CLASSIFYING GENERALIZED QUANTIFIERS

Abstract : Finding a universe $\mathcal{U}$ we prove that any quantifier ranging on a family of n-place relations over $\mathcal{U}$, is bi-expressible with a quantifier ranging over a family of equivalence relations, provided that $V=L$. Most of the analysis is carried assuming ZFC only and for a stronger equivalence relation, also we find independence results in the other direction.

Notation :

1) $\bar{b} \approx_A \bar{c}$ means $\bar{b}=\langle b_i : i<n \rangle, \bar{c}=\langle c_i : i<n \rangle$, and: a) $b_i \in A$ iff $c_i \in A$, b) $b_i \in A$ implies $b_i = c_i$, c) $b_i = b_j$ iff $c_i = c_j$.

2) For a set Δ of $\varphi(\bar{x})$ (φ a formula, $\bar{x}$ a finite sequence of variables including all variables occurring freely in φ),

$$tp_\Delta(\bar{b}, A, M) = \{\varphi(\bar{x}, \bar{a}) : \varphi(\bar{x}, \bar{y}) \in \Delta,\ \bar{a} \subseteq A \ and \ M \models \varphi[\bar{b}, \bar{a}]\}$$

We omit M when its identity is clear, and when $M=(\mathcal{U}, R)$ write R instead of M. Replacing Δ by bs means $\Delta = \{\varphi(\bar{x}) : \varphi$ atomic or negation of atomic formula$\}$. We write φ instead $\{\varphi\}$, and Δ will be always finite.

3) $S_\Delta^m(A, M) = \{tp_\Delta(\bar{b}, A, M) : \bar{b} \subseteq M, l(\bar{b}) = m\}$

Introduction

In [Sh3] we gave a complete classification of a class of second order quantifiers: those which are first-order definable (see below an exact definition). We find that for infinite models up to a very strong notion of equivalence, biinterpretability, there are only four such quantifiers: first order, monadic, one-to-one partial functions, and second-order.

Our aim here is to see what occurs if we remove the restriction that the quantifier is first order definable. As we do not want to replace this by a specific $\mathcal{L}$-definable ($\mathcal{L}$-some logic) we restrict ourselves to a fix infinite universe $\mathcal{U}$. If we then want to restrict ourselves to $\mathcal{L}$-definable quantifiers, we will be able to remove the restriction to a fix universe $\mathcal{U}$.

Let us now make some conventions and definitions.

0.1 convention : 1) $\mathcal{U}$ will be a fix infinite universe

2) K will denote a family of n-place relation over $\mathcal{U}$, (for a natural number $n = n(K)$), closed under isomorphism, i.e. if R_1, R_2 are n-place relations on $\mathcal{U}$, $(\mathcal{U}, R_1) \cong (\mathcal{U}, R_2)$ then $R_1 \in K$ iff $R_2 \in K$.

3) Let $\bar{K}$ denote a finite sequence of such K's.

$$\bar{K} = \langle K_\ell : l < \ell(\bar{K}) \rangle, \quad \bar{K}_i^j = \langle K_{i,l}^j : l < l(\bar{K}_i) \rangle$$

4) Dom $R = \bigcup\{\bar{a} : \models R(\bar{a})\}$, $n = n(R)$ if R is an n-place relation (or predicate; we shall not strictly distinguish).

0.2 Definition : For any K, $\exists_K$ (or Q_K) denote a second order quantifier, intended to vary on members of K. More exactly, $L(\exists_{K_1}, \cdots, \exists_{K_m})$ is defined like the first order logic but we have for each $l = 1, m$ (infinitely many) variables R which serve as $n(K_l)$-place predicates, and we can form $(\exists_{K_l} R)\varphi$ for a formula φ. Defining satisfaction, we look only at models with universe $\mathcal{U}$, and $\models (\exists_{K_\ell} R)\varphi(R, \cdots)$ iff for some $R^0 \in K_\ell$, $\varphi(R^0, \cdots)$.

Remark : Note that quantifiers depending on parameters are not allowed. e.g. on automorphisms; on such quantifiers see [Sh4], [Sh5], [Sh6].

0.3 Definition : We say that K (or Q_K) is $\mathcal{L}$- definable ($\mathcal{L}$-a logic) if there is a formula $\varphi(R) \in \mathcal{L}$, R the only free variable of φ, and is appropriate, i.e. an n (K)-place predicate, such that for any n-place relation R on $\mathcal{U}$

$$(\mathcal{U}, R) \models \varphi(R) \text{ iff } R \in K$$

0.4 Definition : We say that $\exists_{K_1} \leq_{int} \exists_{K_2}$ ($\exists_{K_1}$ is interpretable in $\exists_{K_2}$) **iff** for some first-order formula $\vartheta(\bar{x}, \bar{S}) = \vartheta(x_0, \cdots, x_{n(K_1)-1}, S_0, \cdots, S_{n-1})$, (each S_l an $n(K_2)$-place predicate) the following holds:

(*) for every $R_1 \in K_1$ there are $S_0, \cdots, S_{n-1} \in K_2$ such that $(\mathcal{U}, S_0, \cdots, S_{m-1}) \models (\forall \bar{x})[R_1(\bar{x}) \equiv \vartheta(\bar{x}, S_0, \cdots, S_{m-1})]$

Remark : We can define $\exists_{K_1} \leq^{\mathcal{L}}_{int} \exists_{K_2}$ similarly, by letting $\vartheta \in \mathcal{L}$, but we have no need.

A weaker notion is

0.5 Definition : 1) We say that $\exists_{K_1} \leq_{\exp} \exists_{K_2}$ ($\exists_{K_1}$ is expressible by $\exists_{K_2}$) if there is a formula $\vartheta(\bar{x}, S_0, \cdots, S_{m-1})$ in the logic $L(\exists_{K_2})$ such that:

(*) for every $R_1 \in K_1$, there are $S_0, \cdots, S_{m-1} \in K_2$ such that $(\mathcal{U}, S_0, \cdots, S_{m-1}) \models (\forall \bar{x})[R_1(\bar{x}) \equiv \vartheta(\bar{x}, S_0, \cdots, S_{m-1})]$.

2) We say that $\exists_{K_1} \leq_{inex} \exists_{K_2}$ ($\exists_{K_2}$ is invariantly expressible by $\exists_{K_2}$) if there is a formula $\vartheta(\bar{x}, S_0, \cdots, S_{m-1})$ in the logic $L(\exists_{K_2})$ such that:

(*) for every $R_1 \in K_1$ there are $S_0, \ldots, S_{m-1} \in K_2$ such that for every K_3 which extends K_2, letting ϑ' is ϑ when we replace $\exists_{K_2}$ by $\exists K_3$:

$$(\mathcal{U}, S_0, \ldots, S_{m-1}) \models (\forall \bar{x})[R_1(\bar{x}) \equiv \vartheta'(\bar{x}, S_0, \ldots, S_{m-1})]$$

0.6 Definition : 1) We say $\exists_{K_1} \equiv_{int} \exists_{K_2}$ ($\exists_{K_1}, \exists_{K_2}$ are biinterpratable) if $\exists_{K_1} \leq_{int} \exists_{K_2}$ and $\exists_{K_2} \leq_{int} \exists_{K_1}$.

2) We say $\exists_{K_2} \equiv_{\exp} \exists_{K_2}$ ($\exists_{K_1}, \exists_{K_2}$ are biexpressible) if $\exists_{K_1} \leq_{\exp} \exists_{K_2}$ and $\exists_{K_2} \leq_{\exp} \exists_{K_1}$. Similarly for $\equiv_{inex}$: $\exists_{K_1} \equiv_{inex} \exists_{K_2}$ ($\exists_{K_2}, \exists_{K_1}$ are invariantly biexpressible) if $\exists_{K_1} \leq_{inex} \exists_{K_2}$ and $\exists_{K_2} \leq_{inex} \exists_{K_1}$.

3) We can define $\exists_{K_1} \leq_{int} \{\exists_{K_0}, \cdots, \exists_{K_{k-1}}\}$ as in Def. 0.4, but $S_0, \cdots, \in \bigcup_{i=1}^{k} K_i$, we let $\exists_{\bar{K}}$ stand for $\{ \exists_{K_0}, \cdots, \exists_{K_{k-1}} \}$ where $\bar{K} = \langle K_0, \cdots, K_{k-1} \rangle$; we define $\exists_{\bar{K}^1} \leq_{int} \exists_{\bar{K}^2}$ if $\exists_{K^1_l} \leq_{int} \exists_{\bar{K}^2}$ for each l ; we also define expressible , invariantly expressible, biinterpratable and (invariantly) biexpressible similarly.

0.7 Notation : 1) If R_l is an n_l-place relation let $\sum_{l=0}^{n-1} R_l = \{\bar{a}_o \frown \cdots \frown \bar{a}_{n-1} : \bar{a}_l \in R_l\}$.

2) Let $\sum_{l=0}^{n-1} K_l = \{ \sum_{l=0}^{n-1} R_l : R_l \in K_l \text{ for } l < n \}$.

3) $\exists_R$ stand for $\exists_K$ where $K = \{R_1 : (\mathcal{U}, R^1) \cong (\mathcal{U}, R \cdots)\}$.

0.8 Lemma : 1) $\leq_{int}, \leq_{inex}$ and $\leq_{\exp}$ are partial quasi orders, hence $\equiv_{int}$, $\equiv_{inex}$, $\equiv_{\exp}$ are equivalence relations.

2) $\exists_{\bar{K}_1} \leq_{int} \exists_{\bar{K}_2}$ implies $\exists_{\bar{K}_1} \leq_{inex} \exists_{\bar{K}_2}$ which implies $\exists_{\bar{K}_1} \leq_{\exp} \exists_{\bar{K}_2}$.

3) $\exists_{\bar{K}}$, and $\exists_K$ are biinterpratable if $K=\sum_i K_i$ or $K=\bigcup_i K_i$ ($n(K_i)$ constant in the second case).

0.9 Lemma : 1) If $\bar{K}_1$, $\bar{K}_2$ are $\mathcal{L}$ -definable (i.e. each $K_{l,i}$ is) and $\exists_{\bar{K}_1} \leq_{\exp} \exists_{\bar{K}_2}$ then we can recursively attach to every formula in $\mathcal{L}(\exists_{\bar{K}_1})$ an equivalent formula in $\mathcal{L}(\exists_{\bar{K}_2})$.

2) If $\bar{K}_1$, $\bar{K}_2$ are $\mathcal{L}$-definable, $\exists_{\bar{K}_1} \leq_{\exp} \exists_{\bar{K}_2}$ then the set of valid $\mathcal{L}(\cdots \exists_{\bar{K}_1})$-sentences in recursive in the set of valid $\mathcal{L}(\exists_{\bar{K}_2})$ -sentences.

Remark : The need of " $\mathcal{L}$-definable " is clearly necessary. Though at first glance the conclusions of 0.9 may seem the natural definition of interpratable, I think reflection will lead us to see it isn't.

0.10 Definition : 1) We say that $\exists_{\bar{K}_1} \leq_{int} \exists_{\bar{K}_2}$ for a family of pairs $(\bar{K}_1, \bar{K}_2)$, uniformly, if the formulas $\vartheta_l (l<l(\bar{K}_1))$ depend on the $n(K_{1,l})$, $n(K_{2,j})$, $(i<l(\bar{K}_1), j<l(\bar{K}_2))$ only. (Clearly if we have only finitely many candidates for ϑ_l , it does not matter).

2) We use similar notions for $\leq_{\exp}$, $\leq_{inex}$, $\equiv_{int}$, $\equiv_{\exp}$, $\equiv_{inex}$.

§1 On some specific quantifiers.

1.1 Definition : 1) Let $K_\lambda^{mon} = \{A \subseteq \mathcal{U}: |A|=\lambda \leq |\mathcal{U}-A|\}$

2) but we write Q_λ^{mon} for $\exists_{K_\lambda^{mon}}$, and similarly for the other quantifiers defined below.

3) $K_\lambda^{1-1} = \{f : f$ is a partial one-to-one function.
$|\mathrm{Dom}(f)| = \lambda \leq |\mathcal{U}-\mathrm{Dom}(f)-\mathit{Rang}(f)|\}$.

4) $K_{\lambda,\mu}^{eq} = \{E{:}E$ is an equivalence relation on some $A \subseteq \mathcal{U}$, with λ equivalence classes, each of power μ, and $|\mathcal{U}-A| = |\mathcal{U}|\}$.

5) For $\lambda^+ \geq \mu$, we let
$K_{\lambda,<\mu}^{*eq} = \{E{:}E$ is an equivalence relation, every equivalence class of E has power $<\mu$, for each $\kappa<\mu$, E has exactly λ equivalence classes of power κ, and $|\mathcal{U}-\mathrm{Dom}(E)| = |\mathcal{U}|\}$.

6) $K^{eq}_{\lambda,<\mu} = \{E: E$ is an equivalence relation, with λ equivalence classes, each of power $<\mu$ and $|\mathcal{U}-\text{Dom } E| = |\mathcal{U}|\}$.

7) $K^{mon}_{<\lambda} = \bigcup_{\mu<\lambda} K^{mon}_{\mu}$, $K^{1-1}_{<\lambda} = \bigcup_{\mu<\lambda} K^{1-1}_{\mu}$ and $K^{eq}_{<\lambda,<\mu} = \bigcup_{\chi<\lambda} K^{eq}_{\chi,<\mu}$ and $K^{*eq}_{\lambda,<\mu} = \bigcup_{\substack{\chi<\lambda \\ \chi^+\geq\mu}} K^{*eq}_{\chi,<\mu}$

of course, $K^{eq}_{<\lambda,\mu} = \bigcup_{\chi<\lambda} K^{eq}_{\chi,\mu}$, $K^{eq}_{\lambda} = K^{eq}_{\lambda,\lambda}$, $K^{eq}_{<\lambda} = K^{eq}_{<\lambda,<\lambda}$.

Remark : Of course, always $|\text{Dom } E| \leq |\mathcal{U}|$.

1.2 Claim : Let $\lambda \leq \chi$. All results are uniform.

1) $Q^{mon}_{\lambda} \equiv_{int} Q^{mon}_{<\lambda^+}$ and $Q^{mon}_{<\lambda} \leq_{int} Q^{mon}_{<\chi}$; Q^{mon}_{χ} is $\exists_R$ for some R ; and $Q^{mon}_{<\mu} \equiv_{int} Q^{eq}_{1,<\mu}$.

2) $Q^{1-1}_{\lambda} \equiv_{int} Q^{1-1}_{<\lambda^+}$; $Q^{1-1}_{<\lambda} \leq_{int} Q^{1-1}_{<\chi}$, and $Q^{mon}_{<\lambda} \leq_{int} Q^{1-1}_{<\lambda}$; Q^{1-1}_{λ} is $\exists_R$ for some R , and $Q^{1-1}_{<\mu} \equiv_{int} Q^{eq}_{2,<\mu}$.

1.3 Claim : Let $\lambda \leq \chi$, $\mu \leq \kappa$, all results are uniform.

1) $Q^{eq}_{\lambda,\mu} \leq_{int} Q^{eq}_{\chi,\kappa} \equiv_{int} Q^{eq}_{<\chi^+,<\kappa^+}$ and $Q^{eq}_{<\lambda,<\mu} \leq_{int} Q^{eq}_{<\chi,<\kappa}$.

2) If $\lambda^+ \geq \mu$, $Q^{*eq}_{\lambda,<\mu} \equiv_{int} Q^{eq}_{<\lambda^+,<\mu}$.

3) $\exists_K \leq_{int} Q^{eq}_{<\lambda}$ if $(\forall R \in K)[|\text{Dom } R| < \lambda]$ (when λ is infinite, for λ finite $|\text{Dom } R|^{n(R)} \leq (\lambda-1)^2$ is needed).

4) $Q^{eq}_{<\lambda,<\mu} \leq_{int} \{Q^{eq}_{<\chi_l,<\kappa_l} : l < k\}$ *iff* for some l , $\lambda \leq \chi_l \wedge \mu \leq \kappa_l$ or $\lambda < \aleph_0 \wedge \chi_l > 1 \wedge \mu \leq \kappa_l$ or $\mu < \aleph_0 \wedge \lambda \leq \chi_l \wedge \kappa_l > 1$ or $\lambda < \aleph_0 \wedge \mu < \aleph_0$ (but in the last three cases the interpretation is not uniform.)

Proof : Left to the reader.

1.4 Lemma : The following holds uniformly:

1) $Q^{eq}_{\chi,\chi} \leq_{inex} Q^{eq}_{\lambda,\mu}$ *if* $\lambda \geq \chi$, $\mu \geq \aleph_0$ and $\chi \leq 2^{\mu}$

2) $Q^{eq}_{2^{\mu},\lambda} \leq_{inex} Q^{eq}_{\lambda,\mu}$ if $\lambda \geq 2^{\mu}$, and $\mu \geq \aleph_0$

3) $Q^{1-1}_{<\lambda} \equiv_{inex} Q^{eq}_{<\lambda,\aleph_0} \equiv_{inex} Q^{eq}_{<\lambda,2}$ for $\lambda > \aleph_0$

4) $Q^{1-1}_{<\lambda} \equiv_{inex} Q^{eq}_{<\lambda,<\lambda} \equiv_{inex} Q^{eq}_{<\lambda,2}$ for $\lambda = \aleph_0$

Remark : 1) Clearly in (1) we get biinterpretability.

2) Because of the uniformity e.g. (2) implies $Q^{eq}_{2^{\mu},<\lambda} \leq_{inex} Q^{eq}_{<\lambda,\mu}$ if $\lambda > 2^{\mu}$, $\mu \geq \aleph_0$.

Proof : Repeat the proofs in [Sh1], [Sh2].

1.5 Lemma : 1) For any K consisting of equivalence relations for some n, λ_l, $\mu_l (l < n)$, $\exists_K \equiv_{int} \{Q^{eq}_{<\lambda_l,<\mu_l} : l < n\}$.

2) For any $n, \lambda_l, \mu_l (l<n)$ for some equivalence relation E, $\exists_E \equiv_{int} \{Q^{eq}_{\lambda_l,\mu_l} : l<n\}$.

Remark : This lemma enables us to concentrate on analyzing quantifiers of the form $\exists_R$.

1.6 Lemma: For infinite cardinals λ,μ,χ,κ: $Q^{eq}_{\chi,\kappa} \leq_{inex} Q^{eq}_{\lambda,\mu}$ *iff* $Q^{eq}_{\chi,\kappa} \leq_{\exp} Q^{eq}_{\lambda,\mu}$ *iff* $\chi \leq \lambda \wedge \kappa \leq \mu$ or $\chi+\kappa \leq \lambda \wedge \chi + \kappa \leq 2^\mu$.

Proof: The first condition implies the second trivially the third implies the first by 1.3(1) (if $\chi \leq \lambda \wedge \kappa \leq \mu$) 1.4(1), (if $\chi + \kappa \leq \lambda, 2^\mu$) 1.4(2) (if $2^\mu \leq \lambda$, $\kappa \leq \lambda$ and $\chi \leq 2^\mu$). Now we assume the second is exemplified by $S_0, \ldots, S_{m-1} \in K^{eq}_{\lambda,\mu}$ and suppose $E \in K^{eq}_{\chi,\kappa}$ is definable by an $L(Q^{eq}_{\lambda,\mu})$-formula (with $S_0, \ldots, S_{m-1}$ the only non logical symbols, w.l.o.g. the elements were absorbed). The first case will be $\lambda \geq \mu$. Let E^* be the transitive closure of $\bigvee_{l<m} x\ S_l y$ (with domain $\bigcup_{l<m}$ Dom S_l). Then E^* is an equivalence relation with $\leq \lambda$ equivalence classes, each of power $\leq \mu$, hence $\bigcup_{\ell}$ can be represented as the disjoint union of $A_i (i < \alpha \leq \lambda)$ such that $S_l = \bigcup_{i<\alpha} (S_l \restriction A_i)$. Hence a permutation f is an automorphism of $(\mathcal{U}, S_0, \ldots, S_{m-1})$ iff for some permutation h of α for each i ,$f \restriction A_i$ is an isomorphism from $(A_i, S_0 \restriction A_i, \ldots, S_{m-1} \restriction A_i)$ onto $(A_{h(i)}, S_0 \restriction A_{h(i)}, \ldots, S_{m-1} \restriction A_{h(i)})$.

Let $A_i = \{a_{i,j} : j < j_i \leq \mu\}$ and define E^+: $a_{i_1,j_1} E^+ a_{i_2,j_2}$ iff $j_1 = j_2$ and for some automorphism f of $(\mathcal{U}, S_0, \ldots, S_{m-1})$, $f(a_{i_1,j_1}) = a_{i_2,j_2}$. Clearly E^+ is an equivalence relation on $\bigcup_i A_i$ with $\leq 2^\mu$ equivalence classes, and if B is an E^+-equivalence class then every permutation of it can be extended to an automorphism of $(\mathcal{U}, S_0, \ldots, S_{m-1})$. Let B_i $(i < \gamma \leq 2^\mu)$ list the E^+-equivalence classes.

So if $(\exists x \neq y \in B_i)\ xEy$ then $(\forall x,y \in B_i)\ xEy$.

Let $B^* = \bigcup\{B_i : (\exists x \neq y \in B_i) xEy\}$, $B^{**} = \bigcup B_i : B_i \not\subset B^*, |B_i| > 2\}$, $B^{***} = \bigcup\{B_i : |B_i| \leq 2\}$. So on B^* $(\forall xy \in B^*)(xE^+y \rightarrow xEy)$ i.e. E^+ refine E,,and so E has $\leq 2^\mu$ equivalence classes, each of power $\leq |B^*| \leq |\bigcup_i A_i| \leq \lambda$.

Next on B^{**}, E^{*} refine E: for suppose xEy but $\neg xE^{*}y$, let $i < \gamma$ be such that $y \in B_i$, as $y \in B^{**}$ clearly there is $y' \in B_i$, $\neg xE^{*}y', \neg yEy'$ by a suitable automorphism necessarily xEy' but E is transitive and symmetric contradiction to the definition of B^{**}. So $E \restriction B^{**}$ has $\leq \lambda$ equivalence classes each of power $\leq \mu$. Thirdly on $B^{***}, E \restriction B^{**}$ has $\leq 2^{\mu}$ equivalence classes each of power $\leq 2^{\mu}$, (as $|B^{***}| \leq 2^{\mu}$. Lastly on $\mathcal{U} - B^{*} \cup B^{**} \cup B^{***} = \mathcal{U} - \bigcup_i A_i$, E is the equality.

