpha$ and $\varphi_n(x, \bar{a}^n_\alpha) \equiv \varphi'_n(x, \bar{b}^n_\alpha)$. We can find $b_0 < b$ in $\mathfrak{C}$ such that each $p_\eta \cup \{b_0 < x < b_1\}$ is realized in $\mathfrak{C}$, so without loss of generality $\varphi_n(x, \bar{a}^n_\alpha) \vdash b_0 < x < b_1$ and b_1, b_2 appears in $\bar{a}^n_\alpha$. Also we can restrict ourselves to $\langle \bar{a}^n_\alpha : n < \omega, \alpha \in u_n \rangle$ where $u_n \subseteq \lambda$ is infinite for $n < \omega$. Hence, by the elimination of quantifiers and density of the linear order, without loss of generality, $\varphi_n(x, \bar{y}_n) = (\varphi_{1,n}(x, \bar{y}_n) \vee \varphi_{n,2}(x, \bar{y}_n)) \wedge \varphi_{n,3}(\bar{y}_n)$ where (without loss of generality $\bar{y}_n = \langle y_\ell : \ell = 0, \ldots, \rangle$ but $u(n,1) \subseteq k(n), u(n,2) \subseteq k(n)$)

$$\varphi_{n,1}(x, \bar{y}_n) = \bigvee_{\ell \in u(n,1)} (y_{n,2\ell} < x < y_{n,2\ell+1})$$

$$\varphi_{n,2}(x, \bar{y}_n) = \bigvee_{\ell \in u(n,2)} x = y_{n,\ell}$$

and

$$\varphi_{n,3}(\bar{y}) = \bigwedge_{\ell < k(n)} y_{n,\ell} < y_{n,\ell+1}.$$

For each $\eta \in {}^\omega\lambda, p_\eta$ is consistent (and $\eta \neq \nu \in {}^\omega\lambda \Rightarrow p_\eta, p_\nu$ are contradictory), hence clearly each p_η is not algebraic. From this it follows that (∗) of Def. 3.2(1) is true also if we replace $\langle \varphi_n(x, \bar{y}_n) : n < \omega \rangle$ by $\langle \varphi_{n,1}(x, \bar{y}_n) : n < \omega \rangle$.

Also without loss of generality $\langle \bar{a}^n_\alpha : \alpha < \lambda \rangle$ is indiscernible over $\bigcup\{a^m_\beta : m \neq n, m < \omega \text{ and } \beta < \lambda\}$. Now for some $\langle \ell_n : n < \omega \rangle \in \prod\limits_{n<\omega} k(n)$, we can replace $\varphi_n(x, \bar{y})$ by $\varphi'_n(x, \bar{y}) = y_{n,\ell_n} < x < y_{n,\ell_n+1}$. So without loss of generality $n < \omega \Rightarrow k(n) = 1, \ell_n = 0, \bar{y}_n = (y_{n,0}, y_{n,1})$.

Now $\langle \bar{a}^n_\alpha : \alpha < \lambda \rangle$ is an indiscernible sequence, and $\varphi_n(\mathfrak{C}, \bar{a}^n_\alpha)$ being the open convex sets which $\bar{a}^n_\alpha$ define. Checking by cases (they are $a^n_{\alpha,0} < a^n_{\alpha,1} < a^n_{\alpha+1,0} < a^n_{\alpha+1,1}, a^n_{\alpha,0} < a^n_{\alpha+1,0} < a^n_{\alpha,1} < a^n_{\alpha+1,1}, a^n_{\alpha+1,0} < a^n_{\alpha+1,1} < a^n_{\alpha,0} < a^n_{\alpha,1}, a^n_{\alpha+1,0} < a^n_{\alpha,0} < a^n_{\alpha+1,1} < a^n_{\alpha,1}$. Note that $a^n_{\alpha,0} < a^n_{\alpha+1,0} < a^n_{\alpha+1,1} < a^n_{\alpha,1}$ and $a^n_{\alpha+1,0} < a^n_{\alpha,0} < a^n_{\alpha,1} < a^n_{\alpha+1,1}$ are impossible. Letting $p^n_\beta(x) := \{\varphi(x, \bar{a}^n_\alpha)^{\text{if}(\alpha=\beta)} : \alpha < \lambda\}$ we note that it is a type such that $p^n_\beta(\mathfrak{C})$ is a convex set; obviously it is disjoint from $p^n_\gamma(\mathfrak{C})$ for $\gamma \in \lambda \backslash \{\beta\}$.

Clearly, there are $\alpha \neq \beta < \lambda$ such that $p^0_\alpha(\mathfrak{C}) < p^0_\beta(\mathfrak{C})$ and choose a^* such that $p^0_\alpha(\mathfrak{C}) < a^* < p^0_\beta(\mathfrak{C})$. Now for every $\gamma < \lambda$ we have $p^1_\gamma(\mathfrak{C}) \bigcap p^0_\alpha(\mathfrak{C}) \neq \emptyset$ and $p^1_\gamma(\mathfrak{C}) \bigcap p^0_\beta(\mathfrak{C}) \neq \emptyset$, i.e., $p^1_0(\mathfrak{C})$ is disjoint neither from $p^0_\alpha(\mathfrak{C})$ nor from $p^0_\beta(\mathfrak{C})$ (by the choice of $\bar{\varphi}, \langle \bar{a}^n_\alpha : n < \omega, \alpha < \lambda \rangle$). As $p^1_\gamma(\mathfrak{C})$ is convex, by the choice of a^* necessarily $a^* \in p^1_\gamma(\mathfrak{C})$. As γ was any ordinal $< \lambda$ it follows that $a^* \in \bigcap\{p^1_\gamma(\mathfrak{C}) : \gamma < \lambda\}$, clear contradiction. (In fact, we get contradiction even if we use only $n = 0, 1$, see [Sh:863]). The o-minimal case holds by the same proof. $\blacksquare_{3.3}$

3.4. *Definition:* 1) We say a pair of types $(p(x), q(\bar{y}))$ is a $(1 = \aleph_0)$-pair of types (or $(p(\bar{x}), q(\bar{y}))$ satisfies $1 = \aleph_0$), if there is a set A such that: for every countable set $B \subseteq p(\mathfrak{C})$, there is an element $\bar{a} \in q(\mathfrak{C})$ satisfying $B \subseteq \text{acl}(\{\bar{a}\} \cup A)$. We say $p(x)$ is a $(1 = \aleph_0)$-type if this holds for some $q(\bar{y})$.

1A) If $A = \text{Dom}(p)$ we add purely. We call A a witness to $p(x)$ being a $(1 = \aleph_0)$-type.

2) We say that T is a local $(1 = \aleph_0)$-theory if for some A (the witness) some non-algebraic type p over A is a $(1 = \aleph_0)$-type. If $A = \emptyset$ we say purely.

2A) We say T is a global $(1 = \aleph_0)$-theory when the type $x = x$ is a $(1 = \aleph_0)$-type.

3) We say that a pair $(p(x), q(\bar{y}))$ of types is a semi $(1 = \aleph_0)$-pair of types if: for some set A for every indiscernible sequence $\langle a_n : n < \omega \rangle$ over A satisfying $n < \omega \Rightarrow \bar{a}_n \in p(\mathfrak{C})$ there is $\bar{a} \in q(\mathfrak{C})$ such that $\{\bar{a}_n : n < \omega\} \subseteq \text{acl}(\bar{a} \cup A)$. We say $p(\bar{x})$ is semi $(\aleph_0 = 1)$-type if this holds for some $q(\bar{y})$.

4) We say that the pair $(p(x), q(\bar{y}))$ of types is a weakly $(1 = \aleph_0)$-pair of types if there are $A \supseteq \text{Dom}(p)$ and an infinite indiscernible sequence $\langle a_n : n < \omega \rangle$

over A with each a_n realizing p such that for some $\bar{c} \in q(\mathfrak{C})$ we have $\{a_n : n < \omega\} \subseteq \mathrm{acl}(A \cup \bar{c})$.

5) We say that $p(x)$ is semi/weakly $(\aleph_0 = 1)$-type if some pair (q, p) is semi/weakly $(\aleph_0 = 1)$-pair of types.

6) In (3),(4) we let "purely", "witness" "local"; "global" be defined similarly.

7) Above we can allow $p = p(\bar{x}), \ell g(\bar{x}) = m$.

3.5. Observation: 1) Every algebraic type $p(x)$ is a $(1 = \aleph_0)$-type. If $p \subseteq q$ and p is a $(1 = \aleph_0)$-type, then q is an $(1 = \aleph_0)$-type.

2) If $p(x)$ is a $(1 = \aleph_0)$-type, then $p(x)$ is a semi $(1 = \aleph_0)$-type.

3) If $p(x)$ is a semi $(1, \aleph_0)$-type, then $p(x)$ is a weakly $(1 = \aleph_0)$-type.

4) If $(p(x), q(\bar{y}))$ is [semi][weakly]-$(1 = \aleph_0)$ type in $\mathfrak{C}$, then the same holds in $\mathfrak{C}^{\mathrm{eq}}$. If $p(x)$ is [semi][weakly]-$(1 = \aleph_0)$-type in $\mathfrak{C}^{\mathrm{eq}}$ such that $p(\mathfrak{C}^{\mathrm{eq}}) \subseteq \mathfrak{C}$, then so is the case in $\mathfrak{C}$. We can also keep track of the witness.

5) For some T, T is not locally $(1 = \aleph_0)$-theory but T^{eq} is.

Proof. Easy. ∎

3.6. Claim: 1) *If T is strongly dependent, then no non-algebraic type is a $(1 = \aleph_0)$-type.*

2) *Moreover, no non-algebraic type is a weakly $(1 = \aleph_0)$-type.*

Remark: We can weaken the assumption of 3.6 to: for some ω-sequence of non-algebraic types $\langle p_n(x) : n < \omega \rangle$ over A, for every $\langle b_n : n < \omega \rangle \in \prod_{n<\omega} p_n(\mathfrak{C})$, for some $\bar{c}$ we have $\{b_n : n < \omega\} \subseteq c\ell(A \cup \bar{c})$.

Proof. Let $\lambda > |T|^+$. Assume toward a contradiction that $p(x)$ is a non-algebraic $(1 = \aleph_0)$-type and A a witness for it. As $p(x)$ is not algebraic, we can find $\bar{b}^n = \langle b^n_\alpha : \alpha < \lambda \rangle$ for $n < \omega$ such that

$(*)_1$ b^n_α realizes p;

$(*)_2$ $b^n_\alpha \neq b^n_\beta$ for $\alpha < \beta < \lambda, n < \omega$;

$(*)_3$ $\langle b^n_\alpha : (n, \alpha) \in \omega \times \lambda \rangle$ is an indiscernible sequence over A where $\omega \times \lambda$ is ordered lexicographically.

Let $\bar{a} \in {}^{\omega >}(\mathfrak{C})$ be such that $\{b^n_0 : n < \omega\} \subseteq \mathrm{acl}(A \cup \bar{a})$ so for each n we can find $k_n < \omega, \bar{c}_n \in {}^{\omega >}A$ and a formula $\varphi_n(x, \bar{y}, \bar{z})$ such that

$$\mathfrak{C} \models \varphi(b^n_0, \bar{a}, \bar{c}_n) \quad \text{and} \quad (\exists^{\leq k_n} x)\varphi(x, \bar{a}, \bar{c}_n).$$

By omitting some b^n_α's we have $(n,\alpha) \in \omega \times \lambda \backslash \{(m,\omega) : m < \omega\} \Rightarrow \mathfrak{C} \models \neg\varphi_n[b^n_\alpha, \bar{a}, \bar{c}_n]$.

Let $\bar{a}^n_\alpha = \langle b^n_\alpha \rangle \hat{}\, \bar{c}_n$ and φ_n have already been chosen.

Now check Definition 3.2. $\blacksquare_{3.6}$

3.7. *Definition:* 1) We say T is strongly^2 (or strongly$^+$) dependent when: there is no sequence $\langle \varphi_n(\bar{x}, \bar{y}_0, \dots, \bar{y}_n) : n < \omega \rangle$ and $\bar{a}^n_\alpha \in {}^{\ell g(y_n)}\mathfrak{C}$ for $n < \omega, \alpha < \lambda$ (any infinite λ) such that for every $\eta \in {}^\omega\lambda$ the set $\{\varphi_n(\bar{x}, \bar{a}^0_{\eta(0)}, \dots, a^{n-1}_{\eta(n-1)}, a^n_\alpha)^{\text{if}(\alpha=\eta(n))} : n < \omega, \alpha < \lambda\}$ is consistent.

2) Let $\ell \in \{1,2\}$. We say that T is strongly$^{\ell,*}$ dependent when: if $\langle \bar{\mathbf{a}}_t : t \in I \rangle$ is an indiscernible sequence over $A, t \in I \Rightarrow \ell g(\bar{\mathbf{a}}_t) = \alpha$ (so constant but not necessarily finite) and $m < \omega$ and $\bar{b}_n \in {}^m\mathfrak{C}$ for $n < \omega, \langle \bar{b}_n : n < \omega \rangle$ is an indiscernible sequence over $A \cup \{\bar{\mathbf{a}}_t : t \in I\}$, then we can divide I to finitely many convex sets $\langle I_m : m < k \rangle$ such that for each $m < k, \langle \bar{\mathbf{a}}_t : t \in I_m \rangle$ is an indiscernible sequence over $\cup\{\bar{b}_\alpha : \alpha < \omega\} \cup A \cup \{\bar{a}_s : s \in I \backslash I_m \text{ and } \ell = 2\}$.

3) T is strongly$^\ell$ stable (or strongly$^{\ell,*}$ stable) when it is strongly$^\ell$ dependent (or strongly$^{\ell,*}$ dependent) and stable.

3.8. Claim: *If T is strongly$^+$ dependent then:*

$\circledast_1$ *for any $A \subseteq \mathfrak{C}$, infinite complete linear order I and indiscernible sequence $\langle \bar{a}_t : t \in I \rangle$ over $A, \ell g(\bar{a}_t)$ possibly infinite, for any finite $B \subseteq \mathfrak{C}$, there is a finite $w \subseteq I$ such that: if J is a convex subset of I disjoint to w then $\langle \bar{a}_t : t \in J \rangle$ is indiscernible over $A \cup B \cup \{\bar{a}_s : s \in I \backslash J\}$*

$\circledast_2$ *for any set $A \subseteq \mathfrak{C}$ of cardinality λ and infinite linear orders I_α for $\alpha < \lambda$ and $\bar{a}^\alpha_t (t \in I_\alpha, \alpha < \lambda)$ such that $\langle \bar{a}^\alpha_t : t \in I_\alpha \rangle$ is an indiscernible sequence over $A \cup \{\bar{a}^\beta_s : \beta \in \lambda \backslash \{\alpha\}, s \in I_\beta\}$ and finite $B \subseteq \mathfrak{C}$ there is a finite $u \subseteq \lambda$ and $w_\alpha \in [I_\alpha]^{<\aleph_0}$ for $\alpha \in u$ such that: if $\bar{J} = \langle J_\alpha : \alpha < \lambda \rangle, J_\alpha$ is a convex subset of I_α disjoint to w_α when $\alpha \in u$ then $\langle \bar{a}^\alpha_t : t \in J_\alpha \rangle$ is indiscernible over $A \cup B \cup \{\bar{a}^\beta_s : \beta \in \lambda \backslash \{\alpha\}, s \in J_\beta\}$ for every $\alpha < \lambda$.*

Proof. See this (and more) [Sh:863, §2]. $\blacksquare$

3.9. *Definition:* 1) We say that $\vartheta(x_1, x_2; \bar{c})$ is a finite-to-finite function from $\varphi_1(\mathfrak{C}, \bar{a}_1)$ onto $\varphi_2(\mathfrak{C}, \bar{a}_2)$ when:

(a) if $b_2 \in \varphi_2(\mathfrak{C}, a_2)$ then the set $\{x : \vartheta(x, b_2, \bar{c}) \wedge \varphi_1(x, \bar{a}_1)\}$ satisfies:
   (i) it is finite but

(ii) it is not empty except for finitely many such b_2's

(b) if $b_1 \in \varphi_1(\mathfrak{C}, \bar{a}_1)$, then the set $\{x : \vartheta(b_1, x, \bar{c}) \wedge \varphi_2(x, \bar{a}_2)\}$ satisfies:

(i) it is finite but

(ii) it is not empty except for finitely many such b_1's.

2) If we place "onto $\varphi_2(\mathfrak{C}, \bar{a}_2)$" by "into $\varphi_2(\mathfrak{C}, \bar{a}_1)$" we mean that we require above only clauses $(a)(i), (b)(i), (ii)$.

3) We can replace $\varphi_1(x, \bar{a}_1), \varphi_2(x, \bar{a}_2)$ above by types.

