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MODELS WITH SECOND ORDER PROPERTIES. III. OMITTING TYPES FOR $L(Q)$ *

Saharon Shelah

Abstract

We generalize Keisler's omitting types theorem for $L(Q)$ in the $\aleph_1$-interpretation, to most cases in which Chang's two cardinal theorem applies. As an application we answer positively a question of Magidor and Malitz on the compactness of their logic in cardinalities higher than $\aleph_1$.

1. Introduction

$L(Q)$ is the logic obtained from the usual first order logic by adding a quantifier Q. The λ-interpretation of Q is: $Qx\varphi(x)$ iff $|\{x:\varphi(x)\}| \geqq \lambda$. Keisler [3] studied the $\aleph_1$-interpretation of Q and proved completeness and omitting types theorems for it. If λ is a cardinal such that $\lambda^{<\lambda} = \lambda$, then by Chang's two cardinal theorem [1], Keisler's completeness theorem for the $\aleph_1$-interpretation implies the same theorem for the λ^+-interpretation (for a countable language). But this approach does not yield any omitting types theorem for the λ^+-interpretation.

To prove his omitting types theorem for the $\aleph_1$-interpretation, Keisler built a "strong" model M of T by taking the union of an elementary chain $\langle M_\alpha : \alpha < \omega_1 \rangle$ of countable "weak" models of T. He defined what it is for a theory or a model to *strongly omit* a type p. He showed that if T strongly omits p, then M_0 can be chosen so that it strongly omits p too; moreover if M_α strongly omits p, then we can arrange that $M_{\alpha+1}$ does likewise. Since strongly omitting depends on only finite sets of parameters, it survives at limit ordinals and so M strongly omits p. Thus M omits p.

When we try to replace $\aleph_1$ by λ^+ in this construction, we have to redefine "strongly omitting" so that it depends on $<\lambda$ parameters instead of finitely many. Things immediately go wrong at limit ordinals of cofinality $<\lambda$. We shall repair the proof by using $\diamondsuit_\lambda$. This use of $\diamondsuit_\lambda$, in a construction of length λ^+, is new.

In fact $\diamondsuit_\lambda$ is a very weak condition. We show this in Section 6, where we quote and strengthen a theorem of Gregory [2]. When λ is strongly inaccessible, our theorem can be proved without using $\diamondsuit_\lambda$ at all. Assuming GCH, our theorem applies to

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any λ^+-interpretation when λ is regular and $\neq \aleph_1$. The result was announced in [5, 6].

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2. The Easy Direction

Henceforth λ is a fixed cardinal and T is a complete and consistent theory in $L(Q)$, in a language L of cardinality $\leqq \lambda$. To avoid triviality we assume $T \vdash Qx\, x=x$. All models will be "weak" models of T (see Keisler [3]). A formula $\varphi(x)$ is *small* in T iff $T \vdash \neg Qx\varphi(x)$, and *large* in T iff $T \vdash Qx\varphi(x)$; it is small or large in a model M iff it is small or large in the complete diagram of M. A *type* is a consistent set of formulae; a 1-type with just x free is *small* iff it contains or implies a small formula. A model M is *λ-compact* iff M realises every 1-type p over M with $|p| < \lambda$.

A model M is a *standard* model (for the λ^+-interpretation) iff (1) M is λ-compact, (2) no small formula is realised by $> \lambda$ elements of M, and (3) every non-small 1-type p over M with $|p| < \lambda$ is realised by $> \lambda$ elements of M.

The symbol (R) will stand for well-ordered infinitary quantifiers of length $< \lambda$, of form

$$Qy_0 \exists \bar{z}_0 Qy_1 \exists \bar{z}_1 \ldots Qy_i \exists \bar{z}_i \ldots \quad (i < lh(R)),$$

or derived from such a strong by omitting some variables. Here $\bar{z}_i$ is the sequence $z_i^0 z_i^1 \ldots z_i^j \ldots (j < \lambda)$; the variables z_i^j and y_i will be called the *i-variables*. If p is a set of $< \lambda$ formulae, a *finite approximation* to the infinitary formula $(R) \wedge p$ is a finite conjunction of formulae in p, together with a large enough finite part of (R) to cover the variables of those formulae. We shall freely write $(R) \wedge p$ to mean the set of finite approximations. We shall say

$$T \vdash (R) \wedge p$$

when T entails every finite approximation to $(R) \wedge p$. Similarly with $M \models (R) \wedge p$; we allow p to contain perhaps infinitely many parameters from M.

