ISRAEL JOURNAL OF MATHEMATICS **212** (2016), 259–287
DOI: 10.1007/s11856-016-1286-9

# THE GENERIC PAIR CONJECTURE FOR DEPENDENT FINITE DIAGRAMS*

BY

Itay Kaplan**

*Einstein Institute of Mathematics, The Hebrew University of Jerusalem*
*Edmond J. Safra Campus, Givat Ram, Jerusalem 91904, Israel*
*e-mail: kaplan@math.huji.ac.il*
*URL: https://sites.google.com/site/itay80/*

AND

Noa Lavi

*Einstein Institute of Mathematics, The Hebrew University of Jerusalem*
*Edmond J. Safra Campus, Givat Ram, Jerusalem 91904, Israel*
*e-mail: noa.lavi@mail.huji.ac.il*

AND

Saharon Shelah†

*Einstein Institute of Mathematics, The Hebrew University of Jerusalem*
*Edmond J. Safra Campus, Givat Ram, Jerusalem 91904, Israel*
*and*
*Department of Mathematics, Rutgers, The State University of New Jersey*
*Hill Center-Busch Campus, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, USA*
*e-mail: shelah@math.huji.ac.il*
*URL: http://shelah.logic.at/*

* The research leading to these results has received funding from the European Research Council, ERC Grant Agreement n. 338821.
** The first author would like to thank the Israel Science Foundation for partial support of this research (Grant no. 1533/14).
† No. 1055 on the third author's list of publications.
Received October 20, 2014 and in revised form December 4, 2014

ABSTRACT

This paper generalizes Shelah's generic pair conjecture (now theorem) for the measurable cardinal case from first order theories to finite diagrams. We use homogeneous models in the place of saturated models.

## 1. Introduction

The generic pair conjecture states that for every cardinal $\lambda$ such that $\lambda^+ = 2^\lambda$ and $\lambda^{<\lambda} = \lambda$, a complete first order theory $T$ is dependent if and only if, whenever $M$ is a saturated model whose size is $\lambda^+$, then, after writing

$$M = \bigcup_{\alpha<\lambda^+} M_\alpha$$

where $M_\alpha$ are models of size $\lambda$, there is a club of $\lambda^+$ such that for every pair of ordinals $\alpha < \beta$ of cofinality $\lambda$ from the club, the pair of models $(M_\beta, M_\alpha)$ has the same isomorphism type.

This conjecture is now proved for $\lambda$ large enough. The non-structure side is proved in [She06, She11] and the other direction is proved in [She13, She12], all by the third author. In [She13], the theorem is proved for the case where $\lambda$ is measurable. This is the easiest case of the theorem, and this is the case we will focus on here. In [She12, Theorem 7.3], the conjecture is proved when

$$\lambda > |T|^+ + \beth_\omega^+.$$

The current paper has two agendas.

The first is to serve as an exposition for the proof of the theorem in the case where $\lambda$ is measurable. There are already two expositions by Pierre Simon on some other parts from [She13, She12], which are available on his website[1].

The second is to generalize the structure side of this theorem in the measurable cardinal case to finite diagrams. As an easy byproduct, we also generalize a weak version of the "recounting of types" result [She12, Conclusion 3.13], which states that when $\lambda$ is measurable and $M$ is saturated of cardinality $\lambda$, then the number of types over $M$ up to conjugation is $\leq \lambda$. See Corollary 5.13 below.

A finite diagram $D$ is a collection of types in finitely many variables over $\emptyset$ in some complete theory $T$. Once we fix such a $D$ we concentrate on $D$-models, which are models of $T$ which realize only types from $D$. For instance, in a

[1] `http://www.normalesup.org/~simon/notes.html`

theory with infinitely many unary predicates $P_i$, $D$ could prohibit $x \notin P_i$ for all $i$, thus $D$-models are just the union of the $P_i$'s. In this context, saturated models become $D$-saturated models, which is the same as being homogeneous and realize $D$ (see Lemma 2.3), so our model $M$ will be $D$-saturated instead of saturated.

We propose a definition for when a finite diagram $D$ is dependent. This definition has the feature that if the underlying theory is dependent, then so is $D$, so there are many examples of such diagrams. We also give an example of an independent theory $T$ with some dependent $D$ (Example 2.8).

The proof follows [She13] and also uses constructions from [She12][2]. However, in order to make the proof work, we will need the presence of a strongly compact cardinal $\theta$ that will help us ensure that the types we get are $D$-types and so realized in the $D$-saturated models.

*Organization of the paper.* In Section 2 we expose finite diagrams and prove or cite all the facts we shall need about them and about measurable and strongly compact cardinals. We also give a precise definition of when a diagram $D$ is dependent, and prove several equivalent formulations.

In Section 3 we state the generic pair conjecture in the terminology of finite diagrams, and give a general framework for proving it: we introduce decompositions and good families and prove that if such things exist, then the theorem is true.

Section 4 is devoted to proving that nice decompositions exist. This is done in two steps. In Section 4.1 we construct the first kind of decomposition (tree-type decomposition), which is the building block of the decomposition constructed in Section 4.2 (self-solvable decomposition).

In Section 5 we prove that the family of self-solvable decompositions over a $D$-saturated model form a good family, and deduce the generic pair conjecture.

Acknowledgment. We would like to thank the anonymous referee for his careful reading, for his many useful comments and for finding several inaccuracies.

[2] Instead of “strict decompositions” from [She13] we use $tK$ from [She12].

## 2. Preliminaries

We start by giving the definition of homogeneous structures and of $D$-models.

*Definition 2.1:* Let $M$ be some structure in some language $L$. We say that $M$ is $\kappa$**-homogeneous**[3] if:

- for every $A \subseteq M$ with $|A| < \kappa$, every partial elementary map $f$ defined on $A$ and $a \in M$ there is some $b \in M$ such that $f \cup \{(a,b)\}$ is an elementary map.

We say that $M$ is **homogeneous** if it is $|M|$-homogeneous.

Note that when $M$ is homogeneous, it is also **strongly homogeneous**, meaning that if $f$ is a partial elementary map with domain $A$ such that $|A| < |M|$, $f$ extends to an automorphism of $M$.

Fix a complete first order theory $T$ in a language $L$ with a monster model $\mathfrak{C}$—a saturated model containing all sets and models of $T$, with cardinality $\bar{\kappa} = \bar{\kappa}^{<\bar{\kappa}}$ bigger than any set or model we will consider.

*Definition 2.2:* For $A \subseteq \mathfrak{C}$, let $D(A) = \{\mathrm{tp}(\bar{a}/\emptyset) \,|\, \bar{a} \subseteq A, |\bar{a}| < \omega\}$. A set $D$ of complete $L$-types over $\emptyset$ is a **finite diagram in** $T$ when it is of the form $D(A)$ for some $A$. If $D$ is a finite diagram in $L$, then a set $B \subseteq \mathfrak{C}$ is a $D$**-set** if $D(B) \subseteq D$. A model of $T$ which is a $D$-set is a $D$**-model.**

Let $A \subseteq \mathfrak{C}$ be a $D$-set. Let $p$ be a complete type over $A$ (in any number of variables). We say that $p$ is a $D$**-type** if for every $\bar{c}$ realizing $p$, $A \cup \bar{c}$ is a $D$-set. We denote the set of $D$-types over $A$ by $S_D(A)$ (and as usual we use superscript to denote the number of variables, such as in $S_D^{<\omega}(A)$). We say that $M$ is $(D,\kappa)$**-saturated** if whenever $|A| < \kappa$, every $p \in S_D^1(A)$ is realized in $M$. We say that $M$ is $D$**-saturated** if it is $(D, |M|)$-saturated.

Note that when $D$ is **trivial**, i.e., $D = \bigcup\{D_n(T) \,|\, n < \omega\}$ (with $D_n(T)$ being the set of all complete $n$-types over $\emptyset$), every model of $T$ is a $D$-model.

The connection between $D$-saturation and homogeneity becomes clear due to the following lemma.

Lemma 2.3 ([GL02, Lemma 2.4]): *Let $D$ be a finite diagram. A $D$-model $M$ is $(D,\kappa)$-saturated if and only if $D(M) = D$ and $M$ is $\kappa$-homogeneous.*

[3] In some publications this notion is called $\kappa$-sequence homogeneous, but here we decided upon this simpler notation which is also standard; see [Hod93, page 480, 1.3].

Just as in the first order case, we get the following.

Corollary 2.4: *Let $D$ be a finite diagram. If $M$, $N$ are $D$-saturated of the same cardinality, then $M \cong N$. Furthermore, if $\lambda^{<\lambda} = \lambda$, and there is a $(D,\lambda)$-saturated model, then there exists a $D$-saturated model of size $\lambda$.*

The next natural thing, after obtaining this equivalence, would be to look for monsters. A diagram $D$ is **good** if for every $\lambda$ there exists a $(D,\lambda)$-saturated model (see [She71, Definition 2.1]). We will assume throughout that $D$ is good. By Corollary 2.4, as we assumed that $\bar{\kappa}^{<\bar{\kappa}} = \bar{\kappa}$, there is a $D$-saturated model $\mathfrak{C}_D \prec \mathfrak{C}$ of cardinality $\bar{\kappa}$—the homogeneous monster. From now on we make these assumptions without mentioning them explicitly.

Let us recall the general notion of an average type along an ultrafilter.

*Definition 2.5:* Let $A \subseteq \mathfrak{C}_D$, $I$ some index set, $\bar{a}_i$ tuples of the same length for $i \in I$, and let $\mathcal{U}$ be an ultrafilter on $I$. The **average type** $Av_{\mathcal{U}}(\langle \bar{a}_i | \, i \in I\rangle / A)$ is the type consisting of all the formulas $\phi(\bar{x},\bar{c})$ over $A$ such that

$$\{i \in I | \, \mathfrak{C}_D \models \phi(\bar{a}_i,\bar{c})\} \in \mathcal{U}.$$

When $\mathcal{U}$ is $\kappa$-complete, the average is $< \kappa$ satisfiable in the sequence $\langle \bar{a}_i | \, i \in I\rangle$ (any $< \kappa$ many formulas are realized in the sequence). It follows that the average type is a $D$-type (see below).

Lemma 2.6: *Let $A, I$ be as in Definition 2.5, and let $\mathcal{U}$ be a $\kappa$-complete ultrafilter on $I$, where $\kappa > |T|$. Then $r = Av_{\mathcal{U}}(\langle \bar{a}_i | \, i \in I\rangle / A)$ is a $D$-type.*

*Proof.* We must show that if $\bar{c} \models r$ (in $\mathfrak{C}$), then $A \cup \bar{c}$ is a $D$-set. We may assume that $\bar{c}$ is a finite tuple (and so are the tuples $\bar{a}_i$ for $i \in I$). It is enough to see that if $\bar{c}\bar{a}$ is a finite tuple of elements from $\bar{c} \cup A$, then for some $i \in I$, $\bar{a}_i\bar{a} \equiv \bar{c}\bar{a}$ (i.e., they have the same type over $\emptyset$). For each formula $\varphi(\bar{x},\bar{a})$ such that $\varphi(\bar{c},\bar{a})$ holds, the set $\{i \in I | \, \mathfrak{C}_D \models \varphi(\bar{a}_i,\bar{a})\} \in \mathcal{U}$. Since there are $|T|$ such formulas, by $\kappa$-completeness, there is some $i \in I$ in the intersection of all these sets, so we are done. ∎

Now we turn to Hanf numbers. Let $\mu(\lambda,\kappa)$ be the first cardinal $\mu$ such that if $T_0$ is a theory of size $\leq \lambda$, $\Gamma$ a set of finitary types in $T_0$ (over $\emptyset$) of cardinality $\leq \kappa$, and for every $\chi < \mu$ there is a model of $T_0$ of cardinality $\geq \chi$ omitting all the types in $\Gamma$, then there is such a model in arbitrarily large cardinality. Of course, when $\kappa = 0$, $\mu(\lambda,\kappa) = \aleph_0$. In our context, $T_0 = T$, and $\Gamma = \bigcup\{D_n(T) | n < \omega\} \setminus D$, so we are interested in $\mu(|T|, |\Gamma|)$ which we will denote by $\mu(D)$, the **Hanf number** of $D$. In [She90, Chapter VII, 5] this number is given an upper bound: $\mu(D) \leq \beth_{(2^{|T|})^+}$.

*Definition 2.7:* A finite diagram $D$ has the **independence property** if there exists a formula $\phi(\bar{x}, \bar{y})$ which has it, which means that there is an indiscernible sequence $\langle \bar{a}_i | i < \mu(D) \rangle$ and $\bar{b}$ in $\mathfrak{C}_D$ such that $\mathfrak{C}_D \models \phi(\bar{b}, \bar{a}_i)$ if and only if $i$ is even. Otherwise we say that $D$ is **dependent**.

Of course, if the underlying theory $T$ is dependent, then $D$ is dependent.