3.10. Claim: *If T is strongly$^+$ dependent, then the following are impossible:*

$(St)_1$ *for some $\varphi(x, \bar{a})$*

(a) *$\varphi(x, \bar{a})$ is not algebraic;*

(b) *E is a definable equivalence relation (in $\mathfrak{C}$ by a first order formula possibly with parameters) with domain $\subseteq \varphi(\mathfrak{C}, \bar{a})$ and infinitely many equivalence classes;*

(c) *there is a formula $\vartheta(x, y, \bar{z})$ such that for every $b \in \mathrm{Dom}(E)$ for some $\bar{c}$, the formula $\vartheta(x, y; \bar{c})$ is a finite to finite map from $\varphi(\mathfrak{C}, \bar{a})$ into b/E;*

$(St)_2$ *for some formulas $\varphi(x), xEy, \vartheta(x, y, \bar{z})$ possibly with parameters we have:*

(a) *$\varphi(x)$ is non-algebraic;*

(b) *$xEy \rightarrow \varphi(x) \wedge \varphi(y)$;*

(c) *for uncountably many $c \in \varphi(\mathfrak{C})$ for some $\bar{d}$ the formula $\vartheta(x, y; \bar{d})$ is a finite to finite function from $\varphi(x)$ into xEc;*

(d) *for some $k < \omega$, if $b_1, \ldots, b_k \in \varphi(\mathfrak{C})$ are pairwise distinct then $\bigwedge_{\ell=1}^{k} xEb_\ell$ is algebraic.*

$(St)_3$ *similarly with $\varphi(x, \bar{a})$ replaced by a type, as well as xEy (and x, y, z are replaced by m-tuples and uncountable is replaced by $\bar{\kappa}$).*

Proof. The proof for $(St)_1$ is a special case of the proof of $(St)_2$ and the proof for $(St)_3$ is similar. So it is enough:

Proof of "$(St)_2$ is impossible".

Without loss of generality, in clause (c) of $(St)_2$ we have $\langle c \rangle \triangleleft \bar{d}$ and let $\ell g(\bar{d}) = j$, i.e., $\vartheta = \vartheta(x, y, \bar{z}), \ell g(\bar{z}) = j$; also let $\bar{z}^n = \langle z_{n,0}, \ldots, z_{n,j-1} \rangle$.

Clearly, there is k^* such that

$\boxdot_1$ for some uncountable $C \subseteq \varphi(\mathfrak{C})$ for every $c \in C$ for some $\bar{d}_c \in {}^j\mathfrak{C}$, without loss of generality, $\langle c \rangle \triangleleft \bar{d}_c$ and $\theta(x, y, \bar{d}_c)$ is a finite to finite map from $\varphi(\mathfrak{C})$ into xEc and the size of the finite sets (see Definition 3.9) is $< k^*$;

$\boxdot_2$ moreover, $C = r(\mathfrak{C})$ for some non-algebraic $r(x)$;

$\boxdot_3$ k^* can serve as k in clause (d) of $(St)_2$.

Let $\bar{z}_n = \bar{z}^0 {}^\frown \ldots {}^\frown \bar{z}^{n-1}$. We shall define now, by induction on $n < \omega$, formulas $\varphi_n(x, \bar{z}_n)$ and $\vartheta_n(x_1, x_2, \bar{z}_n)$ also written as $\varphi^n_{\bar{z}_n}(x), \vartheta^n_{\bar{z}_n}(x_1, x_2)$.

Case 1: $n = 0$.

So ($\bar{z}_n = <>$, and) $\varphi_n(x) = \varphi(x)$ and $\vartheta_n(x_1, x_2) = (x_1 = x_2)$.

Case 2: $n = m + 1$.

Let $\varphi^n_{\bar{z}_n}(x) := \varphi^m_{\bar{z}_m}(x) \wedge (\exists x')[x'Ez_m \wedge \vartheta^m_{\bar{z}_m}(x', x)] \wedge \vartheta^n_{\bar{z}_n}(x_1, x_2) := \varphi^m_{\bar{z}_m}(x_2) \wedge \varphi(x_1) \wedge (\exists x')[\vartheta(x_1, x', \bar{z}^m) \wedge \vartheta^m_{\bar{z}_m}(x', x_2) \wedge x'Ez_m]$.

We now prove, by induction on n, that:

$(*)_n$ if $\bar{c} = \langle c_\ell : \ell < n\rangle$ and $c_\ell \in C \backslash ac\ell\{c_k : k < \ell\}$ for $\ell < n$ (so $\vartheta(x, y, \bar{d}_{c_\ell})$ is a finite to finite function from $\varphi(x)$ into xEc_ℓ for $\ell < n$) and $\bar{d} = \bar{d}_{c_0} {}^\frown \bar{d}_{c_1} {}^\frown \ldots {}^\frown \bar{d}_{c_{n-1}}$, then

- (α) $\varphi^n_{\bar{d}}(\mathfrak{C})$ is an infinite subset of $\varphi(\mathfrak{C})$;
- (β) $\vartheta^n_{\bar{d}}(x_1, x_2)$ is a finite to finite function from a co-finite subset of $\varphi(\mathfrak{C})$ into a subset of $\varphi^n_{\bar{d}}(\mathfrak{C})$;
- (γ) if $n = m + 1$ and $e \in \varphi^n_{\bar{d}}(\mathfrak{C})$, then $(\exists c' \in ac\ell(\bar{c} \restriction m \cup \{e\}))[c'Ec_m]$;
- (δ) $m < n \Rightarrow \varphi^m_{\bar{d}\restriction jm}(\mathfrak{C}) \subseteq \varphi^n_{\bar{d}}(\mathfrak{C})$.

This is straightforward. Let I be a linear order such that any interval has $< |T|$ members.

By $\boxdot_2, \boxdot_3$ there are $c_t \in C$ for $t \in I$ pairwise distinct, let $\bar{d}_t = \bar{d}_{c_t}$ so $\theta(x, y, \bar{d}_t)$ is a finite to finite function from $\varphi(x)$ into xEc_t such that $\langle \bar{d}_t : t \in I\rangle$ is an indiscernible sequence (e.g. use $\boxdot$ above).

Now for every $<_I$-increasing sequence $\bar{t} = \langle t_n : n < \omega\rangle$ we consider $\bar{c}^n_{\bar{t}} = \bar{c}^n_{\bar{t}\restriction n} = \bar{d}_{t_0} {}^\frown \ldots \bar{d}_{t_{n-1}}$ and $p_{\bar{t}} = \{\varphi^n_{\bar{c}^n_{\bar{t}\restriction n}}(x) : n < \omega\}$.

Now

$\circledast_1$ for $\bar{t}$ as above $p_{\bar{t}}$ is consistent.

(Why? By $(*)_n(\alpha)$ there is an element $e \in \varphi^n_{\bar{c}^n_{\bar{t}}}(\mathfrak{C})$, by $(*)_n(\delta)$ the element e satisfies $\{\varphi_{\bar{c}^m_{\bar{t}\restriction m}}(x) : m \leq n\}$. As this holds for every n, the set $p_{\bar{t}} = \{\varphi^n_{\bar{d}^n_{\bar{t}\restriction n}}(x) : n < \omega\}$ is finitely satisfiable as required.)

$\circledast_2$ if e realizes $p_{\bar{t}}$, then for every n there is an element e' algebraic over $\{e, \bar{d}_{t_0}, \dots, \bar{d}_{t_{n-1}}\}$ such that $e' E b^1_{t_n}$.

(Why? By $(*)_n(\gamma)$.)

$\circledast_3$ if e realizes $p_{\bar{t}}$ then for every n the set $\{s \in I$: there is e' algebraic over $\{e, \bar{d}^1_{t_0}, \dots, \bar{d}^1_{t_{n-1}}\}$ such that $e' E c^1_s\}$ has $\leq |T|$ members.

(Why? There are $\leq |T|$ such e' and for each e' by clause (d) of $(\mathrm{St})_2$ there are only finitely many such $s \in I$ (if we phrase it more carefully we get that there are $< k_n (< \omega)$ many members).)

This is more than enough to show T is not strongly$^+$ dependent. $\blacksquare_{3.10}$

3.11. Discussion: : We may phrase 3.10 for ideals of small formulas.

3.12. Claim: *If T is strongly^1 dependent and $\ell = 1, 2, 3, 4$, then the statement $\circledast_\ell$ below is impossible where:*

$\circledast_1$ (a) *$\langle \bar{a}_\alpha : \alpha < \lambda \rangle$ is an indiscernible sequence over A;*

(b) *$u_n \subseteq \lambda$ is finite, (non-empty) with $\langle u_n : n < \omega \rangle$ having pairwise disjoint convex hull;*

(c) $\bar{b} \in {}^{\omega>}\mathfrak{C}$

(d) *for each n for some α_n, k and $t^{\mathbf{t}}_{n(0)} < \dots < t^{\mathbf{t}}_{n(k-1)} \in u_n$ for $\mathbf{t} \in \{$ **false, truth**$\}$ and $\bar{c}_n \in {}^{\omega>}A$ and φ we have $\mathfrak{C} \models \varphi(\bar{c}, a_{t^{\mathbf{t}}_{n(0)}}, \dots, a_{t^{\mathbf{t}}_{n(k-1)}}, \bar{c}_n)^{\mathbf{t}}$ for both values of $\mathbf{t}$;*

$\circledast_2$ *like $\circledast_1$ but allows $\bar{a}_\alpha$ to be infinite;*

$\circledast_3$(a) *$\langle \bar{a}^n_\alpha : \alpha < \lambda \rangle$ is an indiscernible sequence over $A \cup \{\bar{a}^a_\beta : m < \omega, m \neq n,\ \beta < \lambda\}$;*

(b) *$\bar{a}^n_\alpha \neq \bar{a}^n_{\alpha+1}$;*

(c) *some $a \in \mathfrak{C}$ satisfies $n < \omega \Rightarrow acl(A \cup \{a\}) \cap \{\bar{a}^n_\alpha : \alpha < \lambda\} \neq \emptyset$;*

$\circledast_4$ *like $\circledast_3$ but replace clause (c) by*

(c)' *for some $\bar{a} \in \mathfrak{C}$ for every n the sequence $\langle \bar{a}^n_\alpha : \alpha < \lambda \rangle$ is not an indiscernible sequence over $A \cup \bar{a}$.*

Proof. Similar to the previous ones. $\blacksquare$

3.13. Discussion: 1) We have asked: show that the theory of the p-adic field is strongly dependent.

Udi Hrushovski has noted that the criterion $(\mathrm{St})_2$ from 3.10 applies so T is not strongly^2 dependent. Namely take the following equivalence relation on $\mathbb{Z}_p$: $\mathrm{val}(x - y) \geq \mathrm{val}(c)$, where c is some fixed element with infinite valuation. Given x, the map $y \mapsto (x+cy)$ is a bijection between $\mathbb{Z}_p$ and the class.

2) By [Sh:863] this theory is strongly[1] dependent.

3) Onshuus shows that the theory of the field of the reals too is not strongly[2] dependent (e.g. though Claim 3.10 does not apply, its proof works, using pairwise not too near $\bar{b}$'s, in general just an uncountable set of $\bar{b}$'s. In [Sh:863] we prove reasonable existence of indiscernibles for strongly dependent T (and in 3.2 we can use the case $\ell g(\bar{x}) = 1$).

3.14. Claim: 1) *If $x = 1, 2, 1*, 2*$, $M \prec \mathfrak{C}$, $A \subseteq \mathfrak{C}$, then (the complete first order) theory $Th(\mathfrak{B}_{M,MA})$ from 1.10(4) is strongly* dependent if and only if T is strongly* dependent; if T is dependent then the theory is equal to $T^*_{M,A}$ see 1.10(4), 1.12(4).*

2) $\kappa_{ict}(T) = \kappa_{ict}(\mathrm{Th}(\mathfrak{B}_{M,MA}))$ *if* $M \prec \mathfrak{C}$, $A \subseteq \mathfrak{C}$;

3) *If $x = 1, 2, 1*, 2*$ and $T_1 \subseteq T_2$ are complete first order theories (so $\tau(T_1) \subseteq \tau(T_2)$), then*

(a) *if T_2 is strongly* dependent then so is T_1*

(b) $\kappa_{ict}(T_1) \le \kappa_{ict}(T_2)$.

4) *If $T_1 \subseteq T_2$ are complete first order and $\tau(T_2) \backslash \tau(T_1)$ consist of individual constants only, then*

(α) *T_2 is strongly* dependent if and only if T_1 is strongly* dependent;*

(β) $\kappa_{ict}(T_1) = \kappa_{ict}(T_2)$.

5) *For $\ell = 1, 2$, T is strongly$^\ell$ dependent if and only if T^{eq} is strongly$^\ell$ dependent; similarly for strongly$^{\ell,*}$.*

6) $\kappa_{ict}(T) = \kappa_{ict}(T^{\mathrm{eq}})$.

Proof. Easy. ∎

4. Definable groups

4.1. Context: (a) T is first order complete

(b) $\mathfrak{C}$ is a monster model of T.

We try here to generalize the theorem on the existence of commutative infinite subgroups for stable T to dependent T. Theorems on definable groups in a monster $\mathfrak{C}$, $\mathrm{Th}(\mathfrak{C})$ stable, are well-known.

4.2. *Definition:* 1) We say that G is a type-definable group (in $\mathfrak{C}$) if $G = (p, *, \mathrm{inv}) = (p^G, *^G, \mathrm{inv}^G)$ where

(a) $p = p(x)$ is a type.

(b) $*$ is a two-place function on $\mathfrak{C}$, possibly partial, definable (in $\mathfrak{C}$), we normally write ab instead of $a * b$ or $*(a, b)$.

(c) $(p(\mathfrak{C}), *)$ is a group, we write $x \in G$ for $x \in p(\mathfrak{C})$.

(d) inv^G is a (partial) unary function, definable (in $\mathfrak{C}$), which on $p(\mathfrak{C})$ is the inverse, so if no confusion arises we shall write $(x)^{-1}$ for $\mathrm{inv}(x)$.

1A) We let B_2^G be the set of parameters appearing in p^G; let B^G be the set of parameters appearing in p^G or in the definition of $*$ or of inv^G.

2) We say that G is a definable group if $p(x)$ is a formula, i.e., a singleton.

3) We say that G is an almost type definable group if $p(x)$ is replaced by $\bar{p} = \langle p_i(x) : i < \delta \rangle, p_i(\mathfrak{C})$ increasing with i and $\bar{p}(\mathfrak{C})$ is defined as $\bigcup \{p_i(\mathfrak{C}) : i < \delta\}$.

Remark: Of course, we can use $p(\bar{x})$ and/or work in $\mathfrak{C}^{\mathrm{eq}}$.

4.3. Claim: *Assume*

(a) *T is dependent;*

(b) *G is a definable group in $\mathfrak{C}$ or just type-definable;*

(c) *$A \subseteq G$ is a set of pairwise commuting elements, D a non-principal ultrafilter on A or just*

$(c)^-$ *$A \subseteq G, D$ a non-principal ultrafilter on A such that*

$$(\forall^D a_1)(\forall^D a_2)(a_1 a_2 = a_2 a_1),$$

where $\forall^D x \varphi(x, \bar{a})$ means $\{b \in \mathrm{Dom}(D) : \mathfrak{C} \models \varphi[b, \bar{a}]\} \in D$.

Then there is a formula $\varphi(x, \bar{a})$ such that:

(α) *$\varphi(x, \bar{a}) \in \mathrm{Av}(\bar{a}, D)$;*

(β) *$G \cap \varphi(\mathfrak{C}, \bar{a})$ is an abelian subgroup of G;*

(γ) *$\bar{a} \subseteq A \cup B^G \cup \{c : c$ realizes $\mathrm{Av}(A \cup B_2^G, D)\}$.*

4.4. *Remark:* 1) If D is a principal ultrafilter, say $\{a^*\} \in D$, then $\varphi(x, \bar{a})$ is essentially $\mathrm{Cm}_G(\mathrm{Cm}_G(a^*))$ so no new point, ($\mathrm{Cm}_G(A) = \{x \in A : x$ commutes with every $a \in A\}$).

2) If D is a non-principal ultrafilter, then necessarily $\varphi(x, \bar{a})$ is not algebraic as it belongs to $\mathrm{Av}(\bar{a}, D)$.

Proof. We try to choose a_n, b_n by induction on $n < \omega$ such that:

(i) a_n, b_n realizes $p_n(x) := \mathrm{Av}(A_n, D)$ where $A_n = A \cup B^G \cup \{a_k, b_k : k < n\}$ so as $A \in D, A \subseteq G$ necessarily $p^G(x) \subseteq p_n(x)$;

(ii) a_n, b_n does not commute (in G, they are in G because $p^G(x) \subseteq p_n$). ∎

Case 1: We succeed.