By 1.3(4) we finish.

§2 Monadic analysis of $\exists_R$

Our aim is to interpret Q^{mon}_{λ} in $\exists_R$ for a maximal λ and show that except on λ elements R is trivial. So continuing later the analysis of $\exists_R$, we can instead analyze $\{Q^{mon}_{\lambda}, \exists_{R_1}\}$ where $|\mathrm{Dom}\ R_1| \leq \lambda$. This is made exact below.

2.1 Definition : For any relation R let

$\lambda_0 = \lambda_0(R) = Min\{|A| : A \subseteq \mathcal{U}$, and for every sequences

$\bar{b}, \bar{c} \in \mathcal{U}$ (of length $n(R)$) . $\bar{b} \approx_A \bar{c}$ implies $R[\bar{b}] \equiv R[\bar{c}]\}$.

where $\bar{b} \approx_A \bar{c}$ iff $tp_{bs}(\bar{b}, A, =) = tp_{bs}(\bar{b}, A, =)$.

Note that $\lambda_0(R) \leq |\mathrm{Dom}\ R|$.

2.2 Theorem : 1) Uniformly $Q^{mon}_{\lambda_0(R)} \leq_{int} \exists_R$.

2) Uniformly $\exists_R \equiv_{int} \{\exists_{R_1}, Q^{mon}_{\lambda_0(R)}\}$ for some R_1, $|\mathrm{Dom}\ R_1| = \lambda_0(R)$, $n(R_1) = n(R)$.

Proof : 1)

Case I : $\lambda_0(R)$ is an infinite regular cardinal.

Let $\bar{R}^m = \langle R^m_l : l < m \rangle$ denote a sequence of $n(R)$-place predicates or relations, $(\mathcal{U}, R_l) \cong (\mathcal{U}, R)$, and $\Delta = \Delta(\bar{R}^m)$ denote a set of formulas of the form $\varphi(\bar{x}, \bar{R}^m)$ closed under permuting the variables and identifying them. Let $k = k(\Delta(\bar{R}), \bar{R})$ be the minimal natural number such that:

(*) there is a formula $\varphi = \varphi(\bar{x}, \bar{y}, \bar{R}) \in \Delta$ with $l(\bar{x}) = l(\bar{y}) = k$, and sequence $\bar{a}, l(\bar{a}) = l(\bar{y})$ such that for every $A \subseteq \mathcal{U}$, $|A| < \lambda_0(R)$ there are sequences $\bar{b}, \bar{c}$ of length k , such that $\varphi(\bar{b}, \bar{a}) \wedge \neg\varphi(\bar{c}, \bar{a}, \bar{R})$ but $\bar{b} \approx_A \bar{c}$.

Let $k(\Delta(\bar{R}^m))$ be the minimal $k(\Delta(\bar{R}^m,)\bar{R}^m)$

By the definition of $\lambda_0(R)$, $k = n(R)$, $\varphi = R(\bar{x})$ satisfies (*) for $\bar{a}$ the empty sequence. By the minimality of k we can assume that $\bar{b}, \bar{c}$ are disjoint to A , and with no repetitions. Clearly as $|A|<\lambda_o(R)\le|\mathcal{U}|$, $\mathcal{U}$ infinite, for any such $A,\bar{b},\bar{c}$ we can find $\bar{b}_l$ $(l=o,2k)$ such that $\bar{b}_o=\bar{b},\bar{b}_{2k}=\bar{c}$ and $\bar{b}_l,\bar{b}_{l+1}$ differ at exactly one coordinate each $\bar{b}_l$ disjoint to A and without repetition. So w.l.o.g. in (*) $\bar{b}=\langle b\rangle\frown\bar{d},\bar{c}=\langle c\rangle\frown\bar{d}$ (and $\bar{d}\frown\langle b,c\rangle$ is disjoint to A and with no repetition, and let $\bar{x}=\langle x\rangle\frown\bar{z}$, so $\varphi=(x,\bar{z},\bar{y})$.) Possibly $\bar{z}$ is empty (i.e. $k=1$) and then our conclusion is immediate as $\{b:\models\varphi[b,\bar{a}]\}$ and $\{c:\models\neg\varphi(c,\bar{a})\}$ has power $\ge\lambda_o(R)$.

By the choice of $k=k\ (\Delta,\bar{R})$ for every $e\in\mathcal{U}$ there is $A_e\subseteq\mathcal{U}$, $|A_e|<\lambda_o(R),e\in A_e$ such that for every $\bar{d}_1\approx_{A_e}\bar{d}_2$ ($\bar{d}_1,\bar{d}_2$ of length $k-1$) $\varphi(e,\bar{d}_1,\bar{a},\bar{R})\equiv\varphi(e,\bar{d}_2,\bar{a},\bar{R})$.

Now we define by induction on l, for $l=0,n(R)-1$ a set of formulas $\Delta_l=\Delta_l(\bar{R}_l)$ where $\bar{R}_l=\langle R_{l,i}:i<2^l\rangle$:

$\Delta_o(\bar{R}_o)=$ the closure of $\{R_{o,o}(x_o,\cdots,x_{n-1})\}$ under permuting and identifying the variables.

$\Delta_{l+1}(\bar{R}_{l+1})=$ the closure of

$$\{(\forall z)[\varphi(z,\bar{x},R_{l+1,0},\cdots,R_{l+1,2^l-1})\equiv\varphi(z,\bar{x},R_{l+1,2^l},\cdots R_{l+1,2^l+2^l-1})]:$$

$$\varphi(z,\bar{x},R_{l,0},\cdots,R_{l,2^l-1})\in\Delta_l(\bar{R}_l)\}\cup\{\varphi(\bar{x},R_{l+1,0},\cdots):\varphi(\bar{x},R_{l,0},\cdots)\in\Delta_l(\bar{R}_l)\}$$

under permuting and identifying the variables.

Now we shall prove by induction on l that

$(**)_l\, k(\Delta_l(\bar{R}_l))\le n-l$.

For $l=0$, as we have mentioned above, this follows from the definition of $\lambda_o(R)$.

So we assume $(**)_l$ and prove $(**)_{l+1}$. As $(*)_l$ holds there are relations R^i $(i<2^l)$, $(\mathcal{U},R^i)\cong(\mathcal{U},R)$ and $k(\Delta_l(\bar{R}_l))=k(\Delta_l(\bar{R}_l),\bar{R})$, where $\bar{R}=\langle R^i:i<2^l\rangle$, and let $\bar{a}_l^*$, $\varphi(\bar{x},\bar{y},\bar{R})$ exemplify (*) for $k=k(\Delta_{l+1}(\bar{R}_l))$. If $k=1$ we finish of course, otherwise we shall prove that $k>k(\Delta_{l+1}(\bar{R}_{l+1}))$; this suffices of course.

Now for every $\psi=\psi(\bar{u},\bar{v},\bar{R})\in\Delta_l(\bar{R})$, $l(\bar{u})<k$, and $\bar{a}\in\mathcal{U}$, there is a set $A_{\psi,\bar{a}}\subseteq\mathcal{U}$ of power $<\lambda_0(R)$ such that: if $\bar{b}\approx_{A_{\psi,\bar{a}}}\bar{c}$, $l(\bar{b})=l(\bar{c})=l(\bar{u})-l(\bar{a})$ then $\models\psi[\bar{b},\bar{a},\bar{R},]\equiv\psi[\bar{c},\bar{a},\bar{R}]$. We can assume $\bar{a}\subseteq A_{\psi,\bar{a}}$.

Now we define by induction on $\alpha<\lambda_0(R)$ $\bar{d}_\alpha$, b_α, c_α as follows. First let $A_\alpha^0=\bigcup_{\beta<\alpha}\bar{d}_\beta\frown\langle b_\beta,c_\beta\rangle\cup\bar{a}^*$, and $A_\alpha=\bigcup\{A_{\psi,\bar{a}}:\bar{a}\subseteq A_\alpha^0,\ l(\bar{a})<k$ and $\psi\in\Delta_l(\bar{R})\}$. Now by the discussion after (*) there is $\bar{d}_\alpha\frown\langle b_\alpha,c_\alpha\rangle$ disjoint to A_α and without repetitions, such that $\models\varphi[b_\alpha,\bar{d}_\alpha,\bar{a}^*]\wedge\neg\varphi[c_\alpha,\bar{d}_\alpha,\bar{a}^*]$

What are the truth values $\mathbf{t}_{\alpha,\beta}$ of $\varphi[b_\alpha,\bar d_\beta,\bar a^*]$ and $\mathbf{s}_{\alpha,\beta}$ of $\varphi[c_\alpha,\bar d_\beta,\bar a^*]$? Clearly if $\alpha=\beta$ then $\mathbf{t}_{\alpha,\beta}$ is truth $\mathbf{s}_{\alpha,\beta}$ is false. If $\alpha>\beta$, then we should remember that $A_{\varphi,\bar d_\beta{}^\frown\bar a^*}\subseteq A_\alpha$, hence $b_\alpha, c_\alpha\notin A_{\varphi,\bar d_\beta{}^\frown\bar a^*}$. Hence $\mathbf{t}_{\alpha,\beta}=t^+_\beta=\mathbf{s}_{\alpha,\beta}=s^+_\beta$. If $\alpha<\beta$ then let $\vartheta(\bar z,x,\bar y,\bar R)=\varphi(x,\bar z,\bar y,R)$, and remembering that $A_{\vartheta,<b_\alpha>^\frown\bar a^*}\subseteq A_\beta, A_{\vartheta,<c_\alpha>^\frown\bar a^*}$, and $\bar d_\alpha$ is disjoint to A_β, it is clear that $\mathbf{t}_{\alpha,\beta}=\mathbf{t}^-_\alpha$, $\mathbf{s}_{\alpha,\beta}=\mathbf{s}^-_\alpha$.

As we can replace $\langle \bar d_\alpha{}^\frown<\beta_\alpha,c_\alpha>:\alpha<\lambda_0(R)\rangle$ by any subsequence of length $\lambda_0(R)$ w.l.o.g. $\mathbf{t}^+_\alpha=\mathbf{t}^+$, $\mathbf{t}^-_\alpha=\mathbf{t}^-$ and $\mathbf{s}^-_\alpha=\mathbf{s}^-$ for every $\alpha<\lambda_0(R)$ we can assume $\mathbf{t}^+$ is truth (otherwise interchange φ and $\neg\varphi$, b_α and c_α in the rest).

Now let h be the following permutation of $\mathcal{U}$: $h(c_{3\alpha+1})=c_{3\alpha+2}$, $h(c_{3\alpha+2})=c_{3\alpha+1}$ and $h(c)=c$ for any other element. Next, let for $2^l\le i<2^{l+1}$, $R^i=h(R^{i-2^l})$ and let $\bar R_{l+1}=\langle R^i:i<z^{l+1}\rangle$, $\bar R^1=\langle R_{i+2^l}:i<2^l\rangle$. Now

(a) $\psi(\bar z,\bar y,\bar R_{l+1})=(\forall x)[\varphi(x,\bar z,\bar y,\bar R)\equiv\varphi(x,\bar z,\bar y,\bar R^1)]$ belong to $\Delta_{l+1}(\bar R_{l+1})$.

(b) $\models\psi(\bar d_{3\beta},\bar a^*,\bar R_{l+1})$ for $\beta<\lambda_0(R)$. This is equivalent to saying that h maps $\{e\in\mathcal{U}:\models\varphi[e,\bar d_{3\beta},\bar a^*,\bar R]\}$ into itself (as $h^{-1}=h$). i.e. we should prove $\models\varphi[e,\bar d_{3\beta},\bar a^*,\bar R]$ implies $\varphi[h(e),\bar d_{3\beta},\bar a^*,\bar R^1]$. If $e=h(e)$ this is trivial. Otherwise, $e=c_{3\alpha+i}$, $i\in\{1,2\}$; and $h(e)=c_{3\beta+(3-i)}$; if $\beta\ge\alpha$ this follows from $\mathbf{s}_{3\alpha+i,3\beta}=\mathbf{s}_{3\alpha+(3-i),\beta}=\mathbf{t}^+$ (as $3\alpha+i$, $3\alpha+(3-i)>3\beta$) ; if $\beta<\alpha$ this follows from $\mathbf{s}_{3\alpha+i,3\beta}=\mathbf{s}_{3\alpha+(3-i),3\beta}=\mathbf{s}^-$ (as $3\alpha+i$, $3\alpha+(3-i)<3\beta$).

(c) $\models\neg\psi[\bar d_{3\beta+1},\bar a^*,\bar R]$ for $\beta<\lambda_0(R)$ Just substitute $x=c_{3\beta+2}$ for the $(\forall x)$ in ψ's definition.

(d) The sequences $\{\bar d_\beta:\beta<\lambda_0(R)\}$ are pairwise disjoint. This is because for $\gamma<\beta$, $\bar d_\beta\subseteq A^0_\beta\subseteq A_\alpha$.

Now (a), (b), (c), (d) together show that $k(\Delta(\bar R_{l+1}),\bar R_{l+1})<k$ Hence $k(\Delta_{l+1}(\bar R_{l+1}))<k\le n-l$ (or $k=1$ and then $h(\Delta_{l+1}(\bar R_{l+1})=k((\Delta_l)(\bar R_l))$. So we have done the induction step in proving (**).

Now $(**)_{n(R)-1}$ show that for $\bar a^*$, $\bar R$ ($(\mathcal{U},R_l)\cong(\mathcal{U},R)$) and $\varphi(x,\bar y,\bar R)$, the powers of $\{c\in\mathcal{U}:\models\varphi[c,\bar a^*,\bar R]\}$ and of $\{c\in\mathcal{U}:\models\neg\varphi[c,\bar a^*,\bar R]\}$ are at least $\lambda_0(R)$, and we get the required interpretation.

Case II : $\lambda_0(R)<|\mathcal{U}|$ (and in particular $\lambda_0(R)$ finite)

Let $A\subseteq\mathcal{U}$ be a set of power $\lambda_0(R)$, such that $\bar b\approx_A\bar c$ implies $R[\bar b]\equiv R[\bar c]$. As $\mathcal{U}$ is infinite, we can find distinct $d_i\in\mathcal{U}-A$ $(i<n(R)^2)$. Define $\bar d=\langle d_i:i<n(R)^2\rangle$, $\varphi^*(x,\bar d,R)=\bigwedge\{(\exists y_0,\ldots,y_{k-1})$ [the elements $y_0,\ldots,y_{k-1},x$ are pairwise distinct and if the elements $y_0,\ldots,y_{k-1},d_m,x$ are pairwise distinct then

$\varphi(x, y_0, \ldots, y_{k-1}) \equiv \varphi(d_m, y_0, \ldots, y_{k-1})]$: $\varphi = \varphi(z_0, \ldots, z_k, R)$

is an atomic formula in $L(R)$(*so* $k+1 \leq n(R)$)*and* $m < n(R)^2$, m, k are natural numbers }. By the choice of A, $x \notin A \Longrightarrow \neg\varphi^*(x, \bar{d})$, hence $B = \{x \in \mathcal{U}: \mathcal{U} \models \varphi^*[x, \bar{d}]\}$ is a subset of A. Clearly $Q^{mon}_{|B|} \leq_{int} \exists_R$ (uniformly) hence it suffices to prove $|B| = \lambda_0(R)$ which follows from

(*) if $\bar{b} \cong_B \bar{c}$ then $R[\bar{b}] \equiv R[\bar{c}]$

for this it suffices to prove:

(**) if $\varphi(\bar{x}, R) \in L(R)$ is atomic, $\bar{b}, \bar{c}$ are sequences of length $l(\bar{x}) \leq n(R)$ without repetition then $\bar{b} \cong_B \bar{c}$ implies $\varphi(\bar{b}, R) \equiv \varphi(\bar{c}, R)$.

Let $\bar{b}^\frown\bar{c}_0, \bar{b}^\frown\bar{c}_1$ be sequences from $\mathcal{U}$, without repetition, $\bar{b} \subseteq B, \bar{c}_0, \bar{c}_1$ disjoint to B ; by the transitivity of $\equiv$, w.l.o.g. $\bar{c}_1$ is disjoint to $\bar{d}$, so for some i, $\langle d_i, d_{i+1}, \ldots, d_{i+k-1}\rangle$ (where $k = l(\bar{c}_0)$) is disjoint to $\bar{c}_0$ (and obviously to $\bar{c}_1$).

Now we shall prove that for every atomic $\varphi(\bar{x}, \bar{y}, R), l(\bar{x}) = k$, $l(\bar{y}) = l(\bar{b}) \models \varphi(\bar{c}_l, \bar{b}, r) \equiv \varphi(\langle d_i, \ldots, \rangle, \bar{b}, R)$ thus finishing. For this we define $\bar{c}_{l,m} (m \leq k)$ such that each $\bar{c}_{l,m}$ is with no repetitions, disjoint to B, $b, \bar{c}_{l,0} = \bar{c}_l$, $\bar{c}_{l,k} = \langle d_i, \ldots, d_{i+k-1}\rangle$, $\bar{c}_{l,m+1}$, $\bar{c}_{l,m}$ are distinct in one place only. By the definition of B (and φ) for every atomic $\varphi(\bar{x}, \bar{y}, R)$, $\models \varphi(\bar{c}_{l,m}, \bar{b}, R) \equiv \varphi(\bar{c}_{l,m+1}, \bar{b}, R)$ so we finish easily.

Case III. $\lambda_0(R)$ a singular cardinal.

We fix the relation R; now for every atomic formula $\varphi(\bar{x}, \bar{y}, R) \in L(R)$ and $\bar{b} \in \mathcal{U}$, $\varphi(\bar{x}, \bar{b}, R)$ define an $l(\bar{x})$-place relation on $\mathcal{U}$, let $\lambda_0(\varphi(\bar{x}, \bar{b}), R))$ be as defined as in Def. 2.1. Clearly the number of atomic $\varphi(\bar{x}, \bar{y}, R)$ (with no dummy variable, $\bar{x}^\frown\bar{y} \subseteq \{x_i : i < m(R)\}$) is finite, and we can find φ and $\bar{b}$ such that $\lambda_0(\varphi(\bar{x}, \bar{b}, R)) = \lambda_0(R)$ and (under this restriction) $l(\bar{x})$ is minimal. Clearly $l(\bar{x}) > 0$ (as $A = \phi$ would serve), if $n = 1$ we finish trivially. So assume $l(\bar{x}) > 1$, let $\bar{x} = \bar{z}^\frown\langle x\rangle$. By the choice of $\varphi, \bar{b}$, for every $c, \lambda_0(\varphi(\bar{z}, c, \bar{b}, R)) < \lambda_0(R)$ and let $A_c \subseteq \mathcal{U}$ be such that

(i) $|A_c| = \lambda_0(\varphi(\bar{z}, c, \bar{b}, R)) \leq |\mathcal{U}|$

(ii) $\bar{d}_1 \sim_{A_c} \bar{d}_2$ implies $\varphi(\bar{d}_1, c, \bar{b}, R) \equiv \varphi(\bar{d}_2, c, \bar{b}, R)$

So $|A_c| = \lambda_0(\varphi(\bar{z},c,\bar{b},R)) < \lambda_0(R) \le |\mathcal{U}|$.

For each c, by case II there are an atomic $\psi_c(x,\bar{y}_1,R)$ and $\bar{d}_c \in \mathcal{U}$ such that $|\{a : \models \psi_c(a,\bar{d}_c,R)\}| = \lambda_0(\varphi(\bar{z},c,\bar{b},R)) = |A_c|$; note there are only finitely many possible ψ_c's

Subcase III a: $\mathrm{Sup}_{c\in\mathcal{U}} |A_c| = \lambda_0(R)$.

Let $\lambda_0(R) = \sum_{\zeta<\kappa} \mu_\zeta$ where $\kappa = cf(\lambda_0(R))$, and each μ_ζ is regular $<\lambda_0(R)$. So assume $|A_{c_i}| \ge \mu_i$, so by Case II applied to $\varphi(\bar{z},c_i,R)$ we can interprete unformly $Q^{mon}_{<\mu_\zeta}$ and even $Q^{mon}_{<\lambda_0(R)}$ and moreover in this case, we have $\bar{a}_\zeta$ $(\zeta<\kappa)$ such that $\mu_\zeta \le |\{e\in\mathcal{U}: \models\varphi[e,\bar{a}_\zeta,\bar{R}]\}| < \lambda_0(R)$ ($\bar{a}_\zeta$ is $\bar{d}_\alpha{}^\frown\bar{a}^*$ for some α). In particular we can interprete Q^{mon}_κ. Let $R=\bigcup_{\zeta<\kappa}\bar{a}_\zeta$ and E be the following equivalence relation on $\mathcal{U}$: bEc iff for every $\bar{a}\subseteq P$, $\varphi[b,\bar{a},\bar{R}]\equiv\varphi[c,\bar{a},\bar{R}]$. Let $\langle A_i : i<\chi\rangle$ be a list of the equivalence classes of E . If $\{i:|A_i|\ge 2\}$ has power $\ge\lambda_0(R)$, we get our conclusion easily; this holds also if there are at least two A_i of power $\ge\lambda_0(R)$, or even if $\mathrm{Sup}_{i<\alpha}|A_i|=\lambda_0(R)$. By the choice of P the only case left is $|\{i: |A_i|=1\}|\ge\lambda_0(R)$. So let $a_\alpha(\alpha<\lambda_0(R))$ be pairwise non E-equivalent $a_i \notin P$. Define a permutation $:h(a_{3\alpha+i})=a_{3\alpha+\beta-i}$ for $i=1,2$ and $h(e)=e$ otherwise. Define $\bar{R}$, $\bar{R}_{l*}$ as in case I and $\varphi^*(x,P,\bar{R})=(\forall x_0,\cdots,x_{k-2})\ [\bigwedge_{i<k} P_i(x_i)\to\varphi(x,x_o,\cdots,x_{k-2},\bar{R})\equiv\varphi(x,x_0,\cdots,x_{k-2},\bar{R}^1)]$ Now we finish: $\varphi^*[a_\beta,P,R]$ iff β is divisible by 3 (for $\beta<\lambda_0(R)$.