*Example 2.8:* Let $L = \{R, P, Q\}$ where $P$ and $Q$ are unary predicates, and $R$ is a binary predicate. Let $T$ be the model completion of the theory that states that $R \subseteq Q \times P$. So $T$ is complete and has quantifier elimination. Let $L' = L \cup \{c_i | i < \omega\}$ where $c_i$ are constant symbols, and let $T'$ be an expansion of $T$ that says that $c_i \in P$ and $c_i \neq c_j$ for $i \neq j$. So $T'$ is also complete and admits quantifier elimination. As $T$ has the independence property, so does $T'$.

Let $p(x) \in S^1(\emptyset)$ say that $x \in P$ and $x \neq c_i$ for all $i < \omega$. Finally, let $D$ be the finite diagram $S^{<\omega}(\emptyset) \setminus \{p\}$. Easily $D$ is good (if $\mathfrak{C}$ is a monster model of $T$, then let $Q^{\mathfrak{C}} \cup \{c_i^{\mathfrak{C}} | i < \omega\}$ be $\mathfrak{C}_D$). It is easy to see that $D$ is dependent.

Recall that a cardinal $\theta$ is **strongly compact** if any $\theta$-complete filter (with any domain) is contained in a $\theta$-complete ultrafilter. For our context we will need to assume that if $D$ is non-trivial, then there is a strongly compact cardinal $\theta > |T|$. Strongly compact cardinals are measurable (see [Kan09, Corollary 4.2]). Recall that a cardinal $\mu$ is **measurable** if it is uncountable and there is a $\mu$-complete non-principal ultrafilter on $\mu$. It follows that there is a **normal** such ultrafilter (i.e., closed under diagonal intersection). See [Kan09, Exercise 5.12]. Measurable cardinals are strongly inaccessible (see [Kan09, Theorem 2.8]), which means that $\theta > \beth_{(2^{|T|})^+} \geq \mu(D)$. Fix some such $\theta$ throughout. If, however, $D$ is trivial, then we do not need a strongly compact cardinal.

We also note here a key fact about measurable cardinals that will be useful later:

*Fact 2.9* ([Kan09, Theorem 7.17]): Suppose that $\mu > |T|$ is a measurable cardinal and that $\mathcal{U}$ is a normal (non-principal) ultrafilter on $\mu$. Suppose that $\langle \bar{a}_i | i < \mu \rangle$ is a sequence of tuples in $\mathfrak{C}$ of equal length $< \mu$. Then for some set $X \in \mathcal{U}$, $\langle \bar{a}_i | i \in X \rangle$ is an indiscernible sequence.

As a consequence (which will also be used later), we have the following.

Corollary 2.10: *If $A = \bigcup_{i<\mu} A_i \subseteq \mathfrak{C}$ is a continuous increasing union of sets where $|A_i| < \mu$, $B \subseteq \mathfrak{C}$ is some set of cardinality $< \mu$, and $\langle \bar{a}_i | \, i < \mu \rangle$, $\mathcal{U}$ are as in Fact 2.9 with $\bar{a}_i$ tuples from $A$, then for some set $X \in \mathcal{U}$, $\langle \bar{a}_i | \, i \in X \rangle$ is* **fully indiscernible over** *$B$ (with respect to $A$ and $\langle A_i | \, i < \mu \rangle$), which means that for every $i \in X$ and $j < i$ in $X$, we have $\bar{a}_j \subseteq A_i$ , and $\langle \bar{a}_j | \, i \leq j \in X \rangle$ is indiscernible over $A_i \cup B$.*

*Proof.* This follows by the normality of the ultrafilter $\mathcal{U}$. First note that if $E \subseteq \mu$ is a club then $E \in \mathcal{U}$ (why? Otherwise $X = \mu \backslash E \in \mathcal{U}$, so the function $f : X \to \mu$ defined by $\beta \mapsto \sup(\beta \cap E)$ is such that $f(\beta) < \beta$, and by Fodor's lemma (which holds for normal ultrafilters), for some $\gamma < \mu$ and $Y \subseteq X$ in $\mathcal{U}$, $f \restriction Y = \gamma$ which easily leads to a contradiction). Hence the set $E = \{i < \mu | \, \forall j < i(\bar{a}_j \subseteq A_i)\}$ is in $\mathcal{U}$. Furthermore, the set of limit ordinals $E'$ is also in $\mathcal{U}$. The promised set $X$ is the intersection of $E \cap E'$ with the diagonal intersection of $X_i$ for $i < \mu$, where $X_i \in \mathcal{U}$ is such that $\langle \bar{a}_i | \, i \in X_i \rangle$ is indiscernible over $A_i \cup B$ (which exists thanks to Fact 2.9). Note that we have $\leq$ and not just $<$ when defining "fully indiscernible", because $\langle A_i | \, i < \mu \rangle$ is continuous and $X$ contains only limit ordinals. ∎

The following demonstrates the need for Hanf numbers and strongly compact cardinals.

Lemma 2.11: *For a finite diagram $D$ the following conditions are equivalent:*

1. *The formula $\phi(\bar{x}, \bar{y})$ has the independence property.*
2. *For any $\lambda$ there is an indiscernible sequence $\langle \bar{a}_i | \, i < \lambda \rangle$ and $\bar{b}$ in $\mathfrak{C}_D$ such that $\mathfrak{C}_D \models \phi(\bar{a}_i, \bar{b})$ iff $i$ is even.*
3. *For any $\lambda$ there is a set $\{\bar{a}_i | \, i < \lambda\} \subseteq \mathfrak{C}_D$ such that for any $s \subseteq \lambda$ there is some $\bar{b}_s \in \mathfrak{C}_D$ such that $\mathfrak{C}_D \models \phi(\bar{a}_i, \bar{b}_s)$ iff $i \in s$.*
4. *The same as (2) but with $\lambda = \theta$.*
5. *The same as (3) but with $\lambda = \theta$.*

*Proof.* (1) $\Rightarrow$ (3): we may assume that $\lambda \geq \mu(D)$. By assumption there is a sequence $\langle \bar{a}_i | \, i < \mu(D) \rangle$ and $\bar{b}$ in $\mathfrak{C}_D$ as in the definition. Let $M \prec \mathfrak{C}_D$ be a model of size $\mu(D)$ containing all these elements. Add to the language $L$ new constants $\bar{c}$ in the length of $\bar{b}$, a new predicate $P$ in the length of $\bar{x}$ and a $2\lg(\bar{x})$-ary symbol $<$, and a function symbol $f$. Expand $M$ to $M'$, a structure of the expanded language, by interpreting $\bar{c}^{M'} = \bar{b}$, $P^{M'} = \{\bar{a}_i |\, i < \mu(D)\}$, $\bar{a}_i <^{M'} \bar{a}_j$ iff $i < j$ and let $f^{M'} : P^{M'} \to M'$ be onto.

Let $T_0 = Th(M')$. By assumption, $T_0$ has a $D$-model of size $\mu(D)$, and so by definition $T_0$ has a $D$-model $N'$ of cardinality $\lambda$ and we may assume that its $L$-part $N$ is an elementary substructure of $\mathfrak{C}_D$. So the elements in $P^{N'}$, ordered by $<^{N'}$, form an $L$-indiscernible sequence, and $|P^{N'}| = \lambda$.

For convenience of notation, let $(I, <)$ be an order, isomorphic to $(P^{N'}, <^{N'})$, and write $P^{N'} = \{\bar{a}_i |\, i \in I\}$. The order $<$ is discrete, so every $i \in I$ has a unique successor $s(i)$, and $N \models \phi(\bar{c}, \bar{a}_i) \leftrightarrow \neg\phi(\bar{c}, \bar{a}_{s(i)})$. Let $Q = \{i \in I |\, N \models \phi(\bar{c}, \bar{a}_i)\}$, so $|Q| = \lambda$. Then, by indiscernibility, for any $R \subseteq Q$,

$$\langle \bar{a}_i |\, i \in Q\rangle \equiv \langle \bar{a}_{s^{R(i)}(i)} |\, i \in Q\rangle$$

where $R(i) = 0$ iff $i \in R$, and $s^0 = \mathrm{id}$, $s^1 = s$. Hence by the strong homogeneity of $\mathfrak{C}_D$, $\{\bar{a}_i |\, i \in Q\}$ satisfies (3).

(2) $\Rightarrow$ (4), (3) $\Rightarrow$ (5), (4) $\Rightarrow$ (1): Obvious.

(5) $\Rightarrow$ (2): We may assume that $\lambda \geq \theta$. Let $\{\bar{a}_i |\, i < \theta\}$ be as in (5). Since $\theta$ is measurable, by Fact 2.9, we may assume that $\langle \bar{a}_i |\, i < \theta\rangle$ is indiscernible. By compactness we can extend this sequence to $\langle \bar{a}_i |\, i < \lambda\rangle$, and let $A = \{\bar{a}_i |\, i < \lambda\}$. Note that by indiscernibility, the set containing all tuples in the new sequence is still a $D$-set, so we may assume that this new sequence lies in $\mathfrak{C}_D$.

Let $O$ be the set of odd ordinals in $\lambda$. By indiscernibility and homogeneity, for each $X \in [\lambda]^{<\theta}$ (i.e., $X \subseteq \lambda$, $|X| < \theta$) there is some $\bar{b}_X$ such that for all $i \in X$, $\mathfrak{C}_D \models \phi(\bar{b}_X, \bar{a}_i)$ iff $i \notin O$. By strong compactness, there is some $\theta$-complete ultrafilter $\mathcal{U}$ on $[\lambda]^{<\theta}$ such that for every $X \in I$ we have $\{Y \in [\lambda]^{<\theta} |\, X \subseteq Y\} \in \mathcal{U}$. Let $\bar{b} \models Av_{\mathcal{U}}(\langle \bar{b}_X |\, X \in [\lambda]^{<\theta}\rangle / A)$ which exists in $\mathfrak{C}_D$ by Lemma 2.6. Then $\mathfrak{C}_D \models \phi(\bar{b}, \bar{a}_i)$ iff $i$ is even. ∎

Dependence gives rise to the concept of the average type of an indiscernible sequence, without resorting to ultrafilters. Let $A \subseteq \mathfrak{C}_D$, let $\alpha$ be an ordinal such that $\mathrm{cof}(\alpha) \geq \mu(D)$, and let $\langle \bar{a}_i |\, i < \alpha\rangle$ be an indiscernible sequence in $\mathfrak{C}_D$. The **average type** of $\langle \bar{a}_i |\, i < \alpha\rangle$ over $A$, denoted by $Av(\langle \bar{a}_i |\, i < \alpha\rangle / A)$, consists of formulas of the form $\phi(\bar{b}, \bar{x})$ with $\bar{b} \in A$, such that for some $i$, $\mathfrak{C}_D \models \phi(\bar{b}, \bar{a}_j)$ for every $j \geq i$. This is well defined as $\mathrm{cof}(\alpha) \geq \mu(D)$ (and as $D$ is dependent): otherwise, we can construct an increasing unbounded sequence of ordinals $j_i < \alpha$, such that $\phi(\bar{b}, \bar{a}_{j_i}) \leftrightarrow \neg\phi(\bar{b}, \bar{a}_{j_{i+1}})$, and the length of this sequence is $\geq \mu(D)$. We show that this type is indeed a $D$-type.

LEMMA 2.12: *Let $A \subseteq \mathfrak{C}_D$ where $D$ is a dependent diagram, $\alpha$ an ordinal such that $\operatorname{cof}(\alpha) \geq \mu(D) + |T|^+$, and let $\langle \bar{a}_i | \, i < \alpha \rangle$ be an indiscernible sequence in $\mathfrak{C}_D$. The average type $r = Av(\langle \bar{a}_i | \, i < \alpha\rangle / A)$ is a $D$-type.*

*Proof.* The proof is similar to that of Lemma 2.6, but here we use the fact that the end-segment filter on $\alpha$ is $\operatorname{cof}(\alpha)$-complete. The main point is that for a formula $\varphi(\bar{x}, \bar{a}) \in r$, there is some $j < \alpha$ such that $\varphi(\bar{a}_i, \bar{a})$ holds for all $i > j$. ∎

## 3. The generic pair conjecture

From this section onwards, fix a dependent diagram $D$. We also fix a strongly compact cardinal $\theta > |T|$. When $D$ is trivial, there is no need for strong compact cardinals, and one can assume $\theta = |T|^+$, and replace $< \theta$ satisfiable by finitely satisfiable. We leave it to the reader to find the precise replacement.