Assume $n < m < \omega, c' \in \{a_n, b_n\}$ and $c'' \in \{a_m, b_m\}$ clearly c', c'' are in G. Now we shall show that they commute because c'' realizes $\mathrm{Av}(A \cup B^G \cup \{c'\}, D)$ and c' realizes $\mathrm{Av}(A \cup B^G_2, D)$ recalling either assumption (c) about commuting in A or assumption $(c)^-$. Hence if $k < \omega, n_0 < \cdots < n_{k-1} < \omega$ and $n < \omega$ then $c := b_{n_0} b_{n_1} \ldots b_{n_{k-1}}$ satisfies: c, a_n commute if and only if $n \notin \{n_0, \ldots, n_{k-1}\}$, so $\varphi(x,y) = [xy = yx]$ has the independence property contradicting assumption (a). So c', c'' actually commute and we are done.

Case 2: We are stuck at $n < \omega$.

So $p_n(x) \cup p_n(y) \vdash (xy = yx)$, hence there is a formula $\psi(x, \bar{a}^*) \in \mathrm{Av}(A_n, D)$ such that

$(*)_1$ $\psi(x, \bar{a}^*) \wedge \psi(y, \bar{a}^*) \vdash xy = yx$ (so both products are well-defined).

Let $p^G(x) = \{\vartheta(x, \bar{a})\}$ or just $p^G(x) \vdash \vartheta(x, \bar{a}), \bar{a} \in B^G_i$ and $\vartheta(x, \bar{a}) \wedge \vartheta(y, \bar{a}) \to$ (xy well defined). Without loss of generality $\bar{a} \trianglelefteq \bar{a}^*$ and $\psi(x, \bar{a}^*) \vdash \vartheta(x, \bar{a})$ and let $\vartheta^*(x, \bar{a}^*) = (\forall y)(\psi(y, \bar{a}^*) \to yx = xy$ (so both well-defined)). So $\psi(x, \bar{a}) \vdash \vartheta^*(x, \bar{a}^*)$.

Let

$$\varphi(x) = \varphi(x, \bar{a}) = \vartheta(x, \bar{a}^*) \wedge (\forall y)[\vartheta^*(y, \bar{a}^*) \to xy = yx \text{ (both well-defined)}].$$

So $\psi(x, \bar{a}^*) \vdash \varphi(x, \bar{a}^*) \vdash \vartheta^*(x, \bar{a}^*)$ hence the formula $\varphi(x, \bar{a}^*)$ belongs to the type $p_n(x)$ which is equal to $\mathrm{Av}(A_n, D)$ hence $\varphi(x, \bar{a}) \in \mathrm{Av}(\bar{a}^*, D)$ and $\bar{a}^* \subseteq A_n \subseteq A \cup B^G \cup \cup\{c : c \text{ realizes } \mathrm{Av}(A \cup B^G, D)\}$.

We are done as $\varphi(\mathfrak{C}, \bar{a}^*) \cap G$ is a subgroup and is abelian by the definition of $\varphi(x)$. $\blacksquare_{4.3}$

4.5. Claim: *Assume*

(a) *G is a definable (infinite) group, (or just type-definable);*

(b) *every element of $G \backslash \{e_G\}$ commutes with only finitely many others;*

(c) *G has infinitely many pairwise non-conjugate members.*

Then T is not strongly$^+$ dependent.

Proof. Assume first $p^G = \{\varphi(x)\}$.

Let $xEy := [x, y$ are conjugates$]$, clearly it is an equivalence relation, and let

$$\vartheta(x_1, x_2, y) := (x_1 = x_2 y x_2^{-1}).$$

Note that: if $M \models \vartheta(x_1, z_1, y) \wedge \vartheta(x_1, z_2, y)$ then $M \models z_1 y z_1^{-1} = z_2 y z_2^{-1}$ hence $M \models (z_2^{-1} z_1)y = y(z_2^{-1} z)$ so $z_2^{-1} z_1 \in \mathrm{Cm}_G(y)$ so $\{z : \vartheta(x_1, z, y)\}$ is finite. Trivially $\{x_1 : \vartheta(x_1, x_2, y)\}$ is finite.

We now get a contradiction by 3.10: $\varphi(x), \vartheta(x_1, x_2, y)$ satisfies the demands in $(St)_1$ there, which is impossible if T is strongly$^+$ dependent; so we are done.

If p^G is a type use $(St)_3$ of 3.10. $\blacksquare_{4.5}$

4.6. *Definition:* 1) A place $\mathbf{p}$ is a tuple

$$(p, B, D, *, \mathrm{inv}) = (p^{\mathbf{p}}, B^{\mathbf{p}}, D^{\mathbf{p}}, *_{\mathbf{p}}, \mathrm{inv}_{\mathbf{p}}) = (p[\mathbf{p}], B[\mathbf{p}], D[\mathbf{p}], *[\mathbf{p}], \mathrm{inv}[\mathbf{p}])$$

such that:

(a) B is a set $\subseteq \mathfrak{C}$, D is an ultrafilter on $B, p \subseteq \mathrm{Av}(B, D)$;

(b) $*$ is a partial two-place function defined with parameters from B; we shall write $a *_{\mathbf{p}} b$ or, when clear from the context, $a * b$ or ab;

(c) inv is a partial unary function definable from parameters in B.

1A) $\mathbf{p}$ is non-trivial if for every A the type $\mathrm{Av}(A, D)$ is not algebraic.

2) We say $\mathbf{p}$ is weakly a place in a definable group G or type definable group G if $\mathbf{p}$ is a place, $p^{\mathbf{p}} \vdash p^G$, the set $B^{\mathbf{p}}$ includes $\mathrm{Dom}(p^G)$ and the operations agree on $p_{\mathbf{p}}[\mathfrak{C}]$ when the place operations are defined.

2A) If those operations are the same, we say that $\mathbf{p}$ is strongly a place in G.

3) We say $\mathbf{p}_1 \le \mathbf{p}_2$ if both are places, $B^{\mathbf{p}_1} \subseteq B^{\mathbf{p}_2}$ and $p^{\mathbf{p}_2} \vdash p^{\mathbf{p}_1}$ and the operations are same.

4) $\mathbf{p} \le_{\mathrm{dir}} \mathbf{q}$ if $\mathbf{p} \le \mathbf{q}$ and $B^{\mathbf{q}} \subseteq A \Rightarrow \mathrm{Av}(A, D^{\mathbf{p}}) = \mathrm{Av}(A, D^{\mathbf{q}})$.

4.7. *Definition:* 1) A place $\mathbf{p}$ is σ-closed when:

(a) σ has the form $\sigma(\bar{x}_1; \ldots; \bar{x}_{n(*)})$, a term in the vocabulary of groups;

(b) if $\bar{a}_\ell \in {}^{(\ell g(\bar{x}_\ell))}\mathfrak{C}$, for $\ell = 1, \ldots, n(*)$ and $B \subseteq A$, then $\sigma(\bar{a}_1, \ldots, \bar{a}_{n(*)})$ is well-defined [1] and realizes $\mathrm{Av}(A, D)$ provided that

$(*)$ $n \le n(*)$ and $\ell < \ell g(\bar{a}_n) \Rightarrow a_{n,\ell}$ realizes

$$\mathrm{Av}(A \cup \bar{a}_1{}^\frown \ldots {}^\frown \bar{a}_{n-1}, D).$$

2) A place $\mathbf{p}$ is $(\sigma_1 = \sigma_2)$-good or satisfies $(\sigma_1 = \sigma_2)$ when

[1] So all the stages in the computation of $\sigma(\bar{a}_0; \ldots; \bar{a}_{n(*)})$ should be well-defined.

(a) $\sigma_\ell = \sigma_\ell(\bar{x}_1, \ldots, \bar{x}_{n(*)})$ a term in the vocabulary of groups for $\ell = 1, 2$ (so e.g. $(x_1x_2)x_3, x_1(x_2x_3)$ are considered as different terms);

(b) if $\bar{a}_\ell \in {}^{(\ell g(\bar{x}_n)}\mathfrak{C}$ for $\ell \leq n$ then $\sigma_1(\bar{a}_1; \ldots; \bar{a}_{n(*)}) = \sigma_2(\bar{a}_1; \ldots; \bar{a}_{n(*)})$ whenever $(*)$ of part (1) holds for $A = B$; so both are well-defined.

3) We can replace σ in part (1) by a set of terms. Similarly in part (2) for a set of pairs.

4) We may write x_ℓ instead of $\langle x_\ell \rangle$. So if we write $\sigma(\bar{x}_1; \bar{x}_2) = \sigma(x_1; x_2) = x_1x_2$ or $\sigma = x_1x_2$ we mean $x_1 = x_{1,0}, x_2 = x_{2,0}, \bar{x}_1 = \langle x_{1,0} \rangle, \bar{x}_2 = \langle x_{2,0} \rangle$. We may use also $\sigma(\bar{x}; \bar{y})$ instead of $\sigma(\bar{x}_1; \bar{x}_2)$ and $\sigma(\bar{x}; \bar{y}; \bar{z})$ similarly.

4.8. *Definition:* 1) We say a place $\mathbf{p}$ is a poor semi-group if it is σ-closed for $\sigma = xy$ and satisfies $(x_1x_2)x_3 = x_1(x_2x_3)$.

2) We say a place $\mathbf{p}$ is a poor group if it is a poor semi-group and is σ-closed for $\sigma = (x_1)^{-1}x_2$.

3) We say a place $\mathbf{p}$ is a quasi semi group if for any semi group term $\sigma_*(\bar{x}), \mathbf{p}$ is σ-closed for $\sigma(\bar{x}; y) = \sigma_*(\bar{x})y$.

4) We say a place $\mathbf{p}$ is a quasi group if for any semi-group terms $\sigma_1(\bar{x}), \sigma_2(\bar{x})$ we place $\mathbf{p}$ is σ-closed for $\sigma(\bar{x}; y) = \sigma_1(\bar{x})y\sigma_2(\bar{x})$.

5) We say $\mathbf{p}$ is abelian (or is commutative) if it is (xy)-closed and satisfies $xy = yx$.

6) We say $\mathbf{p}$ is affine if $\mathbf{p}$ is $(xy^{-1}z)$-closed.

7) We say that a place $\mathbf{p}$ is a pseudo semi-group when: if the terms $\sigma_1(x_1, \ldots, x_n), \sigma_2(x_1, \ldots, x_n)$ are equal in semi-groups then $\mathbf{p}$ satisfies $\sigma_1(x_1, \ldots, x_n) = \sigma_2(x_1, \ldots, x_n)$.

8) We say that a place $\mathbf{p}$ is a pseudo group if any term $\sigma_1(x_1, \ldots, x_n)$, $\sigma_2(x_1, \ldots, x_n)$ which are equal in groups, $\mathbf{p}$ satisfies $\sigma_1(x_1, \ldots, x_n) = \sigma_2(x_1, \ldots, x_n)$.

4.9. *Definition:* We say a place $\mathbf{p}$ is a group if $G = G^{\mathbf{p}} = (\mathrm{Av}(B^{\mathbf{p}}, D), {}^*\mathbf{p}, \mathrm{inv}_{\mathbf{p}})$ is a group. Similarly for a semi-group.

4.10. Claim: 1) *The obvious implications hold.*

2) *If we use $\bar{\mathbf{b}}$ i.e. $\bar{\mathbf{b}}$ is an endless indiscernible sequence $A = \cup\{\bar{b}_t : t \in \mathrm{Dom}(\bar{\mathbf{b}})\}$, D the co-bounded filter on $\bar{\mathbf{b}}$, every $\bar{\mathbf{b}}'$ realizing the same type has the same properties.*

3) *For a place $\mathbf{p}$ the assertion "$\mathbf{p}$ satisfies $\sigma(\bar{x}_1, \ldots, \bar{x}_{n(*)}) = \sigma(\bar{x}_1, \ldots, \bar{x}_{n(*!)})$" means just that in Definition 4.7 the term $\sigma(a_1, \ldots, a_n)$ is well-defined.*

4.11. Claim: 1) *Assume that G is a definable group and $a_n \in p^G[\mathfrak{C}]$ for $n < \omega$. We define $a_{[u]} \in p^G[\mathfrak{C}]$ for any finite non-empty $u \subseteq \omega$ by induction on $|u|$, if $u = \{n\}$, then $a_{[u]} = a_n$, if $|u| > 1$, $\max(u) = n$ then $a_{[u]} = a_{[u \setminus \{n\}]} *^G a_n$ and we are assuming they are all well-defined and $a_{[u_1]} \neq a_{[u_2]}$ when $u_1 \triangleleft u_2$. Then we can find $D^*, \mathbf{q}$ such that:*

(a) *$\mathbf{q}$ is a place inside G;*
(b) *$\mathbf{q}$ is a poor semi-group and non-trivial;*
(c) *$B^{\mathbf{q}} = B^G \cup \bigcup\{a_{[u]} : u \subseteq \omega$ is finite$\}$;*
(d) *D^* is an ultrafilter on $[\omega]^{<\aleph_0}$ such that $(\forall n)([\omega \setminus n]^{<\aleph_0} \in D^*)$ and for every $Y \in D^*$ we can find $Y' \subseteq Y$ from D^* closed under convex union, i.e., if $u, v \in Y'$ and $\max(u) < \min(v)$ then $u \cup v \in Y'$;*
(e) *$D^{\mathbf{q}} = \{\{a_{[u]} : u \in Y\} : Y \in D^*\}$;*
(f) *if the a_n's commute (i.e. $a_n a_m = a_m a_n$ for $n \neq m$), then $\mathbf{q}$ is abelian.*

Proof. By a well-known theorem of Glazer [2], relative of Hindman theorem saying D^* as in clause (d) exists, see Comfort [Cmf77]. $\blacksquare_{4.11}$

4.12. *Remark:* 1) This can be combined naturally with §1.
2) In 4.11, "$u \triangleleft v \Rightarrow a_{[u]} \neq a_{[v]}$" holds if a_n is not in the subgroup generated by $\{a_\ell : \ell < n\}$ (even less).
3) Really in 4.11, G has to be just a type-definable group.

4.13. Claim: 1) *Assume*

(a) *$\mathbf{p}$ is a place in a type-definable group (or much less);*
(b) *the place $\mathbf{p}$ is a semi-group;*
(c) *$\mathbf{p}$ is commutative (in the sense of Definition 4.7 + 4.8, so $\sigma_1(x; y) = [x * y = y * x]$ but not necessarily $\sigma_2(\bar{x}_1) = [x_{1,0} * x_{1,1} = x_{1,1} * x_{1,0}]$);*
(d) *if $A \supseteq B^{\mathbf{p}}$ then for some b, c realizing $\mathrm{Av}(D^{\mathbf{p}}, A)$, $c *_G b, b *_G c$ are (necessarily well-defined, and) distinct.*

Then T has the independence property.

2) *We can weaken clause (a) to*

$(a)'$ *$\mathbf{p}$ is a place such that for $n < \omega$ and $\langle a_1 a'_1 \rangle, \ldots, \langle a_n a'_n \rangle$ are as in Definition 4.7 and $a_\ell \neq a'_\ell \Leftrightarrow \ell = m$, then $a_1, \ldots, a_{m-1}, a_m a_{m+1} \ldots a_n \neq a_1, \ldots, a_{m-1} a'_m a_{m+1} \ldots a_m$.*

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[2] His proof uses the operations from clause (d) of 4.16 and 4.17 below.

Remark: This is related to the well-known theorems on stable theories (see Zilber and Hrushovski's works).

Proof. 1) We choose A_i, b_i, c_i by induction on $i < \omega$. In stage i first let $A_i = B^{\mathbf{p}} \cup \{b_j, c_j : j < i\}$ and add B^G if $B^G \nsubseteq B^{\mathbf{p}}$.

Second, choose b_i, c_i realizing $\mathrm{Av}(A_i, D^{\mathbf{p}})$ such that $b_i * c_i \neq c_i * b_i$.

Now, if $i < j < \omega$ and $a' \in \{b_i, c_i\}, a'' \in \{b_j, c_j\}$ then a' realizes $\mathrm{Av}(A_i, D^{\mathbf{p}})$ and a'' realizes $\mathrm{Av}(A_j, D^{\mathbf{p}})$ which include $\mathrm{Av}(A_i \cup \{a'\}, D^{\mathbf{p}})$. So, by assumption (c), the elements a', a'' commute in G.

So, as is well-known, for $n < \omega, i_0 < i_1 < \cdots < i_n$ the element $b_{i_0} * b_{i_1} * \cdots * b_{i_{n-1}}$ commute in G with a_j if and only if $j \notin \{i_0, \ldots, i_{n-1}\}$, hence T has the independence property.

2) Similarly. $\blacksquare_{4.13}$

Note that 4.14 is interesting for G with a finite bound on the order of elements; as if $a \in G$ has infinite order, then $\mathrm{Cm}_G(\mathrm{Gm}_G(a))$ is as desired.