When we write $\exists x (R) \wedge p$ (as in Lemma 1 below), we mean $(R') \wedge p$ where (R') is $\exists x (R)$; similarly for $Qx(R) \wedge p$, etc. So $\exists x(R) \wedge p$ does not necessarily imply that $(R) \wedge p$ holds for some x, unless we are in a λ-compact structure.

Lemma 1. *Let M be a standard model, $|p| < \lambda$, and $\bar{b}$ a sequence of $< \lambda$ elements of M. Then:*

(1) *$M \models \exists x (R) \wedge p[\bar{b}]$ iff for some $a \in M$, $M \models (R) \wedge p[\bar{b}, a]$.*

(2) *$M \models Qx(R) \wedge p[\bar{b}]$ iff for some $a \in M$ whose complete type over $\bar{b}$ in M is not small,*

$$M \models (R) \wedge p[\bar{b}, a].$$

Proof. The proof of (1) is immediate from the definitions, using the λ-compactness of M. For (2), assume first that $M \models Qx(R) \wedge p[\bar{b}]$. Then $(R) \wedge p$ is a non-small type. Since every non-small type of cardinality $< \lambda$ is realised by at least λ^+ elements of M, there are at least λ^+ elements a such that $M \models (R) \wedge p[\bar{b}, a]$. Now the language L has at most λ small formulae, even allowing the $< \lambda$ parameters $\bar{b}$, and each small formula is satisfied by at most λ elements. Hence there is some a which satisfies no small formula with parameters from $\bar{b}$, such that $M \models (R) \wedge [\bar{b}, a]$. This proves left-to-right in (2). The converse is immediate. □

Let p be a 1-type with x free, and q a type of cardinality $< \lambda$. We call q a *support* of p over T iff

(1) $T \vdash (R) \exists x \bigwedge q$, and
(2) for every $\psi(x) \in p$, $T \nvdash (R) \exists x [\bigwedge q \wedge \neg \psi(x)]$,

where (R) is as described above, and covers all the variables free in q. We say q is a *support* of p over a model M iff the same holds over the complete diagram of M. We say T (or M) *strongly omits* p iff there is no support of p over T (or M).

Main Theorem (Easy half). *If T has a standard model which omits the 1-type p, then T strongly omits p.*

Proof. Let M be a standard model of T, and suppose that q is a support of p over T, so that $T \vdash (R) \exists x \bigwedge q$. Since M is a model of T, we have

$$M \models (R) \exists x \bigwedge q.$$

Choosing witnesses inductively according to Lemma 1, we find a sequence b and an element c in M such that $M \models \exists x \bigwedge q[\bar{b}]$ and

$$M \models \bigwedge q[\bar{b}, c]. \tag{1}$$

Let $\psi(x)$ be in p. Then since q is a support of p and T is complete, there is a finite approximation q' of q such that

$$M \not\models (R') \exists x [\bigwedge q' \wedge \neg \psi(x)].$$

By Lemma 1 in the other direction, we infer

$$M \not\models [\bigwedge q' \wedge \neg \psi(x)] [\bar{b}, c]. \tag{2}$$

From (1) and (2) we derive $M \models \psi[c]$, and this shows that c satisfies p. So p is realised in M. □

3. The Hard Direction: The Framework

Main Theorem (Hard half). *Suppose $\diamondsuit_\lambda$ holds. If T strongly omits p, then T has a standard model which omits p.*

The construction of the model will be based quite closely on Keisler [3]. We shall build an elementary chain $\langle M_\alpha : \alpha < \lambda^+ \rangle$ of models of T such that

I. Every M_α is λ-compact and of cardinality λ.

II. For each α, every non-small 1-type of cardinality $< \lambda$ over M_α has new elements added to it in cofinally many M_β.

Unlike Keisler, we do not require the chain $\langle M_\alpha : \alpha < \lambda^+ \rangle$ to be continuous at limit ordinals. For each α, after $\langle M_\beta : \beta < \alpha \rangle$ has been constructed, we shall suppose someone gives us a set of $\leqq \lambda$ types $\{p_i^\alpha : i < \lambda\}$ over $\bigcup_{\beta<\alpha} M_\beta$ which are strongly omitted by $\bigcup_{\beta<\alpha} M_\beta$ (when $\alpha = 0$, the types are strongly omitted by T). The construction will satisfy:

III. Every type $p_i^\alpha (\alpha < \lambda^+ ; i < \lambda)$ is strongly omitted by $\bigcup_{\alpha<\lambda^+} M_\alpha$.

We shall write $M_{<\alpha}$ for $\bigcup_{\beta<\alpha} M_\beta$, and M^* for $\bigcup_{\alpha<\lambda^+} M_\alpha$.
Let us show first that if I.–III. can be guaranteed, then the theorem follows.