CONJECTURE 3.1 (The generic pair conjecture): *Suppose $D$ is dependent. Assume $\theta < \lambda = \lambda^{<\lambda}$ and $\lambda^+ = 2^\lambda$. Let $\bar{M} = \langle M_\alpha : \alpha < \lambda^+ \rangle$ be an increasing continuous sequence of elementary substructures of $\mathfrak{C}_D$ of cardinality $\lambda$, such that $\mathbf{M} = \bigcup_{\alpha<\lambda^+} M_\alpha$ is $D$-saturated of size $\lambda^+$.*

*Then there exists a club $E \subseteq \lambda^+$ such that*

- *if $\alpha_1 < \beta_1$, $\alpha_2 < \beta_2 \in E$ are all of cofinality $\lambda$, then*

$$(M_{\beta_1}, M_{\alpha_1}) \cong (M_{\beta_2}, M_{\alpha_2}).$$

To give some motivation, note that it is easy to find a club $E_{\text{sat}} \subseteq \lambda^+$ such that for any $\delta \in E_{\text{sat}}$ of cofinality $\lambda$, $M_\delta$ is homogeneous and $D(M_\delta) = D$ (equivalently $D$-saturated by Lemma 2.3). Just let $E_{\text{sat}}$ be the set of ordinals $\delta < \lambda^+$ such that for any $\alpha < \delta$, every $p \in S^1_D(A)$ for any $A \subseteq M_\alpha$ of size $< \lambda$ is realized in $M_\delta$. Then for any $\delta \in E_{\text{sat}}$ of cofinality $\lambda$, $M_\delta$ is $D$-saturated, and any such two are isomorphic (see Corollary 2.4).

In this section we will outline the proof of Conjecture 3.1 under the assumption that a "good family of decompositions" exists.

We call a tuple of the form $\mathbf{x} = (M_{\mathbf{x}}, B_{\mathbf{x}}, \bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}, r_{\mathbf{x}})$ a **$\lambda$-decomposition**[4] when $|M_{\mathbf{x}}| = \lambda$ and $M_{\mathbf{x}} \subseteq \mathfrak{C}$ is a $D$-model, $B_{\mathbf{x}} \subseteq M_{\mathbf{x}}$ has cardinality $< \lambda$,

[4] The idea behind the name "decomposition" will be clearer later, where this notion is used to analyze the type of $\bar{d}$ over $M$.

$\bar{c}_{\mathbf{x}}, \bar{d}_{\mathbf{x}} \in \mathfrak{C}_D^{<\lambda}$ and $r_{\mathbf{x}} \in S_D^{<\lambda}(\emptyset)$ is a complete type in variables $(\bar{x}_{\bar{c}_{\mathbf{x}}}, \bar{x}_{\bar{d}_{\mathbf{x}}}, \bar{x}'_{\bar{c}_{\mathbf{x}}}, \bar{x}'_{\bar{d}_{\mathbf{x}}})$ (where $\bar{x}_{\bar{d}_{\mathbf{x}}}, \bar{x}'_{\bar{d}_{\mathbf{x}}}$ have the same length as $\bar{d}_{\mathbf{x}}$, etc.).

An **isomorphism** between two $\lambda$-decompositions $\mathbf{x}$ and $\mathbf{y}$ is just an elementary map with domain $M_{\mathbf{x}} \cup \bigcup \bar{c}_{\mathbf{x}} \cup \bigcup \bar{d}_{\mathbf{x}}$ which maps all the ingredients of $\mathbf{x}$ onto those of $\mathbf{y}$, and in particular, if $\mathbf{x} \cong \mathbf{y}$ then $r_{\mathbf{x}} = r_{\mathbf{y}}$. A **weak isomorphism** between $\mathbf{x}$ and $\mathbf{y}$ is a restriction of an isomorphism to $(B_{\mathbf{x}}, \bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}, r_{\mathbf{x}})$ (so there exists some isomorphism extending it). We write $\mathbf{x} \leq \mathbf{y}$ when $M_{\mathbf{x}} = M_{\mathbf{y}}$, $B_{\mathbf{x}} \subseteq B_{\mathbf{y}}$, $r_{\mathbf{x}} \subseteq r_{\mathbf{y}}$ (i.e., $r_{\mathbf{y}}$ may add more information on the added variables), $\bar{c}_{\mathbf{x}} \trianglelefteq \bar{c}_{\mathbf{y}}$ (i.e., $\bar{c}_{\mathbf{x}}$ is an initial segment of $\bar{c}_{\mathbf{y}}$) and $\bar{d}_{\mathbf{x}} \trianglelefteq \bar{d}_{\mathbf{y}}$. If $\mathbf{x}$ and $\mathbf{y}$ are $\lambda$-decompositions with $M_{\mathbf{x}} = M_{\mathbf{y}}$ such that for some $\mathbf{z}$, $\mathbf{z} \leq \mathbf{x}, \mathbf{y}$, we will say that they are isomorphic over $\mathbf{z}$ if there is an isomorphism from $\mathbf{x}$ to $\mathbf{y}$ fixing $\bar{d}_{\mathbf{z}}, \bar{c}_{\mathbf{z}}, B_{\mathbf{z}}$.

*Definition 3.2* (A good family): A family $\mathfrak{F}$ of $\lambda$-decompositions is **good** when:

(1) The family $\mathfrak{F}$ is invariant under isomorphisms.

(2) For every $\mathbf{x} \in \mathfrak{F}$, $M_{\mathbf{x}}$ is $D$-saturated.

(3) For every $D$-saturated $M \prec \mathfrak{C}_D$ of size $\lambda$, the "trivial decomposition" $(M, \emptyset, \emptyset, \emptyset, \emptyset) \in \mathfrak{F}$.

(4) For every $\mathbf{x} \in \mathfrak{F}$ and $\bar{d} \in \mathfrak{C}_D^{<\lambda}$ there exists some $\mathbf{y} \in \mathfrak{F}$ such that $\mathbf{x} \leq \mathbf{y}$, and $\bar{d}_{\mathbf{y}} \trianglerighteq \bar{d}_{\mathbf{x}}\bar{d}$.

(5) For every $\mathbf{x} \in \mathfrak{F}$ and $b \in M_{\mathbf{x}}$,
$$(M_{\mathbf{x}}, B_{\mathbf{x}} \cup \{b\}, \bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}, r_{\mathbf{x}}) \in \mathfrak{F}.$$

(6) Suppose that $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1 \in \mathfrak{F}$ where $\mathbf{x}_1 \leq \mathbf{y}_1$ and there exists some isomorphism $f : \mathbf{x}_1 \to \mathbf{x}_2$; then there exists some $\mathbf{y}_2 \in \mathfrak{F}$ such that $\mathbf{x}_2 \leq \mathbf{y}_2$ and $f$ can be extended to an isomorphism $\mathbf{y}_1 \to \mathbf{y}_2$.

(7) Suppose that $\langle \mathbf{x}_i | \, i < \delta \rangle$ is a sequence of $\lambda$-decompositions from $\mathfrak{F}$ such that $\delta < \lambda$ is a limit ordinal and for every $i < j < \delta$ we have $\mathbf{x}_i \leq \mathbf{x}_j$; then $\mathbf{x}_\delta = \sup_{i<\delta} \mathbf{x}_i = (M, \bigcup_{i<\delta} B_{\mathbf{x}_i}, \bigcup_{i<\delta} \bar{d}_{\mathbf{x}_i}, \bigcup_{i<\delta} \bar{c}_{\mathbf{x}_i}, \bigcup_{i<\delta} r_{\mathbf{x}_i}) \in \mathfrak{F}$. Note that as $\lambda$ is regular and $\delta < \lambda$ this makes sense.

(8) Suppose that $\langle \mathbf{x}_i | \, i < \delta \rangle$ and $\langle \mathbf{y}_i | \, i < \delta \rangle$ are increasing sequences of $\lambda$-decompositions from $\mathfrak{F}$ such that $\delta < \lambda$ is a limit ordinal and for each $i < \delta$ there is a weak isomorphism $g_i : \mathbf{x}_i \to \mathbf{y}_i$ such that $g_i \subseteq g_j$ whenever $i < j$. Then the union $\bigcup_{i<\delta} g_i$ is a weak isomorphism from $\mathbf{x} = \sup_{i<\delta} \mathbf{x}_i$ to $\mathbf{y} = \sup_{i<\delta} \mathbf{y}_i$.

(9) For every $D$-model $M$ of cardinality $\lambda$, the number of $\mathbf{x} \in \mathfrak{F}$ with $M_{\mathbf{x}} = M$ up to isomorphism is $\leq \lambda$.

*Remark 3.3:* The roles of $\bar{c}_{\mathbf{x}}$ and $r_{\mathbf{x}}$ will become crucial in the next sections. In this section it is important in order to restrict the class of isomorphisms.

*Remark 3.4:* In Definition 3.2, (6) follows from (1).

*Remark 3.5:* Note that by point (1) in Definition 3.2, and as $\mathbf{M}$ is $D$-saturated of cardinality $\lambda^+$, if $\mathfrak{F}$ is good, then $\mathfrak{F}$ is also good when we restrict it to decompositions contained in $\mathbf{M}$ (rather than $\mathfrak{C}_D$). More precisely, in points (4) and (6), the promised decompositions $\mathbf{y}$ and $\mathbf{y}_2$ respectively can be found in $\mathbf{M}$ if the given decompositions ($\mathbf{x}$, $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{y}_1$) are in $\mathbf{M}$.

Let us give an example of a "baby application" of the existence of a good family before we delve into the generic pair conjecture. This next theorem is a weak version of [She12, Conclusion 3.13].

THEOREM 3.6: *Suppose $\mathfrak{F}$ is a good family. Then, for a $D$-saturated model $M$ of size $\lambda$, the number of types in $S_D^{<\lambda}(M)$ up to conjugation is $\leq \lambda$.*

*Proof.* Suppose $\gamma < \lambda$, and $\langle p_i | \, i < \lambda^+ \rangle$ is a sequence of types in $S_D^{\gamma}(M)$, which are pairwise non-conjugate. Let $\bar{d}_i \models p_i$. By (4) in Definition 3.2, for some $\mathbf{x}_i \in \mathfrak{F}$, $\bar{d}_i \trianglelefteq \bar{d}_{\mathbf{x}_i}$. Obviously, for $i \neq j$, $\operatorname{tp}(\bar{d}_{\mathbf{x}_i}/M)$ and $\operatorname{tp}(\bar{d}_{\mathbf{x}_j}/M)$ are not conjugates. But according to (9), this is impossible. ∎

*Remark 3.7:* Suppose $\mathbf{z}$ is a $\lambda$-decomposition. From (9) in Definition 3.2 it follows that the number of $\mathbf{x} \in \mathfrak{F}$ such that $\mathbf{z} \leq \mathbf{x}$ up to isomorphism over $\mathbf{z}$ is $\leq \lambda$. Indeed, if not there is a sequence $\langle \mathbf{x}_i | \, i < \lambda^+ \rangle$ of $\lambda$-decompositions in $\mathfrak{F}$ containing $\mathbf{z}$ which are pairwise not isomorphic over $\mathbf{z}$. By (9), we may assume that they are pairwise isomorphic, and let $f_i : \mathbf{x}_i \to \mathbf{x}_0$ be isomorphisms. So $f_i$ must fix $\bar{d}_{\mathbf{z}}$ and $\bar{c}_{\mathbf{z}}$ as they are initial segments. In addition, $f_i \restriction B_{\mathbf{z}}$ is a sequence of length $< \lambda$ of elements in $M_{\mathbf{z}}$, and there are $\lambda$ such sequences (as $\lambda^{<\lambda} = \lambda$), so for some $i \neq j$, $f_i \restriction B_{\mathbf{z}} = f_j \restriction B_{\mathbf{z}}$. Hence $f_i^{-1} \circ f_j \restriction B_{\mathbf{z}} = \mathrm{id}$—contradiction.

For a decomposition $\mathbf{x}$, we will write $\mathbf{x} \Subset M$ for $M_{\mathbf{x}} \subseteq M$ and $(\bar{c}_{\mathbf{x}}, \bar{d}_{\mathbf{x}}) \in (M^{<\lambda})^2$.

*Definition 3.8:* Let $\gamma < \lambda^+$, and let $\mathfrak{F}$ be a good family of $\lambda$-decompositions.

(1) We say that $\gamma$ is **$\mathfrak{F}$-complete** if for every $\alpha < \beta < \gamma$ such that $M_\alpha$ is $D$-saturated, $\mathbf{y} \in \mathfrak{F}$ with $M_{\mathbf{y}} = M_\alpha$ and $\bar{d} \in M_\beta^{<\lambda}$ such that $\mathbf{y} \Subset M_\beta$, there exists some $\mathbf{y} \leq \mathbf{x} \in \mathfrak{F}$ such that

$$\bar{d}_{\mathbf{x}} \trianglerighteq \bar{d}\bar{d}_{\mathbf{y}} \quad \text{and} \quad \mathbf{x} \Subset M_\gamma.$$

(2) We say that $\gamma$ is **$\mathfrak{F}$-representative** if for every $\alpha < \beta < \gamma$ such that $M_\alpha$ is $D$-saturated, $\mathbf{y} \in \mathfrak{F}$ with $M_{\mathbf{y}} = M_\alpha$ and every $\lambda$-decomposition $\mathbf{z}$ over $M_\alpha$ such that $\mathbf{z} \Subset M_\beta$ and $\mathbf{z} \le \mathbf{y}$, there exists $\mathbf{x} \in \mathfrak{F}$ such that $M_{\mathbf{x}} = M_\alpha$, $\mathbf{x} \Subset M_\gamma$, $\mathbf{z} \le \mathbf{x}$ and $\mathbf{x}$ is isomorphic to $\mathbf{y}$ over $\mathbf{z}$.