4.14. Conclusion: (T is dependent).

Assume G is a definable group.

1) If $\mathbf{p}$ is a commutative semi-group in G, non-trivial, then for some formula $\varphi(x, \bar{a})$ such that $\varphi(\bar{x}) \vdash$ "$x \in G$" and $\varphi(x, \bar{a}) \in \mathrm{Av}(\bar{a}, D^{\mathbf{p}})$ and $G \restriction \varphi(\mathfrak{C})$ is a commutative place.
2) If G has an infinite abelian subgroup, then it has an infinite definable commutative subgroup.

Proof. 1) By 4.13 for some $A \supseteq B^{\mathbf{p}}$ for every b, c realizing $q := \mathrm{Av}(A, D^{\mathbf{p}})$ we have: the elements of $q(\mathfrak{C})$, which are all in G, pairwise commute. By compactness there is a formula $\varphi_1(x) \in p[\mathbf{p}]$ such that the elements of $\varphi_1(\mathfrak{C}) \cap G$ pairwise commute and, without loss of generality, $\varphi_1(x) \vdash [x \in G]$; note, however, that this set is not necessarily a subgroup. Let $\varphi_2(x) := [x \in G] \wedge (\forall y)(\varphi_1(y) \rightarrow x * y = y * x)$. Clearly, $\varphi_1(\mathfrak{C}) \subseteq \varphi_2(\mathfrak{C}) \subseteq G$ and every member of $\varphi_2(\mathfrak{C})$ commutes with every member of $\varphi_1(\mathfrak{C})$. So $\varphi(z) := [z \in G] \wedge (\forall y)[\varphi_2(y) \rightarrow yz = zy]$ is first order and defines the center of $G \restriction \varphi_2[\mathfrak{C}]$ which includes $\varphi_1(\mathfrak{C})$, so we are done.

2) Let $G' \subseteq G$ be infinite abelian. Choose by induction on $n < \omega, a_n \in G'$ as required in 4.13 and then apply it. $\blacksquare_{4.14}$

4.15. *Remark:* So 4.14 tells us that having some commutativity implies having a lot. If in 4.13 every $a_{[u]}$ is not in any "small" definable set defined with parameters in $B^{\mathbf{p}} \cup \{a_n : n < \max(u)\}$, then also $\varphi(x, \bar{a})$ is not small where small means some reasonably definable ideal.

* * *

4.16. *Definition:* Assume

(a) G is a type definable semi-group;
(b) $M \supseteq B^G$ is $(|T| + |B^G|)^+$-saturated;
(c) $\mathfrak{D}$ is the set of ultrafilters D on M such that $p^G \subseteq \mathrm{Av}(M, D)$;
(d) on $\mathfrak{D} = \mathfrak{D}_{G,M}$ we define an operation;

$$D_1 * D_2 = D_3 \quad \text{if and only if for any} \quad A \supseteq M \text{ and } a \quad \text{realizing} \quad \mathrm{Av}(A, D_1)$$
$$\text{and } b \text{ realizing } \mathrm{Av}(A + a, D_2) \text{ the element } a * b \text{ realizes } \mathrm{Av}(A, D_3).$$

(e) $ID_{G,M} = \{D \in \mathfrak{D}_{G,M} : D * D = D\}$
(f) $H^{\mathrm{left}}_{G,M} = \{a \in G$: for every $D \in \mathfrak{D}$, and $A \supseteq M$ if b realizes $\mathrm{Av}(A+a, D)$ then $a * b$ realizes $\mathrm{Av}(A, D)\}$
(g) $H^{\mathrm{right}}_{G,M}$ similarly using $b * a$
(h) $H_{G,M} = H^{\mathrm{left}}_{G,M} \cap H^{\mathrm{right}}_{G,M}$.

The following fact is as in 4.11.

4.17. Fact: $\mathfrak{D}$ is a semi-group, i.e., associativity holds and the operation is continuous in the second variable hence there is an idempotent (even every non-empty subset closed under $*$ and topologically closed has an idempotent).

4.18. Fact: 1) If G is a group, then

(a) $H^{\mathrm{left}}_{G,M}$ is a subgroup of G, with bounded index, is of the form $\bigcup\{q(\mathfrak{C}) : q \in \mathbf{S}^{\mathrm{left}}_{G,M}\}$ for some $\mathbf{S}^{\mathrm{left}}_{G,M} \subseteq \mathbf{S}(M)$
(b) Similarly $H^{\mathrm{right}}_{G,M}, H_{G,M} = H^{\mathrm{right}}_{G,M} \cap H^{\mathrm{left}}_{G,M}$ with $\mathbf{S}^{\mathrm{right}}_{G,M}, \mathbf{S}_{G,M}$.

2) If $D \in \mathfrak{D}$ is non-principal and $\mathrm{Av}(M, D) \in \mathbf{S}^{\mathrm{right}}_{G,M}$, then for any $A \supseteq M$ and element a realizing $\mathrm{Av}(A, D)$ and b realizing $\mathrm{Av}(A + a, D)$ we have

(α) $a *_G b$ realizes $\mathrm{Av}(A, D)$
(β) also $a^{-1} * b \in D$.

3) $\mathbf{S}^{\mathrm{left}}_{G,M} \subseteq ID_{G,M}$.
4) Similarly for $\mathbf{S}^{\mathrm{right}}_{G,M}, \mathbf{S}_{G,M}$.

5) If $D \in \mathfrak{D}, p = \text{ Av}(M, D) \in \mathbf{S}_{G,M}$ then

(a) $\mathbf{p} = (M, D, *, \text{inv})$ is a quasi group

(b) $\{a^{-1}b : a, b \in p(M)\}$ is a subgroup of G with bound index, in fact is $\{a \in \mathfrak{C} : \text{tp}(a, M) \in \mathbf{S}_{G,M}\}$.

5. Non-forking

5.1. Hypothesis: T is dependent.

5.2. *Definition* ([Sh:93]): 1) An α-type $p = p(\bar{x})$ divides over B if some sequence $\bar{\mathbf{b}}$ and formula $\varphi(\bar{x}, \bar{y})$ witness it which means

(a) $\bar{\mathbf{b}} = \langle \bar{b}_n : n < \omega \rangle$ is an indiscernible sequence over B;

(b) $\varphi(\bar{x}, \bar{y})$ is a formula with $\ell g(\bar{y}) = \ell g(\bar{b}_n)$;

(c) $p \vdash \varphi(\bar{x}, \bar{b}_0)$;

(d) $\{\varphi(\bar{x}, \bar{b}_n) : n < \omega\}$ is contradictory.

1A) Above we say $\varphi(\bar{x}, \bar{b}_0)$ explicitly divide over B.

1B) An α-type $p = p(\bar{x})$ splits strongly over B when for some sequence $\bar{\mathbf{b}}$ and formula $\varphi(\bar{x}, \bar{y})$ witness it which means:

(a),(b) as above;

(c) $\varphi(\bar{x}, \bar{b}_0), \neg\varphi(\bar{x}, \bar{b}_1) \in p$.

2) An α-type p forks over B if for some $\langle \varphi_\ell(\bar{x}, \bar{a}_\ell) : \ell < k \rangle$ we have $p \vdash \bigvee_{\ell<k} \varphi_\ell(\bar{x}, \bar{a}_\ell)$ and $\{\varphi_\ell(\bar{x}, \bar{a}_\ell)\}$ divides over B for each $\ell < k$ (note: though $\bar{x}$ may be infinite, the formulas are finitary).

We say $p(\bar{x})$ exactly forks (or ex-forks) over B <u>when</u> some $\varphi(\bar{x}, \bar{b}) \in p$ does exactly fork over B, <u>which</u> means that for some $\langle \varphi_\ell(\bar{x}, \bar{b}) : \ell < k \rangle$ we have: $\varphi(\bar{x}, \bar{b}) \vdash \bigvee\limits_{\ell<k} \varphi_\ell(\bar{x}, \bar{b})$ and each $\varphi_\ell(\bar{x}, \bar{b})$ explicitly divides over B.

3) We say C/A does not fork over B if letting $\bar{\mathbf{c}}$ list C, $\text{tp}(\bar{\mathbf{c}}, A)$ does not fork over B, or what is equivalent $\bar{c} \in {}^{\omega>}C \Rightarrow \text{ tp}(\bar{c}, A)$ does not fork over B (so below we may write claims for $\bar{c}$ and use them for C).

4) The m-type p is f.s. (finitely satisfiable) in A if every finite $q \subseteq p$ is realized by some $\bar{b} \subseteq A$.

5) The Δ-multiplicity of p over B is $\text{Mult}_\Delta(p, B) = \text{ sup}\{|\{q \restriction \Delta : p \subseteq q \in \mathbf{S}^m(M), q \text{ does not fork over } B\}| : M \supseteq B \cup \text{ Dom}(p)\}$.

Omitting Δ means $\mathbb{L}(\tau_T)$, omitting B we mean $\text{Dom}(p)$.

5.3. *Definition:* 1) Let $p = p(\bar{x})$ be an α-type and Δ be a set of $\mathbb{L}(\tau_T)$-formulas of the form $\varphi(\bar{x}, \bar{y})$ and $k \le \omega$. For a type $p(\bar{x})$ we say that it (Δ, k)-divides over A if some $\bar{\mathbf{b}}, \varphi(\bar{x}, \bar{y})$ witness it which means

(a) $\bar{\mathbf{b}} = \langle \bar{b}_n : n < 2k+1 \rangle$ is Δ-indiscernible;
(b) $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_T)$;
(c) $p \vdash \varphi(\bar{x}, \bar{b}_0)$;
(d) $\{\varphi(\bar{x}, \bar{b}_n) : n < 2k+1\}$ is k-contradictory.

2) For a type $p(\bar{x})$ we say that it (Δ, k)-forks over B if $p \vdash \bigvee_{\ell<n} \varphi_\ell(x, \bar{a}_\ell)$ for some $n, \varphi_\ell(\bar{x}, \bar{y})$ and $\bar{a}_\ell$, where each $\varphi_\ell(\bar{x}, \bar{a}_\ell)$ does (Δ, k)-divide over B.

5.4. Observation: 0) In Definition 5.2(1), if $p = \{\varphi(x, \bar{b})\}$ then without loss of generality $\bar{b} = \bar{b}_0$. If p divides over B then p forks over B.

0A) Forking is preserved by permuting and repeating the variables. If $\text{tp}(\bar{b}\hat{\ }\bar{c}, A)$ does not fork over B then so does $\text{tp}(\bar{b}, A)$ and both do not divide over B. Similarly for dividing and ex-forking and later versions.

1) If $p \in \mathbf{S}^m(A)$ is finitely satisfiable in B, then p does not fork over B; hence every type over M does not fork over M.

2) If $p \in \mathbf{S}^m(A)$ does not fork or just does not divide over $B \subseteq A$, then p does not split strongly over B. (Of course, if p divides over A, then p forks over A).

The type $p(\bar{x})$ divides over B iff for some $k < \omega$ and $\varphi_\ell(\bar{x}, \bar{c}_\ell) \in p(\bar{x})$ for $\ell < k$, letting $\bar{c} = \bar{c}_0\hat{\ } \ldots \hat{\ }\bar{c}_{k-1}$ the formula $\varphi(\bar{x}, \bar{c}) = \bigwedge\limits_{\ell<k} \varphi_\ell(\bar{x}, \bar{c}_\ell)$ explicitly divides over B.

Assume the type $p((\bar{x})$ is $\{\varphi(\bar{x}, \bar{b})\}$ or is complete, i.e. $\in \mathbf{S}(A)$ for some set A or just is directed by $\vdash$, (i.e. for every finite $q(\bar{x}) \subseteq p(\bar{x})$ there is $\psi(\bar{x}, \bar{b}) \in p(\bar{x})$ such that $\psi(\bar{x}, \bar{x}) \vdash q(\bar{x})$); then $p(\bar{x})$ divides over B iff $\psi(\bar{x}, \bar{b})$ explicitly divides over B for some $\psi(\bar{x}, \bar{b}) \in p$, and each of them imply that in Definition 5.1(1), we can choose $\bar{b}_0 = \bar{b}$ and that $\{\varphi(\bar{x}, \bar{b})\}$ forks over B.

If $p(\bar{x}) \in \mathbf{S}^m(A)$ or just $p(\bar{x})$ is closed under conjunctions (or just is directed by $\vdash$), <u>then</u> $p(\bar{x})$ forks over B iff some $\varphi(\bar{x}, \bar{a}) \in p(\bar{x})$ forks over B.

The m-type $p(\bar{x})$ forks over B iff there is $\varphi(\bar{x}, \bar{a})$ which exactly forks over B such that $p(\bar{x}) \vdash \varphi(\bar{x}, \bar{a})$.

3) (Extension property) If an m-type p is over A and does not fork over B, then some extension $q \in \mathbf{S}(A)$ of p does not fork over B.
4) (Few non-forking types) For $B \subseteq A$ the set $\{p \in \mathbf{S}^m(A) : p$ does not fork over B (or just does not split strongly) over $B\}$ has cardinality $\leq 2^{2^{|B|+|T|}}$. If $p(\bar{x})$ does not fork over M, then it does not split over M.
5) (Monotonicity in the sets) If $B_1 \subseteq B_2 \subseteq A_2 \subseteq A_1$ and $p \in \mathbf{S}(A_1)$ does not fork over B_1, then $p \restriction A_2$ does not fork over B_2.
6) (Indiscernibility preservation) If $\bar{\mathbf{b}}$ is an infinite indiscernible sequence over A_1 and $B \subseteq A_1 \subseteq A_2$ and $\bar{\mathbf{b}} \subseteq A_2$ and $\mathrm{tp}(\bar{c}, A_2)$ does not fork over B or just does not divide over B or just does not split strongly over B then $\bar{\mathbf{b}}$ is an (infinite) indiscernible sequence over $A_1 \cup \bar{c}$.
7) (Finite character) If p forks over B then some finite $q \subseteq p$ does; if p is closed under conjunction (up to equivalence suffices) then we can choose $q = \{\varphi\}$. Similarly for divides and the type $p(\bar{x})$ strongly split over A iff some subtype with exactly two members strongly split over A.
8) (Monotonicity in the type) If $p(\bar{x}) \subseteq q(\bar{x})$ or just $q(\bar{x}) \vdash p(\bar{x})$ and $p(\bar{x})$ forks over B then $q(\bar{x})$ forks over B; similarly for divides and for split strongly.
9) An m-type p is finitely satisfiable in A if and only if for some ultrafilter D on ${}^mA$ we have $p \subseteq \mathrm{Av}(\mathrm{Dom}(p), D)$.

Remark: 1) Only parts (2), (4), (6) of 5.4 use "T is dependent".

2) If T is unstable then for every κ there are some A and $p \in \mathbf{S}(A)$ such that p divides over every $B \subseteq A$ of cardinality $< \kappa$ (use a Dedekind cut with both cofinalities $\geq \kappa$).

Proof. 0), 0A), 1) Easy. The proof of part (1) is included in the proof of part (2).

2) Assume toward contradiction that p splits strongly, then for some infinite indiscernible sequence $\langle \bar{b}_n : n < \omega \rangle$ over B and $n < m$ we [3] have $[\varphi(\bar{x}, \bar{b}_n) \equiv \neg\varphi(x, \bar{b}_m)] \in p$ (really $p \vdash [\varphi(\bar{x}, \bar{b}_n) \equiv \neg\varphi(\bar{x}, \bar{b}_m)]$ suffices). By renaming, without loss of generality $n = 0, m = 1$. Let $\bar{c}_n = \bar{b}_{2n}{}^\frown\bar{b}_{2n+1}, \psi(\bar{x}, \bar{c}_n) = [\varphi(\bar{x}, \bar{b}_{2n}) \equiv \neg\varphi(\bar{x}, \bar{b}_{2n+1})]$. Clearly $\langle \bar{c}_n : n < \omega \rangle$ is an indiscernible sequence over B, $p \vdash \psi(\bar{x}, \bar{c}_0)$ and $\{\psi(\bar{x}, \bar{c}_n) : n < \omega\}$ is contradictory as T is dependent. This proves the first sentence. The second is by the definitions and the third sentence.

[3] Recalling $[\varphi_1 \equiv \varphi_2]$ is the formula $(\varphi_1 \wedge \varphi_2) \vee (\neg\varphi_1 \wedge \neg\varphi_2)$.