Subcase III b: $|\bigcup_{c\in\mathcal{U}}(A_c-\{c\}| \ge \lambda_0(R)$ but not II a.

By case II we know A_c is definable (uniformly) from $\bar{b}{}^\frown\langle c\rangle$. Hence we can choose for $i<\lambda_0(R)$ c_i,e_i such that $e_i\in A_{c_i}$, $e_i\ne c_i$, and $c_i,e_i \notin \{c_j,e_j : j<i\}$. By Hajnal free subset theorem (See [H]) w.l.o.g.

$e_i \in A_{c_j}$ iff $i=j$, and e_i,c_i do not appear in $\bar{b}$.

Let g be the following permutation of $\mathcal{U}$

$$g(e_{3i+1}) = e_{3i+2}$$

$$g(e_{3i+2}) = e_{3i+1}$$

$g(e) = e \qquad e \in \mathcal{U} - \{e_{3i+2}, e_{3i+2} : i < \lambda_0(R)\}$

Let $R_0 = R$, $R_1 = g(R)$, then

$$\psi(x,\bar{b},R_0,R_1) \overset{def}{=} (\forall \bar{y})[\varphi(\bar{y},x,b,R_1)]$$

is as required.

Case III c : not III a,b.

So for some $B \subseteq \mathcal{U}$, $|B| < \lambda_0(R)$, $[c \in \mathcal{U} - B \Longrightarrow A_c - \{c\} \subseteq B]$; w.l.o.g. $[c \in B \Longrightarrow A_c \subseteq B]$.

Let $\bar{d} \subseteq \mathcal{U} - B$, $l(\bar{d}) = l(\bar{z})$; now for every set $D \subseteq \mathcal{U}$, $|D| < \lambda_0(R)$ there are $\bar{d}_1{}^\frown \langle c_1 \rangle\ \bar{d}_2{}^\frown, \langle c_2 \rangle$ disjoint to $D \cup B \cup \bar{d}$ without repetition, $\varphi(\bar{d}_1, c_1, \bar{b}, R) \equiv \neg\ \varphi(\bar{d}_2, c_2, \bar{b}, R)$ (by the choice of $\varphi, \bar{b}$). As $A_{c_1} - \{c_1\} \subseteq B$,

$$\varphi(\bar{d}_1, c_1, \bar{b}, R) \equiv \varphi(\bar{d}, c_1, \bar{b}, R).$$

and similarly

$$\varphi(\bar{d}_2, c_2, \bar{b}, R) \equiv \varphi(\bar{d}, c_2, \bar{b}, R)$$

hence

$$\varphi(\bar{d}, c_1, \bar{b}, R) \equiv \neg\ \varphi(\bar{d}, c_2, \bar{b}, R)$$

We can conclude that $\varphi(\bar{d}, x, \bar{b}, R)$ divide $\mathcal{U}$ to two subsets each of cardinality $\geq \lambda_0(R)$.

Remark: In case III the only use of "$\lambda_0(R)$ singular" is $[\sup_{c \in \mathcal{U}} |A_c| < \lambda_0(l) \Longrightarrow \sup_{c \in \mathcal{U}} |A_c|^+ < \lambda_0(R)]$, but with a little more work we can bound the numbers of copies of R used independently of R.

Proof of 2.2(2) :

If $\lambda_0(R) = |\mathcal{U}|$ we choose $R_1 = R$ and have nothing new to prove. If $\lambda_0(R) < |\mathcal{U}|$, let $\varphi_i(\bar{x}_i, \bar{y}_i, R)$ $(i<m)$ list all atomic formulas in $L(R)$, $l(\bar{x}_i) = k_i > 0$, $l(x_i) + l(y_i) \leq n(R)$, and w.l.o.g. $k_i = n(R) \Longrightarrow i = 0$. Let d_i $(0<i<2n(R))$ be distinct element of $\mathcal{U} - B$, B from case II above. Of course, we can concentrate on the case $n(R) > 1$. Let $R_1 = \{\langle a, \ldots, a \rangle : a \in B\} \cup \{\langle a_1, \ldots, a_{n(R)} \rangle : \models R[a_1, \ldots, a_{n(R)}], a_1, \ldots, a_{n(R)}$

are distinct members of $B\}\cup\{\langle d_i, a_1, \ldots, a_{k_i}, d_i, \cdots\rangle : 1 \le i < m,\ a_1, \ldots, a_{k_i}$ distinct members of B , and for all distinct b_l $(l < l(\bar{y}_i))$ from $\mathcal{U}-B$, $\models \varphi_i[\langle a_i, \ldots, a_{k_i}\rangle, \langle b_0, \ldots, \rangle, R]\}$.

Easily $\{\exists_R, Q_{\lambda_0(R)}\}^{mon}\} \le_{int} \exists_{R_1}$, and by case II above $\exists_{R_1} \le_{int} \exists_R$.

§3 The one-to-one function analysis

The aim of this section is similar to the previous one, going one step further, i.e. we want to analyse $\exists_R$, interpreting in it Q_λ^{1-1} for a maximal λ, hoping that " the remainder" has domain $\le\lambda$.

3.1 Definition : Let $\lambda_1 = \lambda_1(R)$ be Sup $\{|\{tp_{bs}(a, A, R) : a \in \mathcal{U}-A\}| : A \subseteq \mathcal{U}\}$

3.2 Fact : $\lambda_1(R) \le \lambda_0(R)$

3.3 Claim : $Q^{1-1}_{\lambda_1(R)} \le_{int} \exists_R$ uniformly, if the sup is obtained.

Proof : Suppose h is a one-to-one, one place 'partial' function from $\mathcal{U}$ to $\mathcal{U}$, with $|\mathrm{Dom}\, h| \le \lambda_1(R)$. Let $A \subseteq \mathcal{U}$ be such that $\{tp_{bs}(a, A, R) : a \in \mathcal{U}-A\}$ has cardinality $\lambda \overset{def}{=} \lambda_1(R)$. So we can find $a_i \in \mathcal{U}-A\ (i<\lambda)$ such that $tp_{bs}(a_i, A, R)$ are pairwise distinct and w.l.o.g. $|\mathcal{U}-\{a_i : i<\lambda\}| = |\mathcal{U}|$. Let $h = \{\langle b_i, c_i\rangle : i<\lambda\}$, w.l.o.g. $b_i, c_i \notin A$ and we can find F_1, F_2 permutation of $\mathcal{U}$ which are the identity on A such that $F_1(a_i) = b_i$, $F_2(a_i) = c_i$. (they exist - see Def. 1.1(3).) Let $R_1 = F_1(R)$, and $R_2 = F_2(R)$ and define the monadic relations $P_0 = A$, $P_1 = \{b_i : i<\lambda\}$, $P_2 = \{c_i : i<\lambda\}$ (all of power $\le\lambda_0(R)$) Let $\varphi(x, y, P_0, P_1, P_2, R_1, R_2)$ "say" that for every atomic $\varphi(x, \bar{z}, R) \in L(R)$ and $\bar{z} \in P_0$, $\varphi(x, \bar{z}, R_1) \equiv \varphi(y, \bar{z}, R_2)$ and $P_1(x), P_2(y)$.

3.4 Lemma : There is a set A such that

1) $|A| \le 5(n(R)+2)n(R)\lambda_1(R)$, furthermore, if the sup is not obtained in the definition of λ_1 then $|A| < \lambda_1$.

2) Let E_A be the equivalence relation:

$tp_{bs}(a, A, R) = tp_{bs}(b, A, R)$

If $\bar{b} \cong_\phi \bar{c}$ and $b_i E_A c_i$ for all $i < l(\bar{b})$ then $R(\bar{b}) \equiv R(\bar{c})$.

Proof : We define by induction on $l \le n(R)+2$ sets A_l such that $[m<l \to A_m \subseteq A_l]$, $|A_l| \le 5ln(R)\lambda_1(R)$, and if the sup is not obtained, $|A_l| < \lambda_1(R)$; we shall show that $A_{n(R)+2}$ satisfies the requirements of the lemma.

Let $A_0 = \phi$

If A_l is given, we define by induction on i a_i^l, A_i^l such that

1) $A_0^l = A^l$

2) A_i^l is increasing, continuous(in i).

3) $a_i^l \notin A_j^l$ for any i, j.

4) $|A_{i+1}^l - A_i^l| \leq 2(n(R)-1)$

5) If $\alpha, \beta < i$, $\alpha \neq \beta$, then $tp_{bs}(a_\alpha^l, A_i^l, R) \neq tp_{bs}(a_\beta^l, A_i^l, R)$ hence $a_\alpha^l \neq a_\beta^l$.

For some $i = i(l)$ (which is necessarily $< \lambda_1(R)^+$) we cannot continue, i.e. A_i^l is defined but not a_i^l, A_{i+1}^l. Define $A_{l+1}' \stackrel{def}{=} A_l \cup \bigcup_{i<i(l)} A_i^l$,

$A_{l+1} \stackrel{def}{=} A_{l+1}' \cup \{a_i^l : i < i(l)\} \cup \{b:$ the basic type realized over $A_l \cup \bigcup_{i<i(l)} A_i^l \cup \{a_i^l : i < i(l)\}$ by b is realized by $\leq 3n(R)$ elements $\}$.

So $|A_{l+1}'| \leq |A_l| + 2(n(R)-1)|i(l)| + |i(l)| = |A_l| + 2n(R)\lambda_1(R)$, and $|A_{l+1}'| \leq |A_l| + 2n(R)\lambda_1(R) + 3n(R)\lambda_1(R) \leq 5n(R)(l+1)\lambda_1(R)$ if the sup in Definition 3.1 is not obtaine the inequality is strict. We prove $A = A_{n(R)+2}$ satisfies the requirements of the lemma. It is easy to see $|A|$ is as required (in demand (1) of 3.4).

Suppose $R(\bar{b})$, $\neg R(\bar{c})$ and $b_m E_A c_m$ for $m < n(R)$ and $\bar{b} \cong_A \bar{c}$. There are at most $n(R)$ l's such that $\bar{b} \cap A_{l+1} \neq \bar{b} \cap A_l$ so we choose l such that $\bar{b} \cap A_{l+1} \subseteq A_l$ and hence $\bar{c} \cap A_{l+1} \subseteq A_l$. Hence we may for simplicity assume:

$\bar{c} \cap A_{l+1} = \bar{b} \cap A_l = \phi$ and $\bar{b}$, $\bar{c}$ are without repetitions.

Let $B = A_l \cup \bigcup_{i<i(l)} A_i^l$ so $\bigwedge_{m<n(R)} b_m E_B c_m$ and $|b_m / E_B| \geq 3n(R)$. Now we can define $\bar{d}_k$ $(k = 0, \ldots, n(R))$, each of length $n(R)$, $\bar{b} = \bar{d}_0$, $\bar{c} = \bar{d}_{n(R)}$, $\bar{d}_k$ with no repetitions, $\bigwedge_{m<n(R)} b_m E_B d_{k,m}$ and $|\{m : d_{k,m} \neq d_{k+1,m}\}| \leq 1$. So, as in proof of the monadic case, we may assume $R(\bar{b}) \wedge \neg R(\bar{c})$, $\bar{b}, \bar{c}$ without repetitions, $\bar{b} = \langle e \rangle ^\frown \bar{a}$, $\bar{c} = \langle f \rangle ^\frown \bar{a}$.

Notice there is $j < i(l)$ such that e, a_j^l realize the same basic type over $\bigcup_{i \leq i(l)} A_i^l$ (as, if not, we could let $A_{i(l)+1}^l = A_{i(l)}^l$ and $a_{i(l)}^l = e$.) w.l.o.g. assume $R(a_j^l, \bar{a})$. (otherwise use $\neg R$) and $R[e_j^l, \bar{a}]$, (otherwise interchange e and f).

3.4 A Claim : We can let $A_{i(l)+1}^l = A_{i(i)}^l \cup \bar{a}$, $a_{i(l)}^l = f$ and hence get a contradiction to the definition of $i(l)$.

Proof : Suppose $tp_{bs}(f, A_{i(l)+1}^l) = tp_{bs}(a_i^l, A_{i(l)+1}^l)$

if $i \neq j$ $tp_{bs}(f, A^l_{i(l)+1}) \supseteq tp_{bs}(f, A^l_{i(l)}) = tp_{bs}(e, A^l_{i(l)}) = tp_{bs}(a^l_j, A_{i(l)}) \neq tp_{bs}(a^l_i, A^l_{i(l)})$ contr.

If $i=j$ use $R(x, \bar{a})$.

So we have proved 3.4A, hence 3.4.

3.4B Claim : The sup is obtained in the definition of $\lambda_1(R)$

Proof : Suppose not, by the lemma, we can find A such that $|A| < \lambda_1(R)$ and $(\forall \bar{b}, \bar{c})[\bigwedge_{i<n(R)} b_i E_A c_i \wedge \bar{b} \cong_{\phi} \bar{c} \rightarrow R(\bar{b}) \equiv R(\bar{c})]$. Clearly $\{a/E_A : a \in \mathcal{U}-A\}$ has power $< \lambda$, then for all B

$|\{tp_{bs}(a,B) : a \in \mathcal{U}-B\}| \leq |A| + |\{tp_{bs}(a, B \cup A) : a \in \mathcal{U}-B\}| < \lambda_1$, contradiction.

3.5 Conclusion : $\{\exists_R, Q^{mon}_{\lambda_0}\}$ is bi-interpretable with $\{Q^{mon}_{\lambda_0}, Q^{1-1}_{\lambda_1}, \exists_{R_1}, \exists_E\}$, where $|\mathrm{Dom}\, R_1| \leq 5(n(R)+2)^2\lambda_1(R)$, E an equivalence relation. This is done uniformly (i.e., the formulas depend on $n(R)$ only).

Proof : We've shown $Q^{1-1}_{\lambda_1(R)} \leq_{int} \{\exists_R, Q^{mon}_{\lambda_0(R)}\}$ (see 3.3). Let $A^1 \cap A = \phi$, $|A^1| = |A| = \lambda_1(R), A$ as in the lemma 3.4, $R_1 = R \restriction (A \cup A^1)$, $A \cup A^1$ includes $\geq Min\{3n(R)\}, |a/E_A|\}$ elements of each E_A equivalence class a/E_A. Now

$$R(x_1, \ldots, x_{n(R)}) \text{ iff}$$
$$(\exists y_1) \cdots (\exists_{n(R)})(\bigwedge_{1 \leq i \leq n(R)} x_i E_A y_i \wedge R_1(\bar{y}))$$

So $\exists_R \leq_{int} \{Q^{1-1}_{\lambda_1}, Q^{mon}_{\lambda_0}, \exists_{R_1}, \exists_{E_A}\}$

Now $\exists_{R_1} \leq_{int} \{\exists_R, Q^{mon}_{\lambda_0}\}$ by the definition of R_1, $\exists_{E_A} \leq_{int} \{\exists_R, Q^{mon}_{\lambda_0}\}$ directly, and $Q^{1-1}_{\lambda_1} \leq \{\exists_R, Q^{mon}_{\lambda_0}\}$ by 3.3, 3.4B. So $\{Q^{1-1}_{\lambda_1}, Q^{mon}_{\lambda_0}, \exists_{R_1}, \exists_{E_A}\} \leq_{int} \{\exists_R, Q^{mon}_{\lambda_0}\}$ and we finish.

3.5A Remark: Note the $Q^{1-1}_{|\mathrm{Dom}\ R_1|}$ is uniformly interpretable (for fixed $n(R)$) in $Q^{1-1}_{\lambda_1}$ including the case λ_1 is finite, so 3.5 holds for it too.

§4 Above the local stability cardinal

We continue our analysis of $\exists_R$. For notational simplicity we make

4.1 Hypothesis : $|\mathrm{Dom}\, R| = \lambda_1(R)$ (or, when $\lambda_1(R)$ is finite, $|\mathrm{Dom}\, R| \leq 5(n(R)+2)\lambda_1(R)$. (and see 3.5A).

Also in this section (as well as in § 5 , § 6) we shall not prove the theorems "uniformly". This can be done, however we feel it will obscure the understanding by making us to deal with too many parameters. We also delay the treatment of the finite cases.

4.2 Definition : An m-type p is called $(\geq\lambda)$ -big if it is realized by λ pairwise disjoint sequences; let λ-big mean $(\geq\lambda^+)$-big. For this section big means $\lambda_2 = \lambda_2(R)$-big (λ_2 is defined below).

Let $(\exists^{\geq\lambda}\bar{x})\varphi(\bar{x},\bar{b})$ mean $\{\varphi(\bar{x},\bar{a})\}$ is $(\geq\lambda)$-big. We define $(\exists^{<\lambda}\bar{x})$, $(\exists^{\leq\lambda}\bar{x})$ similarly [as $\neg(\exists^{\geq\lambda}\bar{x})$, $\neg(\exists^{>\lambda}\bar{x})$, respectively.] Let "small" mean just the negation of big.

4.3 Remark : 1) Since we have monadic relations predicates and 1−1 permutations of power $|\mathrm{Dom}\, R|$ available, we can use one R (copies can be achieved easily).

2) Also, we can code any set of pairwise disjoint n-tuples, or any set of n -tuples forming a Δ-system (of power $\leq\lambda_1(R)$).

4.4 Definition : 1) M is an **admissible model** if it is an expansion of $(\mathcal{U},R)$ by countably many monadics relations and permutations of power $\leq\lambda_1(R)$.

4.5 Definition : $\lambda_2=\lambda_2(R)=$ least λ such that:

1) If M is admissible, Δ is a (finite) set of formulas, $A\subseteq M$, $|A|\leq\lambda$ and $m<\omega$, then $|S^m_\Delta(A,M)|\leq\lambda$.

2) $Q^{eq}_{\lambda,\lambda}\not\leq_{int}\exists_R$

4.6 Remark : the case $\lambda_2=\lambda_1$ is uninteresting as we want to prove now that it suffices to analyze R^*, $|\mathrm{Dom}\, R^*|=\lambda_2$, i.e. for some such R^* and some equivalence relation E

$\{Q^{mon}_{\lambda_0}, Q^{1-1}_{\lambda_1}, \exists_E, \exists_{R^*}\} \equiv_{int} \exists_R$ for some R^*, $|\mathrm{Dom}\; R^*| = \lambda_2(R)$ So we assume $\lambda_2<|\mathcal{U}|$. (but this is not essential).

4.7 Lemma : If M is admissible then there is A^*, $|A^*|\leq\lambda_2$ such that:

a) for any $\bar{a}$, $\bar{a}\cap A^*=\phi$ and finite Δ the type $q=tp_\Delta(\bar{a},A^*)$ is big.

b) For any such q, q is minimal; i.e., there is no $\varphi(\bar{x},\bar{y})\in\Delta$ and $\bar{b}$ such that both $q(\bar{x})\cup\{\pm\varphi(\bar{x},\bar{b})\}$ are big.

c) For any Δ,m, the number of such q's is $<\lambda_2$.

Remark : From c) we shall use only the "$\leq\lambda_2$".

Proof : We define, by induction on $i<\lambda_2$, $A_i\subseteq\mathcal{U}$, $|A_i|\leq\lambda_2$, A_i increasing, continuous, such that for all finite Δ :

0) for every $\bar{a}\in\mathcal{U}$ and $i<\lambda_2$, some $\bar{a}\subseteq A_{i+1}$ realizes $tp_\Delta(\bar{a},A_i)$.

1) If $q_i=tp_\Delta(\bar{a},A_i)$ is not minimal, then some $\varphi_{q_i}(\bar{x},\bar{b}_{q_i})$ witnesses it for some $\varphi_{q_i}\in\Delta$, $\bar{b}_{q_i}\subseteq A_{i+1}$.

2) If $q_i = tp_\Delta(\bar a, A_i)$ is not big, then for some $B_{\bar a} \subseteq A_{i+1}$, no sequence $\bar b$ realizing it is disjoint to B.

Let $A = A_{\lambda_2} \stackrel{def}{=} \bigcup\{A_i : i < \lambda_2\}$. Now 4.7 will follow from 4.8, 4.9.

4.8 Claim : If $A_{\lambda_2} \cap \bar a = \phi$ then for any (finite) Δ for some $i < \lambda_2$, $q_i = tp_\Delta(\bar a, A_i)$ is minimal and big.

Proof : Clearly q_i is big (for every $i < \lambda_2$, by (2)). If q_i is not minimal, take $\varphi_{q_i} \in \Delta$, $\bar b_{q_i} \subseteq A_{i+1}$ witnessing this (by (1)). W.l.o.g., $\varphi_{q_i}(\bar x, \bar b_{q_i}) \in q_{\lambda_2}$ and $q_i \cup \{\neg\varphi_{q_i}(\bar x, \bar b_{q_i})\}$ is realized by the sequences $\langle \bar a_{i,\xi} : \xi < \lambda_2^+ \rangle$ which are pairwise disjoint.

w.l.o.g. $tp_\Delta(\bar a_{i,\xi}, A_{\lambda_2})$ does not depend on ξ, and call it r_i; clearly r_i is λ_2-big.

Also, $r_i \neq r_j$ for $i < j$, since $\varphi_{q_i}(\bar x, \bar b_{q_i}) \in q \restriction A_{i+1} \subseteq q \restriction A_j = q_j \subseteq r_j$

but $\neg\varphi_{q_i}(x, \bar b_{q_i})$ is satisfied by $\bar a_{i,\xi}$. W.l.o.g. $\bar a_{i,\xi}, \bar a_{j,\zeta}$ are disjoint when $\langle i,\xi \rangle \neq \langle j,\zeta \rangle$.

Now we can interpret $Q^{eq}_{\lambda_2,\lambda_2}$: we add a predicate A_{λ_2} and let xEy iff x codes $\bar x$, y codes $\bar y$ (remember 4.3(2)), and $\bar x$ and $\bar y$ realize the same Δ-type over A_{λ_2}. E has $\geq\lambda_2$ equivalence classes of power $\geq\lambda_2$, a contradiction.

4.9 claim : A_{λ_2} satisfies a), b) and c) of the lemma (4.7).

Proof : Let $q = tp_\Delta(\bar a, A^*)$, $\bar a \cap A^* = \phi$. We know that for some $i < \lambda_2$ $q_i = q \restriction A_i$ is big and minimal, hence is realized by pairwise disjoint $\bar c_\xi$ $\xi < \lambda_2^+$.