Proposition 3.9: *Let $\mathfrak{F}$ be a family of good $\lambda$-decompositions. Let $E_{com} \subseteq \lambda^+$ be the set of all $\delta < \lambda^+$ which are $\mathfrak{F}$-complete. Then $E_{com}$ is a club.*

*Proof.* The fact that $E_{\mathrm{com}}$ is a closed is easy. Suppose $\beta < \lambda^+$. Let $\beta < \beta' < \lambda^+$ be such that for every $\alpha < \beta$ such that $M_\alpha$ is $D$-saturated, and every $\bar{d} \in M_\beta^{<\lambda}$ and $\mathbf{y} \in \mathfrak{F}$ with $\mathbf{y} \Subset M_\beta$ and $M_{\mathbf{y}} = M_\alpha$, there is some $\mathbf{y} \le \mathbf{x} \in \mathfrak{F}$ such that $\bar{d}_{\mathbf{x}} \trianglerighteq \bar{d}_{\mathbf{y}}\bar{d}$, $M_{\mathbf{x}} = M_\alpha$ and $\mathbf{x} \Subset M_{\beta'}$. The ordinal $\beta'$ exists because $\lambda^{<\lambda} = \lambda$ (so the number of $\mathbf{y}$'s and the number of $\bar{d}$'s is $\le \lambda$), by (4) of Definition 3.2 and by Remark 3.5. By induction, we can thus define an increasing sequence of ordinals $\beta_i$ for $i < \omega$ where $\beta_0 = \beta$ and $\beta_{i+1} = \beta_i'$. Finally, $\gamma = \beta_\omega \in E_{\mathrm{com}}$. ∎

Proposition 3.10: *Let $\mathfrak{F}$ be a family of good $\lambda$-decompositions. Let $E_{rep} \subseteq \lambda^+$ be the set of all $\delta < \lambda^+$ which are $\mathfrak{F}$-representative. Then $E_{rep}$ is a club.*

*Proof.* The proof is similar to the proof of Proposition 3.9, but now in order to show that $E_{\mathrm{rep}}$ is unbounded, we use Remark 3.7. ∎

Theorem 3.11: *Suppose $\mathfrak{F}$ is a good family. Let*

$$E = E_{sat} \cap E_{rep} \cap E_{com} \subseteq \lambda^+.$$

*This is a club. For every $\alpha_1 < \beta_1$, $\alpha_2 < \beta_2 \in E$ of cofinality $\lambda$ we have $(M_{\beta_1}, M_{\alpha_1}) \cong (M_{\beta_2}, M_{\alpha_2})$. Hence Conjecture 3.1 holds.*

*Proof.* Let $AP$[5] be the collection of tuples of the form

$$p = (\mathbf{x}_p, \mathbf{y}_p, h_p) = (\mathbf{x}, \mathbf{y}, h)$$

where $\mathbf{x}, \mathbf{y} \in \mathfrak{F}$ and $h : \mathbf{x} \to \mathbf{y}$ is a weak isomorphism, such that $M_{\mathbf{x}} = M_{\alpha_1}$, $\mathbf{x} \Subset M_{\beta_1}$, $M_{\mathbf{y}} = M_{\alpha_2}$ and $\mathbf{y} \Subset M_{\beta_2}$. For every $p_1, p_2 \in AP$ we write $p_1 \le_{AP} p_2$ if $\mathbf{x}_{p_1} \le \mathbf{x}_{p_2}$, $\mathbf{y}_{p_1} \le \mathbf{y}_{p_2}$, and $h_{p_1} \subseteq h_{p_2}$.

We proceed to construct an isomorphism by a back and forth argument. In the forth part, we may add an element from $M_{\alpha_1}$ to $B_{\mathbf{x}}$ (thus increasing the $M_{\alpha_1}$-part of the domain of $h$), or an element from $M_{\beta_1}$ to $\bar{d}_{\mathbf{x}}$ (thus increasing the $M_{\beta_1}$-part). We also have to take care of the limit stage.

[5] AP stands for approximations.

As one could take $p$ to be a trivial tuple by (3) in Definition 3.2, and as $\alpha_1, \alpha_1 \in E_{\mathrm{sat}}$ (and their cofinality is $\lambda$ so that $M_{\alpha_1}, M_{\alpha_2}$ are saturated), $AP \neq \emptyset$.

Adding an element from $M_{\alpha_1}$: let $p \in AP$ and $a \in M_{\alpha_1}$. As $h$ is a weak isomorphism, there is some isomorphism $h^+ : \mathbf{x} \to \mathbf{y}$ extending $h$. Let

$$h^+(a) = b \in M_{\alpha_2}.$$

Thus, by (5) in Definition 3.2, we may define $p' = (\mathbf{x}', \mathbf{y}', h')$ by adding $a$ to $B_{\mathbf{x}}$ and $b$ to $B_{\mathbf{y}}$, and defining $h' = h \cup \{(a, b)\}$. Of course, $h'$ is still a weak isomorphism as witnessed by the same $h^+$. It follows that $p \leq_{AP} p'$.

Adding an element from $M_{\beta_1}$: let $d \in M_{\beta_1}$ and $p \in AP$. Since $\mathfrak{F}$ is good, $\alpha_1 \in E_{\mathrm{sat}}$, $\beta_1 \in E_{\mathrm{com}}$, and by (4) in Definition 3.2, there is some $\mathbf{x} \leq \mathbf{x}' \in \mathfrak{F}$ such that $\bar{d}_{\mathbf{x}} d \trianglelefteq \bar{d}_{\mathbf{x}'}$ and $\mathbf{x}' \Subset M_{\beta_1}$ (here we also used the fact that the cofinality of $\beta_1$ is $\lambda$).

Let $h^+ : \mathbf{x} \to \mathbf{y}$ be as above. By (6) in Definition 3.2, $h^+$ extends to an isomorphism $h^{++} : \mathbf{x}' \to \mathbf{y}'$ for some $\mathbf{y}' \in \mathfrak{F}$, such that $\mathbf{y} \leq \mathbf{y}'$ (and we may also assume that $\mathbf{y}$ is contained in $\mathbf{M}$ by Remark 3.5).

Since $\beta_2 \in E_{\mathrm{rep}}$ (and since its cofinality is $\lambda$), there exists some $\mathbf{y}'' \in \mathfrak{F}$ such that $\mathbf{y}'' \Subset M_{\beta_2}$, $\mathbf{y} \leq \mathbf{y}''$, and $\mathbf{y}''$ is isomorphic to $\mathbf{y}'$ over $\mathbf{y}$, as witnessed by $f : \mathbf{y}' \to \mathbf{y}''$ (in particular, $f \restriction M_{\alpha_2}$ is an automorphism of $M_{\alpha_2}$). We have then $p' = (\mathbf{x}', \mathbf{y}'', (f \circ h^{++}) \restriction (B_{\mathbf{x}'}, \bar{d}_{\mathbf{x}'}, \bar{c}_{\mathbf{x}'}, r_{\mathbf{x}'})) \in \mathfrak{F}$ satisfies that $p \leq_{AP} p'$ and $d \in \bar{d}_{\mathbf{x}'}$.

Of course we must also switch the roles of $\mathbf{x}$ and $\mathbf{y}$ in the above steps.

The limit stage: suppose $\langle p_i | \, i < \delta \rangle$ is an increasing sequence of approximation where $\delta < \lambda$ is some limit. Let

$$p = \sup_{i<\delta} p_i = \left( \sup_{i<\delta} \mathbf{x}_{p_i}, \sup_{i<\delta} \mathbf{y}_{p_i}, \bigcup_{i<\delta} h_i \right).$$

This tuple is still in $AP$ by (7) and (8) in Definition 3.2. ∎

## 4. Type decompositions

Section 3 gave the proof of the generic pair conjecture (Conjecture 3.1) by using $\lambda$-decompositions and a good family of these (Definition 3.2). Here we will start to construct what eventually will be the good family. For this we need to define two kinds of decompositions. The first is the tree-type decomposition (explained in Subsection 4.1), which is the basic building block of the self-solvable decomposition which will be introduced in Subsection 4.2. Eventually, the good family will be the family of self-solvable decompositions.

As usual, we assume that $\theta > |T|$ is a strongly compact cardinal (unless $D$ is trivial and then $\theta = |T|^+$, and also replace $< \theta$ satisfiable by finitely satisfiable when appropriate, see the beginning of Section 3), and that $D$ is dependent. Also, assume that $\lambda = \lambda^{<\lambda} > \theta$.

### 4.1. Tree-type decomposition.

*Definition 4.1:* Let $M \prec \mathfrak{C}_D$ be a $D$-model of cardinality $\lambda$. A **$\lambda$-tree-type decomposition** is a $\lambda$-decomposition $(M, B, \bar{d}, \bar{c}, r)$ with the following properties:

(1) The tuple $\bar{c}$ is of length $< \kappa = \theta + |\operatorname{lg}(\bar{d})|^+$ and the type $\operatorname{tp}(\bar{c}/M)$ does not split over $B$. See also Remark 4.4.
(2) For every $A \subseteq M$ such that $|A| < \lambda$ there exists some $\bar{e}_A \in M^{<\kappa}$ such that $\operatorname{tp}(\bar{d}/\bar{e}_A + \bar{c}) \vdash \operatorname{tp}(\bar{d}/A + \bar{c})$. By this we mean that if $\bar{d}' \in \mathfrak{C}_D^{<\lambda}$ realizes the same type as $\bar{d}$ over $\bar{e}_A + \bar{c}$ (which we denote by $\bar{d}' \equiv_{\bar{e}_A \bar{c}} \bar{d}$), then $\bar{d}' \equiv_{A\bar{c}} \bar{d}$. Note: we do not ask that this is true in $\mathfrak{C}$, only in $\mathfrak{C}_D$.

*Remark 4.2:* Why "tree-type"? If $\mathbf{x}$ is a tree-type decomposition such that for simplicity $\operatorname{lg}(\bar{d}) < \theta$, then we may define a partial order on $M^{<\theta}$ by $\bar{e}_1 \le \bar{e}_2$ if $\operatorname{tp}(\bar{d}/\bar{c} + \bar{e}_2) \vdash \operatorname{tp}(\bar{d}/\bar{c} + \bar{e}_1)$. Then this order is $\lambda$-directed (so looks like a tree in some sense).

*Remark 4.3:* If $\operatorname{tp}(\bar{d}/M)$ does not split over a $B$ (where $|B| < \lambda$ as usual), then $(M, B, \bar{d}, \bar{d}, r)$ is a $\lambda$-tree-type decomposition for any $r$: in (2) take $\bar{e}_A = \emptyset$.

*Remark 4.4:* In Definition 4.1 (1), we could ask that $\operatorname{tp}(\bar{c}/M)$ is $< \theta$ satisfiable in $B$ in the sense that any $< \theta$ formulas from this type in finitely many variables are realized in $B$.

*Remark 4.5:* In this section, the role of $\bar{c}$ becomes clearer, but $r$ will not have any role.

*Example 4.6* ([She13, Exercise 2.18]): In DLO—the theory of $(\mathbb{Q}, <)$—suppose $M$ is a saturated model of cardinality $\lambda$, and $d \in \mathfrak{C} \backslash M$ is some point. Let $C_1, C_2$ be the corresponding left and right cuts that $d$ determines in $M$. As $M$ is saturated at least one of these cuts has cofinality $\lambda$. If only one has, then $\operatorname{tp}(d/M)$ does not split over the smaller cut, so $(M, C_i, d, d, \emptyset)$ is a tree-type decomposition for $i = 1$ or $i = 2$. Otherwise for each $A$ of cardinality $< \lambda$, there are $e_1 < d < e_2$ in $M$ such that $C_1 \cap A < e_1 < e_2 < C_2 \cap A$, so $\mathrm{tp}(d/e_1e_2) \vdash \mathrm{tp}(d/e_1e_2A)$. In this case, $(M, \emptyset, d, \emptyset, \emptyset)$ is a tree-type decomposition.