For the third, the "if" part is obvious, hence let us prove the "only if", so assume that $p(\bar{x})$ divides over B, we can find $\varphi(\bar{x}, \bar{b}_0)$, $\langle \bar{b}_n : n < \omega \rangle$ as in Definition 5.2(1), i.e. satisfies clauses (a)-(d) there. As $p(\bar{x}) \vdash \varphi(\bar{x}, \bar{b}_0)$, necessarily there is a finite subset $p'(\bar{x})$ of $p(\bar{x})$ such that $p'(\bar{x}) \vdash \varphi(\bar{x}, \bar{b}_0)$. Let $\langle \varphi_\ell(\bar{x}, \bar{c}_\ell) : \ell < k \rangle$ list $p'(x)$ and as $p(\bar{x})$ is directed by $\vdash$ we can find a formula $\psi(\bar{x}, \bar{c}) \in p(\bar{x})$ such that $\psi(\bar{x}, \bar{c}) \vdash \varphi_\ell(\bar{x}, \bar{c}_\ell)$ for every $\ell < k$ hence $\psi(\bar{x}, \bar{c}) \vdash \varphi(\bar{x}, \bar{b}_0)$. Now for each $n < \omega$, the sequences $\bar{b}_n, \bar{b}_0$ realize the same type over B, hence there is a sequence $\bar{c}^n \in {}^{\ell g(\bar{c})}\mathfrak{C}$ such that the sequences $\bar{b}_0{}^\frown\bar{c}, \bar{b}_n{}^\frown\bar{c}^n$ realize the same type over B By Ramsey theorem and compactness we can find $\langle \bar{d}_n : n < \omega \rangle$ such that $\bar{b}_n{}^\frown\bar{d}_n$ realizes the same type as $\bar{b}_0{}^\frown\bar{c}$ over B and $\langle \bar{b}_n{}^\frown\bar{d}_n : n < \omega \rangle$ is an indiscernible sequence over B. So let F be an automorphism of $\mathfrak{C}$ over B which maps $\bar{b}_0{}^\frown\bar{d}_0$ to $\bar{b}_0{}^\frown\bar{c}$. So $\langle F(\bar{d}_n) : n < \omega \rangle$ is an indiscernible sequence over B and $F(\bar{d}_0) = \bar{c}$ so $\psi(\bar{x}, \bar{d}_0) = \psi(\bar{x}, \bar{c}) \vdash \bigwedge\limits_{\ell<k} \varphi_\ell(\bar{x}, \bar{c}_\ell) \vdash \varphi(\bar{x}, \bar{b}_0) = \varphi(\bar{x}, F(\bar{b}_0))$.

Necessarily also $n < \omega \Rightarrow \psi(\bar{x}, \bar{d}_n) \vdash \varphi(x, \bar{b}_n)$ and as $\{\varphi(\bar{x}, \bar{b}_n) : n < \omega\}$ is contradictory, so is $\{\psi(\bar{x}, \bar{d}_n) : n < \omega\}$. So $\langle F(\bar{d}_n) : n < \omega \rangle$ examplifies that $\psi(\bar{x}, \bar{d}_0) = \psi(\bar{x}, \bar{c})$ explicitly divides over B as promised.

The fourth and fifth sentences are obvious.

3) By the definitions (or see [Sh:93]).

4) Easy or see [Sh:3]; e.g. by part (3) without loss of generality $B = M$, $A = |N|$ is $\|M\|^+$-saturated. Now if $\bar{a}_\ell \in {}^mN$ realizes the same type over M for $\ell = 1, 2$ then for some $\bar{c}_n \in {}^mN$ for $n = 1, 2, \ldots, \langle \bar{a}_\ell \rangle{}^\frown\langle \bar{c}_1, \bar{c}_2, \ldots \rangle$ is indiscernible over M.

5) Easy.

6) By part (2) and transitivity of "equality of types" and Fact 5.5 below.

7), 8), 9) Easy. $\blacksquare_{5.4}$

We implicitly use the trivial.

5.5. Fact: 1) If I is a linear order, $\bar{s}_0, \bar{s}_1$ are increasing n-tuples from I then

$\circledast_\omega$ there is a linear order $J \supseteq I$ such that for $\ell \in \{0, 1\}$ there is an indiscernible sequence $\langle \bar{t}^\ell_k : k < \omega \rangle$ of increasing n-tuples from J such that $\bar{t}^0_{k+1} = \bar{t}^1_{k+1}$ for $k < \omega$ and $\ell = 0, 1 \Rightarrow \bar{s}_\ell = \bar{t}^\ell_0$; indiscernible means for quantifier free formulas in the order language, i.e., in the vocabulary $\{<\}$ is satisfaction in J. If I has no last element or no first element then we can take $I = J$.

2) Similarly, for $\langle \bar{b}_t : t \in I \rangle$ an infinite indiscernible sequence over A in $\mathfrak{C}$.

Proof. 1) Let $J \supseteq I$ be with no last element. Choose for $k = 1, 2, \ldots$ an increasing sequence $\bar{t}_k$ of length n from J such that $2 \le k < \omega \Rightarrow \mathrm{Rang}(\bar{s}_0\hat{\ }\bar{s}_1) < \mathrm{Rang}(\bar{t}_k) < \mathrm{Rang}(\bar{t}_{k+1})$. So $\langle \bar{s}_\ell \rangle\hat{\ }\langle \bar{t}_1, \bar{t}_2, \ldots \rangle$ is an indiscernible sequence in J for $\ell = 0, 1$.

2) Easy. $\blacksquare_{5.5}$

5.6. *Definition:* 1) Let p be an m-type, $p \restriction B_2 \in \mathbf{S}^m(B_2)$. We say that p strictly does not divide over (B_1, B_2), (when $B_1 = B_2 = B$ we may write "over B") if :

(a) p does not divide over B_1;

(b) if $\langle \bar{c}_n : n < \omega \rangle$ is an indiscernible sequence over B_2 such that $\bar{c}_0$ realizes p and A is any set satisfying $\mathrm{Dom}(p) \cup B_2 \subseteq A$, then there is an indiscernible sequence $\langle \bar{c}'_n : n < \omega \rangle$ over A such that $\bar{c}'_0$ realizes p and $\mathrm{tp}(\langle \bar{c}_n : n < \omega \rangle, B_2) = \mathrm{tp}(\langle \bar{c}'_n : n < \omega \rangle, B_2)$.

1A) "Strictly divide" is the negation.

2) We say that p strictly forks over (B_1, B_2) if and only if $p \vdash \bigvee_{\ell < n} \varphi_\ell$ for some $\langle \varphi_\ell : \ell < n \rangle$ such that $(p \restriction B_2) \cup \{\varphi_\ell\}$ strictly divides over (B_1, B_2) for each $\ell < n$.

3) An m-type $p(\bar{x})$ strictly does not fork over (B_1, B_2) <u>when</u>: the type $p(\bar{x})$ does not fork over B_1 and $p(\bar{x}) \restriction B_2 \in \mathbf{S}^M(B_2)$ and if $\langle \bar{c}_n : n < \omega \rangle$ is an indiscernible sequence over B_2 of sequences realizing $p(\bar{x})$ and $C \supseteq B_1 \cup \mathrm{Dom}(p(\bar{x}))$ and $q(\bar{x}) \in \mathbf{S}^m(C)$ extend $p(\bar{x})$ and does not fork over B_1 <u>then</u> there is an indiscernible sequence $\langle \bar{c}'_n : n < \omega \rangle$ over C realizing $\mathrm{tp}(\langle \bar{c}_n : n < \omega \rangle, B_2)$ such that $\bar{c}'_0$ realizes $q(\bar{x})$; note that "strictly does not fork" is not defined as "does not strictly forks"; to stress we may write "strictly* does not fork".

We shall need some statements concerning "strictly does not fork" parallel to those on "does not fork".

5.7. Observation: 0) In clause (b) of Definition 5.6(1) we can weaken the assumption "$\bar{c}_0$ realizes p" to "$\bar{c}_0$ realizes $p \restriction B_2$".

1) "Strictly does not divide/fork over (B_1, B_2)" is preserved by permuting the variables, repeating variables and by automorphisms of $\mathfrak{C}$ and if it holds for $\mathrm{tp}(\bar{b}\hat{\ }\bar{c}, A)$, then it holds for $\mathrm{tp}(\bar{b}, A)$. Similarly for does not strictly fork.

1A) The m-type $p(\bar{x})$ strictly does not divide over (B_1, B_2) <u>iff</u> $p(\bar{x}) \restriction B_2 \in \mathbf{S}^m(B_2)$ and $(p(\bar{x}) \restriction B_2) \cup q(\bar{x})$ strictly does not divide over (B_1, B_2) for every finite $q(\bar{x}) \subseteq p(\bar{x})$.

2) If p strictly does not fork over (B_1, B_2) then p does not strictly fork over B which implies p strictly does not divide over (B_1, B_2).
3) If p strictly does not divide over B, then p does not divide over B.
4) If p does not strictly fork over B, then p does not fork over B.
5) If p is an m-type which strictly does not fork over (B_1, B_2) and $\mathrm{Dom}(p) \subseteq A$, then there is $q \in \mathbf{S}^m(A)$ extending p which strictly does not fork over (B_1, B_2). If $p_1(\bar{x}) \subseteq p_2(\bar{x})$ and $p_1(\bar{x})$ strictly does not fork over (B_1, B_2) and $p_2(\bar{x})$ does not fork over B_1 then $p_2(\bar{x})$ does not strictly fork over (B_1, B_2).
6) If $B_1 \subseteq B_1' \subseteq B_2' = B_2$ and $p(\bar{x}) \vdash p'(\bar{x})$ and $p(\bar{x})$ strictly does not divide/fork over (B_1, B_2) and $p' \restriction B_2$ is complete then $p'(\bar{x})$ strictly does not divide/fork over (B_1', B_2').
7) In Definition 5.6, clause (b) the case $A = \mathrm{Dom}(p) \cup B_2$ suffices.
8) If p strictly forks over (B_1, B_2), then for some finite $q \subseteq p$ the type $q \cup (p \restriction B_2)$ strictly forks over (B_1, B_2). Moreover, for some finite $B_2' \subseteq B_2$, (p is an m-type), if $B_1 \cup B_2' \subseteq B_2''$ and p' is an m-type extension of q and $p' \restriction B_2 \in \mathbf{S}^m(B_2)$, then p' strictly forks over (B_1, B_2''). Similarly for strictly divide.
9) If $M \subseteq A$, $p = \mathrm{tp}(\bar{b}, A)$ and $\mathrm{tp}(A, M + \bar{b})$ is finitely satisfiable in M, then p strictly does not fork over M.

Proof. Easy, e.g.,

0) The new version is stronger hence it implies the one from the definition.

So assume that p is an m-type, $p \restriction B_2 \in \mathbf{S}^m(B_2)$ and p strictly does not divide over (B_1, B_2) and we shall prove the new version of clause (b). I.e., we have $\langle \bar{c}_n : n < \omega \rangle$ is an indiscernible sequence over B_2 and $\bar{c}_0$ realizes $p \restriction B_2$. Let $\bar{c}_0'' \in {}^m\mathfrak{C}$ realizes p hence it realizes $p \restriction B_2$, but $p \restriction B_2 \in \mathbf{S}^m(B_2)$ so $\mathrm{tp}(\bar{c}_0, B_2) = \mathrm{tp}(\bar{c}_0'', B_2)$. We can deduce that there is an automorphism F of $\mathfrak{C}$ over B_2 which maps $\bar{c}_0$ to $\bar{c}_0''$, and define $\bar{c}_n'' = F(\bar{c}_n)$.

Now $\langle F(\bar{c}_n) : n < \omega \rangle$ satisfies the assumption of clause (b) from Definition 5.6(1) hence there is an indiscernible sequence $\langle \bar{c}_n' : n < \omega \rangle$ over A such that $\mathrm{tp}(\langle \bar{c}_n' : n < \omega \rangle, B_2) = \mathrm{tp}(\langle F(\bar{c}_n') : n < \omega \rangle, B_2)$, but the latter is equal to $\mathrm{tp}(\langle c_n : n < \omega \rangle, B_2)$ so we are done.

6) Without loss of generality $\mathrm{Dom}(p) \cup B_2' \subseteq A$. Recall that by part (0) in Claim 5.5, clause (b) we can demand only "$\bar{c}_0$ realizes $p \restriction B_2$" and for any such $\langle \bar{c}_n : n < \omega \rangle$ there is $\bar{c}_0''$ realizing p hence $\bar{c}_0$ and $\bar{c}_0''$ realizes the same type over

B_2 hence there is automorphism F of $\mathfrak{C}$ over B_2 mapping $\bar{c}_0$ to $\bar{c}''_0$ and use the definition for $\langle F(\bar{c}_n) : n < \omega\rangle$.

7) By Ramsey theorem and compactness.

9) Use an ultrafilter D. $\blacksquare_{5.7}$

The next claim is a parallel of: every type over A does not fork over some "small" $B \subseteq A$. If we have "p is over A implies p does not fork over A" we could have improvement.

More elaborately, note that if M is a dense linear order and $p \in \mathbf{S}(M)$, then p actually corresponds to a Dedekind cut of M. So though in general p is not definable, $p \restriction \{c \in M : c \notin (a,b)\}$ is definable whenever (a,b) is an interval of M which includes the cut. So p is definable in large pieces. The following (as well as 5.20) realizes the hope that something in this direction holds for every dependent theory.

5.8. Claim: *If $p \in \mathbf{S}^m(A)$ and $B \subseteq A$, then we can find $C \subseteq A$ of cardinality $\leq |T|$ such that:*

- $\circledast$ *if $\langle \bar{a}_n : n < \omega\rangle$ is an indiscernible sequence over $B \cup C$ such that $\bar{a}_0 \subseteq A$ and $\mathrm{tp}(\bar{a}_0, B \cup C)$ strictly does not fork over B and $\{\varphi(\bar{x}, \bar{a}_n) : n < \omega\}$ is contradictory or just $\varphi(\bar{x}, \bar{a}_0)$ ex-forks over $B \cup C$, then $\neg\varphi(x, \bar{a}_0) \in p$.*

5.9. Conclusion: 1) For every $p \in \mathbf{S}^m(A)$ and $B \subseteq A$, we can find $C \subseteq A, |C| \leq |T|$ such that:

- $\circledast$ if $\langle \bar{a}_n : n < \omega\rangle$ is an indiscernible sequence over $B \cup C$ satisfying $\bar{a}_0 \cup \bar{a}_1 \subseteq A$ and $\mathrm{tp}(\bar{a}_0 \char`^ \bar{a}_1, B \cup C)$ strictly does not fork over B, then for any φ
  - $(*)$ $\varphi(x, \bar{a}_0) \in p$ if and only if $\varphi(x, \bar{a}_1) \in p$.

2) For every $\bar{x} = \langle x_\ell : \ell < m\rangle$ and formula $\varphi = \varphi(\bar{x}; \bar{y})$ for some finite $\Delta \subseteq \mathbb{L}(T)$ we have:

> if $p \in \mathbf{S}^m(A), B \subseteq A$, then for some finite $C \subseteq A$ (in fact $|C| < f(m, \varphi, T)$ for some function f), we have:
>
> if $\langle \bar{a}_\ell : \ell < k\rangle$ is Δ-indiscernible sequence over $B \cup C$ and $\mathrm{tp}_\Delta(\bar{a}_0 \char`^ \bar{a}_1, B \cup C)$ strictly does not fork over A, then $\varphi(x, \bar{a}_0) \in p \Leftrightarrow \varphi(x, \bar{a}_1) \in p$.

3) The local version of 5.8 holds with a priori finite bound on C.

Proof of 5.8. By induction on $\alpha < |T|^+$ we try to choose $C_\alpha, \bar{a}_\alpha, k_\alpha$ and $\langle \bar{a}^k_{\alpha,n} : n < \omega\rangle$ and $\varphi_\alpha(\bar{x}, \bar{y}_\alpha), \varphi_{\alpha,k}(\bar{x}, \bar{y}_{\alpha,k})$ such that:

(a) $C_\alpha = \bigcup\{\bar{a}_\beta : \beta < \alpha\} \cup B$;
(b) $\langle \bar{a}^k_{\alpha,n} : n < \omega\rangle$ is an indiscernible sequence over C_α for $k < k_\alpha$;
(c) $\bar{a}_\alpha \subseteq A$ and $\bar{a}_\alpha = \bar{a}^k_{\alpha,0}$ for $k \leq k_\alpha$;
(d) $\varphi_\alpha(\bar{x}, \bar{a}_\alpha) \in p$;
(e) $\{\varphi_{\alpha,k}(\bar{x}, \bar{a}^k_{\alpha,n}) : n < \omega\}$ is contradictory;
(f) $\mathrm{tp}(\bar{a}_\alpha, B \cup C_\alpha)$ strictly does not fork over B;
(g) $\varphi_\alpha(\bar{x}, \bar{a}_\alpha) \vdash \bigvee_{k<k_\alpha} \varphi_{\alpha,k}(\bar{x}, \bar{a}^k_{\alpha,0})$.

If for some $\alpha < |T|^+$ we are stuck, $C = C_\alpha \backslash B$ is as required. So assume that we have carried the induction and we shall eventually get a contradiction.