For every $\varphi(\bar x, \bar b) \in q$, $q_i \cup \{\varphi(\bar x, \bar b)\}$ is big [as $q_i \cup \{\varphi(\bar x, \bar b)\} \in q_j$, for some large enough $j < \lambda_2$], hence $q_i \cup \{\neg\varphi(\bar x, \bar b)\}$ is not big. There are $\leq\lambda_2$ such $\varphi(\bar x, \bar b)$, so omitting any tuple realizing any of them from our sequence $\langle \bar c_\xi : \xi < \lambda_2^+ \rangle$ still leaves λ_2^+ many, so each realizes q hence q is big.

If q is not minimal, then q_i is not minimal, contradiction. If 4.7(c) fails, we can interpret $Q^{eq}_{\lambda_2,\lambda_2}$ by taking Δ witnessing the fact that c) fails and defining E as before. This finishes lemma 4.7.

4.10 The Symmetry Lemma : There are no $\varphi(\bar x, \bar y, \bar z)$, $\bar d, \bar a_\alpha, \bar b_{\alpha,\beta} (\alpha, \beta < \lambda_2^+)$ such that:

1) for every $\alpha, \beta < \lambda_2^+$, $\models \varphi(\bar b_{\alpha,\beta}, \bar a_\alpha, \bar d)$ and these three sequences are disjoint.

2) For fixed α, $\bar b_{\alpha,\beta} (\beta < \lambda_2^+)$ are disjoint.

3) The $\bar a_\alpha$'s are disjoint.

4) $\varphi(\bar b_{\alpha,\beta}, \bar x, \bar d)$ is not big.

Proof : We can throw away many $\bar a_\alpha$'s , $\bar b_{\alpha\beta}$'s, as long as their number remains so w.l.o.g. all the sequence $\bar{b}_{\alpha,\beta}(\alpha,\beta < \lambda_2^+)$ are pairwise disjoint. Let $\Delta=\{\varphi\}$.

We may assume that for each α, all $\bar{b}_{\alpha,\beta}$ realize the same Δ-type over $\{\bar{a}_\gamma:\gamma\leq\alpha\}\cup\bar{d}$ (use part (1) of Def. 4.5 to thin the set $\{\bar{b}_{\alpha,\beta}:\beta<\lambda_2^+\}$). Similarly, we may assume that $\bar{a}_{\alpha_1}, \bar{a}_{\alpha_2}$ realize the same Δ-type over $\{\bar{a}_\gamma:\gamma<\alpha_1\cap\alpha_2\}\cup\{\bar{b}_{\gamma,i}:\gamma<\alpha_1\cap\alpha_2,\ i<\lambda_2\}\cup\bar{d}$ (use 4.7).

What is the truth value of $\varphi(\bar{b}_{\alpha,\beta},\bar{a}_\gamma,\bar{d})$ for $\alpha,\beta,\gamma<\lambda_2$?

True, if $\gamma=\alpha$.

False when $\gamma>\alpha$ (note that we have assumed $\alpha,\beta,\gamma<\lambda_2$ and not $\alpha,\beta,\gamma<\lambda_2^+$; note that $\varphi(\bar{b}_{\alpha,\beta},\bar{x},\bar{d})$ is not big, so only few $\bar{a}_i$ realize it, so no $\bar{a}_i$ realizes it for $i>\alpha$ (as then all such $\bar{a}_i$'s realize it).)

If $\gamma<\alpha$, the answer does not depend on β.

Let a_α code $\bar{a}_\alpha$, $b_{\alpha,\beta}$ code $\bar{b}_{\alpha,\beta}$. For notational simplicity we ignore the coding.

Let $P=\{a_\alpha:\alpha<\lambda_2\}$.

Let xEy iff $(\forall z\in P)(\varphi(x,z,\bar{d})\equiv\varphi(y,z,\bar{d}))$.

4.11 Fact : For $\alpha_1,\alpha_2,\beta_1,\beta_2<\lambda_2$, $b_{\alpha_1,\beta_1}E\ b_{\alpha_2,\beta_2}$ iff $\alpha_1=\alpha_2$.

Proof : We have just shown ($\Longleftarrow$).

Conversely, say $\alpha_1<\alpha_2$

$\varphi(b_{\alpha_1,\beta_1},a_{\alpha_2},\bar{d})$ is false but $\varphi(b_{\alpha_2,\beta_2},a_{\alpha_2},\bar{d})$ is true, so ($\Longrightarrow$) is clear.

So we have interpreted $Q^{eq}_{\lambda_2,\lambda_2}$, contradiction, hence we have proven 4.10.

4.12 Lemma : For any admissible M , and any $\varphi(x,\bar{y})$, there is an admissible expansion M^* of M, and $\psi(\bar{y})$ such that $M^*\models(\exists^{\leq\lambda_2}x)\varphi(x,\bar{y})\equiv\psi(\bar{y})$.

Proof : We define $A_i\subseteq M$ $i<\lambda_2^+$ increasing, continuous, $|A_i|\leq\lambda_2$. Take A_0 to witness lemma 4.7.

A_{i+1} realizes all Δ-types over A_i for all finite Δ, and $\langle A_{i+1};A_0,\ldots,A_i\rangle$ is an elementary substructure of $\langle M;A_0,\ldots,A_i\rangle$ even allowing the quantifier $\exists^{\leq\lambda_2}$.

Let E be the equivalence relation on $\mathcal{U}-A_0$: x_1Ex_2 iff $(\forall\bar{y}\subseteq A_0)(\varphi(x,\bar{y})\equiv\varphi(x_2,\bar{y}))$.

Clearly, every E- equivalence class is represented in each $A_{i+1}-A_i$.

We say that (i,j) is a **good pair** if $i<j$ and for any $\bar{a}$ such that $\bar{a}\cap(A_j-A_i)=\phi$, and $c\in A_j-A_i$, $\varphi(c,\bar{a})\equiv(\exists^{>\lambda_2}x)(xEc\wedge\varphi(x,\bar{a}))$.

4.13 Claim : If there are $i_0<j_0<i_1<j_1<\cdots<i_n<j_n$, $n>l(\bar{y})$, (i_l,j_l) good, then the lemma holds with $M^*=(M,A_{i_0},A_{j_0},\ldots,A_{i_n},A_{j_n})$ and $\psi(\bar{y})=\bigwedge_{l=0}^{n}$ [if $\bar{y}$ is disjoint from $A_{j_l}-A_{i_l}$ then there is no $c\in A_{j_l}-A_{i_l}$ such that $\varphi(c,\bar y)$.]

Proof : Suppose $M^*\models\psi(\bar y)$. Then for some l, $\bar y$ is disjoint to $A_{j_l}-A_{i_l}$, hence $M^*\models$ "there is no $c\in A_{j_l}-A_{i_l}$ such that $\varphi(c,\bar y)$".

By the definition of a good pair, and as every E-equivalence class is represented in $A_{j_l}-A_{i_l}$, and there are $\leq\lambda_2$ E-equivalence classes, clearly $M^*\models(\exists^{\leq\lambda_2}x)\varphi(x,\bar y)$.

For the converse, suppose $M^*\models(\exists^{\leq\lambda_2}x)\varphi(x,\bar y)$, and suppose $\bar y$ is disjoint to $A_{j_l}-A_{i_l}$ but $(\exists c\in A_{j_l}-A_{i_l})\varphi(c,\bar y)$. This contradicts the definition of a good pair. So we have proved 4.13.

Now we assume there are few good pairs (i,j) i.e. there are no i_m,j_m as in 4.13 and get a contradiction, thus finishing the proof of 4.12.

For a club set $C\subseteq\lambda_2^+$, the following holds:

(*) $\delta\in C, i<\delta$ implies $\delta>sup\{j:(i,j)$ is a good pair $\}$ if the sup is $<\lambda$.

By the choice of the A_i's (and see [Sh4], beginning of §2 (or guarantee this in the A_i's definition) also e.g. 4.15 is a repetition of this):

(**) if $\langle c\rangle^\frown\bar b_1\subseteq A_\delta$, $\bar b_2\cap A_\delta=\phi$, $M^*\models\varphi_1(c,\bar b_1{}^\frown\bar b_2)$ but $M^*\models(\exists^{\leq\lambda_2}x)(xEc\wedge\varphi_1(x,\bar b_1{}^\frown\bar b_2))$, φ_1 is gotten from φ by permuting the variables **then** for every $\beta>\delta$ (but $\beta<\lambda_2^+$) there is such a $\bar b_2$ with $\bar b_2\cap A_\beta=\phi$.

Let K be the set of $\langle c\rangle^\frown\bar b_1$ such that $\varphi(c,\bar b,\bar z)$ is big (when $\langle c\rangle^\frown\bar b_1\in\bigcup A_i$ this is equivalent to: for arbitrarily large β there is $\bar b_2$ as in the antecedent (above), $\bar b_2\cap A_\beta=\phi$.)

So again by the A_i's choice, if $\delta\in C$, $\bar b_1\subseteq A_\delta, c\notin A_\delta, c\in\bigcup_{i<\lambda_2^+}A_i=_{df}A_{\lambda_2^+}$, $\langle c\rangle^\frown\bar b_1\in K$ then $(\forall\beta<\lambda_2^+)(\exists c^1\in A_{\lambda_2^+})(\langle c^1\rangle^\frown\bar b_1\in K\wedge c^1\notin A_\beta)$. (This is by a similar hand-over-hand construction.)

Now if $\delta_1<\delta_2\in C$, (δ_1,δ_2) not good, we can contradict lemma 4.7, 4.10.

4.14 Lemma : For M^* rich enough, for every $\varphi(x_1,\ldots,x_{n+1})$ there are $\vartheta_{i,j}$ $(i\leq n,\ j<k)$ such that:

1) If $(\exists^{\leq\lambda_2}y)\varphi(x_1,\ldots,x_n,y)\wedge\varphi(x_1,\ldots,x_n,x_{n+1})$, then $\bigvee_{i=1}^{n}\bigvee_{j=1}^{k}\vartheta_{i,j}(x_{n+1},x_i)$.

2) $\forall x\exists^{\leq\lambda_2}y\,\vartheta_{i,j}(y,x)$.

Proof: : It suffices to find $\vartheta_{i,j}$ such that $\exists^{\leq\lambda_2}y\,\varphi(x_1,\ldots,x_n,y)\wedge\varphi(x_1,\ldots,x_{n+1})\to\bigvee_{i,j}[\vartheta_{i,j}(x_{n+1},x_i)\wedge\exists^{\leq\lambda_2}y\,\vartheta_{i,j}(y,x_i)]$, as then the formulas $\vartheta_{i,j}(x_{n+1},x_i) \wedge \exists^{\le\lambda_2} y\, \vartheta_{i,j}(y,x_i)$ witness the lemma (using 4.12).

We prove by induction on n.

For $n=0,1$ trivial.

Assume for n, and we shall prove for $n+1$. We assume M^* is rich enough to contain the unary predicate A^* as in lemma 4.7 and the formulas ψ as in lemma 4.12. We shall define $n^{**}=4$ and (latter) a sequence of finite sets of formulas

Δ_i ($i \le n^{**}$), $\Delta_i \subseteq \Delta_{i+1}$, $\varphi \in \Delta_0$.

Remark : The $n^{**}=4$ is somewhat misleading: in a sense it is large compared to $n(R)$ but this is absorbed by some w.l.o.g. below. What is the point in having those Δ_i? Lemma 4.10 gives us a kind of symmetry (if a depends on b then b depends on a). But this is not true if we restrict ourselves to dependency witnessed by a formula from a finite Δ_i. but if we have long enough increasing sequence of Δ_i for some i, Δ_i-dependency is equivalent to Δ_{i+1}-dependency (for those sequences).

So suppose $\varphi(a_1,\ldots,a_{n+1},c) \wedge \exists^{\le\lambda_2} x\, \varphi(a_1,\ldots,a_{n+1},x)$. We want to prove that some $\vartheta \in \Delta_{n^{**}}$ satisifies $\vartheta(c,a_i) \wedge \exists^{\le\lambda_2} x\, \vartheta(x,a_i)$, for some i . W.l.o.g., there are no repetitions in $\langle a_1,\ldots,a_{n+1},c \rangle$. (If $a_i=a_j$, use induction hypothesis on n; if $c=a_j$, we are done because we could have chosen to have $x=y \in \Delta_0$). W.l.o.g., no a_i satisfies any $\vartheta(x) \in \Delta_{n^{**}}$ such that $\exists^{\le\lambda_2} x\, \vartheta(x)$

Similarly for next observation, as then we use the induction hypothesis with the formula $\exists x [\varphi(a_1,\ldots,a_{i-1},x,a_{i+1},\ldots,a_{n+1},c) \wedge$

$$(\exists^{\le\lambda_2} y) \qquad \varphi(a_1,\ldots,a_{i-1},x,a_{i+1},\ldots,a_{n+1},y)$$
$$\wedge\, \vartheta(a_1,\ldots,a_{i-1},x,a_{i+1},\ldots,a_{n+1}) \wedge (\exists^{\le\lambda_1} z)\vartheta(a_1,\ldots,a_{i-1},z,a_{i+1},\ldots,a_{n+1})]:$$

W.l.o.g., for no $\vartheta \in \Delta_{n^{**}}$

$$\vartheta(a_1,\ldots,a_{i-1},a_i,a_{i+1},\ldots,a_{n+1}) \wedge \exists^{\le\lambda_2} x\, \vartheta(a_1,\ldots,a_{i-1},x,a_{i+1},\ldots,a_{n+1}).$$

Let $\bar a = \langle a_1,\ldots,a_n \rangle$ $\psi_i = \bigwedge \{\chi(\bar x,y,z) : \chi \in \Delta_i, \models \chi[\bar a,a_{n+1},c]\}$.

We know that $\exists^{\le\lambda_2} x\, \psi_i(\bar a,a_{n+1},x) \wedge \psi_i(\bar a,a_{n+1},c)$ for $i \le n^{**}$; we say a_{n+1} i-depends on $\bar a$ if $(\exists^{\le\lambda_2} y)(\exists x)(\psi_i(\bar a,y,x)) \wedge \exists x\, \psi_i(\bar a,a_{n+1},x))$.

We can assume that for $i<n^{**}$, a_{n+1} does not i-depend on $\bar a$ by putting $\psi_i(\bar a,y,x)$ into Δ_{i+1}. Similarly, $[i<n^{**} \Longrightarrow c$ does not i-depend on $\bar a]$.

Now by 4.10 (and the assumption, as Δ_1 is large enough and the uniformity of

4.10):
(*) for some $\vartheta^* \in \Delta_1$, $M^* \models \vartheta^*[a_1, \ldots, a_{n+1}, c]$ and $M^* \models (\exists^{\le\lambda_2} \langle x_1, \ldots, x_{n+1} \rangle) \vartheta^*(x_1, \ldots, x_{n+1}, c)$.
If $M^* \models (\exists^{>\lambda_2} \bar{x}) \psi_1(\bar{x}, a_{n+1}, c)$ then by (*) the formula $\vartheta^a(y,z) = (\exists^{>\lambda_2} \bar{x}) \psi_1(\bar{x}, y, z) \in \Delta_2$ necessarily satisfied $M^* \models \vartheta^a(a_{n+1}, c)$ and $M^* \models (\exists^{\le\lambda_2} y) \vartheta^a(y, c)$ hence for some $\vartheta \in \Delta_3, \vartheta(a_{n+1}, c) \wedge (\exists^{\le\lambda_2} z) \vartheta(a_{n+1}, z)$ contradiction. So assume $M^* \models \neg \vartheta^a(a_{n+1}, c)$. Also we can assume that $M^* \models (\exists^{>\lambda_2} \bar{x})(\exists y) \psi_2(\bar{x}, y, c)$ (otherwise use 4.10 and then the induction hypothesis on n) hence
$M^* \models (\exists^{>\lambda_2} \bar{x})(\exists y)[\psi_1(\bar{x}, y, c) \wedge \neg \vartheta^a(y, c)]$, hence there are pairwise disjoint $\bar{a}_\alpha (\alpha < \lambda_2^+)$ and elements b_α, such that $M^* \models \psi_1(\bar{a}_\alpha, b_\alpha, c) \wedge \neg \vartheta^a(b_\alpha, c)$. If there are λ_2^+ distinct b_α's, we easily contradict (*); so w.l.o.g. $b_\alpha = b$ for every α. But then $M^* \models \psi_1(\bar{a}_\alpha, b, c)$,$(\alpha < \lambda_2^+)$ implies $M^* \models (\exists^{>\lambda_2} \bar{x}) \psi_1(\bar{x}, b, c)$ contradicting $M^* \models \neg \vartheta^a(b, c)$.

This proves lemma 4.14.

* * *

4.15 Lemma : For any $\varphi(\bar{x}, \bar{y})$ there are $\vartheta_i(z, \bar{x})$ such that:
1) If $\exists^{\le\lambda_2} \bar{y}\, \varphi(\bar{a}, \bar{y}) \wedge \varphi(\bar{a}, \bar{b})$ then $\bigvee_i \vartheta_j(b_j, \bar{a})$

2) $\exists^{\le\lambda_2} z\, \vartheta_j(z, \bar{a})$ for every $\bar{a}$

Proof : By induction on the length of $\bar{y}$ and of $\bar{x}$.
Instead of one ϑ we can produce a finite set. We shall define $\Delta_i (i < n^{**})$ be finite, increasing. $\varphi(\bar{x}, \bar{y}) \in \Delta_0$
Assume $\models \varphi[\bar{a}, \bar{b}]$, $\varphi(\bar{a}, \bar{y})$ small.
We can make similar assumptions as in the proof of the previous lemma and define ψ_i similarly.
Let $\bar{b} = \bar{c} ^\frown \langle d \rangle$
By the induction hypothesis, for $i < n^{**}$ there are pairwise disjoint $\bar{c}^\alpha (\alpha < \lambda_2^+)$ such that $\exists z\, \psi_i(\bar{a}, \bar{c}^\alpha, z)$.
Say $\psi_i(\bar{a}, \bar{c}^\alpha, d^\alpha)$
if there are λ_2^+ distinct d^α's, we get a contradiction because $(\exists^{\le\lambda_2} \bar{y}) \varphi(\bar{a}, \bar{y})$. So w.l.o.g. $d^\alpha = d^0$ all $\alpha < \lambda_2^+$
So $(\exists^{\ge\lambda_2^+} \bar{w}) \psi_{i-1}(\bar{a}, \bar{w}, d)$ is one of the conjuncts of ψ_i

If $(\exists^{\geq\lambda_2^+}\overline{w})\psi_i(\overline{a},\overline{w},x)$ is not small we first define distinct d_i $(i<\lambda_2^+)$ such that $(\exists^{\geq\lambda_2^+}\overline{w})\psi_i(\overline{a},\overline{w},d_i)$, so $(\exists^{\geq\lambda_2^+}\overline{w})\psi_{i-1}(\overline{a},\overline{w},d_i)$ then define $\bar{c}^{i,\alpha},d^i$ pairwise disjoint for $\alpha<\lambda_2^+$ such that $\psi_{i-2}(\overline{a},\bar{c}^{i,\alpha},d_i)$. This shows $(\exists^{>\lambda_2}\overline{y})\psi_{i-1}(\overline{a},\overline{y})$. Contradiction.

If $(\exists^{\geq\lambda_2^+}\overline{w})\psi_i(\overline{a},\overline{w},x)$ is small, we get the desired conclusion.

4.16 Lemma : Every formula is equivalent to a Boolean combination of formulas of the form: $\bigwedge_{i=1}^{n}\vartheta_i(y_i,y_0)\wedge\psi(y_0,\ldots,y_n,F_1(y_0,\ldots,y_n),\ldots,F_k(y_0,\ldots,y_n))$ such that :

$\forall y\exists^{\leq\lambda_2}z\,\vartheta_i(z,y)\wedge\forall z\exists^{\leq\lambda_2}y\,\vartheta_i(z,y)$ and for some ϑ^1 we have

$(\forall x_0,\cdots)\vartheta^1(F_j(x_0,\cdots))\wedge(\exists^{\leq\lambda_2}x)\vartheta^1(x)$ and the F_i's are definable functions.

Proof : Let $\varphi(x_1,\ldots,x_n)$, $\langle a_1,\ldots,a_n\rangle$ be given. We define $n^{**}<\omega$, a sequence, increasing, of finite sets of formulas $\Delta_i(i\leq n^{**})$. Let

$I_i=\{l : a_l$ realizes a non-big formula in $\Delta_i\}$.

$T_i=\{\langle l,m\rangle:\vartheta(a_l,a_m)\neg\exists^{\leq\lambda_2}x\,\vartheta(x,a_m)$, for some $\vartheta\in\Delta_i\}$.

n^{**} is chosen big enough so that for some $i<j-8-2n(R)$, $j<n^{**}$, $I_i=I_j$, $T_i=T_j$. Note that T_i is an equivalence relation on $\{l:1\leq l\leq n, l\notin I_i\}$ when the appropriate ϑ's from the conclusion of lemma 4.10 and 4.12 are included in Δ_i $l+1$ for each l;

Since $T_i=T_j$, $j>i+8$, the necessary witnesses already appear in Δ_j. So T_j is an equivalence relation, as claimed.

Let $\overline{a}=\overline{a}_0\frown\bar{b}_1\frown\cdots\frown\bar{b}_m$ where $\overline{a}_0=\langle a_l:l\in I_i\rangle$ and such that a_l and a_{l^1} appear in the same $\bar{b}_j$ iff $a_l,a_{l^1}\in\overline{a}$ and $\langle l,l^1\rangle\in T_i$

We may assume that for each Δ_k there is a predicate A_k^* as in lemma 4.7and $A_k^*(x)\in\Delta_{k+1}$, so $|A_k^*|\leq\lambda_2$ and every complete Δ_k-type over A_k^* is minimal.