Our aim now is to prove that when $M$ is a $D$-model, then for every $\bar{d} \in \mathfrak{C}_D^{<\lambda}$ there exists a tree-type decomposition $\mathbf{x}$ such that $\bar{d} = \bar{d}_{\mathbf{x}}$. In fact, we can start with any tree-type decomposition $\mathbf{x}$, for instance the trivial one $(M, \emptyset, \emptyset, \emptyset, \emptyset)$, and find some tree-type decomposition $\mathbf{y} \geq \mathbf{x}$ such that $\bar{d}_{\mathbf{y}} = \bar{d}_{\mathbf{x}}\bar{d}$. In a sense, we decompose the type of $\bar{d}$ over $M$ into two parts: the invariant one and the "tree-like" one.

*Definition 4.7:* Let $M \prec \mathfrak{C}_D$ be of size $\lambda$, $\bar{d} \in \mathfrak{C}_D^{<\lambda}$ and $C \subseteq \mathfrak{C}_D$ be of size $< \kappa = |\lg(\bar{d})|^+ + \theta < \lambda$. The class $\mathbf{K}_{\lambda,\theta}^{M,C,\bar{d}}$ contains all pairs $\mathfrak{a} = (B_\mathfrak{a}, \bar{c}_\mathfrak{a}) = (B, \bar{c})$ such that:

(1) $\bar{c} = \langle (\bar{c}_{i,0}, \bar{c}_{i,1}) | \, i < \gamma \rangle \in (\mathfrak{C}_D^{<\omega} \times \mathfrak{C}_D^{<\omega})^\gamma$, and $B \subseteq M$, $|B| < \lambda$.
(2) $\gamma < \kappa$.
(3) For all $i < \gamma$, $\mathrm{tp}(\bar{c}_i/MC + \bar{c}_{<i})$ is $< \theta$ satisfiable in $B$ where $\bar{c}_i$ is $\bar{c}_{i,0} \frown \bar{c}_{i,1}$. Abusing notation, we identify $\bar{c}$ with the concatenation of $\bar{c}_i$ for $i < \gamma$. It follows that $\mathrm{tp}(\bar{c}/MC)$ does not split over $B$.
(4) For every $i < \gamma$, $\mathrm{tp}(\bar{c}_{i,0}/MC + \bar{c}_{<i}) = \mathrm{tp}(\bar{c}_{i,1}/MC + \bar{c}_{<i})$ and in particular they are of the same (finite) length, and
$$\mathrm{tp}(\bar{c}_{i,0}/MC + \bar{c}_{<i} + \bar{d}) \neq \mathrm{tp}(\bar{c}_{i,1}/MC + \bar{c}_{<i} + \bar{d}).$$

The class $\mathbf{MxK}_{\lambda,\theta}^{M,C,\bar{d}}$ consists of all the maximal elements in $\mathbf{K}_{\lambda,\theta}^{M,C,\bar{d}}$ with respect to the order $<$ defined by $\mathfrak{a} < \mathfrak{b}$ iff $B_\mathfrak{a} \subseteq B_\mathfrak{b}$, $\bar{c}_\mathfrak{a} \trianglelefteq \bar{c}_\mathfrak{b}$ and $\bar{c}_\mathfrak{a} \neq \bar{c}_\mathfrak{b}$. That is, it contains all $\mathfrak{a} \in \mathbf{K}_{\lambda,\theta}^{M,C,\bar{d}}$ such that there is no $\mathfrak{b} \in \mathbf{K}_{\lambda,\theta}^{M,C,\bar{d}}$ with $B_\mathfrak{a} \subseteq B_\mathfrak{b}$ and $\bar{c}_\mathfrak{a}$ is a strict first segment of $\bar{c}_\mathfrak{b}$.

Theorem 4.8: *For every $\bar{d} \in \mathfrak{C}_D^{<\lambda}$, $C$ and $M$ as in Definition 4.7, if $\mathfrak{a} \in \mathbf{K}_{\lambda,\theta}^{M,C,\bar{d}}$ then there exists some $\mathfrak{b} \in \mathbf{MxK}_{\lambda,\theta}^{M,C,\bar{d}}$ such that $\mathfrak{a} \leq \mathfrak{b}$.*

*Proof.* Let $\bar{c} = \bar{c}_\mathfrak{a} = \langle (\bar{c}_{i,0}, \bar{c}_{i,1}) | \, i < \gamma \rangle$. We try to construct an increasing sequence $\langle \mathfrak{a}_\alpha | \, \gamma \leq \alpha < \kappa \rangle$ of elements in $\mathbf{K}_{\lambda,\theta}^{M,C,\bar{d}}$, where $\kappa = |\lg(\bar{d})|^+ + \theta < \lambda$, as follows:

(1) $\mathfrak{a}_\gamma = \mathfrak{a}$.
(2) If $\alpha$ is limit then $\mathfrak{a}_\alpha = \sup_{\beta<\alpha} \mathfrak{a}_\beta$, i.e., $B_{\mathfrak{a}_\alpha} = \bigcup_{\beta<\alpha} B_{\mathfrak{a}_\beta}$ and $\bar{c}_{\mathfrak{a}_\alpha} = \bigcup_{\beta<\alpha} \bar{c}_{\mathfrak{a}_\beta}$. Note that this is well defined, i.e., $\mathfrak{a}_\alpha \in \mathbf{K}_{\lambda,\theta}^{M,C,\bar{d}}$.

(3) Suppose $\alpha = \beta + 1$ and $\mathfrak{a}_\beta$ has been constructed. Let

$$\mathfrak{a}_\alpha = (B_\alpha, \bar{c}_{\mathfrak{a}_\beta} \frown (\bar{c}_{\beta,0}, \bar{c}_{\beta,1}))$$

just in case there are $\bar{c}_{\beta,0}, \bar{c}_{\beta,1} \in \mathfrak{C}_D^{<\omega}$, $B_{\mathfrak{a}_\beta} \subseteq B_\alpha \subseteq M$ such that $\mathfrak{a}_\alpha \in \mathbf{K}_{\lambda,\theta}^{M,C,\bar{d}}$.

If we got stuck somewhere in the construction it must be in the successor stage $\alpha$, and then $\mathfrak{a}_\alpha \in \mathbf{MxK}_{\lambda,\theta}^{M,C,\bar{d}}$ is as requested. So suppose we succeed: we constructed $\langle (\bar{c}_{\alpha,0}, \bar{c}_{\alpha,1}) \,|\, \alpha < \kappa \rangle$. As usual we denote $\bar{c}_\alpha = \bar{c}_{\alpha,0} \frown \bar{c}_{\alpha,1}$.

By the definition of $\mathbf{K}_{\lambda,\theta}^{M,C,\bar{d}}$, it follows that for every $\alpha < \kappa$, there are $\bar{a}_\alpha \in A^{<\omega}$ where $A = MC$, $\bar{b}_\alpha \in C_\alpha^{<\omega}$ where $C_\alpha = \bigcup_{\beta<\alpha} \bar{c}_\beta$, and a formula $\varphi_\alpha(\bar{x}_{\bar{d}}, \bar{w}_\alpha, \bar{y}_\alpha, \bar{z}_\alpha)$ such that $\mathfrak{C}_D \models \varphi_\alpha(\bar{d}, \bar{c}_{\alpha,0}, \bar{a}_\alpha, \bar{b}_\alpha)$ but

$$\mathfrak{C}_D \models \neg\varphi_\alpha(\bar{d}, \bar{c}_{\alpha,1}, \bar{a}_\alpha, \bar{b}_\alpha).$$

(The variables are all in the appropriate length, but only finitely many of them appear in the formula.)

For every $\alpha < \kappa$, let $f(\alpha)$ be the maximal ordinal $< \alpha$ such that $\bar{b}_\alpha$ intersects $\bar{c}_{f(\alpha)}$. By Fodor's Lemma, There exists some cofinal set $S \subseteq \kappa$ and $\beta < \kappa$ such that for every $\alpha \in S$ we have $f(\alpha) = \beta$. By restricting to a smaller set, we may assume that for any $\alpha \in S$, $\alpha > \beta$ and $\varphi_\alpha = \varphi$ is constant.

As $\bar{c}_{\alpha,0} \equiv_{A\bar{c}_{<\alpha}} \bar{c}_{\alpha,1}$ and as $\mathrm{tp}(\bar{c}_\alpha / A + \bar{c}_{<\alpha})$ does not split over $A$, it follows that $\mathrm{tp}(\langle \bar{c}_{\alpha,\eta(\alpha)} \,|\, \alpha \in S \rangle / AC_{\beta+1})$ does not depend on $\eta$ when $\eta : S \to 2$. To prove this it is enough to consider a finite subset $S_0 \subseteq S$, and to prove it by induction on its size. Indeed, given $S_0 = \{\alpha_0 < \cdots < \alpha_{n+1}\}$, and any $\eta : S_0 \to 2$,

$$\begin{aligned}\langle \bar{c}_{\alpha,\eta(\alpha)} \,|\, \alpha \in S_0 \rangle &\equiv_{AC_{\beta+1}} \langle \bar{c}_{\alpha,\eta(\alpha)} \,|\, \alpha \in S_0 \backslash \{\alpha_{n+1}\} \rangle \frown \langle \bar{c}_{\alpha_{n+1},0} \rangle \\ &\equiv_{AC_{\beta+1}} \langle \bar{c}_{\alpha,0} \,|\, \alpha \in S_0 \rangle.\end{aligned}$$

It follows by homogeneity that for any subset $R$ of $S$ there is some $\bar{d}_R \in \mathfrak{C}_D^{<\lambda}$ such that $\mathfrak{C}_D \models \varphi(\bar{d}_R, \bar{c}_{\alpha,0}, \bar{a}_\alpha, \bar{b}_\alpha)$ iff $\alpha \in R$. But this is a contradiction to the fact that $D$ is dependent, see Lemma 2.11 (5). ∎

*Definition 4.9:* Suppose $p(\bar{x}), q(\bar{y}) \in S_D(A)$ for some $A \subseteq \mathfrak{C}_D$. We say that $p$ is **orthogonal**[6] to $q$ if there is a unique $r(\bar{x}, \bar{y}) \in S_D(A)$ which extends $p(\bar{x}) \cup q(\bar{y})$.

[6] Usually this notion is called **weakly orthogonal**, as the notion of orthogonal types already has meaning in stable theories. However, here we have no room for confusion, so we decided to stick with the simpler term.

*Definition 4.10:* Suppose that $M \prec \mathfrak{C}_D$, and $C \subseteq \mathfrak{C}_D$ is some set. Let $p \in S_D(MC)$. We say that $p$ is **tree-like** (with respect to $M$,$C$) if it is orthogonal to every $q \in S_D^{<\omega}(MC)$ for which there exists some $B \subseteq M$ with $|B| < |M|$ such that $q$ is $< \theta$ satisfiable in $B$.

Proposition 4.11: *Let $M, C$ be as in Definition 4.10. Suppose that $p \in S_D^\alpha(MC)$ is tree-like and that $|C| < \kappa = \theta + |\alpha|^+$. Then for every $B \subseteq M$ such that $|B| < |M|$ there exists some $E \subseteq M$ with $|E| < \kappa$ such that $p|_{CE} \vdash p|_{CB}$.*

*Proof.* It is enough to show that for any formula $\varphi(\bar{x}, \bar{y}, \bar{c})$ where $\bar{c}$ is a finite tuple from $C$, there is some $E_\varphi \subseteq M$ such that $|E_\varphi| < \theta$ and

$$p|_{E_\varphi C} \vdash (p \restriction \varphi)|_B = \{\varphi(\bar{x}, \bar{b}, \bar{c}) \in p | \, \bar{b} \in B^{\lg(\bar{y})}\}$$

(because then we let $E = \bigcup_\varphi E_\varphi$).