Lemma 2. *Assume M_α is λ-compact, and let $\varphi(x)$ be a small formula over M_α. Then M_α strongly omits the type*

$$\{\varphi(x)\} \cup \{x \neq a : a \in M_\alpha\}.$$

Proof (cf. Keisler [3, Lemma 4.4]). If not, then this type has a support q over M_α. So

$$M_\alpha \models (R)\exists x \bigwedge q, \qquad M_\alpha \not\models (R)\exists x[\bigwedge q \wedge \neg\varphi(x)]. \tag{1}$$

Then for some finite approximation $(R')\exists x \bigwedge q'$,

$$M_\alpha \models \neg(R')\exists x[\bigwedge q' \wedge \neg\varphi(x)]. \tag{2}$$

We claim that

$$M_\alpha \models (R)\exists x[\bigwedge q \wedge \varphi(x)]. \tag{3}$$

For otherwise there is a finite approximation $(R'')\exists x \bigwedge q''$, which can be assumed to include $(R')\exists x \bigwedge q'$, such that

$$M_\alpha \models \neg(R'')\exists x[\bigwedge q'' \wedge \varphi(x)]. \tag{4}$$

But from (2), (4) and the law $\vdash Qx[\psi \vee \chi] \to Qx\psi \vee Qx\chi$ (cf. Keisler [3, Lemma 1.9]), we deduce

$$M_\alpha \models \neg (R'')\exists x \bigwedge q'',$$

which contradicts (1). This proves (3). Now Keisler showed that

$$\vdash (R)\exists x\psi \to [\exists x(R)\psi \vee Qx\exists \bar{u}\psi]$$

[where $\bar{u}$ are the variables in (R)]; by this and the fact that $\varphi(x)$ is small, (3) implies

$$M_\alpha \models \exists x(R)[\bigwedge q \wedge \varphi(x)].$$

So, since M_α is λ-compact, there is $a \in M_\alpha$ such that

$$M_\alpha \models (R)\exists x[\bigwedge q \wedge x = a],$$

which contradicts the choice of q. □

To prove the theorem, let the p_i^0 be p, and for each α let the $p_i^{\alpha+1}$ be the types described in Lemma 2. Then all these types are strongly omitted by M^*, according to III., and hence omitted by M^*. Let $\varphi(x)$ be small over M^*. Then $\varphi(x)$ is small over some M_α. By I. and Lemma 2, M_α strongly omits the type p' of an element which satisfies $\varphi(x)$ but is not one of the elements of M_α. So p' is one of the $p_i^{\alpha+1}$ and hence is omitted in M^*. This means that $\varphi(x)$ gains no new elements after M_α, and so at most λ elements satisfy it in M^*. By I. and II. it is clear that M^* is a standard model of T, and we are done.

The rest of this section will prove I and II, using $\lambda^{<\lambda} = \lambda$ (which follows from $\diamondsuit_\lambda$). We return to III. in Section 4.

To construct the M_α, we define their complete diagrams, by induction on α. New constants are needed: for each $\alpha < \lambda^+$, we introduce new constants b_α and $c_\alpha^j (j < \lambda)$, to be known as the *α-constants*. Write L as the union of a continuous increasing chain of languages of cardinality $< \lambda$, $L = \bigcup_{i<\lambda} L_i$. For each $\alpha < \lambda^+$, L_i^α shall be L_i enriched with the constants b_β and $c_\beta^j (\beta \leqq \alpha; j < i)$. Thus for each α, L_i^α is continuous increasing in i. We write L^α for $\bigcup_{i<\lambda} L_i^\alpha$, $L^{<\alpha}$ for $\bigcup_{\beta<\alpha} L^\beta$, and L^* for $L^{<\lambda^+}$.

We shall define an increasing sequence of theories T^α such that each T^α is complete and consistent in the language L^α, and has an α-constant as witness for each existentially quantified sentence in it. M_α will always be the model whose complete diagram is T^α, so that we automatically have an elementary chain $\langle M_\alpha : \alpha < \lambda^+ \rangle$. We write $T^{<\alpha}$ for $\bigcup_{\beta<\alpha} T^\beta$, except that $T^{<0}$ is T; we write T^* for $T^{<\lambda^+}$. Each T^α will be constructed as the union of an increasing chain, $T^\alpha = \bigcup_{i<\lambda} T_i^\alpha$. We write $T_{<i}^\alpha$ for $\bigcup_{j<i} T_j^\alpha$. We shall impose the condition:

IV. For each $i < \lambda$ and each $\beta < \lambda^+$, $T_i^\beta - T^{<\beta}$ has cardinality $< \lambda$.

List as $\langle r_\alpha : \alpha < \lambda^+ \rangle$ all sets of $<\lambda$ formulae of L^* with just x free, so that each such set occurs cofinally often in the list. This uses $\lambda^{<\lambda} = \lambda$. At each α, if r_α is a non-small type over $T^{<\alpha}$ in $L^{<\alpha}$, take T_0^α to be $T^{<\alpha} \cup \{\psi(b_\alpha) : \psi(x) \in r_\alpha\}$; otherwise take T_0^α to be $T^{<\alpha}$. In both cases, write Γ_α for the set

$$\{\neg \psi(b_\alpha) : \psi(x) \in L^{<\alpha} \text{ and } T^{<\alpha} \vdash \neg Qx\psi(x)\}.$$

Lemma 3. *Assume $T^{<\alpha}$ is complete and consistent in $L^{<\alpha}$. For every set $\Phi(b_\alpha, \bar{c}_\alpha)$ of sentences of L^α, $\Gamma_\alpha \cup \Phi(b_\alpha, \bar{c}_\alpha)$ is consistent iff $T^{<\alpha} \vdash Qy\exists \bar{z} \wedge \Phi(y, \bar{z})$.*

Proof. (cf. the proof of Lemma 2.7 in Keisler [3]).
$\Rightarrow$. If $T^{<\alpha} \nvdash Qy\exists\bar{z} \wedge \Phi(y, \bar{z})$, then since $T^{<\alpha}$ is complete, there is some finite approximation $Qy\exists\bar{z}' \wedge \Phi'(y, \bar{z}')$ such that

$$T^{<\alpha} \vdash \neg Qy\exists\bar{z}' \wedge \Phi'(y, \bar{z}'),$$

so that $\neg\exists\bar{z}' \wedge \Phi'(b_\alpha, \bar{z}') \in \Gamma_\alpha$. Then $\Gamma_\alpha \cup \Phi(b_\alpha, \bar{c}_\alpha)$ is inconsistent.
$\Leftarrow$. Suppose $\Gamma_\alpha \cup \Phi(b_\alpha, \bar{c}_\alpha)$ is inconsistent. Then there are $\neg\psi(b_\alpha) \in \Gamma_\alpha$ and a finite conjunction $\varphi(b_\alpha, \bar{c}_\alpha)$ of elements of $\Phi(b_\alpha, \bar{c}_\alpha)$, such that $\vdash \varphi(b_\alpha, \bar{c}_\alpha) \to \psi(b_\alpha)$, and hence

$$\vdash Qy\exists\bar{z}\varphi(y, \bar{z}) \to Qy\psi(y).$$

Since $T^{<\alpha} \vdash \neg Qy\psi(x)$, we infer that $T^{<\alpha} \vdash \neg Qy\exists\bar{z}\varphi(y, \bar{z})$, which contradicts the right-hand side in the lemma. □

Now it is obviously possible to choose the T_{i+1}^α so as to build up T^α into a complete theory with witnesses, such that M_α is λ-compact. This assures I. We shall require:

V. Each T_i^β is consistent with Γ_β.

By Lemma 3 this is no problem. Then $\Gamma_\beta \subseteq T^\beta$, so that b_β represents a new element of M_β. This assures II. Finally there is no difficulty in preserving IV. at successor i. The only restrictions we have placed on T_i for i a limit ordinal are those in IV. and V. These can be achieved by putting $T_i^\beta = T_{<i}^\beta$. But in the next section we shall do something different in order to get III. as well.

4. The Hard Direction: Omitting Types

The intuitive idea will be as follows. Suppose at the end of the construction, a type p which was strongly omitted over $M_{<\alpha}$ gains a support $\varphi = (R)\exists x \wedge q$. We wish to have prevented this. By adding irrelevant parts of T^*, we can assume without loss that φ is complete for some sublanguage of L^*; so that if $\varphi' = (R')(R)\exists x \wedge q'(c_\alpha, \bar{b}_\alpha)$ is the result of quantifying out all β-constants ($\beta > \alpha$) in φ, then φ' is just $T_{<\delta}^\alpha - T^{<\alpha}$ for some δ, up to logical equivalence. Then by Lemma 3,

$$T^{<\alpha} \vdash Qy_\alpha \exists\bar{z}_\alpha (R')(R)\exists x \wedge q'(y_\alpha, \bar{z}_\alpha).$$

The displayed formula cannot be a support of p over $T^{<\alpha}$, so we can find ψ in p such that

$$T^{<\alpha}\vdash Qy_\alpha\exists\bar{z}_\alpha(R')(R)\exists x[\bigwedge q'\wedge\neg\psi(x)].$$

At $T^\alpha_{<\delta}$ and subsequent $T^\beta_{<\delta}$ we use Lemma 3 to remove first $Qy_\alpha\exists\bar{z}_\alpha$ and then the remainder of (R'), until T^* contains $(R)\exists x[\bigwedge q\wedge\neg\psi(x)]$. This shows that φ is not after all a support of p.