Also, using a few permutations, we have in some admissible expansion of M the predicates R_l^k such that R_l^k codes $\{\bar{b}_{l,\xi}^k:\xi<\xi_l<\lambda_2\}$, pairwise disjoint sequences of length $l=l(\bar{b}_l^k)$ such that R_l^k contains exactly one code for a sequence from each complete big type in $S_{\Delta_k}^l(A_k^*)$, and $\bar{b}_{l,\xi}^k\in A_{k+1}^*$ and it realizes a big Δ_k-type over $A_k^*\cup\bigcup_{\zeta<\xi}\bar{b}_{l,\zeta}^k$. So we may assume $R_l^k(x)\in\Delta_{k+1}$ $(l=1,\ldots,m)$. Similarly, we may assume for the functions F_l mapping sequences of the appropriate length which realize some big type in $S_{\Delta_k}(A_k^*)$ satisfying $\bigwedge_{i,j=1}^{n} \vartheta^*(b_i,b_j)$ [for $\vartheta^*(x,y)=\bigvee \{\vartheta(x,y)\in\Delta_i: \forall x \exists^{\leq\lambda_2} y\, \vartheta(x,y)$ and $\forall y \exists^{\leq\lambda_2} y\, \vartheta(x,y)\}$ which is in Δ_{k+1}] to the unique sequence in R_l^k realizing the same Δ_k- type over $A_k^*\}$, (i.e. $(F(\bar{x}_1) = y) \in \Delta_{k+1}$)

This proves lemma 4.16.

4.17 Theorem : Q_R is bi-interpretable with $\{Q_{\lambda_0}^{mon}, Q_{\lambda_1}^{1-1}, Q_E, Q_{R^*}\}$ with E an equivalence relation, and $|\mathrm{Dom}\ R^*|\leq\lambda_2$.

Proof : By what we already know, we may assume $|\mathrm{Dom}\ R|\leq\lambda_1$. We know $R(x_1,\ldots,x_{n(R)})$ is equivalent to some Boolean combination of formulas as in the statement of the previous lemma. There appear there formulas $\psi_l, \vartheta_l(x), \vartheta_l(x,y)$. Without loss of generality, $\forall x \exists^{\leq\lambda_2} y\, \vartheta_l(y,x) \wedge \forall y \exists^{\leq\lambda_2} x\, \vartheta_l(y,x)$.

Let $B^0=\{x; \bigvee_l \vartheta_l(x)\}$. Let $\vartheta^1(y,x)=\bigvee_l [\vartheta_l(y,x) \vee \vartheta_l(x,y)] \vee x=y$ (so ϑ^1 is symmetric but not necessarily transitive.)

On $\mathcal{U}-B^0$ we have the equivalence relation $E^0=$ the transitive closure of $\vartheta^1(y,x)$.

By our assumption, each equivalence class of E^0 has power $\leq\lambda_2$.

Let $B^1=B^0 \cup \{x: (\exists^{\leq\lambda_2} y)(|y/E^0|=|x/E^0|)\}$ and let $B^2=\{x\in\mathcal{U}-B^1 : (\exists y)(yE^0x \wedge (\exists^{\geq\aleph_0} z)[\vartheta^1(z,x)\vee\vartheta^1(x,z)])\}$

$E^1=E^0 \restriction B^2$

We want to interpret E^1 and analyze $E^0 \restriction (\mathcal{U}-B^0 \cup B^1)$. Note that if we want to "express" our life will be much easier. For each equivalence class C of E^1 we do the following:

Case I : There is $b_C\in C$ such that $|\{x\in\mathcal{U}-B^0: \vartheta^1(x,b_C)\}|=|C|$.

Let $D_C=\{x\in C: x\neq b_C, \vartheta^1(x,b_C)\}$.

Case II : Not I , so $|C|$ is singular.

Choose a regular $\lambda_C<|C|$ in such a way that ($\forall\mu\leq\lambda_2, \mu$ singular) ($\forall\lambda<\mu, \lambda$ regular) [$|\{C: |C|=\mu, C$ an E^1-equivalence class $\}| = |\{C: |C|=\mu,\ \lambda_C=\lambda,\ C$ an E^1-equivalence class $\}|$.]

This is possible as

$\lambda_2<|\{C: |C|=\mu\}|$ (else $C\subseteq B^1$); and choose $b_C\in C$, $D_C=\{x\in C: x\neq b_C, \vartheta^1(x,b_C)\}$ such that $|D_C|\geq\lambda_C$.

Let $P=\{b_C: C$ an E^1 equivalence class $\}$

$Q=\bigcup\{D_C : C$ an E^1 equivalence class $\}$.
W.l.o.g., P and Q are predicates of M, as $|P|\le\lambda_0$, $|Q|\le\lambda_0$.
Let yE^*z iff
$Q(y)\wedge Q(z)\wedge(\forall x)(P(x)\rightarrow\vartheta^1(y,x)\equiv\vartheta^1(z,x))$. The E^* equivalence classes are the sets D_C.

E^* will serve as the E mentioned in 4.17, so we have proved $Q_E\le_{int} Q_R$. Now we shall start to prove the other direction (we still have to define R^*).
We shall now interpret E^1.
Take some isomorphic copies of E^*, say E_0^*, E_1^*, such that for each E^1-equivalence class C satisfying $|C|=\mu$ is singular, E_0^* decomposes C into cf μ equivalence classes, each of power $<\mu$; and some E_1^* equivalence class includes exactly one element from each and is disjoint from all other C's, and E_0^*, E_1^* refine E_1.
For $|C|$ regular, C is an E_0^* equivalence class and an E_1^* equivalence class. So we have interpreted E^1.
(If $\lambda_1=|\mathcal{U}|$, it may happen that such a choice of E_0^* and E_1^* is not possible, but then split $\mathcal{U}$ into two parts closed under E^1 and do this on each part.)
Let $\Delta=\{\psi_l,\vartheta_l(x),\vartheta_l(x,y) : l\}$.
Let $S=\bigcup_{k=1}^{n(R)} S_\Delta^k(B^1,M)$
For each $p\in S$, choose $\bar{x}_p$ to realize p.
Let $B_2=B^1\cup\bigcup_{p\in S}\bar{x}_p$.
Let $R^*=R\restriction B_2$.
Suppose $|C|=\mu$, C an E^1-equivalence class. Then E^1 has $\ge\lambda_2^+$ equivalence classes of power μ, else C would be contained in B^1. So we can use several copies of E^1 to code whatever we want on C (for all C's simultaneously). In particular, we can have elements of C code sequences from C. We can also interpret the equivalence relation $xEy \overset{def}{=}$"x *and* y code sequences realizing the same Δ-type over B^1."
Use another few copies of E^1 together with 1−1 functions of power $\lambda_1\ge|\mathrm{Dom}\, R|$ to interpret the functions F_l (for coding F_l it is enough to have Rang (F_l) and E_{F_l} : $xE_{F_l}y$ iff $x=y$ $\vee(\exists\bar{z})[(y$ *code* $\bar{z})\wedge y\in B_2\wedge(x=F_l(y)\vee y=F_l(x)\vee F_l(x)=F_l(y))]$.)

Def. 5.1. So λ_2^+ satisfies 5.3A below. So from 5.10, 5.12 it follows that $|A| \le \lambda_2^+ \Longrightarrow |S^m_\Delta(A)| \le \lambda_2^+$. So λ_3^+ satisfies the demands of λ_2, hence $\lambda_2 \le \lambda_3^+$.

So there are only two possibilities: $\lambda_2=\lambda_3$ or $\lambda_2=\lambda_3^+$. For this section:

5.3 Hypothesis : $\lambda_2=\lambda_3^+$. Let $\lambda=\lambda_2$.

We shall eventually prove that $\{\exists_R, Q^{mon}_\lambda, Q^{1-1}_\lambda\}$ ($|\mathrm{Dom}\, R|=\lambda$) is bi-interpretable with Q^{word}_λ = { well orderings of A of order type λ: $|A|=\lambda\le|\mathcal{U}-A|$ }. Together with the preceding theorems, this completely analyzes the case $\lambda_2\ne\lambda_3$.

However we want to do this in a somewhat more general case, so for the rest of this section:

5.3A Hypothesis : λ is regular , $\lambda\le\lambda_2$ and for finite Δ, m and admissible M, if $A\subseteq\mathcal{U}, |A|<\lambda$, then $|S^m_\Delta(A,M)|<\lambda$ (hence as in 5.2's proof, $Q^1_{\lambda,\lambda} \not\le_{int} Q_R$).

5.4 Definition : We say q is a pure extension of p (both are m-types) if $x_l=c\in q \Longrightarrow x_l=c\in p$; we write $p\subseteq_{pr} q$. We call p pure if $\phi\subseteq_{pr}p$.

5.5 Definition : For every admissible M, $|A|<\lambda$, $p\in S^m_\Delta(A,M)$ we define rank $Rk(p)=\langle\alpha,\beta\rangle$ (α,β *may be* ∞) (really we should write Rk^m_Δ):

$Rk(p)\ge\langle 0,0\rangle$ if p is realized by some $\bar{a}$.

$Rk(p)\ge\langle\alpha,\gamma\rangle$ ($0<\gamma<\infty$) if for every $\beta<\gamma$, and $A^1\supseteq A$ such that $|A^1|<\lambda$, p has an extension $q\in S^m_\Delta(A^1,M)$ such that $Rk(q)\ge\langle\alpha,\beta\rangle$ and if $\alpha>0$, q is a pure extension of p.

$Rk(p)\ge\langle\alpha,\infty\rangle$ if $Rk(p)\ge\langle\alpha,\gamma\rangle$ for every γ.

$Rk(p)\ge\langle\alpha,0\rangle$ when $\alpha>0$, if for every $\beta<\alpha$ there are $A^1\supseteq A$, $|A^1|<\lambda$, and $[q_1\supseteq_{pr}p$ and $q_2\supseteq_{pr}p]$ or $[\alpha=1, q_1\supseteq p, q_2\supseteq p]$ such that $q_1,q_2\in S^m_\Delta(A^1,M)$, $q_1\ne q_2$, $Rk(q_1)\ge\langle\beta,\infty\rangle$, $Rk(q_2)\ge\langle\beta,\infty\rangle$.

Now $Rk(p)=\langle\alpha,\beta\rangle$ iff $Rk(p)\ge\langle\alpha,\beta\rangle$ and $Rk(p)\not\ge\langle\alpha,\beta+1\rangle$.

$Rk(p)=\langle\alpha,\infty\rangle$ iff $Rk(p)\ge\langle\alpha,\beta\rangle$ all β, $Rk(p)\not\ge\langle\alpha+1,0\rangle$.

$Rk(p)=\langle\infty,\infty\rangle$ if $Rk(p)\ge\langle\alpha,\beta\rangle$ for all α,β.

5.6 Remark : We can show that if $Rk(p)=\langle\alpha,\beta\rangle$ then $\beta\in\{0,\infty\}$. Note that there is no connection between ranks for different Δ's.

5.7 Claim: : Fix Δ, m, p,q will be complete $\Delta-m-$types.

0) $p\subseteq_{pr}q \Longrightarrow Rk(p)\ge Rk(q)$

It is easy to interpret R.

The analysis is complete, getting the biinterpretability except that we have forgotten B^3=Dom $R-B^1\cup B^2$. On B^3 , E^0 may have countable equivalence classes but $(\forall x\in B^3)(\exists^{<\aleph_0}y)\vartheta^1(y,x)$. We shall deal with the new points only.

First we can define a partition of B^3 to $B_l^3(l=0,1,2,3)$ such that $\vartheta^1(x,y)$, $x\in B_l^3$ implies $y\in B_{l-1}^3\cup B_l^3\cup B_{l+1}$ (where $l-1,l+1$ is computed mod 4) [e.g. choose $x_C\in C$ from each E^0-equivalence class C ($\subseteq B^3$) and let $y\in C$ be in B_l^3 if $d(y,x)\equiv l$ mod 4 where $d(y,x)=Min\{k$: there are $z_1,\dots,z_k$, $y=z_1,x=z_k,\vartheta^1(z_i,z_{i+1})$ for each $i\}$].

Next for $x\in B^3$ let $\lambda_x=|\{y:\vartheta^1(y,x)\}|$ and $\mu(\lambda)=|\{x\in B^3:\lambda_x\geq\lambda\}$ (for $\lambda<\aleph_0$.) We can assume each $\mu(\lambda)$ is 0 or $\geq\aleph_0$, (and even $\geq\lambda_2^+$) and note $\mu(\lambda)$ is decreasing in λ, hence eventually constant, say for $k\leq\lambda<\aleph_0$.

Now we can interprate $\exists_E$, E an equivalence relation which for $\chi\leq k$ has exactly $\mu(\lambda)$ classes.

For the converse, let us e.g. interprate $\vartheta_i(x,y)$. It suffices to code for $l<4$, $S=\{\langle x,y\rangle:\vartheta_1(x,y)\wedge(x\in B_l^3)\}$ Note that $|B_l^3|=|B^3|>\lambda_2$ (by the definition of B^1).

Let F be a one-to-one function from S into B_{l+2}^3, and let E_1,E_2 be equivalence relations. The E_1-equivalence classes are $\{x\}\cup\{F(\langle x,y\rangle):\langle x,y\rangle\in S\}$, for $x\in B_l^3$, and the E_2 equivalence classes are $\{y\}\cup\{F(\langle x,y\rangle):\langle x,y\rangle\in S\}$. (we can assume B_{l+2}^3 has the right cardinality as we are dealing with $\geq\lambda_2^+$ equivalence classes hence could have chosen it suitably). Together with monadic predicates the reconstruction is easy; as well as dealing with the ψ's.

§5 In the first stability cardinal

5.1 Definition :

Let $\lambda_3=\lambda_3(R)$ be the least λ such that

$(\forall A\subseteq\mathfrak{M})(|A|\leq\lambda\rightarrow|S_\Delta^m(A,M)|\leq\lambda$

Δ finite, M admissible.

5.2 Fact : λ_2 is λ_3 or λ_3^+

Proof : Clearly $\lambda_3\leq\lambda_2$.

Suppose $\lambda_3\neq\lambda_2$. We cannot interpret $Q^{eq}_{\lambda_3^+,\lambda_3^+}$ because otherwise for some admissible M, finite Δ, $A\subseteq M$, $|A|=\lambda_3$ we would have $|S_\Delta(A,M)|=\lambda_3^+$, contradiction to

1) $Rk(p) \geq \langle 0,0 \rangle$ iff $Rk(p) \geq \langle 0,\infty \rangle$.

2) If p is realized by no λ pairwise disjoint m-tuples outside Dom p then $Rk(p) \leq \langle 1,0 \rangle < \langle 1,\infty \rangle$.

3) If p is realized by $\geq\lambda$ pairwise disjoint outside Dom p m-tuples then $Rk(p) \geq \langle 1,\infty \rangle$.

Proof : 0) is obvious.

1) Let $\bar{a}$ realize p. Suppose $Rk(p) \geq \langle 0,\beta \rangle$ all $\beta<\alpha$. Suppose $A^1 \supseteq A$ is given, $|A^1|<\lambda$. then $q=tp_\Delta(\bar{a},A^1)$ extends p, so $Rk(p) \geq \langle 0,\alpha \rangle$.

2) If p is realized by no λ pairwise disjoint m-tuples, let A^1 be such any no sequence disjoint to A^1-Dom p realize p, Dom $p \subseteq A^1$, and $|A^1|<\lambda$. There is no $q \supseteq_{pr} p$, $q \in S^m_\Delta(A^1,M)$, hence $Rk(q) \not\geq \langle 1,1 \rangle$. So $Rk(p) \leq Rk(q) \leq \langle 1,0 \rangle$.

3) Suppose p is realized by $\geq\lambda$ pairwise disjoint outside Dom p sequences, pairwise disjoint outside Dom p. We prove $Rk(p) \geq \langle 1,\gamma \rangle$ by induction on γ.

$\gamma=0$: Let $\bar{a} \neq \bar{b}$ realize p. Let $A^1=A \cup \bar{a} \cup \bar{b}$ and let $q_1=tp_\Delta(\bar{a},A^1)$ $q_2=tp_\Delta(\bar{b},A^1)$. Easily $\{q_1,q_2\}$ witnesses $Rk(p) \geq \langle 1,0 \rangle$.

$\gamma>0$: Let $A^1 \supseteq A$, $|A^1|<\lambda$, $\beta<\gamma$. We know $|S^m_\Delta(A^1,M)|<\lambda$.

So by Hypothesis 5.3A p has $<\lambda$ extensions in $S^m_\Delta(A^1,M)$. Since p is realized by λ pairwise disjoint outside Dom p sequences, some extension q of p in $S^m_\Delta(A^1,M)$ is realized by λ pairwise disjoint sequences, by regularity of λ.

By the induction hypothesis, $Rk(q) \geq \langle 1,\beta \rangle$, as required. This proves the claim.

5.8 Claim : Assume $p \in S^m_\Delta(A,M)$, $Rk(p) = \langle \alpha,\infty \rangle$, $0<\alpha<\infty$, $A \subseteq B$, $|B|<\lambda$. Then p has one and only one pure extension $q \in S^m_\Delta(B,M)$ of the same rank.

: **Proof** : Take γ^* so large that

$Rk(p^1) \geq \langle \alpha,\gamma^* \rangle \Rightarrow Rk(p^1) \geq \langle \alpha,\infty \rangle$ (possible, as there are only set-many types). We know $Rk(p) \geq \langle \alpha,\gamma^*+1 \rangle$ so p has a pure extension $q \in S^m_\Delta(B,M)$ with $Rk(q) \geq \langle \alpha,\gamma^* \rangle$. Hence $Rk(q) \geq \langle \alpha,\infty \rangle$. If there are two such q, then $Rk(p) \geq \langle \alpha+1,0 \rangle$, contradiction.

5.9 Claim : 1) If $\lambda>\aleph_0$, then for any A of cardinality less than λ, and finite Δ,m, there is $B \supseteq A$, $|B|<\lambda$, such that

$$p \in S^m_\Delta(B,M) \Rightarrow [Rk(p) = \langle \alpha,\infty \rangle \text{ for some } \alpha<\infty \text{ or } Rk(p \restriction A) = \langle \infty,\infty \rangle]$$

2) We can do (1) simultaneously for all Δ.

Proof : 1) Define A_n $(n \in \omega)$ by induction:

$A_0=A$.

Suppose A_n has been defined. For each $p\in S^m_\Delta(A_n,M)$ such that $Rk(p)=\langle\alpha,\gamma\rangle$, $\gamma<\infty$, take $B_p\supseteq A_n$ $|B_p|<\lambda$ such that p has no extension in $S^m_\Delta(B_p,M)$ of rank $\langle\alpha,\gamma\rangle$. (i.e. B_p witnesses $Rk(p)\not\geq\langle\alpha,\gamma+1\rangle$).

Let $A_{n+1}=A_n\cup\bigcup\{B_p:\ p\in S^m_\Delta(B_p,M)\}$.

Let $p\in S^m_\Delta(\bigcup_n A_n,M)$. Since $Rk(p\restriction A_0)\geq Rk(p\restriction A_1)\geq Rk(p\restriction A_2)\geq\cdots$, we can find N such that $Rk(p\restriction A_N)=Rk(p\restriction A_{N+1})=\cdots$. Suppose $n>N$ and $Rk(p\restriction A_n)=\langle\alpha,\gamma\rangle$, $\gamma\neq\infty$. $Rk(p\restriction A_n)=Rk(p\restriction A_{n-1}), A_{n-1}\subseteq A_{n-1}\cup B_{p\restriction A_{n-1}}=B_{p\restriction A_{n-1}}\subseteq A_n$ so $p\restriction B_{p\restriction A_{n-1}}$ is an extension of $p\restriction A_{n-1}$ of the same rank, contradicting the definition of B_p. So $n>N\rightarrow[Rk(p\restriction A_n)=\langle\alpha,\infty\rangle$ *for some* $\alpha]$.

Let $A^*=\bigcup_n A_n$.

If $Rk(p\restriction A_N)\neq\langle\infty,\infty\rangle$, take $\alpha<\infty$ such that $Rk(p\restriction A_n)=\langle\alpha,\infty\rangle$ for every large enough n. $p\restriction A_N$ has a unique extension $q\in S^m_\Delta(A^*,M)$ such that $Rk(q)=\langle\alpha,\infty\rangle$. Also $p\restriction A_N$ has a unique extension in $S^m_\Delta(A_n,M)$ of rank $\langle\alpha,\infty\rangle$, but $q\restriction A_n$, $p\restriction A_n$ are such extension for large n.

So $p\restriction A_n=q\restriction A_n$ for large n, so $p=q$, so $Rk(p)=\langle\alpha,\infty\rangle$.

2) Same proof.

5.10 Fact : If $Rk(p)<\langle\infty,\infty\rangle$ for every $p\in S^m_\Delta(A,M)$, $|A|<\lambda$, then $\lambda_2\leq\lambda$ (hence w.l.o.g. $|\mathrm{Dom}\ R|\leq\lambda$.

Proof : Easy noting $A\subseteq B\Longrightarrow|S^m_\Delta(A)|\leq|S^m_\Delta(B)|$.

5.11 Lemma : Suppose for some large enough finite Δ, for each Δ-type p in m variables $Rk(p)<\langle 2,\infty\rangle$. Then,

1) $\{\exists_R, Q^{mon}_{\lambda_0}, Q^{1-1}_{\lambda_1}\}$ can be analyzed as before (in §4, with λ for λ_2^+), and $\{\exists_R, Q^{mon}_{\lambda_0}, Q^{1-1}_{\lambda_1}\}\equiv_{int}\{Q^{mon}_{\lambda_0}, Q^{1-1}_{\lambda_1},\exists_E,\exists_{R^*}\}$ for some equivalence relation E, and relation R^*, $|\mathrm{Dom}\ R^*|<\lambda$.

or 2) $\{\exists_R, Q^{1-1}_\lambda\}$ is bi-interpretable with $\{Q^{word}_\lambda, Q^{eq}_{\kappa,\lambda}\}$, some κ.

or 3) $\{\exists_R, Q^{1-1}_{\lambda_1}\}$ is bi-interpretable with $\{\exists_{R^*},\exists_E\}$, E an equivalence relation $|\mathrm{Dom}\ R^*|<\lambda=\aleph_0$.

Proof : Notice that if $p\in S^1_\Delta(A,M)$ has rank $\langle 1,\infty\rangle$, then p is minimal big.

We shall determine Δ later.

Let A^* be as in the previous (of 5.9), so the rank of any Δ-type in one variable over A^* is either $\langle 0,\infty\rangle$ or $\langle 1,\infty\rangle$. (If $\lambda=\aleph_0$ we can still get this by Konig Lemma and 5.7(1).