Suppose not. Let $I = [M]^{<\theta}$ (all subsets of $M$ of size $< \theta$); then for every $E \in I$ there exists some $\bar{d}_1^E, \bar{d}_2^E \in \mathfrak{C}_D^\alpha$, $\bar{b}_E \in B^{\lg(\bar{y})}$ such that $\bar{d}_1^E, \bar{d}_2^E$ realize $p|_{EC}$ and $\mathfrak{C}_D \models \varphi(\bar{d}_1^E, \bar{b}_E, \bar{c}) \wedge \neg\varphi(\bar{d}_2^E, \bar{b}_E, \bar{c})$. By strong compactness, there is some $\theta$-complete ultrafilter $\mathcal{U}$ on $I$ such that for every $X \in I$ we have

$$\{Y \in I | \, X \subseteq Y\} \in \mathcal{U}.$$

By Lemma 2.6,

$$r = Av_{\mathcal{U}}(\langle \bar{d}_1^E \bar{d}_2^E \bar{b}_E | \, E \in I\rangle / MC) \in S_D(MC).$$

Let $\bar{d}_1, \bar{d}_2 \in \mathfrak{C}_D^\alpha$ and $\bar{b} \in \mathfrak{C}_D^{<\omega}$ be such that $\bar{d}_1 \bar{d}_2 \bar{b}$ is a realization of $r$. Now, $r' = \operatorname{tp}(\bar{b}/MC)$ is $< \theta$ satisfiable in $B$, $\bar{d}_1, \bar{d}_2$ realize $p$ (by our choice of $\mathcal{U}$) but

$$\operatorname{tp}(\bar{d}_1/\bar{b}\bar{c}) \neq \operatorname{tp}(\bar{d}_2/\bar{b}\bar{c})$$

(as witnessed by $\varphi$). Hence $p$ is not orthogonal to $r'$, which is a contradiction. ∎

*Remark 4.12:* Let $A \subseteq B \subseteq C \subseteq \mathfrak{C}_D$. If $p \in S_D^n(B)$ is $< \theta$ satisfiable in $A$ and $n < \omega$, then there is an extension $p \subseteq q \in S_D^n(C)$ which is $< \theta$ satisfiable in $A$. Indeed, let $\mathcal{U}_0 = \{\varphi(A^n) | \, \varphi \in p\}$, note that it is $\theta$-complete, and extend it to a $\theta$-complete ultrafilter $\mathcal{U}$ on all subsets of $A^n$. Let

$$q = \{\varphi(\bar{x}, \bar{c}) | \, \bar{c} \subseteq C, \varphi(A^n, \bar{c}) \in \mathcal{U}\}.$$

Now, as $|T| < \theta$ this type is a $D$-type: for any finite tuple $\bar{c}$ from $C$, $q|_{\bar{c}}$ is realized by some tuple from $A^n$ (as in the proof of Lemma 2.5).

Theorem 4.13: *Let*

$$M \prec \mathfrak{C}_D, \quad \bar{d} \in \mathfrak{C}_D^{<\lambda}, \quad \text{and} \quad \bar{c}' \in \mathfrak{C}_D^{<\lambda}$$

*be of length* $< \kappa = |\lg(\bar{d})|^+ + \theta$. *Let* $C = \bigcup \bar{c}'$ *and suppose that* $\mathfrak{a} \in \mathbf{MxK}_{\lambda,\theta}^{M,C,\bar{d}}$ *and that* $\mathrm{tp}(\bar{c}'/M)$ *does not split over* $B_{\mathfrak{a}}$. *Then for any* $r$,

$$\mathbf{x} = (M, B_{\mathfrak{a}}, \bar{d}, \bar{c}'\bar{c}_{\mathfrak{a}}, r)$$

*is a* $\lambda$*-tree-type decomposition (see Definition 4.1).*

*Proof.* As $\mathrm{tp}(\bar{c}'/M)$ does not split over $B_{\mathfrak{a}}$, and $\mathrm{tp}(\bar{c}_{\mathfrak{a}}/MC)$ does not split over $B_{\mathfrak{a}}$, it follows that $\mathrm{tp}(\bar{c}'\bar{c}_{\mathfrak{a}}/M)$ does not split over $B_{\mathfrak{a}}$. Let $\bar{c} = \bar{c}'\bar{c}_{\mathfrak{a}}$. We are left to check that for every $A \subseteq M$ such that $|A| < \lambda$ there exists some $\bar{e}_A \in M^{<\kappa}$ where $\kappa = |\lg(\bar{d})|^+ + \theta$, such that $\mathrm{tp}(\bar{d}/\bar{e}_A + \bar{c}) \vdash \mathrm{tp}(\bar{d}/A + \bar{c})$.

By Proposition 4.11 it is enough to prove that $p(\bar{x}) = \mathrm{tp}(\bar{d}/M + \bar{c})$ is tree-like (with respect to $M$, $\bar{c}$). Let $q(\bar{y}) \in S_D^{<\omega}(M + \bar{c})$ be some type which is $< \theta$ satisfiable in some $B \subseteq M$ with $|B| < \lambda$. Suppose that $p$ is not orthogonal to $q$. This means that there are $\bar{d}_1, \bar{d}_2, \bar{b}_1, \bar{b}_2$ in $\mathfrak{C}_D$ such that $\bar{d}_1, \bar{d}_2 \models p$, $\bar{b}_1, \bar{b}_2 \models q$ and $\bar{d}_1\bar{b}_1 \not\equiv_{M\bar{c}} \bar{d}_2\bar{b}_2$. By homogeneity, we may assume $\bar{d}_1 = \bar{d}_2 = \bar{d}$. Let $q'(\bar{y}) \in S_D^{<\omega}(M + \bar{c}\bar{b}_1\bar{b}_2)$ be an extension of $q$ which is $< \theta$ satisfiable in $B$ (which exists by Remark 4.12), and let $\bar{b} \models q'$. Then for some $i = 1, 2$, it must be that $\bar{d}\bar{b}_i \not\equiv_{M\bar{c}} \bar{d}\bar{b}$. Let $\mathfrak{b} \geq \mathfrak{a}$ be $(B_{\mathfrak{a}} \cup B, \bar{c}_{\mathfrak{a}} \frown (\bar{b}_i, \bar{b}))$, then easily $\mathfrak{b} \in \mathbf{K}_{\lambda,\theta}^{M,C,\bar{d}}$, which contradicts the maximality of $\mathfrak{a}$. ∎

By Theorems 4.8 and 4.13, we get that:

Corollary 4.14: *Suppose* $\mathbf{x}$ *is a* $\lambda$*-tree-type decomposition, and* $\bar{d}_0 \in \mathfrak{C}_D^{<\lambda}$. *Then there exists some* $\lambda$*-tree-type decomposition* $\mathbf{y} \geq \mathbf{x}$ *such that* $\bar{d}_{\mathbf{x}}\bar{d}_0 = \bar{d}_{\mathbf{y}}$ *and* $r_{\mathbf{y}} = r_{\mathbf{x}}$.

*Proof.* Apply Theorem 4.8 with $\bar{d} = \bar{d}_{\mathbf{x}}\bar{d}_0$, $C = \bigcup \bar{c}_{\mathbf{x}}$, $M = M_{\mathbf{x}}$ and $\mathfrak{b} = (B_{\mathbf{x}}, \emptyset)$, to get some $\mathfrak{b} \leq \mathfrak{a} \in \mathbf{MxK}_{\lambda,\theta}^{M,C,\bar{d}}$. Now apply Theorem 4.13 with $\bar{c}' = \bar{c}_{\mathbf{x}}$, $\mathfrak{a}$ and $r_{\mathbf{x}}$. ∎

## 4.2. Self-solvable decomposition.

*Definition 4.15:* Let $M \prec \mathfrak{C}_D$ be a $D$-model of cardinality $\lambda$. A **$\lambda$-self-solvable decomposition**[7] is a $\lambda$-tree-type decomposition $(M, B, \bar{d}, \bar{c}, r)$ such that for every $A \subseteq M$ with $|A| < \lambda$ there exists some $\bar{c}_A\bar{d}_A \in M^{<\lambda}$ with the following properties:

[7] In [She12, Definition 3.6], this is called **tK**.

(1) The tuple $\bar{c}_A$ has the same length as $\bar{c}$ (so $< \kappa = |\lg(\bar{d})|^+ + \theta$) and $\bar{d}_A$ has the same length as $\bar{d}$.
(2) $(\bar{c}_{\mathbf{x}}, \bar{d}_{\mathbf{x}}, \bar{c}_A, \bar{d}_A)$ realize $r_{\mathbf{x}}(\bar{x}_{\bar{c}_{\mathbf{x}}}, \bar{x}_{\bar{d}_{\mathbf{x}}}, \bar{x}'_{\bar{c}_{\mathbf{x}}}, \bar{x}'_{\bar{d}_{\mathbf{x}}})$.
(3) $(\bar{c}_A, \bar{d}_A)$ realize $\mathrm{tp}(\bar{c}_{\mathbf{x}}\bar{d}_{\mathbf{x}}/A)$.
(4) The main point is that we extend point (2) from Definition 4.1 by demanding that

$$\mathrm{tp}(\bar{d}_{\mathbf{x}}/\bar{c}_A + \bar{d}_A + \bar{c}_{\mathbf{x}}) \vdash \mathrm{tp}(\bar{d}_{\mathbf{x}}/A + \bar{c}_{\mathbf{x}} + \bar{c}_A + \bar{d}_A).$$

The first thing we would like to show is that under the assumption that $\lambda$ is measurable, a $\lambda$-self-solvable decomposition exists. In the first order case one can weaken the assumption to ask that $\lambda$ is weakly compact (see [She12, Claim 3.27]). However, we do not know how to extend this results to $D$-models, so we omit it.

Note that the trivial decomposition

$$(M, \emptyset, \emptyset, \emptyset, \emptyset)$$

is a $\lambda$-self-solvable decomposition.

Proposition 4.16: *Let $M$ be a $D$-saturated model of cardinality $\lambda$, with $\lambda > \theta$ measurable. Let $\mathcal{U}$ be a normal non-principal $\lambda$-complete ultrafilter on $\lambda$. Let $\mathbf{x}$ be a $\lambda$-self-solvable decomposition with $M_{\mathbf{x}} = M$, and let $\bar{d} \in \mathfrak{C}_D^{<\lambda}$. Also write $M$ as an increasing continuous union $\bigcup_{\alpha<\lambda} M_\alpha$ where $M_\alpha \subseteq M$ is of size $< \lambda$. Finally, let $\kappa = |\lg(\bar{d}_{\mathbf{x}}\bar{d})|^+ + \theta$.*

*Then for any $n<\omega$, there is a set $U_n \in \mathcal{U}$, a sequence $\langle(\bar{c}_{\alpha,n}, \bar{d}_{\alpha,n}) | \alpha \in U_n \cup \{\lambda\}\rangle$, a type $r_n$ and a set $B_n \subseteq M$ with $|B_n| < \lambda$ such that the following holds:*

(1) *For each $n < \omega$, $U_{n+1} \subseteq U_n$, $\mathbf{x}_n = (M, B_n, \bar{c}_{\lambda,n}, \bar{d}_{\lambda,n}, r_n)$ is a $\lambda$-tree-type decomposition, $\mathbf{x} \leq \mathbf{x}_n \leq \mathbf{x}_{n+1}$ and $\bar{d}_{\mathbf{x}}\bar{d} \trianglelefteq \bar{d}_{\lambda,n}$. Also,*

$$\lg(\bar{d}_{\lambda,n}), \lg(\bar{c}_{\lambda,n}) < \kappa.$$

(2) *For each $n < \omega$ and $\alpha \in U_n \cup \{\lambda\}$, $\bar{c}_{\alpha,n-1}, \bar{d}_{\alpha,n-1} \trianglelefteq \bar{c}_{\alpha,n}, \bar{d}_{\alpha,n}$, and when $\alpha < \lambda$ they are in $M$, $(\bar{c}_{\lambda,n}, \bar{d}_{\lambda,n}, \bar{c}_{\alpha,n}, \bar{d}_{\alpha,n}) \models r_n$ (so $r_n$ is increasing) and $\bar{c}_{\alpha,n}\bar{d}_{\alpha,n}$ realizes $\mathrm{tp}(\bar{c}_{\lambda,n}\bar{d}_{\lambda,n}/M_\alpha)$ (where $\bar{c}_{\alpha,-1}, \bar{d}_{\alpha,-1} = \emptyset$).*
(3) *For each $n < \omega$ and $\alpha \in U_n$, $\mathrm{tp}(\bar{c}_{\lambda,n}, \bar{d}_{\lambda,n}, \bar{c}_{\alpha,n}, \bar{d}_{\alpha,n})$ contains $r_{\mathbf{x}}$ (when restricted to the appropriate variables).*
(4) *For each $n < \omega$ and $\alpha \in U_n$,*

$$\mathrm{tp}(\bar{d}_{\lambda,n}/\bar{c}_{\lambda,n} + \bar{d}_{\alpha,n+1}) \vdash \mathrm{tp}(\bar{d}_{\lambda,n}/\bar{c}_{\lambda,n} + \bar{c}_{\alpha,n} + \bar{d}_{\alpha,n} + M_\alpha).$$

*Proof.* The construction is by induction on $n$.

Assume $n=0$. Let $\bar{d}_{\lambda,0}=\bar{d}_{\mathbf{x}}\bar{d}$ and let $\bar{c}_{\lambda,0}\in\mathfrak{C}_D^{<\kappa}$, $B_0$ be such that

$$\mathbf{x}\le(M,B_0,\bar{d}_{\lambda,0},\bar{c}_{\lambda,0},r_{\mathbf{x}})$$

is a $\lambda$-tree-type decomposition (which exists by Corollary 4.14). For $\alpha<\lambda$, as $\mathbf{x}$ is a self-solvable decomposition, there are $\bar{c}_{\alpha,\mathbf{x}},\bar{d}_{\alpha,\mathbf{x}}$ in $M$ which realize $\mathrm{tp}(\bar{c}_{\mathbf{x}},\bar{d}_{\mathbf{x}}/M_\alpha)$ such that $(\bar{c}_{\mathbf{x}},\bar{d}_{\mathbf{x}},\bar{c}_{\alpha,\mathbf{x}},\bar{d}_{\alpha,\mathbf{x}})\models r_{\mathbf{x}}$.