The problem lies in knowing which φ to operate on at each $T^\alpha_{<\delta}$. We know of course that φ' is logically equivalent to $T^\alpha_{<\delta}-T^{<\alpha}$, but we do not know the appropriate choice of (R'), since this amounts to a prediction of later stages of the construction. We shall use $\diamondsuit_\lambda$ to ensure that we choose the right (R') at least once during the construction of T^α. Some elaborate coding is needed to ensure that we tackle the same (R') at the right stage in the construction of subsequent T^β.

For each $\alpha<\lambda^+$, fix an enumeration $\langle\alpha_i:i<|\alpha+1|\rangle$ of the elements of $\alpha+1$ without repetition, so that $\alpha_0=\alpha$. For each $j<\lambda$, put $t(\alpha,j)=\{\alpha_i:i<j\}$. Let $L(\alpha,j)$ be the largest sublanguage of L_j such that for every β-constant which occurs in $L(\alpha,j)$, $\beta\in t(\alpha,j)$. We say that the pair (α,j) is *full* iff for each $\beta\in t(\alpha,j)$,

(1) $t(\beta,j)=t(\alpha,j)\cap(\beta+1)$,

(2) $T^\beta_{<j}-T^{<\beta}$ is a complete theory in the language $L(\beta,j)$.

Clearly we know whether (α,j) is full as soon as $T^\alpha_{<j}$ has been constructed. Also when T^α has been constructed, we can define D_α to be the set of $j<\lambda$ such that $t(\alpha,j)$ is full.

Lemma 4. *For each $\alpha<\lambda^+$, D_α is closed unbounded.*

Proof. Clear. □

Now suppose that for each limit ordinal $\delta<\lambda$, there are given us an infinitary formula $(R_\delta)\exists x\bigwedge q_\delta$ in the vocabulary of L, with $|q_\delta|<\lambda$, and a pair $\langle h_\delta,k_\delta\rangle\in\lambda\times\lambda$. The choice of the formulae and the ordinals will be discussed later. For the moment they are given on a plate, and we complete the definition of T^*. The only restriction at this stage is that there is a least ordinal $g_\delta<\lambda$ such that every free variable of $(R_\delta)\exists x\bigwedge q_\delta$ is an i-variable for some $i<g_\delta$.

Consider an ordinal $\alpha<\lambda^+$ and a limit ordinal $\delta<\lambda$. Enumerate $t(\alpha,\delta)$ in increasing order as $\langle\beta_i:i\leqq e(\alpha,\delta)\rangle$. Writing e for $e(\alpha,\delta)$ when α and δ are fixed, we have $\beta_e=\alpha$. For each $m<\lambda$, let (R^m) be the quantifier prefix

$$Qy_m\exists\bar{z}_m\ldots Qy_i\exists\bar{z}_i\ldots\quad(m\leqq i<g_\delta).$$

If $m\geqq g_\delta$, (R^m) is empty. For each $m\leqq e+1$, let q^m_δ be q_δ with each i-variable $(i<m)$ replaced by the corresponding β_i-constant. (y corresponds to b and z to c.)

We say that the pair (α,δ) is *veridical* iff (α,δ) is full and

$$(R^{e+1})(R_\delta)\exists x\bigwedge q^{e+1}_\delta=T^\alpha_{<\delta}-T^{<\alpha}$$

(up to choice of bound variables).

Lemma 5. *Assume we have succeeded in constructing $T^\alpha_{<\delta}$, and (α, δ) is veridical. Then for every $\beta \in t(\alpha, \delta)$, (β, δ) is veridical.*

Proof. It is clear that for every $\beta \in t(\alpha, \delta)$, (β, δ) is full. Since $T^\alpha_{<\delta}$ is consistent (by assumption V.), we have

$$T^\beta_{<\delta} - T^{<\beta} = (T^\alpha_{<\delta} - T^{<\alpha}) \cap L(\beta, \delta).$$

Hence it remains to show that if β is β_i, then

$$(R^{i+1})(R_\delta)\exists x \wedge q^{i+1}_\delta = (T^\alpha_{<\delta} - T^{<\alpha}) \cap L(\beta_i, \delta).$$

The inclusion $\supseteq$ is immediate. The inclusion $\subseteq$ is proved, for each separate finitary formula on the left, by assumption V and repeated applications of Lemma 3. □

We may now define T^α_δ for limit ordinals δ, assuming that $T^\alpha_{<\delta}$ has been defined and that IV. and V. hold so far. There are three cases.