Let $\mathcal{P}_m=\{p\in S^m_\Delta(A^*,M):Rk(p)=\langle 1,\infty\rangle\}$, $\kappa_m=|\mathcal{P}_m|$ and $\kappa=\kappa_{m(*)}$ where $m(*)$ is large enough with respect to Δ let $\mathcal{P}=\bigcup_{m\le m(*)}\mathcal{P}_m$. We can interpret $Q^{eq}_{\kappa_m,\lambda}$ - in fact, for $m=1$, the equivalence relation of realizing the same Δ-type over A^* with domain $\{a:a$ realizes some $p\in\mathcal{P}_m$, $a\in\mathrm{Dom}\,(R)\}$, is an equivalence relation of this form. For $m>1$, remember we can code sets of $\le\lambda$ pairwise disjoint sequences so we can interpret $Q^{eq}_{\kappa,\lambda}$.

Define $A_i\,(i<\lambda)$ continuous, increasing, such that:

1) $A_0=A^*$.

2) $A_{i+1}\supseteq A_i\cup\bigcup_p B_p$ where B_p is defined as before.

3) $|A_i|<\lambda$ for $i<\lambda$.

4) $\mathrm{Dom}\,(R)=\bigcup_{i<\lambda}A_i$ (see 5.10).

We know that every $p\in\mathcal{P}_m$ has a unique pure extension $p^{[i]}\in S^m_\Delta(A_i,M)$ of the same rank. We shall show that every pure $p\in S^m_\Delta(A_i,M)$ is of this form, provided that Rank $(p)=\langle 1,\infty\rangle$.

If $p\restriction A_0\not\in\mathcal{P}_m$, then it has rank $\langle 0,\infty\rangle$, so $Rk(p)\le Rk(p\restriction A_0)=\langle 0,\infty\rangle<\langle 1,\infty\rangle=Rk(p)$, contradiction.

If $p\restriction A_0=q\in\mathcal{P}_m$ but $p\ne q^{[i]}$, then for some $\varphi\in q^{[i]}$, $\neg\varphi\in p$. But by Def. 5.5, $p,q^{[i]}$ exemplify $Rk(q)\ge(2,0)$, contradiction.

This proves every p of rank $\langle 1,\infty\rangle$ in $S^m_\Delta(A_i,M)$ is $q^{[i]}$ for some $q\in\mathcal{P}_m$.

We assume for a while:

Hypothesis A $:(\forall i)(\exists j>i)(\exists m\le m(*))(\exists p_i\in S^m_\Delta(A_j,M))\,[\,Rk(p_i)\ge\langle 1,\infty\rangle$ and p_i

Δ_1 -splits over A_i] ($\Delta_1\supseteq\Delta$ to be determined.)

where we define: $p\in S^m_\Delta(A,M)$ Δ_1-splits over $B\subseteq A$ if there are $\bar b,\bar c\subseteq A$ realizing the same Δ_1-type over B and there is $\varphi\in\Delta$ such that $\varphi(\bar x,\bar b)\in p$, $\neg\varphi(\bar x,\bar c)\in p$.

Clearly for all i, $p_i=q_i^{[j]}$ for some $j=j_i$, and some $q_i\in\mathcal{P}$ (when we restrict ourselves to Δ-types in one variable.) As $|\mathcal{P}_{m(*)}|<\lambda$, we may assume all q_i are the same, $q=q_i$, and q is pure. For notational simplicity, let $j_i=i+1$.

For each i, let $\bar a_i\subseteq A_{i+1}-A_i$ realize $q^{[i]}$ and $\bar b_i,\bar c_i\subseteq A_{i+1}$ be such that

$\varphi_i(x,\bar{b}_i)$, $\neg\varphi_i(x,\bar{c}_i)\in q^{[i+1]}$. We may assume all the $\varphi_i(\bar{x},\bar{y})$ are the same, $\varphi_i=\varphi$. Now $\varphi(\bar{a}_\alpha,\bar{b}_\beta)\wedge\neg\varphi(\bar{a}_\alpha,\bar{b}_\beta)$ holds whenever $\alpha>\beta$ (as $\bar{a}_\alpha$ realizes $q^{[\alpha]}\supseteq q^{[\beta+1]}\supseteq\{\varphi(\bar{x},\bar{b}_\beta),\neg\varphi(\bar{x},\bar{c}_\beta)\}$) and $\varphi(\bar{a}_\alpha,\bar{b}_\beta)\wedge\neg\varphi(\bar{a}_\alpha,\bar{c}_\beta)$ fails if $\alpha<\beta$ when we choose Δ_1 appropriately, namely, when we ensure $\bar{b}_\beta$ and $\bar{c}_\beta$ realize the same $\{\psi(\bar{y};\bar{x})\}$-type over A_β where $\psi(\bar{y},\bar{x})=\varphi(\bar{x},\bar{y})$..
So some formula well-orders $\{\bar{a}_\alpha{}^\frown\bar{b}_\alpha{}^\frown\bar{c}_\alpha:\alpha<\lambda\}$. There is a subset of power λ which is a Δ-system, (as λ is regular $\geq\aleph_0$) so we can code the elements of that subset (with a few permutations) by elements of M and thus interpret Q^{word}_λ so $\{Q^{eq}_{\kappa,\lambda},Q^{word}_\lambda\}\leq_{int}\{Q_R,Q^{1-1}_\lambda\}$.
To see $Q_R\leq_{int}\{Q^{eq}_{\kappa,\lambda},Q^{word}_\lambda\}$, for simplicity we show that this holds when R is binary. ($|\mathrm{Dom}\,R|=\lambda$, of course). With a well-order and a set we code an equivalence relation E whose equivalence classes are $A_{i+1}-A_i$. Recall $\kappa=|\mathcal{P}_{m(*)}|$. On each E-equivalence class C, we can code (by more well orderings) $R\restriction C$ and for every $q\in\mathcal{P}$ and $a\in C$ we have to say whether a realizes q and whether $R(x,a)\in q^{[i+1]}$. We can do this with $Q^{eq}_{\kappa,\lambda}$ and Q^{word}_λ. So we have proved the desired conclusion (5.11(2)).

So we finish the case Hypothesis A holds, so assume

Hypothesis B : Hypothesis A is false.
By relabelling and taking A_i for some large i as our A_0, we can assume no $q^{[i]}$ Δ_1-splits over A_0. (for every $q\in\mathcal{P}$).
Now we ask:

If $A_0\subseteq A_1\subseteq A_2$ are as above, $\bar{a}_1\subseteq A_2$, $\bar{b}_1\subseteq A_2$, $\bar{a}_2\subseteq\mathcal{U}-A^*_2$, $\bar{b}_2\subseteq\mathcal{U}-A^*_2$,

$p_2=tp_{\Delta_1}(\bar{a}_2,A_2)\supseteq p_1=tp_{\Delta_1}(\bar{a}_1,A_1)$

$q_2=tp_{\Delta_1}(\bar{b}_2,A_2)\supseteq q_1=tp_{\Delta_1}(\bar{b}_1,A_1)$.

$Rk_{\Delta_1}(p_1)=Rk_{\Delta_1}(p_2)$, $Rk_{\Delta_1}(q_1)=Rk_{\Delta_1}(q_2)$.

Must $tp_\Delta(\bar{a}_1{}^\frown\bar{b}_2,A_1)=tp_\Delta(\bar{a}_2{}^\frown\bar{b}_1,A_1)$?

(Caution: Unlike first order types, the answer may depend on the specific $\bar{a}_l,\bar{b}_l$ used and not just the types they realize.)
If the answer is yes, (for every Δ, for some A_0 for every A_1,A_2), then we can essentially copy the analysis (in § 4) of reducing from $|\mathrm{Dom}\,R|=\lambda_1$ to $|\mathrm{Dom}\,R|=\lambda_2$ and get the desired conclusion (5.11(1) if $\lambda>\aleph_0$, or 5.11(3) if $\lambda=\aleph_0$),

If the answer is no (for some Δ, for every A_0), then by inductively choosing counter- examples, thinning to a Δ-system, and coding via permutations, we can interpret Q_λ^{word} and, as before, we get $\{Q_R, Q_\lambda^{1-1}\} \equiv_{int} \{Q_{\lambda,k}^{eq}, Q_\lambda^{word}\}$. This proves lemma 5.11.

Now we are reduced to the case that $Rk(p)=\langle\infty,\infty\rangle$ for some p or $Rk(p)\geq\langle 2,0\rangle$ for some Δ-type p in one variable.

5.12 Lemma : For no $p\in S_\Delta^m(A,M)$ is $Rk(p)=\langle\infty,\infty\rangle$,$(|A|<\lambda)$.

Proof : We assume $Rk(p)=\langle\infty,\infty\rangle$ and reach a contradiction by interpreting $Q_{\lambda,\lambda}^{eq}$.

5.13 Definition : Suppose (by adding dummy variable) that Δ is a (finite) set of formulas of the form $\varphi(\bar{x},\bar{y})$ (with a fixed $\bar{y}$) and p is a Δ-type in the sequence of variables $\bar{x}$. Let Δ^c be the set of formulas obtained by reversing the role of $\bar{x}$ and $\bar{y}$; i.e. a Δ^c-type would consist of formulas $\varphi(\bar{a},\bar{y})$.

5.13A Fact : If $p=tp_\Delta(\bar{a},A)$, $Rk(p)=\langle\infty,\infty\rangle$, then for some $B\supseteq A$, $|B|<\lambda$, $q=tp_\Delta(\bar{a}^1,B)\supseteq_{pr}p$, $Rk(q)=\langle\infty,\infty\rangle$ and q Δ^c-splits over A.

Proof : Choose $B_0\supseteq A, |B_0|<\lambda$ such that every Δ^c-type over A realized in M is realized in B_0.

We take $p_0\in S_\Delta^m(B_0,M)$, $p_0\supseteq_{pr}p$, $Rk(p_0)=\langle\infty,\infty\rangle$. So there exists $\varphi(\bar{x},\bar{b})$ such that both $p_0\cup\{\varphi(\bar{x},\bar{b})\}$ and $p_0\cup\{\neg\varphi(\bar{x},\bar{b})\}$ can be completed to Δ-types of rank $\langle\infty,\infty\rangle$.

So there is $\bar{c}\subseteq B_0$, $tp_{\Delta^c}(\bar{c},A)=tp_{\Delta^c}(\bar{b},A)$.

Without loss of generality, $\varphi(\bar{x},\bar{c})\in p_0$.

So $p_0\cup\{\varphi(x,\bar{c})\neg\varphi(x,\bar{b})\}$ can be completed to a Δ -type rank $\langle\infty,\infty\rangle$ which Δ^c-splits over A (and is a pure extension of p_0).

5.13B Fact : We can interpret $Q_{\lambda,<\lambda}^{eq}$.

Proof : Take $A_i,(i<\lambda)$ as in lemma the proof of 5.11.

For each i, take $p_i=tp_\Delta(\bar{a}_i,A_i)$ to have rank $\langle\infty,\infty\rangle$ and w.l.o.g. is pure. By fact 5.13A we can take $\bar{b}_i$, $\bar{c}_i$ such that $tp_{\Delta^c}(\bar{b}_i,A_i)=tp_{\Delta^c}(\bar{c}_i,A_i)$ and $p_i\cup\{\varphi_i(\bar{x},\bar{b}_i),\neg\varphi_i(\bar{x},\bar{c}_i)\}$ has a pure completion of rank $\langle\infty,\infty\rangle$.

So for some $\psi_i(\bar{x},\bar{d}_i)$ both $p_i\cup\{\varphi_i(\bar{x},\bar{b}_i),\neg\varphi_i(\bar{x},\bar{c}_i),\pm\psi_i(\bar{x},\bar{d}_i)\}$ have completions of rank $\langle\infty,\infty\rangle$.

Let $\bar{a}_{i,\alpha}^l$ be pairwise disjoint, $\bar{a}_{i,\alpha}^l\cap A_i=\phi$ $(\alpha<i$, $l=0,1)$, $\bar{a}_{i,\alpha}^l$ realize $p_i\cup\{\varphi_i(\bar{x},\bar{b}_i),\neg\varphi_i(\bar{x},\bar{c}_i)\}$ and $\psi_i(\bar{a}_{i,\alpha}^l,\bar{d}_i)$ iff $l=0$.

Without loss of generality, $\bar{d}_i{}^\frown\bar{b}_i{}^\frown\bar{c}_i{}^\frown\bar{a}^l_{i,\alpha}\subseteq A_{i+1}$. for $\alpha<|A_{i+1}|$; w.l.o.g., $\varphi_i=\varphi$, $\psi_i=\psi$ do not depend on i. W.l.o.g., $\bar{b}_i{}^\frown\bar{c}_i{}^\frown\bar{d}_i\cap A_i$ is constant, (if $\lambda>\aleph_0$, by applying Fodor's theorem to $F(i)=$ at least j such that $A_i\cap(\bar{b}_i{}^\frown\bar{c}_i{}^\frown\bar{d}_i)\subseteq A_j$; and then using that there are λ-many i but less than λ-many finite sequences from A_j; if $\lambda=\aleph_0$, by the Δ-system lemma and renaming.) W.l.o.g. , $\bar{a}^l_{i,\alpha}$ is disjoint to $\bar{b}_i{}^\frown\bar{c}_i{}^\frown\bar{d}_i$.

We can interpret $P=\{\bar{b}_i{}^\frown\bar{c}_i{}^\frown\bar{d}_i;\ i<\lambda\}$ since we have arranged that they form a Δ-system.

Let f be the permutation

$f(\bar{a}^l_{i,\alpha})=\bar{a}^{1-l}_{i,\alpha}$, f is the identity elsewhere.

When does $\varphi(\bar{a}^l_{i,\alpha},\bar{b}_j)\wedge\neg\varphi(\bar{a}^l_{i,\alpha},\bar{c}_j)\wedge[\psi(\bar{a}^l_{i,\alpha},\bar{d}_j)\equiv\neg\psi(f(\bar{a}^l_{i,\alpha}),\bar{d}_j)]$ hold? For $i=j$, the formula is true by inspection.

For $i<j$, the answer is no, as $\bar{b}_j,\bar{c}_j$ realize the same Δ^c-type over $A_j\supseteq\bar{a}^l_{i,\alpha}$.

For $i>j$, the answer is no; since $\bar{d}_j\subseteq A_i$, and $\bar{a}^l_{i,\alpha},f(\bar{a}^l_{i,\alpha})$ realize p_i which is a complete Δ-type over A_i, contrary to the third conjunct.

So we can interpret E with domain $\{\bar{a}^0_{i,\alpha}:\ \alpha<i<\lambda\}$, $\bar{a}^0_{i_1,\alpha_1}E\ \bar{a}^0_{i_2,\alpha_2}\equiv_{df} i_1=i_2$. (using P and f to do so, remember 4.3(2)).

But E is in $Q^{eq}_{\lambda,<\lambda}$.

5.13C Fact : We can interpret Q^{word}_λ.

Proof : By Fact 5.13B we can interprete an equivalence relation E with equivalence classes $A_{i+1}-A_i$.

Let E_i be the equivalence relation on finite sequences of suitable length m from $A_{i+1}-A_i$:

$\bar{a}_1E_i\bar{a}_2$ iff $\bar{a}_1,\bar{a}_2$ realize the same Δ^c-type over A_i.

We can code $\bigcup_{i<\lambda}E_i=E^1$ by fact 5.13B, since $|A_{i+1}-A_i|<\lambda$.

Let $x\in A_{i+1}-A_i$ $y\in A_{j+1}-A_j$.

If $j<i$,

$\vartheta(x,y)\overset{def}{=}$" If $\{b_1,\ldots,b_m,c_1,\ldots,c_m,x\}$ are in the same E-equivalence class and $\bar{b}E^1\bar{c}$ then $(\forall\bar{z}\ \textit{such that}\ z_l Ey)\bigwedge_{\varphi\in\Delta}(\varphi(\bar{z},b_1,\ldots,b_m)\equiv\varphi(\bar{z},c_1,\ldots,c_m))$ holds."

Obviously if $j<i$, $\vartheta(x,y)$ holds.

If $j>i$, $p_i\cup\{\varphi_i(\bar{x},\bar{b}_i),\neg\varphi_i(\bar{x},\bar{c}_i)\}$ has a completion of rank $\langle\infty,\infty\rangle$, so w.l.o.g. it is realized in $A_{j+1}-A_j$, so $\vartheta(x,y)$ fails.

This proves Fact 5.13C.

5.13D Fact : We can interpret $Q^{eq}_{\lambda\lambda}$.

Proof : We can find $\bar{x}_{i,j},\bar{y}_{i,j}\subseteq A_{j+1}-A_j$, $(i<j)$ such that $tp_\Delta(\bar{x}_{i,j},A_i)=tp_\Delta(\bar{y}_{i,j},A_i)$, $tp_\Delta(\bar{x}_{i,j},A_{i+1})\neq tp_\Delta(\bar{y}_{i,j},A_{i+1})$ for all $i<j<\lambda$.
(we can take them to realize $p_i\cup\{\pm\psi_i(\bar{x},\bar{d}_i)\}$ from the proof of 5.13A).
In fact there are $|A_i|$ such pairwise disjoint pairs.
So, w.l.o.g., $\bar{x}_{i_1,j_1},\bar{x}_{i_2,j_2},\bar{y}_{i_1,j_1},\bar{y}_{i_2,j_2}$ are all disjoint for $(i_1,j_1)\neq(i_2,j_2)$. Since we can interpret Q^{word}_λ, we can interpret the equivalence relation $\bar{x}_{i_1,j_1}E\bar{x}_{i_2,j_2}$ iff $i_1=i_2$.
So we have proved 5.13D, hence 5.12.

5.14 Lemma : For no p (and Δ) $Rk(p)>\langle 2,\infty\rangle$).

Proof : We know $(\forall p)(Rk(p)<\langle\infty,\infty\rangle)$. If the lemma fails we shall interpret $Q^{eq}_{\lambda\lambda}$ getting a contradiction. By Def. 5.5 we can find p $Rk(p)=\langle 2,\infty\rangle$.
We can define A_i $(i<\lambda)$, A_i increasing continuous as in 5.11's proof, $|A_i|<\lambda$, $p_0\in S^m_\Delta(A_0)$, $Rk(p_0)=\langle 2,\infty\rangle$; $p_i=p_0^{[i]}\in S^m_\Delta(A_i)$ $p_0\subseteq p_i$, $Rk(p_i)=\langle 2,\infty\rangle$, $p_i\subseteq_{pr}q_i\in S^m_\Delta(A_{i+1})$, $Rk(q_i)=\langle 1,\infty\rangle$, $\bar{a}_{i,j}\subseteq A_{j+1}-A_j$, $tp_\Delta(\bar{a}_{i,j},A_j)\supseteq_{pr}q_i$ has rank $\langle 1,\infty\rangle$, and $\varphi_i(\bar{x},\bar{b}_i)\in q_i,\neg\varphi_i,(\bar{x},\bar{b}_i)\in p_{i+1}$. W.l.o.g. the $\langle\bar{b}_i:i<\lambda\rangle$ form a Δ-system. And even $\bar{a}_{i,j}$, $\bar{b}_i$ are pairwise disjoint outside some $\bar{b}^*$.
If for every i for some j, $p_0^{[j]}$ Δ^c-split over A_i, we can easily interpret Q^{word}_λ. Otherwise we can easily interpret first $Q^{eq}_{\lambda,<\lambda}$, with which we can code $\{A_{i+1}-A_i: i<\lambda\}$ and relation over the $A_{i+1}-A_i$; so we can again code Q^{word}_λ. (really the first case occurs as for every $i\le\lambda$, there are $i<j_1<j_2<\lambda$, $tp_{\Delta^c}(\bar{b}_{j_1},A_i)=tp_{\Delta^c}(\bar{b}_{j_2},A_i)$ In both cases we finish as in 5.13D.

Now 5.11, 5.12, 5.14 give a complete analysis of the case $\lambda_2\neq\lambda_3$

* * *

During our investigations, we came across the following quantifier:

5.14A Definition : Let $K^{word}_\alpha=\{R:R$ **a two place relation, (** Dom R,R **) is a well ordering of order-type** $\alpha\}$.

5.15 Claim : 1) If $\alpha\le\beta$ **then** $Q^{word}_\alpha\le Q^{word}_\beta$.

2) $Q^{eq}_{|\alpha|,<|\alpha|}\le_{int}Q^{word}_\alpha$ **for infinite** α **(hence** $Q^{1-1}_\lambda\le Q^{word}_\lambda$).

3) $Q^{eq}_{\kappa,\lambda}\le_{int}Q^{word}_{\lambda\kappa}$, **(** $\lambda>\kappa\ge\aleph_0$ **are cardinals).**

4) For λ **singular** $Q^{word}_\lambda\equiv_{int}Q^{eq}_{\lambda,\lambda}$.

5) If $\alpha=\lambda^2$ **then** $Q^{eq}_{\lambda,\lambda}\le Q^{word}_\alpha$.

Proof : Easy (for (4) use 6.4).

5.16 Lemma : 1) If λ is regular, $\kappa<\mu\leq\lambda$, $\mu\leq 2^\kappa$, then $Q^{eq}_{\mu,\lambda} \leq_{int} \{Q^{word}_\lambda, Q^{eq}_{\kappa,\lambda}\}$

2) If λ is regular $\kappa\leq\mu\leq\lambda$, $\mu\leq\beth_n(\kappa)$ then $Q^{eq}_{\mu,\lambda} \leq_{int}\{Q^{word}_\lambda, Q^{eq}_{\kappa,\lambda}\}$

3) Assume $\alpha<\lambda$, $\lambda\alpha\leq\beta<\lambda(\alpha+1)$, λ regular, $|\alpha|=\kappa$. Then $Q^{word}_\beta \equiv_{int} \{Q^{word}_\lambda, Q_{\kappa,\lambda}\}$.

Proof : 1) Let $S=\{\delta<\lambda:\delta$ divisible by $\kappa\}$, and let E be an equivalence relation on S with $\leq\mu$ equivalence classes each of power λ (we shall define and interpret him). As the number of models $(\kappa,<,P)$ is 2^κ, we can find $P\subseteq\lambda$ such that:

(*) for $\delta_1,\delta_2\in S, \delta_1 E\delta_2$ iff for every $i<\kappa$, $\delta_1+i\in P \Leftrightarrow \delta_2+i\in P$.