Go on to find $\bar{c}_{\alpha,\mathbf{x}},\bar{d}_{\alpha,\mathbf{x}}\trianglelefteq\bar{c}_{\alpha,0},\bar{d}_{\alpha,0}$ in $M$ which realize $\mathrm{tp}(\bar{c}_{\lambda,0}\bar{d}_{\lambda,0}/M_\alpha)$ (exists as $M$ is $D$-saturated). By Corollary 2.10, we can find $U_0$ such that $\langle\bar{c}_{\alpha,0}\bar{d}_{\alpha,0}|\,\alpha\in U_0\rangle$ is a fully indiscernible sequence over $\bar{c}_{\lambda,0}+\bar{d}_{\lambda,0}$. Let

$$r_0=\mathrm{tp}(\bar{c}_{\lambda,0},\bar{d}_{\lambda,0},\bar{c}_{\alpha,0},\bar{d}_{\alpha,0}/\emptyset),$$

where $\alpha\in U_0$.

Assume $n=m+1$. Note that $\kappa=|\lg(\bar{d}_{\lambda,m})|^++\theta$. For $\alpha\in U_m$, let

$$\bar{e}_{\alpha,m}\in M^{<\kappa}$$

be such that $\mathrm{tp}(\bar{d}_{\lambda,m}/\bar{c}_{\lambda,m}+\bar{e}_{\alpha,m})\vdash\mathrm{tp}(\bar{d}_{\lambda,m}/\bar{c}_{\lambda,m}+\bar{d}_{\alpha,m}+\bar{c}_{\alpha,m}+M_\alpha)$, which exists as $\mathbf{x}_m$ is a tree-type decomposition. As $\kappa<\lambda$, by restricting $U_m$, we may assume that $\bar{e}_{\alpha,m}$ has a constant length, independent of $\alpha$. Further, let us assume that $\langle\bar{d}_{\alpha,m}\bar{c}_{\alpha,m}\bar{e}_{\alpha,m}|\,\alpha\in U_m\rangle$ is fully indiscernible. Let $\bar{e}_{\lambda,m}$ be such that $\bar{d}_{\lambda,m}\bar{c}_{\lambda,m}\bar{e}_{\lambda,m}\models\bigcup\{\mathrm{tp}(\bar{d}_{\alpha,m}\bar{c}_{\alpha,m}\bar{e}_{\alpha,m}/M_\alpha)|\,\alpha\in U_m\}$. This is a type by full indiscernibility, and such a tuple can be found in $\mathfrak{C}_D$ since $\bar{d}_{\lambda,m}\bar{c}_{\lambda,m}$ already realize this union when we restrict to the appropriate variables, by point (2).

Now we essentially repeat the case $n=0$, applying Corollary 4.14 with $\bar{d}_0,\mathbf{x}$ there being $\bar{e}_{\lambda,m},\mathbf{x}_m$ to find $B_n$ and $\bar{c}_{\lambda,n}$, but now we want that $\bar{c}_{\alpha,m}\trianglelefteq\bar{c}_{\alpha,n}$ and $\bar{d}_{\alpha,m}\bar{e}_{\alpha,m}=\bar{d}_{\alpha,n}$ for $\alpha\in U_m\cup\{\lambda\}$, so we find these tuples and find $U_n$ such that $\langle\bar{c}_{\alpha,n}\bar{d}_{\alpha,n}|\,\alpha\in U_n\rangle$ is fully indiscernible over $\bar{c}_{\lambda,n}\bar{d}_{\lambda,n}$ and we let $r_n=\mathrm{tp}(\bar{c}_{\lambda,m},\bar{d}_{\lambda,m},\bar{c}_{\alpha,n},\bar{d}_{\alpha,n}/\emptyset)$.

(In fact, in the proof we did not need full indiscernibility at any stage. In the case $n=0$ and the last stage of the successor step, we only needed that $\mathrm{tp}(\bar{c}_{\lambda,m},\bar{d}_{\lambda,m},\bar{c}_{\alpha,n},\bar{d}_{\alpha,n}/\emptyset)$ is constant, and in the construction of the $\bar{e}_{\lambda,m}$ we only needed that the types $\mathrm{tp}(\bar{d}_{\alpha,m}\bar{c}_{\alpha,m}\bar{e}_{\alpha,m}/M_\alpha)$ are increasing with $\alpha$.) ∎

COROLLARY 4.17: *Let $M$ be a $D$-saturated model of cardinality $\lambda$, where $\lambda\ge\theta$ is measurable, and let $\bar{d}\in\mathfrak{C}_D^{<\lambda}$. Let $\mathbf{x}$ be some $\lambda$-self-solvable decomposition, possibly trivial. Then there exists some $\lambda$-self-solvable decomposition $\mathbf{x}\le\mathbf{y}$ such that $\bar{d}_{\mathbf{x}}\bar{d}\trianglelefteq\bar{d}_{\mathbf{y}}$.*

*Proof.* Write

$$M = \bigcup_{\alpha<\lambda} M_\alpha$$

where $M_\alpha \subseteq M$ are of cardinality $< \lambda$ and the sequence is increasing and continuous. Also choose some normal ultrafilter $\mathcal{U}$ on $\lambda$. Now we apply Proposition 4.16, to find $U_n$, $B_n$, $r_n$ and $\langle(\bar{c}_{\alpha,n}, \bar{d}_{\alpha,n}) \mid \alpha \in U_n \cup \{\lambda\}\rangle$. Let $\bar{d}_\lambda = \bigcup_{n<\omega} \bar{d}_{\lambda,n}$, $\bar{c}_\lambda = \bigcup_{n<\omega} \bar{c}_{\lambda,n}$, $B = \bigcup_{n<\omega} B_n$ and $r = \bigcup_{n<\omega} r_n$ (note that this is indeed a $D$-type). Also, let $U = \bigcap_{n<\omega} U_n \in \mathcal{U}$ (as $\mathcal{U}$ is $\lambda$-complete).

Then $(M, B, \bar{d}_\lambda, \bar{c}_\lambda, r)$ is a $\lambda$-self-solvable decomposition: first of all it is a tree-type decomposition, as $\mathrm{tp}(\bar{c}_\lambda/M)$ does not split over $B$. Also, $\kappa = |\lg(\bar{d}_{\mathbf{x}}\bar{d})|^+ + \theta$ is regular of cofinality $> \aleph_0$, so $\lg(\bar{c}_\lambda) < \kappa = |\lg(\bar{d}_\lambda)|^+ + \theta$. For each $A \subseteq M$ of size $< \lambda$, there is some $\alpha \in U$ such that $M_\alpha$ contains $A$. Let

$$\bar{c}_A, \bar{d}_A = \bigcup_{n<\omega} \bar{c}_{\alpha,n}, \bigcup_{n<\omega} \bar{d}_{\alpha,n}.$$

Then, it follows from point (2) in Proposition 4.16 that $(\bar{c}_\lambda, \bar{d}_\lambda, \bar{c}_A, \bar{d}_A) \models r$ and that $(\bar{c}_A\bar{d}_A)$ realize $\mathrm{tp}(\bar{c}_\lambda\bar{d}_\lambda/A)$. Also, note that $r_{\mathbf{x}} \subseteq r$, $B_{\mathbf{x}} \subseteq B$, $\bar{c}_{\mathbf{x}}, \bar{d}_{\mathbf{x}} \trianglelefteq \bar{c}_\lambda, \bar{d}_\lambda$.

Finally, we must check that

$$\mathrm{tp}(\bar{d}_\lambda/\bar{c}_A + \bar{d}_A + \bar{c}_\lambda) \vdash \mathrm{tp}(\bar{d}_\lambda/A + \bar{c}_\lambda + \bar{c}_A + \bar{d}_A).$$

This holds since formulas have finitely many variables. ∎

## 5. Finding a good family

In this section we will show that the family of $\lambda$-self-solvable decompositions is a good family of $\lambda$-decompositions whenever $\lambda > \theta$ is measurable (note that in that case $\lambda^{<\lambda} = \lambda$). This will conclude the proof of Conjecture 3.1 in this case. So let $\mathfrak{F}$ be the family of $\lambda$-self-solvable decompositions $\mathbf{x}$ such that $M_{\mathbf{x}}$ is $D$-saturated of cardinality $\lambda$. Let us go over Definition 3.2, and prove that each clause is satisfied by $\mathfrak{F}$.

Claim 5.1: *Points* (1), (2), (3), (4), (5) *and* (6) *are satisfied by* $\mathfrak{F}$.

*Proof.* Everything is clear, except (4), which is exactly Corollary 4.17. ∎

We now move on to point (7), but for this we will need the following lemma.

Lemma 5.2: *Suppose that $(I,<)$ is some linearly ordered set. Let $\langle \bar{a}_i | \, i \in I\rangle$ be a sequence of tuples of the same length from $\mathfrak{C}_D$, and let $B \subseteq \mathfrak{C}_D$ be some set. Assume the following conditions.*

(1) *For all $i \in I$, $\bar{a}_i = \bar{c}_i \bar{d}_i$.*
(2) *For all $i \in I$, $\mathrm{tp}(\bar{a}_i/B_i)$ is increasing with $i$, where $B_i = B \cup \{\bar{a}_j | \, j < i\}$.*
(3) *For all $i \in I$, $\mathrm{tp}(\bar{c}_i/B_i)$ does not split over $B$.*
(4) *For every $j < i$ in $I$, $\mathrm{tp}(\bar{d}_i/\bar{c}_i + \bar{a}_j) \vdash \mathrm{tp}(\bar{d}_i/\bar{c}_i + \bar{a}_j + B_j)$.*
(5) *For every $i_1 < i_2$, $j_1 < j_2$ from $I$, $\mathrm{tp}(\bar{a}_{i_2}\bar{a}_{i_1}/\emptyset) = \mathrm{tp}(\bar{a}_{j_2}\bar{a}_{j_1}/\emptyset)$.*

*Then $\langle \bar{a}_i | \, i \in I\rangle$ is indiscernible over $B$.*

*Proof.* We prove by induction on $n$ that $\langle \bar{a}_i | \, i \in I\rangle$ is an $n$-indiscernible sequence over $B$.

For $n = 1$ it follows from (2).

Now suppose that $\langle \bar{a}_i | \, i \in I\rangle$ is $n$-indiscernible over $B$. Let $i_1 < \cdots < i_n < i_{n+1} \in I$ and $j_1 < \cdots < j_n < j_{n+1} \in I$ be such that, without loss of generality, $i_{n+1} \leq j_{n+1}$. By (2), we know that $\bar{a}_{i_1} \cdots \bar{a}_{i_n} \bar{a}_{i_{n+1}} \equiv_B \bar{a}_{i_1} \cdots \bar{a}_{i_n} \bar{a}_{j_{n+1}}$. By (3) and the induction hypothesis, we know that $\bar{a}_{i_1} \cdots \bar{a}_{i_n} \bar{c}_{j_{n+1}} \equiv_B \bar{a}_{j_1} \cdots \bar{a}_{j_n} \bar{c}_{j_{n+1}}$. Combining, we get that

$$\bar{a}_{i_1} \cdots \bar{a}_{i_n} \bar{c}_{i_{n+1}} \equiv_B \bar{a}_{j_1} \cdots \bar{a}_{j_n} \bar{c}_{j_{n+1}}.$$

Suppose that $\varphi(\bar{d}_{i_{n+1}}, \bar{c}_{i_{n+1}}, \bar{a}_{i_n}, \ldots, \bar{a}_{i_1}, \bar{b})$ holds where $\bar{b}$ is a finite tuple from $B$. Let $r(\bar{x}_{\bar{d}}, \bar{x}_{\bar{c}}, \bar{x}_{\bar{a}}) = \mathrm{tp}(\bar{d}_{i_{n+1}}, \bar{c}_{i_{n+1}}, \bar{a}_{i_n}/\emptyset)$. By (4),

$$r(\bar{x}_{\bar{d}}, \bar{c}_{i_{n+1}}, \bar{a}_{i_n}) \vdash \varphi(\bar{x}_{\bar{d}}, \bar{c}_{i_{n+1}}, \bar{a}_{i_n}, \ldots, \bar{a}_{i_1}, \bar{b}).$$

Applying the last equation, we get that

$$r(\bar{x}_{\bar{d}}, \bar{c}_{j_{n+1}}, \bar{a}_{j_n}) \vdash \varphi(\bar{x}_{\bar{d}}, \bar{c}_{j_{n+1}}, \bar{a}_{j_n}, \ldots, \bar{a}_{j_1}, \bar{b}).$$