Case 1. (α, δ) is not veridical, or $h_\delta > e$. Then put $T^\alpha_\delta = T^\alpha_{<\delta}$. Clearly IV. and V. are preserved.

Case 2. (α, δ) is veridical and $h_\delta = e$. Then

$$T^{<\alpha} \vdash (R^e)(R_\delta)\exists x \wedge q^e_\delta \tag{1}$$

by Lemma 3, and $p^\alpha_{k_\delta}$ is a type p' which is strongly omitted over $T^{<\alpha}$. Hence there is some formula $\psi(x) \in p'$ such that

$$T^{<\alpha} \vdash (R^e)(R_\delta)\exists x[\wedge q^e_\delta \wedge \neg\psi(x)].$$

Choose such a ψ, call it $\psi_{\alpha,\delta}$, and put

$$T^\alpha_\delta = T^\alpha_{<\delta} \cup (R^{e+1})(R_\delta)\exists x[\wedge q^{e+1}_\delta \wedge \neg\psi_{\alpha,\delta}(x)].$$

IV. clearly remains true. V. holds by (1), Lemma 3 and the veridicality of (α, δ).

Case 3. (α, δ) is veridical and $h_\delta < e$. Let $\langle \beta_i : i \leqq e \rangle$ once again be the enumeration of $t(\alpha, \delta)$ in increasing order. We put

$$T^\alpha_\delta = T^\alpha_{<\delta} \cup (R^{e+1})(R_\delta)\exists x[\wedge q^{e+1}_\delta \wedge \neg\psi_{\beta_{h_\delta},\delta}(x)].$$

Again IV. holds at T^α_δ. Now for each i $(h_\delta \leqq i < e)$, (β_i, δ) is veridical by Lemma 5, and we have

$$T^{\beta_i} \vdash (R^{i+1})(R_\delta)\exists x[\wedge q^{i+1}_\delta \wedge \neg\psi_{\beta_{h_\delta},\delta}(x)]. \tag{2}$$

This is by Case 2 at β_i when $i = h_\delta$, and by Case 3 when $i > h_\delta$. It follows that

$$T^{<\alpha} \vdash (R^e)(R_\delta)\exists x[\wedge q^e_\delta \wedge \neg\psi_{\beta_{h_\delta},\delta}(x)].$$

(If e is a limit ordinal, consider finite approximations.) Hence V. holds at T_δ^α by Lemma 3.
This completes the construction. It achieves the following.

Lemma 6. *Suppose* (α, δ) *is veridical; we write* $h = h_\delta$, $k = k_\delta$, *etc. If* $g = e + 1 \geqq h + 1$, *then for some* $\psi(x) \in p_k^{\beta_h}$, $M^* \models (R_\delta)\exists x[\bigwedge q_\delta^{e+1} \wedge \neg\psi(x)]$. $\square$

It remains only to choose the formulae $(R_\delta)\exists x \bigwedge q_\delta$ and the pairs $\langle h_\delta, k_\delta\rangle$ so as to ensure III. For this we use $\diamondsuit_\lambda$ in the form which says there is a family $\langle (S_\delta : \delta \to \delta) : \delta < \lambda\rangle$ of maps such that if $F : \lambda \to \lambda$ then $F | \delta = S_\delta$ for a stationary set of δ.
Let us say φ is a *diagram* iff φ is an infinitary sentence $(R)\exists x \bigwedge q$ for the language of L, in which q is a set of $< \lambda$ finitary formulae and the set $\{i$: an i-variable occurs free in $q\}$ is an initial segment s of λ; we write length$(\varphi) = s$. Since $\lambda^{<\lambda} = \lambda$, we can list as $A^\gamma = \langle \varphi^\gamma, h^\gamma, k^\gamma, <^\gamma\rangle$ $(\gamma < \lambda)$ all quadruples such that φ^γ is a diagram, h^γ is an ordinal $<$ length(φ^γ), k^γ is an ordinal $< \lambda$, and $<^\gamma$ is a map which assigns to each ordinal $i \in$ length(φ^γ) a well-ordering $<_i^\gamma$ of $i + 1$.
We say $A^\gamma \prec A^\delta$ iff there is an order-preserving injection f:length$(\varphi^\gamma) \to$ length(φ^δ) such that

(1) if $f(\varphi^\gamma)$ is φ^γ with each free i-variable replaced by the corresponding $f(i)$-variable, then $f(\varphi^\gamma) \subseteq \varphi^\delta$;
(2) $f(h^\gamma) = h^\gamma$ and $k^\gamma = k^\delta$;
(3) each $<_j^\gamma$ is taken isomorphically to an initial segment of $<_{f(j)}^\delta$ by f.