By induction on $\alpha < |T|^+$ we choose $D_\alpha, F_\alpha, \bar{b}_\beta, \langle \bar{b}^k_{\beta,n} : n < \omega\rangle$ for $\beta < \alpha$ such that (but $\bar{b}^k_{\alpha,n}$ are defined in the $(\alpha+1)$-th stage):

(α) F_α is an elementary mapping, increasing continuous with α;
(β) $\mathrm{Dom}(F_\alpha) = C_\alpha$, $\mathrm{Rang}(F_\alpha) \subseteq D_\alpha$;
(γ) $D_\alpha = \mathrm{Rang}(F_\alpha) \cup \bigcup\{\bar{b}^k_{\beta,n} : \beta < \alpha, k < k_\alpha$ and $n < \omega\}$ so $D_\alpha \subseteq \mathfrak{C}$ is increasing continuous;
(δ) $\langle \bar{b}^k_{\alpha,n} : n < \omega\rangle$ is an indiscernible sequence over $D_\alpha \supseteq F_\alpha(C_\alpha)$ and $\bar{b}^k_{\alpha,0} = \bar{b}_\alpha$;
(ε) $F_{\alpha+1}(\bar{a}_\alpha) = \bar{b}_\alpha$ and $\mathrm{tp}(\bar{b}_\alpha, D_\alpha)$ does not fork over $F_\alpha(B)$;
(ζ) Some automorphism $F^k_{\alpha+1} \supseteq F_{\alpha+1}$ of $\mathfrak{C}$ maps $\bar{a}^k_{\alpha,n}$ to $\bar{b}^k_{\alpha,n}$ for $n < \omega, k < k_\alpha$.

For $\alpha = 0, \alpha$ limit this is trivial. For $\alpha = \beta+1$, clearly $F_\alpha(\mathrm{tp}(\bar{a}_\alpha, C_\alpha))$ is a type in $\mathbf{S}^{<\omega}(F(C_\alpha))$ which strictly does not fork over $F_\alpha(B) = F_0(B)$ hence has an extension $q_\alpha \in \mathbf{S}^{<\omega}(D_\alpha)$ which does not fork over $F_0(B)$ and let $\bar{b}_\alpha$ realize it. Let $F_{\alpha+1} \supseteq F_\alpha$ be the elementary mapping extending F_α with domain $C_{\alpha+1}$ mapping $\bar{a}_\alpha$ to $\bar{b}_\alpha$. Let $F^k_{\alpha+1} \supseteq F_\alpha + 1$ be an automorphism of $\mathfrak{C}$ as required by clauses $(\delta)+(\zeta)$; F^k_{a+1} exists as $\mathrm{tp}(\bar{a}_\alpha, B \cup C_\alpha)$ strictly* does not fork over B; and let $\bar{b}^k_{\alpha,n} = F^k_{\alpha+1}(\bar{a}^k_{\alpha,n})$ for $n < \omega, k < k_\alpha$. So $D_{\alpha+1}$ and $F_{\alpha+1}$ are well-defined.

Having carried the induction let $F \supseteq \bigcup\{F_\alpha : \alpha < |T|^+\}$ be an automorphism of $\mathfrak{C}$. We claim that for each $\alpha < |T|^+$ and $k < k_\alpha$, for every $\beta \in [\alpha, |T|^+]$ we have

$(*)_{\alpha,\beta}$ $\langle \bar{b}^k_{\alpha,n} : n < \omega\rangle$ is an indiscernible sequence over $D_\alpha \cup \bigcup\{\bar{b}_\gamma : \gamma \in [\alpha+1, \beta)\}$.

We prove this by induction on β. For $\beta = \alpha$ this holds by clause (δ), for $\beta \equiv \alpha+1$ this is the same as for $\beta = \alpha$. For β limit use the definition of indiscernibility. For $\beta = \zeta+1$ use the fact that $\mathrm{tp}(\bar{b}_\zeta, D_\zeta)$ does not fork over

$F_0(B)$ hence over $D_\alpha \cup \{F_{\gamma+1}(\bar b_\gamma) : \alpha < \gamma < \zeta\}$ by 5.4(5); so by the induction hypothesis and 5.4(6) clearly $(*)_{\alpha,\beta}$ holds.

¿From $\alpha < |T|^+ \Rightarrow (*)_{\alpha,|T|^+}$ we can conclude

$(**)$ for any $n < \omega$ and $\alpha_0 < \cdots < \alpha_{n-1} < |T|^+$ and $\nu \in \prod_{\ell<n} k_{\alpha_\ell}$ and $\eta \in {}^n2$ the sequences

$$\bar b^{\nu(0)}_{\alpha_0,0} {}^\frown \bar b^{\nu(1)}_{\alpha_1,0} {}^\frown \ldots {}^\frown \bar b^{\nu(n-1)}_{\alpha_{n-1},0} \quad \text{and} \quad \bar b^\nu_\eta := \bar b^{\nu(0)}_{\alpha_0,\eta(0)} {}^\frown \bar b^{\nu(1)}_{\alpha_1,\eta(1)} {}^\frown \ldots {}^\frown \bar b^{\nu(n-1)}_{\alpha_{n-1},\eta(n-1)}$$

realize the same type over B.

(Why? By induction on $\max\{\ell : \eta(\ell) = 1 \text{ or } \ell = -1\}$. If $\ell(*) = -1$ then the two sequences are the same so $(**)$ holds trivially. Let $\rho \in {}^n2$ is defined by: $\rho(\ell)$ is 0 if $\ell = \ell(*)$ and is $\eta(\ell)$ otherwise. So the induction hypothesis applied to ρ, hence it suffice to prove that the sequences $\bar b^\nu_\rho, \bar b^\nu_\eta$ realize the same type over B . Now assume $\ell(*) \in \{0, \ldots, n-1\}$ and use $(*)_{\alpha_{\ell(*)},|T|^+}$ for $k = \nu(\ell(*))$, it says that the sequence $\langle \bar b^{\nu(\ell(*))}_{\alpha_{\ell(*)},n} : n < \omega\rangle$ is an indiscernible sequence over $D_{\alpha_{\ell(*)}} \cup \{\bar b_\gamma : \gamma \in [\alpha_{\ell(*)+1}, |T|^+)\}$.

The second part in the union includes

$$\{\bar b^{\nu(\ell(*)-1)}_{\alpha_{\ell(*)+1},0}, \ldots, \bar b^{\nu(n-1)}_{\alpha_{n-1},0}\} = \{\bar b^{\nu(\ell(*))}_{\alpha_{\ell(*)+1},\eta(\ell(*)+1)}, \ldots, \bar b^{\nu(n-1)}_{\alpha_{n-1},\eta(n-1)}\}$$

by the choice of $\ell(*)$, and the first part of the union includes the rest. So it suffice to show that $\bar b^{\nu(\ell(*))}_{\alpha_{\ell(*)}} = \bar b^{\nu(\ell(*))}_{\alpha_{\ell(*)},0} = \bar b^{\nu(\ell(*)))}_{\alpha_{\ell(*)},\rho(\ell(*))}$ and $\bar b^{\nu(\ell(*))}_{\alpha_{\ell(*)},1} = \bar b^{\nu(\ell(*))}_{\alpha_{\ell(*)},\eta(\ell(*))}$ realize the same type over $D_{\alpha_{\ell(*)}} \cup \{\bar b_\gamma : \gamma = \alpha_{\ell(*)+1}, \ldots, \alpha_{n-1}\}$ which has been proved above.)

Let $\bar c$ realize p. For each $\alpha < |T|^+, \varphi_\alpha(x, \bar a_\alpha) \in p$, hence $\varphi_\alpha(x, \bar b_\alpha) \in F(p)$. Also $\varphi_\alpha(x, \bar a_\alpha) \vdash \bigvee_{k<k_\alpha} \varphi_{\alpha,k}(x, a^k_{\alpha,0})$, hence by clause (ζ)

$$\varphi_\alpha(x, \bar b_\alpha) \vdash \bigvee_{k<k_\alpha} \varphi_{\alpha,k}(x, a^k_{\alpha,0}),$$

hence we can choose $k(\alpha) < k_\alpha$ such that $\mathfrak{C} \models \varphi[\bar c, \bar a^{k(\alpha)}_{\alpha,0}]$.

Now as $\{\varphi_{\alpha,k(\alpha)}(x, \bar b^{k(\alpha)}_{\alpha,n}) : n < \omega\}$ is contradictory there is $n = n[\alpha] < \omega$ such that $\mathfrak{C} \models \neg\varphi_{\alpha,k(\alpha)}(\bar c, \bar b^{k(\alpha)}_{\alpha,n})$, whereas $\mathfrak{C} \models \varphi_{\alpha,k(\alpha)}[\bar c, \bar b^{k(\alpha)}_{\alpha,0}]$; by renaming without loss of generality $\mathfrak{C} \models \neg\varphi_{\alpha,k(\alpha)}[c, \bar b^{k(\alpha)}_{\alpha,n}]$ for $\alpha < |T|^+, n \in [1,\omega)$. Now if $n < \omega$, $\alpha_0 < \cdots < \alpha_{n-1} < |T|^+$ and $\eta \in {}^n2$, then $\mathfrak{C} \models \bigwedge_{\ell<m} \varphi_{\alpha_\ell,k(\alpha_\ell)}(\bar c, \bar b^{k(\alpha_\ell)}_{\alpha_\ell,\eta(\ell)})^{\text{if}(\eta(\ell)=0)}$ hence $\mathfrak{C} \models (\exists \bar x)[\bigwedge_{\ell<n} \varphi_{\alpha_\ell,k(\alpha_\ell)}(\bar x, \bar b_{\alpha_\ell,\eta(\ell)})^{\text{if}(\eta(\ell)=0)}]$ hence by $(**)$ we have

$\mathfrak{C} \models (\exists \bar x)[\bigwedge_{\ell<n} \varphi_{\alpha_\ell,k(\alpha_\ell)}(\bar x, \bar b^{k(\alpha_\ell)}_{\alpha_\ell,0})^{\text{if}(\eta(\ell)=0)}]$.

Hence the independence property holds, contradiction. $\blacksquare_{5.8}$

Proof of 5.9. 1) Follows from 5.8 by 5.4(2).

2) By 5.8 and compactness or repeating the proof.

3) Similarly. $\blacksquare_{5.9}$

5.10. Claim: 1) *Assume p is a type, $B \subseteq M, \mathrm{Dom}(p) \subseteq M$ and M is $|B|^+$-saturated. Then*

(A) *p does not fork over B if and only if p has a complete extension over M which does not fork over B if and only if p has a complete extension over M which does not divide over B if and only if p has a complete extension over M which does not split strongly over B;*

(B) *if $p = \mathrm{tp}(\bar{c}, M)$ and $\varphi(\bar{x}, \bar{a}) \in p$ forks over B, then for some $\langle \bar{a}_n : n < \omega \rangle$ indiscernible over $B, \{\bar{a}_n : n < \omega\} \subseteq M, \bar{a}_0 = a$ and $\neg\varphi(\bar{x}, \bar{a}_1) \in p$, and of course, $\varphi(\bar{x}, \bar{a}_0) \in p$.*

2) *Assume $\mathrm{tp}(C_1/A)$ does not fork over $B \subseteq A$ and $\mathrm{tp}(C_2, (A \cup C_1))$ does not fork over $B \cup C_1$. Then $\mathrm{tp}(C_1 \cup C_2), A)$ does not fork over B.*

Proof. 1) Read the definitions.

Clause (A):

First implies second by 5.4(3), second implies third by Definition 5.2 or 5.4(2), third implies fourth by 5.4(2). If the first fails, then $p \vdash \bigvee_{\ell<k} \varphi_\ell(\bar{x}, \bar{a}_\ell)$ for some k where each $\varphi_\ell(\bar{x}, \bar{a}_\ell)$ divides over B; let $\langle \bar{a}_{\ell,n} : n < \omega \rangle$ witness this hence by 5.4(2) without loss of generality $\bar{a}_\ell = \bar{a}_{\ell,0}$. As M is $|B|^+$-saturated, without loss of generality $\bar{a}_{\ell,n} \subseteq M$. So for every $q \in \mathbf{S}^m(M)$ extending p, for some $\ell < k, \varphi_\ell(\bar{x}, \bar{a}_\ell) \in q$ but for every large enough $n, \neg\varphi_\ell(\bar{x}, \bar{a}_{\ell,n}) \in q$, so q splits strongly, i.e., fourth fails. So fourth implies first, "closing the circle".

Clause (B): Similar.

2) Let M be $|B|^+$-saturated model such that $A \subseteq M$. By 5.4(3) there is an elementary mapping f_1 such that $f_1 \restriction A = \mathrm{id}_A$ and $\mathrm{Dom}(f_1) = C_1 \cup A$ and $f_1(C_1)/M$ does not fork over B. Similarly we can find an elementary mapping $f \supseteq f_1$ such that $\mathrm{Dom}(f) = C_1 \cup C_2 \cup A$ and $f(C_2)/(M \cup f(C_1))$ does not fork over $A \cup f(C_1)$. By 5.4(2), $f_1(C_1)/M$ does not split strongly over B. Again by 5.4(2), $f(C_2)/(M \cup f_1(C_1))$ does not split strongly over $B \cup f_1(C_1)$. Together they imply that if $\bar{\mathbf{b}} \subseteq M$ is an infinite indiscernible sequence over B then it is an indiscernible sequence over $f(C_1) \cup B$ and even over $f(C_2) \cup (f(C_1) \cup B)$ (use the two previous sentences and 5.4(6)). But this means that $f(C_1) \cup f(C_2)/M$ does not split strongly over B, (here the exact version of strong splitting we choose is immaterial as M is $|B|^+$-saturated). So by 5.10(1) we get that $f(C_1) \cup f(C_2)/M$

does not fork over B hence $f(C_1 \cup C_2)/A$ does not fork over B but $f \supseteq \mathrm{id}_A$ so also $C_1 \cup C_2/A$ does not split strongly over B. $\blacksquare_{5.10}$

5.11. CONCLUSION: 1) If M is $|B|^+$-saturated and $B \subseteq M$ and $p \in \mathbf{S}^n(M)$ then p does not fork over B if and only if p does not strongly split over B.

2) If $A = |M|$, then in Conclusion 5.9(1) we can replace strong splitting by dividing.

Proof. 1) By 5.10. 2) By part (1). $\blacksquare_{5.11}$

5.12. *Definition:* 1) We say $\langle \bar{a}_t : t \in J \rangle$ is a non-forking sequence over (B, A) when $B \subseteq A$ and for every $t \in J$ we have

$$\mathrm{tp}(\bar{a}_t, A \cup \bigcup \{\bar{a}_s : s <_J t\})$$

does not fork over B.

2) We say that $\langle \bar{a}_t : t \in J \rangle$ is a strict non-forking sequence over (B_1, B_2, A) if $B_1 \subseteq B_2 \subseteq A$ and for every $t \in J$ the type $\mathrm{tp}(\bar{a}_t, A \cup \bigcup \{\bar{a}_s : s <_J t\})$ strictly does not fork over (B_1, B_2), see Definition 5.6. If $B_1 = B_2$ we may write (B_1, A) instead of (B_1, B_1, A).

3) We say $\mathscr{A} = (A, \langle (\bar{a}_\alpha, B_\alpha) : \alpha < \alpha^* \rangle)$ is an $\mathbf{F}^f_\kappa$-construction or $\langle (\bar{a}_\alpha, B_\alpha) : \alpha < \alpha^* \rangle$ an $\mathbf{F}^f_\kappa$-construction over A if $B_\alpha \subseteq A_\alpha := A \cup \bigcup \{\bar{a}_\beta : \beta < \alpha\}$ has cardinality $< \kappa$ and $\mathrm{tp}(\bar{a}_\alpha, A_\alpha)$ does not fork over B_α.

4) We can above replace $\bar{a}_t$ by A_t meaning for some/every $\bar{a}_t$ listing A_t the demand holds.

5.13. CLAIM: 1) *Assume*

(a) *$\langle \bar{a}_t : t \in J \rangle$ is a strictly non-forking sequence over (B, B, A);*

(b) *$\langle \bar{b}_{t,n} : n < \omega \rangle$ is an indiscernible sequence over B each $\bar{b}_{t,n}$ realizing $\mathrm{tp}(\bar{a}_t, B)$ for each $t \in J$.*

Then we can find $\bar{a}_{t,n}$ for $t \in J, n < \omega$ such that

(α) *$\langle \bar{a}_{t,n} : n < \omega \rangle$ is an indiscernible sequence over $A \cup \{\bar{a}_{s,n} : n < \omega, s \in J \backslash \{t\}\}$;*

(β) *$\mathrm{tp}(\langle \bar{b}_{t,n} : n < \omega \rangle, B) = \mathrm{tp}(\langle \bar{a}_{t,n} : n < \omega \rangle, B)$;*

(γ) *$\bar{a}_{t,0} = \bar{a}_t$.*

Proof. We prove by induction on $|J|$.