Now let E_0 be $\{\langle\delta_1+i,\delta_2+i\rangle : \delta_1\in S$, $\delta_2\in S$, $i<\kappa\}$. Easily we can interpret E by $<,P$ and E_0, all interpretable by $\{Q^{word}_\lambda, Q^{eq}_{\kappa,\lambda}\}$.

2) By induction on n.

3) Easy.

5.17 Lemma : 1) $Q_{\aleph_0,\lambda}\not\leq_{int}Q^{word}_\lambda$ for λ regular.

2) $Q^{eq}_{\kappa,\lambda}\not\leq_{int}\{Q^{word}_\lambda, Q^{eq}_{\mu,\lambda}\}$ for λ,κ regular, $\lambda\geq\kappa\geq\aleph_0$ and $\mu\geq\beth_\omega(\kappa)$.

Proof : 1) We can prove that if $\bigwedge_{l=1}^{n}(Q^{word}_\lambda R)(\forall\bar{x})[R(\bar{x})\equiv R_l(\bar{x})]$, then the model $M=(\bigcup_{l=1}^{n}\text{Dom } R_l, R_1, \ldots, R_n)$ can be represented as $\sum_{i<\lambda} M_i$ where:

(A) each M_i is a model of power $<\lambda$.

(B) the $|M_i|$ are pairwise disjoint

(C) the meaning of $M=\sum_{i<\lambda} M_i$ is that if $\bar{a}_l\subseteq M_{i(l)}, i(1)<\cdots<i(k)$, we can compute the basic type of $\bar{a}_1{}^\frown\cdots{}^\frown\bar{a}_n$ in M from the basic types of $\bar{a}_l$ in $M_{i(l)}$ (not depending on the particular $i(l)$'s.)

Now by Feferman-Vaught theorem the conclusion follows.

2) Like (1); but for formulas to depth n, we use the F.V. theorem for formulas of $L_{\infty,\kappa}$ of (quantifier) depth $\leq n$.

5.18 Lemma : $Q^{word}_\lambda \equiv_{\exp} Q^{eq}_{\lambda,\lambda}$.

Proof : Clearly $Q^{word}_\lambda \leq_{\exp} Q^{eq}_{\lambda,\lambda}$ (in fact $Q^{word}_\lambda \leq_{int} Q^{eq}_{\lambda,\lambda}$). Ordinal addition on λ gives a pairing function, and on a subset of cardinality λ, and we can define addition as we can quantify over one-to-one functions.

§6 Below the first Stability cardinal

Hypothesis : We now assume $\lambda=\lambda_2=\lambda_3$.

We try to approach λ from below. W.l.o.g. $|\mathrm{Dom}\, R|\le\lambda$, and we analyze $\{\exists_R, Q^{1-1}_\lambda\}$.

6.1 Construction : Let M be admissible, rich enough, Δ finite large enough. We define by induction on i:

$\bar d^l_{i,\alpha}, \bar a^l_{i,\alpha}$ $(l=0,1\ \alpha<i)$, $\varphi_i(\bar x,\bar y,\bar z,\bar b_i)$, $\psi_i(\bar x,\bar c_i)$ such that , for

$A_i=\{\bar d^l_{i,\alpha}, \bar a^l_{j,\alpha}, \bar b_j, \bar c_j : \alpha<j<i, l<2\}$:

a) $\varphi_i, \psi_i\in\Delta$.

b) $\varphi_i(\bar x,\bar y,\bar z,\bar b_i)$ is not realized in A_i or at least by no $\bar a^0_{j,\alpha}{}^\frown\bar a^1_{j,\alpha}{}^\frown\bar d_{j,\alpha}$ $(i>j>\alpha)$.

c) $\varphi_i(\bar x,\bar y,\bar z,\bar b_i)\to(\psi_i(\bar x,\bar c_i)\wedge\neg\psi_i(\bar y,\bar c_i))$.

d) $\bar a^0_{i,\alpha}$, $\bar a^1_{i,\alpha}$ realize the same Δ-type over A_i.

e) $\varphi_i(\bar a^0_{i,\alpha}, \bar a^1_{i,\alpha}, \bar d_{i,\alpha}, \bar b_i)$ for all $\alpha<i$.

f) all the sequences $\{\bar d_{j,\alpha}{}^\frown\bar a^0_{j,\alpha}{}^\frown a^1_{j,\alpha} : \alpha<j\le i\}$ are pairwise disjoint.

Continue until i^*, when the process breaks down.

Let $\chi=$ card (i^*), $A^0=A_{i^*}$ so $|A^0|=\chi$ if χ is infinite $|A^0|<2^{2^\chi}$ if χ is finite.

6.2 Claim 1) We can interpret $Q^{eq}_{\chi,<\chi}$ if χ is infinite.

2) We can interpret $Q^{eq}_{\chi_1,<\chi_1}$ if χ is finite, $\chi_1=\chi^{1/2^{m(\Delta)}}$.

Proof : 1) If χ is regular, we can make the parameters $(b_i{}^\frown\bar c_i)$ into a Δ-system and proceed as in fact 5.13B previously.

So suppose $\kappa=cf\,\chi<\chi=\sum_{i<\kappa}\chi_i$, $\kappa<\chi_i$ all;

We can find a subsequence of $\langle \bar b_i{}^\frown\bar c_i : i<i^*\rangle$ of length κ which is a Δ-system, and with it interpret an equivalence relation E with κ equivalence classes of arbitrarily large powers less than χ.

For each i, there is a set $S_i\subseteq\chi_i^+$, of cardinality χ_i^+ such that $\langle \bar b_j{}^\frown\bar c_j : j\in S_i\rangle$ is a Δ-system. Let $\bar e_i$ be the heart of this Δ-system.

There is $T\subseteq\kappa$, $|T|=\kappa$, such that $\langle \bar e_i : i\in T\rangle$ is a Δ-system with heart $\bar e$.

Let $\gamma_i\in S_i$ for each i.

By hand over hand thinning of each S_i we may assume $\bar b_\alpha{}^\frown\bar c_\alpha\cap\bar b_{\alpha^1}{}^\frown\bar c_{\alpha^1}\subseteq\bar e$ if $\alpha\in S_i$, $\alpha^1\in S_j$, $i\ne j$.

We may assume $S_i\cap S_j=\phi$ for $i\ne j$, $i,j\in T$. Let $i(\alpha)$ be the unique i, such that

$\alpha\in S_i$. By permutations, we can code $\{\bar{e}_i : i\in T\}$ and $\{\bar{b}_\alpha{}^\frown\bar{c}_\alpha-\bar{e}_{i(\alpha)} : \alpha\in S_i\}$ and $\{\bar{b}_{\gamma_i}{}^\frown\bar{c}_{\gamma_i} : i\in T\}$.

We need to code the equivalence relation E^1: $\bar{b}_\alpha{}^\frown\bar{c}_\alpha{}^\frown\bar{e}_{i(\alpha)}E^1\ \bar{b}_\beta{}^\frown\bar{c}_\beta{}^\frown\bar{e}_{i(\beta)}$ iff $i(\alpha)=i(\beta)$.

By our reduction, this can be accomplished if we can do it for singletons rather than sequences. This we can do, with the equivalence relation E.

2) Left to the reader (really we need just that χ_1 as a function of χ diverge to $\aleph_0$)

6.3 Conclusion : $\chi\le\lambda$.

6.4 Claim : 1) If χ is singular, $Q^{eq}_{\chi,\chi}\equiv_{int}Q^{eq}_{\chi,<\chi}$.

2) $Q^{eq}_{\lambda,\chi}\equiv_{int}Q^{eq}_{\lambda,<\chi}$ if $\lambda>\chi$, χ singular.

3) If χ is finite then $Q^{eq}_{\chi/2,\chi/2}\le_{int}Q^{eq}_{\chi,<\chi}\le_{int}Q^{eq}_{\chi,\chi}$.

Proof : 1) Now we know $Q^{eq}_{\chi,<\chi}\le_{int}Q^{eq}_{\chi,\chi}$. Let us do the other inequality. Say E is the following equivalence relation on $\{\langle i,j\rangle : i<j<\chi\}$:

$\langle i,j\rangle E\langle k,l\rangle$ iff $j=l$; clearly $E\in Q^{eq}_{\chi,<\chi}$.

Let $\langle \chi_i : i<\kappa\rangle$ be as before, and let $E^1\in Q^{eq}_{\chi,<\chi}$ be an equivalence relation on $\{\langle 0,j\rangle : 0<j<\chi\}$ with χ equivalence classes each equivalence class unbounded in χ of power less than χ.

$xE^*y\equiv\exists x^1\exists y^1(x^1Ex\wedge y^1Ey\wedge x^1E_1y^1)$ is an equivalence relation with χ classes of power χ.

2) Similarly.

3) Easy.

6.5 Claim : At least one of the following occurs (if χ finite, we should use $3n(R)\chi$) (A^0 etc are from 6.1):

(1) For no $m<m(\Delta),l(\bar{x})=m,\varphi(\bar{x},\bar{y})\in\Delta,\bar{a}$ (finite) and pure $p\in S^m_\Delta(A^0,M)$, are both $p\cup\{\pm\varphi(\bar{x},\bar{a})\}$ realized by χ pairwise disjoint sequences.

(2) For no $\varphi(\bar{x},\bar{y})\in\Delta$ and $\bar{a}$ is $\varphi(\bar{x},\bar{a})$ realized by no $\bar{b}\subseteq A^0$, but $(\exists^{\ge\chi}\bar{x})[\varphi(\bar{x},\bar{c})\wedge\bar{x}\cap A^0=\phi]$.

Proof : Suppose $\varphi(\bar{x},\bar{b}),p\in S^m_\Delta(A^0,M)$ exemplify (1) fail, i.e. there are $\bar{a}^l_\alpha(l<2,\alpha<i^*)$ realizing p, pairwise disjoint and disjoint to A^0 (as p is pure) such that $\models\varphi(\bar{a}^l_\alpha,\bar{b})$ iff $l=0$.

Suppose further that $\psi(\bar{x},\bar{c})$ exemplify (2) fail i.e. there are $\bar{d}_\alpha(\alpha<3n(R)i^*)$, pairwise disjoint and disjoint to A^0 such that $\models\psi[\bar{d}_\alpha,\bar{c}]$ and $\neg(\exists\bar{x}\subseteq A)\psi(\bar{x},\bar{c})$.

We now can, by thinning, have $\bar{a}^l_\alpha, \bar{d}_\alpha (\alpha < i^*)$ which are pairwise disjoint. However we could have chosen $a^l_{i^*,\alpha} = a^l_\alpha$ ($l<2, \alpha<i^*$), $\bar{b}_{i^*} = \bar{b}$, $\varphi_{i^*} = \varphi(\bar{x}, \bar{b})$, $\bar{d}_{i,\alpha} = \bar{d}_\alpha$, $\psi_{i^*} = \psi(\bar{z}, \bar{c}_i)$, $\bar{c}_{i^*} = \bar{c}$, contradicting the choice of i^*.

6.6 Lemma : Suppose that 6.5(1) holds, and let $A^1 = \{a \not\in A^i$: $tp_\Delta(a, A_0)$ is realized by exactly one element $\}$.

$A^2 = \mathcal{U} - A^0 \cup A^1$.

Then

1) If $A^0 \subseteq B$, $p = tp_\Delta(\bar{a}, A^0, M)$ then the number of $q \in S^m_\Delta(B, M)$ extending p is at most $|B|$ (or $\le \chi\, |B|^{m(\Delta)} |\Delta|$ when χ is finite).

2) $|S^{m(\Delta)}_\Delta(B, M)| \le |S^1_\Delta(A^0, M)| + |B|$ (or $\le S^1_\Delta(A^0, M)\, \chi |\Delta|\, |B|^{m(\Delta)}$ if χ is finite).

3) $\lambda \le 2^\chi$ and $\lambda = |S^{\le m(\Delta)}_\Delta(A^0, |M|)|$ for χ infinite.

Proof : 1) Immediate from 6.5(1): let B be infinite ($\supset A^0$) and suppose, $tp_\Delta(\bar{a}_\alpha, B)$ $(\alpha < |B|^+)$ are distinct and

(*) $\bar{a}_\alpha (\alpha < |B|^+)$ realizes p.

W.l.o.g. $\bar{a}_\alpha = \bar{a}^+_\alpha \frown \bar{a}^*$, where $\bar{a}^* \subseteq B$, $\bar{a}^+_\alpha \cap B = \phi$, and the $\bar{a}^+_\alpha (\alpha < |B|^+)$ are pairwise disjoint. As 6.5(1) holds, for every $\bar{c}, \psi$ one of the sets $\{\alpha < |B|^+ : \models \psi(a, \bar{c})\}$, $\{\alpha < |B|^+ : \models \neg \psi(\bar{a}, \bar{c})\}$ has cardinality $< \chi$. Now we get contradiction to (*).

2) Follows from (1), as w.l.o.g. we can count pure types only.

3) Clearly $|S^{\le m(\Delta)}_\Delta(B, M)| \le 2^\chi$ for B of power $\le 2^\chi$. This is closely related to the definition of $\lambda_3 = \lambda$ but there is a difference: M and Δ are here fixed. But we could have repeat §4 , §5 for a fix larged enough Δ, M (with Δ depending on $n(R)$ and not on R). If λ is regular use §5 with hypothesis 5.3A, for λ singular 6.2, 6.4. (alternatively repeat this section for any Δ).

6.7 Lemma : Suppose 6.5(2), and that Δ is closed under permuting the variables.

(1) There is $A^1, A^0 \subseteq A^1, |A^1| \le |S^{\le m(\Delta)}_\Delta(A^0, M)|\; m(\Delta)$, such that for every B extending A^1, and $\bar{b}$ disjoint to B, (of length $\le m(\Delta)$) $tp(\bar{b}, B)$ does not Δ-split over A^0.

2) $\lambda \le |S^{\le m(\Delta)}_\Delta(A^1, M)|$

3) $\lambda \le 2^\chi$ when χ is infinite.

Proof : 1) Immediate.

2) Follows from (1), as in 6.6 (as by (1) every pure $p \in S^{\le m(\Delta)}_\Delta(B)$ is

determined by $p \restriction A^l$)

3) By (1), (2) it suffices to prove that if $A^0 \subseteq B$, $|B| \leq 2^\chi$, $m \leq m(\Delta)$ then $|S^m_\Delta(B,M)| \leq 2^\chi$. Suppose B,m form a counterexample. Then for some $\bar{e} \subseteq B, \varphi(\bar{x},\bar{y},\bar{e}) \in \Delta$, $|S_{\{\varphi(\bar{x};\bar{y},\bar{e})\}}(B,M)| > 2^\chi$, and we choose an example with minimal $l(\bar{x} \frown \bar{y})$ and so there are $\bar{b}_i$ disjoint to B, for $i < (2^\chi)^+$, with $tp_{\{\varphi(\bar{x};\bar{y},\bar{e})\}}(\bar{b}_i, B, M)$ distinct, $l(\bar{b}_i) = l(\bar{x})$, and w.l.o.g. $tp(\bar{b}_i, A^0) = p$ for a fixed p. If for some $i \neq j$, there are $\bar{a}_\alpha (\alpha < \chi)$ pairwise disjoint, disjoint to A^0 such that $\varphi(\bar{b}_i, \bar{a}_\alpha) \wedge \neg\varphi(\bar{b}_j, \bar{a}_\alpha)$, we get contradiction to 6.5(2).

So for every $j > 0$ there is $B_j \subseteq B, |B_j| < \chi$ such that if $\bar{a} \subseteq B - B_j \cup A^0$ then $\varphi(\bar{x},\bar{a}) \in p_0 \Longleftrightarrow \varphi(x,\bar{a}) \in p_j$. The number of possible B_j is $\leq |B|^{<\chi}$ so w.l.o.g. $B_j = B_i$ for $j > 0$. But now let $\{\varphi_\alpha(\bar{x},\bar{y}_\alpha,\bar{e}_\alpha) : \alpha < \chi\}$ be the formulas we get from $\varphi(\bar{x},\bar{y},\bar{e})$ by substituting one member of $\bar{y}$ by a member of $A^0 \cup B_1$. Clearly $0 < i < j$ implies $\bigvee_{\alpha < \chi} tp_{\varphi_\alpha(\bar{x};\bar{y}_\alpha,\bar{e}_\alpha)}, (\bar{a}_i, B, M) \neq tp_{\varphi_\alpha(\bar{x};\bar{y},\bar{e}_\alpha)}(\bar{a}_j, B, M)$. Hence for some α $|S_{\varphi_\alpha(\bar{x};\bar{y}_\alpha,\bar{e}_\alpha)}(B,M)| > 2^\chi$, contradicting the minimality of $l(\bar{x} \frown \bar{y})$.

6.8 Theorem : There is a function $f : \omega \to \omega$, diverging to infinity such that: if $\chi = \chi(R)$ is finite, then for some R^* and equivalence relation E, $\{\exists_R, Q^{1-1}_\lambda\} \equiv_{int} \{\exists_{R^*}, \exists_E\}$, $n = |\text{Dom } R^*|$ finite, and $Q^{eq}_{f(n),f(n)} \leq_{int} \exists_R$.

Proof : Combine the previous lemmas.

By 6.6 or 6.7 there is A^1, with $|A^1|$ not too large than χ, such that every pure $p \in S^{\leq m(\Delta)}_\Delta(A^1,M)$ has no two explictely contradicting χ-big extensions. Now as in §5, we can apply §4 to get $\exists_R \equiv_{int} \{\exists_{R^*}, \exists_E\}$ with $|\text{Dom } R^*|$ not too large than $|A^2|$.

As for $Q^{eq}_{f(n),f(n)} \leq_{int} \exists_R$, use 6.2(2).

6.9 Claim : We can interpret $Q^{eq}_{\chi,\chi}$ if χ is infinite.

Proof : We are done if χ is singular by 6.2, 6.4. So we assume χ is regular. If $|S^{\leq m(\Delta)}_\Delta(A,M)| < \chi$ whenever $|A| < \chi$, we repeat the case $\lambda_2 = \lambda_3^+$ with λ_2 replaced by χ everywhere.

So we assume $\kappa = |A^1| < \chi$, $|S^1_\Delta(A^m,M)| \geq \chi$. Now $|S^1_\Delta(A^1,M^1)| \geq \chi$, for some A^1 of cardinality $< \chi$, and some admissible M^1 [as if $tp_\Delta(\bar{a}_i, A) \in S^m_\Delta(A,M)$ are distinct $(i < \chi)$, then w.l.o.g. $\{\bar{a}_i\}$ form a Δ-system. We can expand A to include its heart and use permutations to get distinct elements of $S^1_\Delta(A^1,M^1)$]

Since Δ is finite, there is $\varphi(x,\bar{y})$ such that $|S^1_{\{\varphi\}}(A^1,M)| \geq \chi$.

Let m = length $(\bar{y})$.

Let $I=\{\bar{a}\subseteq A^1:\varphi(x,\bar{a})$ belongs to at least χ types $p\in S^1_{\{\varphi\}}(A^1,M)$, and $\neg\varphi(x,\bar{a})$ belongs to at least χ types $p\in S^1_{\{\varphi\}}(A^1,M)\}$.
Let $S_\varphi(I)=\{p\cap\{\pm\varphi(x,\bar{a}) : \bar{a}\in I\}:p\in S_{\{\varphi\}}(A^1,M)\}$.
Note that $|S_\varphi(I)|\geq\chi$, as follows:
Let $F:S^1_\varphi(A^1,M^1)\rightarrow S_\varphi(I)$ be the obvious projection.
If $|S^1_\varphi(I)|<\chi$, we take $q\in S_\varphi(I)$ with $\geq\chi$ pre-images p_i $(i<\chi)$, $p_j\neq p_j$, so each, except possibility one contains a formula which belongs to fewer than χ of the p_i. But there are fewer that χ-many formulas in all, contradiction.
Also note that for every $\bar{a}\in I$: $|\{p\in S_\varphi(I):\varphi(x,\bar{a})\in p\}|\geq\chi$ and
$|\{p\in S_\varphi(I):\neg\varphi(x,\bar{a})\in p\}|\geq\chi$.
(otherwise the pre-image under F of a set of size $<\chi$ would have cardinality $\geq\chi$, but we just showed any $q\in S_\varphi(I)$ can have only $<\chi$ elements in its pre-image).
On I, define the following equivalence relation E:
$\bar{a}E\bar{b}$ iff $|\{p\in S_\varphi(I):\varphi(x,\bar{a})\in p\equiv\varphi(x,\bar{b})\not\in p\}|<\chi$.
Let $J\subseteq I$ be a set of representatives and $G:S_\varphi(I)\rightarrow S_\varphi(J)$ the natural map.

6.10 Fact : $|G^{-1}(q)|<\chi$ for any q.

Proof : Suppose $G(p_i)=q$ and $p_i\neq p_j$ for $i<j<\chi$. For each i, take $\bar{b}_i\in I$, such that $(\varphi(x,\bar{b}_i)\in p_i)\Longleftrightarrow(\varphi(x,\bar{b}_i)\not\in p_{i+1})$.
Let $\bar{a}_i\in J$, $\bar{a}_iE\bar{b}_i$.
Since $|J|\leq|I|\leq|A^1|<\chi$, there are $\bar{a}^*$, and $\bar{b}^*$, such that $S=\{i:\bar{a}_i=\bar{a}^*$ and $\bar{b}_i=\bar{b}^*\}$ has cardinality $=\chi$, so $\bar{a}^*E\bar{b}^*$. W.l.o.g., $\varphi(x,\bar{a}^*)\in q$.
For all $i<\chi$, $\varphi(x,\bar{a}^*)\in p_i$ and $\bar{a}^*E\bar{b}^*$ so for all but fewer than χ ordinals $i\in S$, $\varphi(x,\bar{b}^*)\in p_i$. Similarly, for all but $<\chi$ ordinals $i\in S$, $\varphi(\bar{x},\bar{b}^*)\in p_{i+1}$. So for some $i\in S$, $\varphi(x,\bar{b}^*)\in p_i$ and $\varphi(\bar{x},\bar{b}^*)\in p_{i+1}$, but $\bar{b}^*=\bar{b}_i$ contradiction. So 6.10 holds.