Let δ be a limit ordinal $< \lambda$. If there is an A^γ such that for a final segment $\theta_0, \theta_1, \ldots$ of δ,

$$A^{S_\delta(\theta_0)} \preceq A^{S_\delta(\theta_1)} \preceq \ldots$$

converging to A^γ, then clearly A^γ is uniquely determined (up to bound variables), and we put

$$(R_\delta)\exists x \bigwedge q_\delta = \varphi^\gamma, \quad h_\delta = h^\gamma, \quad k_\delta = k^\gamma.$$

For other δ, choose $(R_\delta)\exists x \bigwedge q_\delta$, h_δ and k_δ arbitrarily.

Lemma 7. *With the above definitions,* III. *holds.*

Proof. Suppose not; suppose p_k^β first gains a support q in M_α. Then

$$M_\alpha \models (R)\exists x \bigwedge q.$$

Let E be the set of limit ordinals $\delta \in D_\alpha$ such that q lies inside $L(\alpha, \delta)$ and $\beta \in t(\alpha, \delta)$. By Lemma 4, E is closed unbounded. For each $\delta \in E$, listing $t(\alpha, \delta)$ in increasing order as $\langle \beta_i : i \leqq e\rangle$, let $F(\delta)$ be γ such that

φ^γ is $(R)\exists x \bigwedge [q \cup (T_{<\delta}^\alpha - T^{<\alpha})]$ with each β_i-constant replaced by a free occurrence of the corresponding i-variable;
$\beta_{h^\gamma} = \beta$, $k^\gamma = k$;
$<_i^\gamma$ orders $t(\beta_i, \delta)$ as in the enumeration of $\beta_i + 1$ fixed near the beginning of this section.

For other $i \in \lambda$, let $F(i)$ be $\cup\{F(\delta):\delta \in E \cap i\}$. By $\diamondsuit_\lambda$ there is a limit point δ of E such that $F|\delta = S_\delta$. Since δ is a limit point of E, it follows that $\varphi^{F(\delta)}$ is $(R_\delta)\exists x \bigwedge q_\delta$ (up to choice of bound variables) and hence that (α, δ) is veridical. We can also verify that $\beta_{h_\delta} = \beta$, $k_\delta = k$, and $g_\delta = e(\alpha, \delta) + 1 \geqq h_\delta + 1$. Hence by Lemma 6,

$$M^* \models (R_\delta)\exists x[\bigwedge q_\delta^{e+1} \wedge \neg \psi(x)] \quad \text{for some} \quad \psi(x) \in p_k^\beta.$$

But q_δ^{e+1} includes q, so this implies that q is not after all a support of p_k. This proves the lemma and the theorem. □

5. Remarks

1. By Section 6 below, $\diamondsuit_\lambda$ holds for all successor cardinals $\lambda > \aleph_1$ if we assume the GCH. If λ is strongly inaccessible, then the theorem can be proved without the hypothesis of $\diamondsuit_\lambda$.
2. The proof in fact shows how to omit λ^+ types at the same time. Moreover λ of these types can be added at each stage of the chain, so we don't have to know all the types in advance.
3. Chang's two cardinal theorem $\langle \aleph_1, \aleph_0 \rangle \to \langle \lambda^+, \lambda \rangle$ is equivalent to "Every theory in $L(Q)$ with a model for the $\aleph_1$-interpretation has a model for the λ^+-interpretation". So the above proof also gives a new proof of Chang's theorem. Unlike Chang, we do not expand the language; so we can get some new results. For example in [6, Theorem 10] we can omit "and (T, W) satisfies Chang's condition". A systematic application of this to e.g. existence theorems for models with few automorphisms will appear elsewhere.
4. In [4], Magidor and Malitz define a family of languages $\{L^n : n < \omega\}$. L^n is obtained from the usual first order language by adding a new quantifier Q^n. In the κ-interpretation, a model M satisfies $Q^n x_1 \ldots x_n \varphi(x_1, \ldots, x_n)$ iff there is a set $X \subseteqq M$ of cardinality κ such that $M \models \varphi[a_1, \ldots, a_n]$ whenever $a_1, \ldots, a_n$ are distinct elements of X. Assuming $\diamondsuit_{\omega_1}$, Magidor and Malitz prove compactness for the $\aleph_1$-interpretation. They ask [4, Problem 3] under what set-theoretic assumptions one can have compactness for the κ-interpretation of L^n. Our method of proof can be used quite straightforwardly to show that L^n is compact in the λ^+-interpretation whenever $\lambda^{<\lambda} = \lambda$ and $\diamondsuit_{\lambda^+}(E)$ holds, where E is the set of limit ordinals $< \lambda^+$ of cofinality λ.