CASE 1: J is finite.

We prove this by induction on n, for $n = 0, 1$ this is trivial; assume we have proved for n and we shall prove for $n+1$. Let $\lambda = (|A| + |T|)^+$.

So let $J = \{t_\ell : \ell \leq n\}$ t_ℓ increasing with ℓ. First we can find an indiscernible sequence $\langle \bar{a}_{t_0,\alpha} : \alpha < \lambda\rangle$ over A such that $\bar{a}_{t_0,0} = \bar{a}_{t_0}$ and for some automorphism F of $\mathfrak{C}$ over B we have $k < \omega \Rightarrow F(\bar{b}_{t_0,k}) = \bar{a}_{t_0,k}$. Let $A' := A \cup \{\bar{a}_{t_0,\alpha} : \alpha < \lambda\}$. (This is possible by Definition 5.6.)

Second, we can choose $\bar{a}'_{t_\ell}$, by induction on ℓ, such that $\bar{a}'_{t_0} = \bar{a}_{t_0}$ and if $\ell > 0$ then $\mathrm{tp}(\bar{a}'_{t_\ell}, A' \cup \bigcup\{\bar{a}'_{t_m} : m = 1, \dots, \ell-1\})$ strictly does not fork over B and the two sequences $\bar{a}_{t_0}{}^\frown \dots {}^\frown \bar{a}_{t_\ell}, \bar{a}'_{t_0}{}^\frown \dots {}^\frown \bar{a}'_{t_\ell}$ realizes the same type over A. We can do it by 5.6(5) and "strictly does not fork" being preserved by elementary mapping. By 5.10(2) the type $\mathrm{tp}(\bar{a}'_{t_1}{}^\frown \dots {}^\frown \bar{a}'_{t_n}, A'\})$ does not fork over B hence by 5.4(6) the sequence $\langle \bar{a}_{t_0,\alpha} : \alpha < \lambda\rangle$ is an indiscernible sequence over $A \cup (\bar{a}'_{t_1}{}^\frown \dots {}^\frown \bar{a}'_{t_n})$. As $\mathrm{tp}(\bar{a}_{t_\ell}, A \cup \{\bar{a}_{t_m} : m < i\})$ strictly does not fork over (B, A) without loss of generality $\langle \bar{b}_{t_\ell,m} : m < \omega\rangle$ is an indiscernible sequence over A' such that each $\bar{b}_{t_\ell,m}$ realizes $\mathrm{tp}(\bar{a}'_{t_\ell}, A)$.

Now we use the induction hypothesis with $B, A', \langle \bar{a}'_{t_\ell} : \ell = 1, \dots, n\rangle, \langle \bar{b}_{t_\ell,m} : m < \omega\rangle$ for $\ell = 1, \dots, n$ and let $\langle \bar{a}'_{t_\ell,n} : n < \omega\rangle$ for $\ell = 1, \dots, n$ be as in the claim.

By [Sh:715] for some $\alpha^* < \lambda$ the sequence $\langle \bar{a}'_{t_0,\alpha} : \alpha \in [\alpha^*, \alpha^* + \omega)\rangle$ is an indiscernible sequence over $A \cup \bigcup\{\bar{a}'_{t_\ell,m} : m < \omega, \ell = 1, \dots, n\}$ and as $A' = A \cup \{\bar{a}'_{t_0,\alpha} : \alpha < \lambda\}$ clearly for $\ell = 1, \dots, n$ the sequence $\langle \bar{a}'_{t_\ell,m} : m < \omega\rangle$ is indiscernible over $A \cup \bigcup\{\bar{a}'_{t_k,m} : k \in \{1, \dots, n\} \backslash \{\ell\}$ and $m < \omega\} \cup \bigcup\{\bar{a}'_{\alpha^*+m} : m < \omega\}$. But we know that $\langle \bar{a}'_{t_0,\alpha} : \alpha < \alpha^* + \omega\rangle$ is an indiscernible sequence over $A \cup \{\bar{a}'_{t_\ell} : \ell = 1, \dots, n\}$, hence the sequence $\bar{a}'_{t_\alpha,\alpha^*}{}^\frown \bar{a}'_{t_1}{}^\frown \dots {}^\frown \bar{a}'_{t_n}$ realizes over A the same type as $\bar{a}'_{t_0,0}{}^\frown \bar{a}'_{t_1}{}^\frown \dots {}^\frown \bar{a}'_{t_n}$ hence it realizes over A the same type as $\bar{a}_{t_0}{}^\frown \dots {}^\frown \bar{a}_{t_n}$ over A. So for some automorphism F of $\mathfrak{C}, F \restriction A = \mathrm{id}_A, \ell = 1, \dots, n \Rightarrow \bar{a}_{t_\ell} = F(\bar{a}'_{t_\ell,0})$ and $\bar{a}_{t_0} = F(\bar{a}'_{t_0,\alpha^*})$ and let $\bar{a}_{t_\ell,m} = F(\bar{a}'_{t_\ell,m})$ for $\ell = 1, \dots, n$ and $m < \omega$ and $\bar{a}_{t_0,m} = F(\bar{a}'_{t_0,\alpha^*+m})$.

So we are done.

Case 2: J infinite.

By Case 1 + compactness. $\blacksquare_{5.13}$

Remark: Can we use just no dividing?

5.14. Claim: 1) *Assume* $\langle A_t : t \in J\rangle$ *is a non-forking sequence over* (B, A) *and* $C_t \subseteq \mathfrak{C}$ *for* $t \in J$. *Then we can find* $\langle f_t : t \in J\rangle$ *such that*

(a) f_t *is an elementary mapping with domain*

$$A \cup A_t \cup C_t;$$

(b) $f_t \restriction (A \cup A_t)$ *is the identity;*

(c) $\mathrm{tp}(A_t, A \cup \bigcup\{A_s \cup f_s(C_s) : s < t\}$ *does not fork over* B.

2) *If, in addition,* $\mathrm{tp}(C_t, A \cup A_t)$ *does not fork over* $A \cup A_t$, *then we can add*

(c)$^+$ $\langle A_t \cup f_t(C_t) : t \in J\rangle$ *is a non-forking sequence over* (B, A).

5.15. *Remark:* 1) We may consider $\mathbf{F}^f$-construction, i.e., $\mathscr{A} = (A, \langle a_\alpha^{B_i} : \alpha < \alpha^*\rangle)$ is an $\mathbf{F}^f$-construction, when

(a) $B_i \subseteq A_i := A \cup \{a_j : j < i\}$;

(b) $\mathrm{tp}(a_i, A_i)$ does not fork over B_i;

(c) $|B_i| < \kappa$.

1A) We may replace α above by a linear order I, not necessarily well-founded.

2) In 5.14(2) we may weaken the assumption to: for every $A' \supseteq A$, $A_t \cup C_t/A$ can be embedded to a complete non-forking type over A'.

Proof. 1) As in the proof of 5.8.

2) Similarly. ∎

5.16. Claim: 1) *Assume*

(a) $\langle A_t : t \in J\rangle$ *is a non-forking sequence over* (B, A).

Then for any initial segment I *of* J, $\mathrm{tp}(\bigcup\{A_t : t \in J\backslash I\}, A \cup \bigcup\{A_t : t \in I\})$ *does not fork over* B.

2) *Assume (a) and*

(*b*) $\langle \bar{a}_{t,n} : n < \omega\rangle$ *is an indiscernible sequence over* A;

(*c*) $\bar{a}_{t,n} \in {}^{\omega>}(A_t)$;

(*d*) $\langle \bar{a}_{t,n} : n < \omega\rangle$ *is an indiscernible sequence over* $A \cup \bigcup\{A_s : s <_J t\}$.

Then $\langle\langle \bar{a}_{t,n} : n < \omega\rangle : t \in J\rangle$ *are mutually indiscernible over* A. *Also for any non-zero* $k < \omega$ *and* $t_0 < \cdots < t_{k-1}$ *in* J *the sequences* $\langle \bar{a}_{t_\ell,n} : n < \omega\rangle$ *for all* $\ell < k$ *are mutually indiscernible over* $A \cup \bigcup\{A_s : \neg(t_0 \leq s \leq t_{k-1})\}$.

5.17. QUESTION: If $n_\ell < \omega$ for $\ell < n$ do the sequences

$$\langle \bar a_{t_0,n_0} \hat{}\, \bar a_{t_1,n_1} \hat{}\, \ldots \hat{}\, \bar a_{t_{k-1},n_{k-1}} \rangle \quad \text{and} \quad \langle \bar a_{t_0,0} \hat{}\, \bar a_{t_1,0} \hat{}\, \ldots \hat{}\, \bar a_{t_{k-1},0} \rangle$$

realize the same type over $A \cup \bigcup\{A_s : s <_J t_0 \text{ or } s_J > t_{k-1}\}$. Need less?

Remark: A statement similar to 5.16(1) for $\mathbf{F}^f_\kappa$-construction holds.

Proof. 1) If $J\backslash I$ is finite, we prove this by induction on $|J\backslash I|$ using 5.10(2). The general case follows by 5.4(7).

2) It is enough to prove the second sentence. For $k = 1$ this follows by 5.4(6) and 5.10(2) using part (1) with $A \cup \bigcup\{A_s : s < t\}, \langle A_r : r \in J, r > t\rangle$ instead $A, \langle A_r : r \in J\rangle$.

For $k + 1 > 1$, let $t_0 <_J \cdots <_J t_k$ be given. Use the case $k = 1$ for each t_ℓ and combine. $\blacksquare_{5.16}$

Recall by 5.10

5.18. *Remark:* 1) Recall that by 5.10 if $p \in \mathbf{S}^m(M)$, M is quite saturated, then dividing is the same as forking for the type p.

5.19. CLAIM: *Assume that for every set B, if $p(\bar x) \in \mathbf{S}^m(B)$ then p does not fork over B. Assume that $\langle \bar a_t : t \in J\rangle$ is a non-forking sequence over (B, A) and $A = |M|$.*

1) *For every (finite sequence) $\bar b$ the set $\{t : \bar b/(A \cup \bar a_t)$ forks over $\bigcup_{s<t} a_s \cup A$ or if A is a model over $A\}$ has cardinality $\leq |T|$.*

2) *For each $\varphi(\bar x, \bar y, \bar z)$ and $k < \omega$ for some $n = n_{\varphi(\bar x,\bar y),k}$ the set $W^\varphi_{\bar b} :=$ $\{t : \mathrm{tp}_\varphi(\bar b, A \cup \bar a_t)$ has a subset with $\leq k$ members which forks over $\cup_{s<\tau} \bar a_s \cup A\}$ has $\leq n$ members.*

Proof. 1) By (2).

2) Fix k. Assume toward contradiction that this fails for n. We can find $t_0 <_I t_1 <_J \cdots <_J t_{n-1}$ from $W^\varphi_{\bar b}$.

Now, for every $u \subseteq \{0, \ldots, n-1\}$ there is $\bar b_u$ realizing $\mathrm{tp}(\bar b, A \cup \{\bar a_{t_\ell} : \ell \in u\})$ such that $\mathrm{tp}(\bar b_u, \mathrm{A} \cup \{\bar a_{t_\ell} : \ell < n\})$ does not fork over $A \cup \{\bar b_{t_\ell} : \ell \in u\}$. For each $\ell < n$ we can find $q_\ell \subseteq \mathrm{tp}_\varphi(\bar b, A \cup \bar a_{t_\ell})$ and with $\leq k$ members which forks over A. Let $A_\ell = A \cup \bar a_{t_0} \cup \cdots \cup \bar a_{t_{\ell-1}}$. Clearly, $\ell \in u \Rightarrow q_\ell \subseteq \mathrm{tp}(\bar b_u, A \cup \{\bar a_{t_m} : m < n\})$. If $\ell \in n\backslash u$, let $i_{\ell,0} < \cdots < i_{\ell,m(\ell)-1} < n$ list $u\backslash\ell$ so $\mathrm{tp}(\bar a_{i_{\ell,m}}, A_\ell \cup \bar a_{t_\ell} \cup \bar a_{t_{i_{\ell,0}}} \cdots \cup \bar a_{t_{i_{\ell,m-1}}})$ does not fork over A for $m < m(\ell)$ and $\mathrm{tp}(\bar b_u, A_\ell \cup \bar a_{t_\ell} \cup \bar a_{t_{i_{\ell,0}}} \cup \cdots \cup \bar a_{t_{i_{\ell,m(\ell)-1}}})$ does not fork over $A \cup \{\bar a_{t_k} : k \in u\} \subseteq$

$A_\ell \cup \{\bar{a}_{t_{i_{\ell,0}}}, \cdots, \bar{a}_{t_{i_{\ell,m(\ell)-1}}}\}$ hence by 5.10(2) + 5.4(0) $\mathrm{tp}(\bar{b}_u, \mathrm{A} \cup \bar{a}_{t_\ell})$ does not fork over A_ℓ, hence $\neg \wedge q_\ell$ belongs to it. As for our fixed k this holds for every n, we get that T has the independence property contradiction. $\blacksquare_{5.19}$

5.20. Claim: *Assume that $p(\bar{x})$ is a type of cardinality $< \kappa$ which does not fork over A. then for some $B \subseteq A$ of cardinality $< \kappa + |T|^+$, the type p does not fork over B.*

Proof. Without loss of generality p is closed under conjunction.

For any finite sequence $\bar{\varphi} = \langle (\varphi_\ell(\bar{x}, \bar{y}_\ell), n_\ell) : \ell < n\rangle$ and formula $\psi(x, \bar{c}) \in p$ and set $B \subseteq A$ we define

$$\begin{aligned}\Gamma_{B,\bar{\varphi},\psi(\bar{x},\bar{c})} = \{(\forall x)(\psi(x,\bar{c}) \rightarrow \bigvee_{\ell<n} \varphi_\ell(\bar{x}, \bar{y}_{\ell,0}))\}& \\ \cup \{\neg(\exists \bar{x}) \bigwedge_{n \in w} \varphi_\ell(\bar{x}, y_{\ell,n}) : \ell < n \text{ and } w \in [\omega]^{n_\ell}\}& \\ \cup \{\vartheta(y_{\ell,m_1}, \ldots, y_{\ell,m_k}, \bar{b}) = \vartheta(y_{\ell,0}, \ldots, y_{\ell,k}, \bar{b}) :& \\ \bar{b} \subseteq B, \vartheta \in \mathbb{L}(\tau_T), m_1 < \cdots < m_k < \omega\}.&\end{aligned}$$

Now as p does not fork over A, clearly for any $\bar{\varphi}$ as above and $\psi(\bar{x}, \bar{c}) \in p$ the set $\Gamma_{A,\bar{\varphi},\psi(\bar{x},\bar{c})}$ is inconsistent. Hence for some finite set $B = B_{\bar{\varphi},\psi(x,\bar{c})} \subseteq A$ the set $\Gamma_{B,\bar{\varphi},\psi(x,\bar{c})}$ is inconsistent. Now $B^* = \bigcup\{B_{\bar{\varphi},\psi(\bar{x},\bar{c})} : \psi(\bar{x}, \bar{c}) \in p$ and $\bar{\varphi}$ is as above$\}$ is as required. $\blacksquare_{5.20}$

The following is another substitute for "every type p does not fork over a small subset of $\mathrm{Dom}(p)$".

5.21. Claim: *Assume that for every set B, if $p \in \mathbf{S}^{<\omega}(B)$ then p does not fork over B. Assume $p \in \mathbf{S}^m(M)$ and $B \subseteq M$. Then we can find C such that*

$(*)_1$ *$C \subseteq M$ and $|C| \leq |T|$ and*

$(*)^p_{M,B,C}$ *if $D \subseteq M$ and $\mathrm{tp}(D/B \cup C)$ does not fork over B, then $p \restriction (B \cup D)$ does not fork over $B \cup C$.*

Proof. Follows by 5.19. $\blacksquare$

* * *

5.22. *Definition:* Assume that $C = |M|, M$ is κ-saturated $A \subseteq M, |A| < \kappa$ and $p \in \mathbf{S}^m(M)$ does not split over A. For any set $B (\subseteq \mathfrak{C})$ let $p^{[A,B]}$ be $q \restriction B$, where $q \in \mathbf{S}^m(M \cup B)$ is the unique type in $\mathbf{S}^m(M \cup B)$ which does not split over A.

5.23. Observation: 1) In Definition 5.22, $p^{[A,B]}$ is well-defined.

2) In 5.22 instead "C is κ^+-saturated; $|A| < \kappa$" it suffices to assume that every $q \in \mathbf{S}^{<\omega}(B)$ is realized in C.