Thus $|S_{\{\varphi\}}(J)|\geq\chi$.
We define B_i $(i<\chi)$ by induction; such that
1) B_i is disjoint from $A^1\cup\bigcup_{j<i}B_j$.
2) $|B_i|=\kappa<\chi$ (remember $|J|\leq|I|\leq|A^1|^m=\kappa$).
3) No two elements of $\bigcup_{j<i}B_j$ realize the same $S^1_\varphi(J,M^1)$ type. (Possible, as $|\bigcup_{j<i}B_j|<\chi\leq|S^1_\varphi(J)|$).

4) If $\bar{a} \neq \bar{b}$ and $\bar{a}, \bar{b} \in J$, then for some admissible $c \in B_i$, $\varphi(c, \bar{a}) \equiv \neg\varphi(c, \bar{b})$ (possible, as $|J| \leq \kappa = |B_i|$ and the choice of J).
Since $Q^{eq}_{\chi,\kappa} \leq_{int} Q^{eq}_{\chi,<\chi}$, we can interpret an equivalence relation E^1 relation with equivalence classes the B_i's. Also, since $|J| \leq \kappa$, we can code sequences from J by single elements.
Let $\langle c_{i,\alpha} : \alpha < \kappa \rangle$ enumerate B_i, and $\{\bar{a}_\alpha : \alpha < \kappa\}$ enumerate J.
With equivalence relations from $Q^{eq}_{\chi,<\chi}$ we can code pairs from B_i by elements of B_i, so with a monadic predicate we can interpret
$Q = \{\langle c_{i,\alpha}, c_{i,\beta} \rangle : \varphi(c_{i,\alpha}, \bar{a}_\beta)\}$. Now we can interpret $S = \{\langle c_{i,\alpha}, \bar{a}_\alpha \rangle : i < \chi, \alpha < \kappa\}$ by the formula $\Theta(x, \bar{y}) \equiv x \in \bigcup_{i<\chi} B_i \wedge \bar{y} \in J \wedge (\forall z)(z E^1 x \rightarrow (\varphi(z, \bar{y}) \equiv Q(z, x)))$.
Suppose $R = \{\langle d^1_\xi, d^2_\xi \rangle : \xi < \chi\}$ is a binary relation on $\bigcup_{i<\chi} B_i$. Let $P^l = \{c_{\xi,\alpha} : \varphi(d^l_\xi, \bar{a}_\alpha), \xi < \chi, \alpha < \kappa\}$ for $l = 1,2$.
Now $\langle b^1, b^2 \rangle \in R$ iff $(\exists x \in \bigcup B_i)(\forall y \in x/E^1) \bigwedge_{l=1}^{2} (y \in P^l$ *iff* $(\exists \bar{z} \in J)(\langle y, \bar{z} \rangle \in S \wedge \varphi(b^l, \bar{z})))$.

(We are coding b^l by the $\{\varphi\}$-type it realizes over J. Even though b^l might be in some other B_n, the code is on level B_ξ for b^l in the ξth pair of R).
So $Q^{eq}_{\chi,\chi} \leq_{int} Q_R$.

6.10 Remark : So we have proved that if in $(\mathcal{U}, R)$ for some $A, \mu = |S^1_\Delta(A, M)| > |A| = \kappa \geq \aleph_0$ (κ minimal) then $Q^{eq}_{\mu,\mu} \geq_{int} \{\exists_R, Q^{eq}_{\mu,\kappa}\}$.

6.11 Theorem : Suppose χ is infinite. Then
1) $\exists_R \equiv_{int} \{Q^{eq}_{\chi,\chi}, \exists_E, \exists_{R_1}\}$
Where E is an equivalence relation, $|\mathrm{Dom}\, R_1| \leq 2^\chi$.
2) Also for some M and (finite) Δ, there are A^0, A^1, $|A^0| = \chi$, $|S^{m(0)}_\Delta(A^0, M)| \geq |A^1|$. $|S^{m(1)}_\Delta(A^1, M)| = |\mathrm{Dom}\, R_1|$.

Proof : Combine the previous proofs.

§7 Summing Positive Results.

7.1 Theorem : If $V=L$, then any R is uniformly invariantly bi-expressible with Q_E, where E is some equivalence relation, or with $\{Q^{mon}_{\lambda_0}, \exists_{R_1}\}$, Dom R_1 finite.

Proof : Clearly $\lambda_3 = \lambda_2 \leq 2^\chi$ is the only remaining case. We can find A such that $|A| = \chi$, $|S^1_\Delta(A, M)| = \chi^+$, $Q^{eq}_{\chi,\chi} \leq_{int} \exists_R$. ($\Delta = \{$ atomic and negated atomic formulas in the language of $M\}$ (and is finite).

We shall show that we can express $Q^{word}_{\chi^+}$.

On A we can interpret the structure $\langle L_\chi, \in \rangle$. For every a, $tp_\Delta(a, A, M)$ can be viewed as a subset of L_χ.

We express an ordering

$a \le b$ *iff* ($\exists$ well-founded $\mathcal{L}$) [$\mathcal{L} \models$ "I am an L_α for some α" and $\mathcal{L}$ extends (as $\langle L_\chi, \in \rangle$ already interpreted) and the subsets of L_χ which a and b represent appear in $|\mathcal{L}|$ and the subset representing a occurs earlier].

$\le$ is a well-founded linear quasi-ordering.

Use a monadic predicate to pick out one element from each of the induced equivalence classes. This gives us a well-ordering of order-type χ^+. By 5.18 we finish. Q.E.D.

7.2 Conclusion : ($V=L$) for every K either for some family $\mathbf{E}$ of equivalence relation, $\exists_K, \exists_{\mathbf{E}}$ are uniformly invariantly bi-expressible or for some finite family K_f of finite relations and λ, $\exists_K, \{Q^{mon}_{<\lambda}, \exists_{K_f}\}$ are uniformly invariantly bi-expressible (if we omit uniformly we can omit the second case.)

Remark : On analysing $\mathbf{E}$ see 1.5.

* * *

For some χ we can close the gap (χ, λ) more easily, so such χ are impossible.

7.3 Lemma : Suppose M is admissible. And for some finite Δ, m and $A, |A| = \chi$, $|S^m_\Delta(A)| = \mu > \chi$ and $B \subseteq A$, $[|B| < \chi \Longrightarrow |S^m_\Delta(B)| \le \kappa]$, $\chi \le \kappa < \mu$ and χ is singular,

Then 1) $\{\exists_R, Q^{eq}_{\kappa,\kappa}, Q^{eq}_{\mu, cf\chi}\} \le_{int} Q^{eq}_{\mu,\mu}$

2) If $\aleph_0 < cf\chi$, $2^{cf\chi} < \chi$, μ regular, then $\{\exists_R, Q^{eq}_{\kappa,\kappa}\} \le_{int} Q^{word}_\mu$.

Remark : For (1) note that if $cf\chi = \aleph_0$, then $Q^{eq}_{\mu,\aleph_0} \le_{inex} Q^{1-1}_\mu$.

Proof : We can interpret in (an admissible expansion of) M, a tree T of power κ, with $cf\chi$ levels, and μ branches $\{B_i : i < \mu\}$ (of order type $cf\chi$).

If μ is regular, we can assume that $x \in B_i \Longrightarrow |\{j : x \in B_j\}| = \mu$ so each B_i can be coded by a set W_i of length $cf\chi$ branches, as its limit, with $[i \ne j \Longrightarrow B_i \cap B_j = \phi]$ and (1) follows.

If μ is singular, we can similarly code $Q_{\mu,<\mu}$ and finish (1) by 6.4.

For (2) we can consider $\{B_i:i<\mu\}$ as a set of function in $\kappa^{cf\chi}$, which are pairwise eventually distinct. By [Sh 7] for some ultrafilter D over $cf\,\chi$ and $I\subseteq\mu$, $\{B_i:i\in I\}$ is well ordered by $<_D$.

§8 Complementary Independence Results

8.1 Lemma : Suppose $\lambda=\lambda^{<\lambda}>\aleph_0$, $\mu>\lambda$, and P is the forcing for adding μ functions $F_i:\lambda\to 2$ $(i<\mu)$ (equivalently a function $F:\mu\times\lambda\to 2$, $F(i,\alpha)=F_i(\alpha)$ by conditions of power $<\lambda$) . Let for $S\subseteq\mu$, $\underset{\sim}{R}_S$ be the following partial order $<_S$ on ${}^{\lambda>}2\cup S$: $x\leq y$ iff $(\exists\alpha<\lambda)(y\in{}^{\alpha}2\wedge y=x\restriction\alpha)$ $\vee$ $(\exists\alpha<\lambda)(x\in{}^{\alpha}2\wedge y\in S\wedge x\subseteq F_y)$ (so $\underset{\sim}{R}_S$ is defined in V^P). We let for $x\in S$ and $\alpha<\lambda$, $x\restriction\alpha=^{def}F_i\restriction\alpha$; for x an ordinal $\notin S$, let $x\restriction\alpha=\langle -1\rangle$.
Then $Q^{word}_{\lambda^+}\not\leq_{int}Q_{R_S}$ (we assume $\{\alpha\cup{}^{\lambda>}2=\mathcal{U}$ for some ordinal α such that $S\subset\alpha$).

Proof : Suppose not, then for some $p\in P$ and first order $\varphi(x,y,\bar{c},\bar{H},R_S)$,

$p\Vdash_P$ " $\bar{c}$ a finite sequence of elements of $\mathcal{U}$,$\underset{\sim}{\bar{H}}=\langle\underset{\sim}{H}_l: l<n^*\rangle$ a finite sequence of permutations of $\mathcal{U}$, and $\{\langle x,y\rangle:\varphi(x,y,\underset{\sim}{c},\underset{\sim}{\bar{H}},\underset{\sim}{R}_S)\}$ is a well ordering of order-type λ^+ and w.l.o.g. $\{\underset{\sim}{H}_l^{-1}:l<n^*\}=\{\underset{\sim}{H}_l:l<n^*\}$."

As P satisfies the λ^+-chain condition, there is $S_0\in V$, $S_0\subseteq\mathcal{U}$, $|S_0|\leq\lambda$ such that

$p\Vdash_P$ " ${}^{\lambda>}2\cup S_0$ is closed under $\underset{\sim}{H}_l$ for $l<n^*$ ".

Let $\underset{\sim}{M}=(\mathcal{U},\underset{\sim}{R}_S,\underset{\sim}{H}_0,\ldots,\underset{\sim}{H}_{n^*-1})$. For notational simplicity let $\mathcal{U}={}^{\lambda>}2\cup S$. Let

$K=\{I$:I a model of the form $(|I|,f^I_0,\ldots,f^I_{n^*-1})$, each f_l a permutation of $|I|$, and I has no proper (non empty) submodel $\}$.

Clearly, $K\in V$, and each $I\in K$ as cardinality $\leq\aleph_0$.

We let $K^1\subseteq K$ be a set of representatives of the isomorphism types, and $I\in K^1\Longrightarrow|I|\leq\aleph_0$ hence $K^1\in V$. In V^P we define, for each $I\in K$:

(a) we call $\langle x_t:t\in I\rangle$ a component if $H_l(x_t)=x_{f^I_l(t)}$.

(b) $A_I=\{\langle\eta_t:t\in I\rangle$: $\eta_t\in{}^{\lambda>}2$ for every $t\in I$, and there are function $G_i:I\to\mathcal{U}(i<\lambda^+)$ with pairwise disjoint ranges, such that $\eta_t=G_i(t)\restriction l(\eta_t)$ and $\bigwedge_{i<\lambda^+}(\forall t\in I)[\bigwedge_{l<n^*}G_i(f^I_l(t))=H_l(G_i(t))]\}$.

Note that $\mathcal{U}(={}^{\lambda>}2\cup S)$ is partitioned into components. (x,y are in the same components iff $x=z_o$, $y=z_k$ and $\bigwedge_{m<k}\bigvee_{l<n^*} z_{n+1}=H_l(z_m)$ for some k and $\langle z_m : 0<m<k\rangle$).
So in V^P there is S_1, $|S_1|\le\lambda$, ${}^{\lambda>}2\subseteq S_1$ and for every $x\in\mathcal{U}-S_1$, its component $\langle x_t : t\in I\rangle$ is disjoint to $\mathcal{U}-S_1$, and for every $\alpha<\lambda$, $\langle x_t\restriction\alpha : t\in I\rangle\in A_I$.
So again as P satisfies the λ^+- chain condition, we can assume $S_1\in V$, and that forcing of $\langle F_i : i\in S_1\rangle$ also determines $\langle A_I : I\in K^1\rangle$. So let $V^1=V[\langle F_i : i\in S_1\rangle]$, P^2 the quotient forcing (which is just forcing $\langle \underset{\sim}{F}_i : i\in\mu-S_1\rangle$ by approximations of power $<\lambda$.)
Notice that in order to know in $V^1[G^2]$ that $\models\varphi[x,y]$ (which holds), $(x,y\in S)$ it is not enough to know $\langle x\restriction\alpha, y\restriction\alpha\rangle$ for large enough α, though it is enough to know $p_1\in G$ for some large enough $p_1\in P^2$, which force it! $V^1[G^2]$). A simple example is $\psi(x,y)=[H_0(x)=y]$. But something similar and more general holds.

Fact : 1) If ψ is a formula from L_{ω_1,ω_1}, $\langle x_t^\xi : t\in I_\xi\rangle$ for $\xi<\xi_0$ distinct components disjoint to S_o, and $\bar y$ is a countable sequence from S_1,$(I_\xi\in K^1)$ $\models\psi[\cdots x_t^\xi,\ldots,\bar y]_{\xi<\xi_o,t\in I_\xi}$ for $t\in I_\xi$, $\xi<\xi_0$
then for some $\alpha<\lambda$:

(*) if $z_t^\xi\in\mathcal{U}-S_1$, $\langle z_t^\xi : t\in I_\xi\rangle$ distinct components and $z_t^\xi\restriction\alpha=x_t^\xi\restriction\alpha$ **then** $\models\psi[\cdots z_t^\xi,\ldots,\bar y]_{\xi<\xi_0,t\in I_\xi}$.

2) We could also have assumed that for any such ψ, $I_\xi(\xi<\xi_0)$ and $\bar y$, the P-name

$T_{\psi,<I_\xi:\xi<\xi_0>} = \{\langle\ldots,\eta_t^\xi,\ldots\rangle : t\in I_\xi,\ \xi<\xi_0$ for λ^+ pairwise distinct components, $\langle\cdots x_t^{\xi,i}\cdots : t\in I_\xi\rangle$,

$\models\psi[\cdots x_t^{\xi,i},\ldots,\bar y]_{\xi<\xi_0,t\in I_\xi}$

and for some γ, $x_t^{\xi,i}\restriction\gamma=\eta_t^\xi$ for every $t\in I_\xi,\xi<\xi_0,i<\lambda^+\}$
depend only on $\langle \underset{\sim}{F}_i : i\in S_1\rangle$

Proof: For 2) close S_1 λ or just $\aleph_1$ times, and if $\langle\eta_t^\xi : t\in I_\xi,\ \xi<\xi_0\rangle$ is not in $T_{\psi,\langle I_\xi:\xi<\xi_0\rangle}$, but $\models\psi[\cdots x_t^\xi,\ldots,\bar y]$, $x_t^\xi\restriction\gamma=\eta_t^\xi$ for every $t\in I_\xi,\xi<\xi_0$ then $\{x_t^\xi : t\in I_\xi,\ \xi<\xi_0\}\cap S_1\neq\phi$.

Now we can prove by induction on the depth of ψ (in V^1) that the fact is forced (i.e. $\Vdash_{P^z}$)

From the fact, and the Tarski-Vaught criterion we can conclude (in V^P) that if $S_1 \subseteq \mathcal{U}_1 \subseteq \mathcal{U}$, $\mathcal{U}_1$ closed under $H_l(l<n^*)$, then $M \restriction \mathcal{U}_1$ is an L_{ω_1,ω_1}- elementary submodel of M. By increasing S_1 further we get that this holds for any $\mathcal{U}_1 \subseteq \mathcal{U}$, extending S_1 and closed under $H_l(l<n^*)$.

Now if $Q^{word}_{\lambda^+} \leq_{int} Q_{R_S}$ then we can find $I \in K$ $\psi \in L_{\omega_1,\omega}, \bar{y} \subseteq S_1$, and distinct components $\langle X^{\xi}_t : t \subseteq I \rangle$ disjoint to S_1 such that for $\xi,\zeta<\lambda^+$, $M \models \psi(\cdots x^{\xi}_t, \ldots ; \ldots, x^{\zeta}_t, \ldots, \bar{y})$ iff $\xi<\zeta$.

Now, for some $\xi_0<\lambda^+$, for every $\xi \in (\xi_0, \lambda^+)$ and $\alpha<\lambda$ for arbitrarily large $\zeta<\lambda^+$, $\bigwedge_{t \in I} x^{\xi}_t \restriction \alpha = x^{\zeta}_t \restriction \alpha$. Using the fact, the contradiction is easy.

8.2 Lemma : 1) In 8.1 if $S, S^1 \subseteq M$, $|S-S^1|>\lambda$, then $\exists_{R_S} \not\leq_{int} \exists_{R_{S^1}}$.

2) If $\kappa<\lambda$, in 8.1 we can get $Q^{word}_{\lambda^+} \not\leq_{int} \{\exists_{R_S}, Q^{eq}_{|\mathcal{U}|,\kappa}\}$ and even $\exists_{R_S} \leq_{int} \{\exists_{\underset{\sim}{R}_{S^1}}, Q^{eq}_{|\mathcal{U}|,\kappa}\}$.

3) If $\kappa<\lambda$, in 8.1 we get $Q^{eq}_{\lambda^+,\kappa^+} \not\leq_{int} \{\exists_{\underline{R}_S}, Q^{eq}_{|\mathcal{U}|,\kappa}\}$.

Proof : 1) Similar.

2) We use L_{κ^+,κ^+} instead L_{ω_1,ω_1}, and κ permutations $H_i(i<\kappa)$ (instead n^*), and repeat the previous proofs - but any $E \in K^{eq}_{|\mathcal{U}|,\kappa}$ can be defined by an L_{κ^+,κ^+}-formula using suitable κ permutations.

3) Similar proof.

8.3 Lemma : (G.C.H.) If $\lambda>\aleph_1$ is regular $\mu=\lambda^+$, we can build $\langle F_i : i<\mu \rangle$ as required in 8.1, 8.2 without forcing.

Proof : See [Sh 6], 2.1.

The following lemma shows that we cannot prove 8.2 without some set-theoretic hypothesis.

8.4 Lemma : Suppose $V \models \chi=\chi^{<\chi} \wedge \chi<\lambda \wedge cf\,\lambda=\lambda$, then for some χ-complete forcing notion P of power $\lambda^{<\chi}$, satisfying the χ^+- c.c., $\Vdash_P$ "$2^\chi \geq \lambda$", and

1) $\Vdash_P$ "if $S \subseteq {}^{\chi}2$, $cf\,|S| \neq \lambda$, R_S as in 8.1, then $Q^{eq}_{|S|,|S|} \leq_{int} \{\exists_R, Q^{eq}_{\chi,\chi}\}$".

2) in V^P, $\chi(R)=\chi_1<\lambda_3(R)$ implies $\lambda_3(R)=\chi^+=\lambda$.

Proof : We let P be the limit of the χ-support iteration $\langle P_i, \underset{\sim}{Q}_i : i<\lambda \rangle$ where $\underset{\sim}{Q}_i \in V^{P_i}$ is defined as follows:

let $({}^{\lambda}2)^{V^{P_i}}=\{f^i_\alpha:\alpha<(2^\chi)^{V^{P_i}}\}$ and

$Q_i=\{(F,A)$: F a function from a subset of ${}^{\chi>}2$ of power $<\chi$, into χ, $A\subseteq(\chi)^{V^{P_i}}$, $|A|<\chi$, and for $\beta,\gamma\in A$, $\alpha<\chi$, if $f^i_\beta\restriction\alpha=f^i_\gamma\restriction\alpha$, then $f^i_\beta\restriction\alpha\in\mathrm{Dom}\ F\}$,

$(F_1,A_1)\le(F_2,A_2)$ *iff* $F_1\subseteq F_2, A_1\subseteq A_2$ and if $\eta\ne\nu$ are in A_2 and

$[\beta\in A_1\wedge\gamma\in A_1\wedge\alpha<\chi\wedge f^i_\beta\restriction\alpha\ne f^i_\gamma\restriction\alpha\wedge\beta<\gamma$
$\wedge f^i_\gamma\restriction\alpha\notin\mathrm{Dom}\ F_1\Longrightarrow F_2(f^i_\beta\restriction\alpha)<F_2(f^i_\gamma\restriction\alpha)]$.

8.6 Conjecture : It is consistent with ZFC that every $\exists_K$ is biinterpratable with some $\exists_{\mathbf{E}}$, $\mathbf{E}$ a family of equivalence classes.

8.7 Question: Prove it is consistent with ZFC that some $\exists_K$ is not bi-expressible with any $\exists_{\mathbf{E}}$,$\mathbf{E}$ a family of equivalence classes.

References.

[BS] J. Baldwin and S. Shelah, Classification of theories by second order quantifiers, Proc. of the 1980/1 Jerusalem model theory year, **Notre Dame J. of Formal logic**. to appear.

[H] A. Hajnal, Proof of a conjecture of S. Ruziewicz, *Fund. Math.* 50 (1961/62) 123-128.

[Sh1] S. Shelah, First order theory of permutation groups, **Israel J. Math** 14 (1973), 149-162.

[Sh2] S. Shelah, Errata to: First order theory of permutation groups, **Israel J. Math** 14 (1973) 262-279.

[Sh3] S. Shelah, There are just for second order quantifiers. **Israel J. Math** 15 (1973) 282-300.

[Sh4] S. Shelah, Models with second order properties I, **Annals of Math Logic**, 14 (1978) 52-72.

[Sh5] S. Shelah, Models with second order properties III, omitting types in λ^+ for $L(Q)$ **Archiv. Math Logic**, 21 (1981) 1-11.

[Sh6] S. Shelah, Models with second order properties IV **Annals of Math Logic**. 25 (1983), 183-212.

[Sh7]

S. Shelah, Powers of singular cardinals, **Notre Dame J. of Formal logic**.

[Sh8]

S. Shelah, The monadic theory of order, **Annals of Math**, 102 (1975) 379-419.