6. $\diamondsuit_\lambda$ is Not a Strong Demand

Let λ be a regular cardinal and E a stationary subset of λ. Jensen defined:

$\diamondsuit^*(E)$ means there is $\langle W_\alpha : \alpha \in E \rangle$ such that except for a bounded set of α, each W_α is a family of $\leqq |\alpha|$ subsets of α, and for every $X \subseteqq \lambda$ there is a closed unbounded $C \subseteqq \lambda$ such that $X \cap \alpha \in W_\alpha$ for all $\alpha \in C \cap E$.

$\diamondsuit_\lambda(E)$ means there is $\langle S_\alpha : \alpha \in E \rangle$ such that $S_\alpha \subseteqq \alpha$ and for every $X \subseteqq \lambda$, $\{\alpha : X \cap \alpha = S_\alpha\}$ is stationary in λ.

Kunen (unpublished) showed that $\diamondsuit^*_\lambda(E)$ implies $\diamondsuit_\lambda(E)$, and that if $E_1 \subseteqq E_2$ are stationary, then $\diamondsuit_\lambda(E_1)$ implies $\diamondsuit_\lambda(E_2)$ and $\diamondsuit^*_\lambda(E_2)$ implies $\diamondsuit^*_\lambda(E_1)$.
Write $E(\kappa)$ for the set of all $\alpha < \lambda$ such that $\mathrm{cf}\alpha = \kappa$. If $\kappa < \lambda$, then $E(\kappa)$ is a stationary subset of λ.

Theorem. *Suppose* $\lambda = 2^\mu = \mu^+$ *and* κ *is a regular cardinal* $< \mu$. *Then each of* (i), (ii) *below implies* $\diamondsuit^*_\lambda(E(\kappa))$.
(i) *(Gregory [2, Lemma 2.1])* $\mu^\kappa = \mu$.
(ii) μ *is singular*, $\mathrm{cf}\mu \neq \kappa$, *and for every* $\delta < \mu$, $\delta^\kappa < \mu$.

Proof. Let $\langle A_\alpha : \alpha < \lambda \rangle$ be a list of all the bounded subsets of λ. (There are λ such subsets as $\lambda = 2^\mu = \mu^+$.)
(i) For each $\alpha \in E(\kappa)$, let W_α be the set of all sets of form $\cup Y$ where $Y \subseteqq \mathscr{P}(\alpha) \cap \{A_\beta : \beta < \alpha\}$ and $|Y| \leqq \kappa$. Given any $X \subseteqq \lambda$, let $C = \{\alpha_i : i < \lambda\}$ be defined as follows. α_0 is any successor ordinal $< \lambda$. For limit δ, put $\alpha_\delta = \bigcup_{\beta < \delta} \alpha_\beta$. Put α_{i+1} = the least $\alpha > \alpha_i$ such that for some $\gamma < \alpha$, $A_\gamma = X \cap \alpha_i$.
(ii) For each $\alpha \in E(\kappa)$, fix an increasing sequence $\langle a_i : i < \kappa \rangle$ cofinal in α. Also fix an increasing sequence of sets $\langle V^\alpha_j : j < \alpha \rangle$ such that $\alpha = \bigcup_{j<\alpha} V^\alpha_j$ and each $|V^\alpha_j| < \mu$. Let W_α be the set of all sets of form $\cup Y$ where $|Y| \leqq \kappa$ and for some $j < \alpha$, Y is a set of A_δ with $\delta \in V^\alpha_j$. Given any $X \subseteqq \lambda$, let $f : \lambda \to \lambda$ be such that each $X \cap \alpha = A_{f(\alpha)}$, and let C be the set of $\alpha < \lambda$ such that $\beta < \alpha$ implies $f(\beta) < \alpha$. Then if $\alpha \in C \cap E(\kappa)$, there is j such that V^α_j contains κ $f(\alpha_i)$ $(i < \kappa)$, and so $X \cap \alpha = \bigcup \{X \cap \alpha_i : i < \kappa$ and $f(\alpha_i) \in V^\alpha_j\} \in W_\alpha$. □

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Saharon Shelah
Department of Mathematics
Hebrew University
Jerusalem
Israel