3) $p^{[A,B_1]} \subseteq p^{[A,B_2]}$ if $B_1 \subseteq B_2$.

5.24. Claim: 1) *If the triple (A, C, p) is as in 5.23(2), $A \subseteq A_0$ and $\bar{a}_n \in {}^m\mathfrak{C}$ realizes $p^{[A,A_n]}$ for $n < \omega$ where $A_n = A_0 \cup \bigcup\{\bar{a}_\ell : \ell < n\}$, then $\langle \bar{a}_n : n < \omega\rangle$ is an indiscernible sequence over A_0. Also $\mathrm{tp}(\langle \bar{a}_n : n < \omega\rangle, A_0)$ is determined by (A, C, p, A_0) and we call it $p^{[A,A_o,\omega]}$.*

Proof. See [Sh:c, II,§1] or [Sh:3]. ∎

5.25. Claim: *Assume that*

(a) *(C, A) is as in 5.23(2)*
(b) *$p_0, p_1 \in \mathbf{S}^m(C)$ does not split over A*
(c) *$p_0^{[A,A,\omega]} = p_1^{[A,A,\omega]}$.*

Then $p_0 = p_1$.

Proof. Let $\langle \bar{a}^\ell_n : n < \omega\rangle$ realize $p_\ell^{[A,A,\omega]}$ so by clause (c) of the assumption

$(*)_1$ $\bar{a}^0_0, \dots, \bar{a}^0_{n-1}$ and $\bar{a}^1_0{}^\frown \dots {}^\frown \bar{a}^1_{n-1}$ realizes the same type over A.

If the conclusion fails, we can find $\bar{c}$ and $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_T)$ such that

$(*)_2$ $\neg\varphi(\bar{x}, \bar{c}) \in p_0$ and $\varphi(\bar{x}, \bar{c}) \in p_1$ so $\bar{c} \in {}^{\ell g(\bar{y})}C$.

Now we choose by induction on n a sequence $\bar{a}_n$ such that

$(*)_3$ if $\ell < 2$ and $n = \ell$ mod 2 and we let $A_n = A \cup \bigcup\{\bar{a}_n, \dots, \bar{a}_{n-1}\}$ then $\mathrm{tp}(\bar{a}_n, A_{k,n} \cup \bar{c}) = p_\ell^{[A,A_n \cup \bar{c}]}$.

Now we can prove by induction on $n < \omega$ that

$(*)_4$ the sequences $\bar{a}^0_0{}^\frown \dots {}^\frown a^0_{n-1}, \bar{a}^1_0{}^\frown \dots {}^\frown \bar{a}^1_{n-1}$ and $\bar{a}_0{}^\frown \dots {}^\frown \bar{a}_{n-1}$ realizes the same type over A.

(Why? The first two sequences realizes the same type by $(*)_1$. For the induction step, if $n = \ell$ mod 2, by the definition 5.22, we have $\bar{a}^\ell_0{}^\frown \dots {}^\frown \bar{a}^\ell_{n-1}{}^\frown \bar{a}^\ell_n$ and $\bar{a}_0{}^\frown \dots {}^\frown \bar{a}_{n-1}{}^\frown \bar{a}_n$ realizes the same type over A.)

So $\langle \bar{a}_n : n < \omega\rangle$ is an indiscernible sequence and $\mathfrak{C} \models \varphi[\bar{a}_n, \bar{c}]$ if and only if n is odd, contradiction to "T dependent". $\blacksquare_{5.25}$

5.26. Conclusion: 1) If $A \subseteq C$ and every $p \in \mathbf{S}^{<\omega}(A)$ is realized in C then $\{p \in \mathbf{S}^m(C) : p$ does not split over $A\}$ has cardinality $\leq |\mathbf{S}^\omega(A)|$ which is

$\leq (\mathrm{Ded}_r(|A| + |T|))^{|T|} \leq 2^{|A|+|T|}$ recalling $\mathrm{Ded}_r(\mu) = \mathrm{Min}\{\lambda : \lambda$ is regular and every linear order of density $\leq \mu$ has cardinality $\leq \lambda\}$.

2) Also, for any finite $\Delta \subseteq \mathbb{L}(\tau_T)$, the set $\{p \restriction \Delta : p \in \mathbf{S}^m(C)$ does not split over $A\}$ has cardinality $\leq \mathrm{Ded}_r(C)$.

3) If $p \in \mathbf{S}^m(C)$ is finitely satisfiable in $A \subseteq C$ then p does not split over A.

Proof. Should be clear. ∎

* * *

5.27. *Definition:* For $\ell \in \{1, 2\}$, we say $\{\bar{a}_\alpha : \alpha < \alpha^*\}$ is ℓ-independent over A if we can find $\bar{a}_{\alpha,n}$ (for $\alpha < \alpha^*, n < \omega\rangle$ such that:

(a) $\bar{a}_\alpha = \bar{a}_{\alpha,0}$;

(b) $\langle \bar{a}_{\alpha,n} : n < \omega\rangle$ is an indiscernible sequence over $A \cup \bigcup\{\bar{a}_{\beta,m} : \beta \in \alpha^* \setminus \{\alpha\}$ and $m < \omega\}$;

(c) (α) if $\ell = 1$, then for some $\bar{b}_n \in A$ $(n < \omega)$ for every $\alpha < \alpha^*$ we have $\langle \bar{b}_n : n < \omega\rangle\hat{\ }\langle \bar{a}_{\alpha,n} : n < \omega\rangle$ is an indiscernible sequence;

(β) if $\ell = 2$, then for some $\bar{b}_{\alpha,n} \subseteq A$ (for $\alpha < \alpha^*, n < \omega$),

$$\langle \bar{b}_{\alpha,n} : n < \omega\rangle\hat{\ }\langle \bar{a}_{\alpha,n} : n < \omega\rangle$$

is an indiscernible sequence.

We now show that even a very weak version of independence has limitations.

5.28. Claim: 1) *For every finite $\Delta \subseteq \mathbb{L}(\tau_T)$ there is $n^* < \omega$ such that we cannot find $\bar{\varphi} = \langle \varphi_n(\bar{x}, \bar{a}_n) : n < n^*\rangle$ for which*

$(*)_{\bar{\varphi}}$ *for each $n < n^*$ there are $m_n < \omega$ and $\langle \bar{b}^n_{m,\ell} : \ell < \omega, m < m_n\rangle$ and $\langle \psi^n_m(\bar{x}, \bar{y}_n) : m < m_n\rangle$ such that*

(α) *$\langle \bar{b}^n_{m,\ell} : \ell < \omega\rangle$ is an indiscernible sequence over $\cup\{\bar{a}_k : k < n^*, k \neq n\}$;*

(β) *$\bar{b}^n_{m,0} = \bar{a}_n$;*

(γ) *$\{\psi^n_m(x, \bar{b}^n_{m,\ell}) : \ell < \omega\}$ is contradictory for each n and $m < m_n$;*

(δ) *$\psi^n_m(\bar{x}, \bar{y}_n) \in \Delta$;*

(ε) *$\varphi_n(\bar{x}, \bar{a}_n) \vdash \bigvee_{m<m_n} \psi^n_m(\bar{x}, \bar{a}_n)$;*

(ζ) *$\models (\exists \bar{x}) \bigwedge_{n<n^*} \varphi_n(\bar{x}, \bar{a}_n)$.*

2) *We weaken (α) above to $\mathrm{tp}(\bar{b}^n_{m,\ell}, \bigcup\{\bar{a}_k : k < n^*, k \neq n\}) = \mathrm{tp}(\bar{a}_n, \bigcup\{\bar{a}_k : k < n^*, k \neq n\})$.*

3) *For some finite* $\Delta^+ \subseteq \mathbb{L}(\tau_T)$, *we can in* (α) *demand only* Δ^+*-indiscernible; also without loss of generality* $\varphi_n(\bar{x}, \bar{y}_n) = \bigvee_{m<m_n} \psi^n_m(\bar{x}, \bar{y}_n)$.

Proof. 1) [Close to 5.8.] Note

$\circledast$ if $\bar{c} \in {}^{\ell g(\bar{x})}(\mathfrak{C})$ and $n < n^*$ and $\models \varphi_n(\bar{c}, \bar{a}_n)$ then for some $\bar{c}' \in {}^{\ell g(\bar{x})}(\mathfrak{C})$ we have

(i) $\mathrm{tp}(\bar{c}', \bigcup\{\bar{a}_k : k < n^*, k \neq n\}) = \mathrm{tp}(\bar{c}, \bigcup\{\bar{a}_k : k < n^*, k \neq n\})$;

(ii) $\mathrm{tp}_\Delta(\bar{c}, \bar{a}_n) \neq \mathrm{tp}_\Delta(\bar{c}', \bar{a}_n)$.

(Why $\circledast$ holds? Clearly it is enough to find $\bar{b}'_n$ such that

(i) $\bar{b}_n, \bar{b}'_n$ realize the same type over $\bigcup\{\bar{a}_k : k < n^*, k \neq n\}$

(ii) for some $m < m_n$ we have $\psi^n_m(\bar{b}_n, \bar{a}_n) \wedge \neg\psi^n_m(\bar{b}'_n, \bar{a}_n)$.

Why does $\bar{b}'_n$ exist? As $\models \varphi_n[\bar{c}, \bar{a}_n]$ by (ε) for some $m < m_n, \models \psi^n_n[\bar{c}, \bar{a}_n]$ and by $(\alpha) + (\gamma)$, for some $\ell < \omega, b'_n = \bar{b}^n_{m,\ell}$ is as required.)

By repeated use of $\circledast$ we get $m^*_\ell < m_\ell$ such that $\langle \psi^n_{m^*_\ell}(\bar{x}, \bar{a}_n) : n < n^* \rangle$ is independent but $\psi^n_{m^*_\ell}(\bar{x}, \bar{y}_n) \in \Delta$ is finite, so n^* as required exists.

2),3) Similarly. $\blacksquare_{5.28}$

5.29. Claim: *Assume*

(a) $\langle \bar{b}_n : n < \omega \rangle$ *is indiscernible over* M*;*

(b) $\{\varphi(\bar{x}, \bar{b}_n) : n < \omega\}$ *is contradictory;*

(c) $M \prec N, p \in \mathbf{S}(N), \varphi(\bar{x}, \bar{b}_0) \in p$ *and* $\neg\varphi(x, \bar{b}_n) \in p$ *for* $n > 0$*;*

(d) N *is* $\|M\|^+$*-saturated.*

then for some $\langle b'_n : n < \omega \rangle$ *we have*

(α) $\langle \bar{b}'_n : n < \omega \rangle$ *is indiscernible over* M *based on* M, $\bar{b}'_n \subseteq N$*;*

(β) $\bar{b}'_0 \in \{\bar{b}_0, \bar{b}_1\}$*;*

(γ) $\varphi(\bar{x}, \bar{b}'_0) \equiv \neg\varphi(\bar{x}, \bar{b}'_1)$ *belongs to* p.

5.30. *Definition:* 1) For $p \in \mathbf{S}^m(M)$ let $\mathscr{E}(p)$ be the set of pairs $(\varphi(\bar{x}, \bar{y}), \mathbf{e})$ such that

(a) $\mathbf{e}$ is a definable equivalence relation on ${}^{\ell g(\bar{y})}M$ in M

(b) if $\bar{b}_1 \mathbf{e} \bar{b}_2$ then $\varphi(\bar{x}, \bar{b}_1) \in p \Leftrightarrow \varphi(\bar{x}, \bar{b}_2) \in p$.

2) $\mathscr{E}'_{\mathrm{tp}}(p)$ is defined similarly by $\mathbf{e}$ is definable by types.

5.31. Claim: *Assume* $\varphi = \varphi(x, y), M \prec N, N$ *is* $\|M\|^+$*-saturated and* $p \in \mathbf{S}(N)$. *Then we cannot find* $\{D_i : i < n_\varphi\}$*, a set of ultrafilters over* (N) *pairwise* *orthogonal (as below) with* $p_i = \mathrm{Av}(M, D_i)$ *such that* $p(x) \cup p_i(\bar{y}_0) \cup p_i(\bar{y}_1) \cup \{\varphi(x, \bar{y}_1), \neg\varphi(x, \bar{y}_0)\}$ *is consistent for* $i < n_\varphi$.

Now we deal with orthogonality.

5.32. *Definition:* Definition 1) Two complete types $p(\bar{x}), q(\bar{y})$ over A are weakly orthogonal if $p(\bar{x}) \cup q(\bar{y})$ is a complete type over A.

2) Assume $\bar{\mathbf{b}}_1, \bar{\mathbf{b}}_2$ are endless indiscernible sequences. We say $\bar{\mathbf{b}}_1, \bar{\mathbf{b}}_2$ are orthogonal and write $\bar{\mathbf{b}}_1 \perp \bar{\mathbf{b}}_2$ if:

> for every set A which includes $\bar{\mathbf{b}}_1 \cup \bar{\mathbf{b}}_2$, $\mathrm{Av}(A, \bar{\mathbf{b}}_1)$, $\mathrm{Av}(A, \bar{\mathbf{b}}_2)$ are weakly orthogonal

3) $\bar{\mathbf{b}}_1$ is strongly orthogonal to $\bar{\mathbf{b}}_2, \mathbf{b}_1 \underset{\mathrm{st}}{\perp} \mathbf{b}_2$ if it is orthogonal to every endless indiscernible sequence $\bar{\mathbf{b}}_2'$ of finite distance from $\bar{\mathbf{b}}_2$ (see [Sh:715, 1.11](2).

4) An endless indiscernible sequence $\bar{\mathbf{b}}_1$ is orthogonal to $\varphi(x, \bar{a})$ if it is orthogonal to every endless indiscernible sequence $\bar{\mathbf{b}}_2 = \langle b_{2,\alpha} : \alpha < \delta \rangle$ such that $b_{2,\alpha} \in \varphi(\mathfrak{C}, \bar{a})$ for every $\alpha < \delta$.

5) $\bar{\mathbf{b}}$ is based on A if $\bar{\mathbf{b}}$ is an indiscernible sequence and $C_A(\bar{\mathbf{b}})$ (see [Sh:715] or [Sh:93]) has boundedly many conjugations over A.

6) If $\bar{\mathbf{b}}_1 {}_{\mathrm{st}} \rightarrow\perp \bar{\mathbf{b}}_2$ and $\bar{\mathbf{b}}_\ell'$ is a neighbour (see [Sh:715, 1.11=np1.4B]) to $\bar{\mathbf{b}}_\ell$ then $\bar{\mathbf{b}}_1'$ is strongly orthogonal to $\bar{\mathbf{b}}_2'$.

5.33. Claim: 1) *Orthogonality is symmetric relation.*

2) *If* $\mathbf{b}_1, \mathbf{b}_2$ *are orthogonal, then they are perpendicular (see Definition 2.2).*

5.34. *Example:* In $\mathrm{Th}(\mathbb{R}, <)$, different initial segments are orthogonal, even two disjoint intervals. For $(\mathbb{R}, 0, 1, +, \times)$ the situation is different: any two non trivial intervals are "the same".

5.35. Claim: 1) *Assume* $\lambda = \lambda^{<\lambda}, I$ *is a dense linear order with neither first nor last element and* $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I \rangle$ *an indiscernible sequence. If* $|I| = \lambda$, *then there is* $M \supseteq \bar{\mathbf{b}}$ *which is* λ*-saturated and* λ*-atomic over* $\bar{\mathbf{b}}$.

2) *If* $p \in \mathbf{S}^m(\bar{\mathbf{b}})$ *is* λ*-isolated then it is* $|T|^+$*-isolated.*

5.36. Question: If $\mathrm{Av}(M, \bar{\mathbf{b}}_1)$, $\mathrm{Av}(M, \bar{\mathbf{b}}_2)$ (or with D's) are weakly orthogonal and are perpendicular, then they are orthogonal.

5.37. Question: On $\mathrm{Av}(\bar{\mathbf{b}}_1, \bar{\mathbf{b}}), \bar{\mathbf{b}}$ endless indiscernible sequence, can we define a dependence relation exhausting the amount of indiscernible sets like dependence?

5.38. Question: For each of the following conditions can we characterize the dependent theories which satisfy it?

(a) for any two non-trivial indiscernible sequences $\bar{\mathbf{b}}_1, \bar{\mathbf{b}}_2$, we can find $\bar{\mathbf{b}}'_\ell$ of finite distance from $\bar{\mathbf{b}}_\ell$ (see [Sh:715], for $\ell = 1, 2$) such that $\bar{\mathbf{b}}'_1, \bar{\mathbf{b}}'_2$ are not orthogonal

(b) any two non-trivial indiscernible sequences of singletons have finite distance?

(c) T is $\mathrm{Th}(\mathbb{F}), \mathbb{F}$ a field (so this class includes the p-adics various reasonable fields with valuations and closed under finite extensions).

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