By (5), $r = \mathrm{tp}(\bar{d}_{j_{n+1}}, \bar{c}_{j_{n+1}}, \bar{a}_{j_n}/\emptyset)$, so $\bar{d}_{j_{n+1}}$ satisfies the left-hand side, and so also the right-hand side, and so $\varphi(\bar{d}_{j_{n+1}}, \bar{c}_{j_{n+1}}, \bar{a}_{j_n}, \ldots, \bar{a}_{j_1}, \bar{b})$ holds and we are done. ∎

Corollary 5.3: *Suppose that $\mathbf{x} \in \mathfrak{F}$, and let $M = M_{\mathbf{x}}$. Let $B \supseteq B_{\mathbf{x}}$ be any subset of $M$ of cardinality $< \lambda$, and let $\alpha \leq \lambda$. For $i < \alpha$, let $\bar{a}_i$ be such that $\bar{a}_0 = \bar{c}_B \bar{d}_B$ (see Definition 4.15), and for $i > 0$, $\bar{a}_i = \bar{c}_{B_i} \bar{d}_{B_i}$ where $B_i = B \cup \{\bar{a}_j | \, j < i\}$. Then $\langle \bar{a}_i | \, i < \alpha\rangle \frown \langle \bar{c}_{\mathbf{x}} \bar{d}_{\mathbf{x}}\rangle$ is an indiscernible sequence over $B$.*

*Proof.* Apply Lemma 5.2 with $I = \alpha + 1$ (so that $\bar{a}_\alpha = \bar{c}_{\mathbf{x}}\bar{d}_{\mathbf{x}}$). Let us check that the conditions there hold. (1) is obvious. (2) holds as

$$\operatorname{tp}(\bar{a}_i/B_i) = \operatorname{tp}(\bar{c}_{\mathbf{x}}\bar{d}_{\mathbf{x}}/B_i) \supseteq \operatorname{tp}(\bar{c}_{\mathbf{x}}\bar{d}_{\mathbf{x}}/B_j)$$

when $\alpha + 1 > i \geq j$. (3) holds as $\operatorname{tp}(\bar{c}_{\mathbf{x}}/B_i)$ does not split over $B$, so the same is true for $\bar{c}_i$. (5) holds because for $i_1 < i_2 < \alpha + 1$,

$$\begin{aligned}\operatorname{tp}(\bar{a}_{i_2}\bar{a}_{i_1}/\emptyset) &= \operatorname{tp}(\bar{c}_{\mathbf{x}}\bar{d}_{\mathbf{x}}\bar{c}_{i_1}\bar{d}_{i_1}/\emptyset) = r_{\mathbf{x}}\\ &= \operatorname{tp}(\bar{c}_{\mathbf{x}}\bar{d}_{\mathbf{x}}\bar{c}_{j_1}\bar{d}_{j_1}/\emptyset) = \operatorname{tp}(\bar{a}_{j_2}\bar{a}_{j_1}/\emptyset).\end{aligned}$$

Finally, (4) holds because $\operatorname{tp}(\bar{d}_{\mathbf{x}}/\bar{c}_{\mathbf{x}} + \bar{a}_j) \vdash \operatorname{tp}(\bar{d}_{\mathbf{x}}/\bar{c}_{\mathbf{x}} + \bar{a}_j + B_j)$, and as $\bar{c}_{\mathbf{x}}\bar{d}_{\mathbf{x}} \equiv_{B_{j+1}} \bar{c}_{B_i}\bar{d}_{B_i} = \bar{a}_i$, we can replace $\bar{c}_{\mathbf{x}}\bar{d}_{\mathbf{x}}$ by $\bar{c}_{B_i}\bar{d}_{B_i}$ in this implication by applying an automorphism of $\mathfrak{C}_D$. ∎

Lemma 5.4: *Suppose that $\mathbf{x}_1 \leq \mathbf{x}_2$ are two $\lambda$-decompositions from $\mathfrak{F}$. Then for every subset $A$ of $M$ of size $< \lambda$ containing $B_{\mathbf{x}_1}$, and for any choice of $\bar{c}_A, \bar{d}_A$ which we get when we apply Definition 4.15 on $\mathbf{x}_2$, their restrictions to $\lg(\bar{c}_{\mathbf{x}_1}), \lg(\bar{d}_{\mathbf{x}_1})$ satisfy all the conditions in Definition 4.15.*

*Proof.* Denote these restrictions by $\bar{c}'_A, \bar{d}'_A$. As $r_{\mathbf{x}_1} \subseteq r_{\mathbf{x}_2}$, we get Clause (2) of Definition 4.15 immediately. Clause (3) is also clear, so we are left with (4). Since $\mathbf{x}_1 \in \mathfrak{F}$, there are some $\bar{c}''_A, \bar{d}''_A$ in $M$ in the same length as $\lg(\bar{c}_{\mathbf{x}_1}), \lg(\bar{d}_{\mathbf{x}_1})$, which we get when applying Definition 4.15 on $\mathbf{x}_1$. It is enough to show that $\bar{d}_{\mathbf{x}_1}\bar{c}_{\mathbf{x}_1}\bar{c}''_A\bar{d}''_A \equiv_A \bar{d}_{\mathbf{x}_1}\bar{c}_{\mathbf{x}_1}\bar{c}'_A\bar{d}'_A$. Note first that $\bar{c}''_A\bar{d}''_A \equiv_A \bar{c}'_A\bar{d}'_A$ by (3), and as $\operatorname{tp}(\bar{c}_{\mathbf{x}_1}/M)$ does not split over $A$, we also get $\bar{c}_{\mathbf{x}_1}\bar{c}''_A\bar{d}''_A \equiv_A \bar{c}_{\mathbf{x}_1}\bar{c}'_A\bar{d}'_A$. So suppose that $\mathfrak{C}_D \models \varphi(\bar{d}_{\mathbf{x}_1}, \bar{c}_{\mathbf{x}_1}, \bar{c}''_A, \bar{d}''_A, \bar{a})$ where $\bar{a}$ is a finite tuple from $A$. By (4) and (2), $r_{\mathbf{x}_1}(\bar{c}_{\mathbf{x}_1}, \bar{x}_{\bar{d}_{\mathbf{x}_1}}, \bar{c}''_A, \bar{d}''_A) \vdash \varphi(\bar{x}_{\bar{d}_{\mathbf{x}_1}}, \bar{c}_{\mathbf{x}_1}, \bar{c}''_A, \bar{d}''_A, \bar{a})$, and applying the last equation, we get that $r_{\mathbf{x}_1}(\bar{c}_{\mathbf{x}_1}, \bar{x}_{\bar{d}_{\mathbf{x}_1}}, \bar{c}'_A, \bar{d}'_A) \vdash \varphi(\bar{x}_{\bar{d}_{\mathbf{x}_1}}, \bar{c}_{\mathbf{x}_1}, \bar{c}'_A, \bar{d}'_A, \bar{a})$, but as $\bar{d}_{\mathbf{x}_1}$ satisfies the left hand side (because $r_{\mathbf{x}_1} \subseteq r_{\mathbf{x}_2}$), we are done. ∎

Theorem 5.5: *Suppose $\delta < \lambda$ is a limit ordinal. Let $\langle \mathbf{x}_j \,|\, j < \delta \rangle$ be an increasing sequence of decompositions from $\mathfrak{F}$. Then $\mathbf{x} = \sup_{j<\delta} \mathbf{x}_j \in \mathfrak{F}$. Hence point (7) of Definition 3.2 is satisfied by $\mathfrak{F}$.*

*Proof.* Easily $\mathbf{x}$ is a $\lambda$-decomposition (i.e., $|B_{\mathbf{x}}| < \lambda$ and $r_{\mathbf{x}}$ is well defined). Also, $\operatorname{tp}(\bar{c}_{\mathbf{x}}/M)$ does not split over $B_{\mathbf{x}} = \bigcup B_{\mathbf{x}_i}$, where we let $M = M_{\mathbf{x}}$.

Let $A \subseteq M$ be of cardinality $< \lambda$ and without loss of generality suppose $B_{\mathbf{x}} \subseteq A$.

In order to prove the theorem, we need to find some $\bar{c}, \bar{d} \in M^{<\lambda}$ in the same length as $\bar{c}_{\mathbf{x}}, \bar{d}_{\mathbf{x}}$ such that $\mathrm{tp}(\bar{c}\bar{d}/A) = \mathrm{tp}(\bar{c}_{\mathbf{x}}\bar{d}_{\mathbf{x}}/A)$, $\mathrm{tp}(\bar{c}_{\mathbf{x}}, \bar{d}_{\mathbf{x}}, \bar{c}, \bar{d}) = r_{\mathbf{x}}$, and $\mathrm{tp}(\bar{d}_{\mathbf{x}}/\bar{c}_{\mathbf{x}} + \bar{c} + \bar{d}) \vdash \mathrm{tp}(\bar{d}_{\mathbf{x}}/\bar{c}_{\mathbf{x}} + \bar{c} + \bar{d} + A)$.

Let us simplify the notation by letting $\beta_j = \lg(\bar{c}_{\mathbf{x}_j}), \gamma_j = \lg(\bar{d}_{\mathbf{x}_j})$. Note that when $\beta_j$ and $\gamma_j$ are constant from some point onwards, finding such $\bar{c}, \bar{d}$ is done by just applying Definition 4.15 to some $\mathbf{x}_j$, so although the following argument works for this case as well, it is more interesting when $\beta_j$ and $\gamma_j$ are increasing.

For every $i < \delta$, let $\bar{c}_i, \bar{d}_i = \bar{c}_{A_i}\bar{d}_{A_i}$ be as in Definition 4.15 applied to $\mathbf{x}_i$ (so their length is $\beta_i, \gamma_i$), where $A_i = A \cup \{\bar{c}_j, \bar{d}_j \,|\, j < i\}$. Now repeat this process starting with $A_\delta$ to construct $\bar{c}_i, \bar{d}_i$ for $\delta \leq i < \delta + \delta$.

Now we repeat this process $\kappa + 1$ times, for $\kappa = \mu(D)^+ + |T|^+ < \lambda$, to construct $\bar{c}_i, \bar{d}_i$ and $A_i$ for $\delta + \delta \leq i < \delta \cdot \kappa + \delta$. For $j < \delta$, let $O_j \subseteq \delta \cdot \kappa + \delta$ be the set of all ordinals $i$ such that $i \pmod{\delta} \geq j$. By Corollary 5.3 and Lemma 5.4, for each $j < \delta$, the sequence $I_j = \langle (\bar{c}_i \restriction \beta_j, \bar{d}_i \restriction \gamma_j) \,|\, i \in O_j \rangle \frown \langle (\bar{c}_{\mathbf{x}_j}\bar{d}_{\mathbf{x}_j}) \rangle$ is an indiscernible sequence over $A$.

Let $O'_j = O_j \cap \delta \cdot \kappa$, $O''_j = O_j \cap [\delta \cdot \kappa, \delta \cdot \kappa + \delta)$, and let $I'_j = I_j \restriction O'_j$, $I''_j = I_j \restriction O''_j$. As $O'_j$ has cofinality $\kappa$ (suppose $X \subseteq O'_j$ is unbounded, then the set $\{i < \kappa \,|\, X \cap [\delta \cdot i, \delta \cdot i + \delta) \neq \emptyset\}$ is unbounded, so has cardinality $\kappa$, so $|X| \geq \kappa$, but easily, the set $\{\delta \cdot i + j \,|\, i < \kappa\}$ is cofinal in $O'_j$), we can apply Lemma 2.12, and consider the type $q_j(\bar{x}) = Av(I'_j/A_{\delta \cdot \kappa + \delta})$, which is a complete $D$-type. So each $q_j$ is a type in $\beta_j + \gamma_j$ variables.

Claim: *For $j_1 < j_2$, $q_{j_1} \subseteq q_{j_2}$.*

*Proof.* Suppose $\varphi(\bar{y}, \bar{a}) \in q_{j_1}$, where $\bar{a}$ is a finite tuple from $A_{\delta \cdot \kappa + \delta}$ and $\bar{y}$ is a finite subtuple of variables of $\bar{x}$. By definition, it means that for large enough $i \in O'_{j_1}$, $\varphi(\bar{c}_i \restriction \beta_{j_1}, \bar{d}_i \restriction \gamma_{j_1}, \bar{a})$ holds (where we restrict $\bar{c}_i, \bar{d}_i$ to $\bar{y}$, of course). But $j_2 > j_1$, so $O'_{j_2} \subseteq O'_{j_1}$, so the same is true for $O'_{j_2}$, and so $\varphi(\bar{y}, \bar{a}) \in q_{j_2}$. ∎

Let $q = \bigcup_{j<\delta} q_j$. As $\delta$ is limit, it follows that $q$ is also a $D$-type over $A_{\delta \cdot \kappa + \delta}$. Let $\bar{c}', \bar{d}' \models q$, and for each $j < \delta$, let $\bar{c}'_j = \bar{c} \restriction \beta_j, \bar{d}'_j = \bar{d}' \restriction \gamma_j$. It now follows that for each $j < \delta$, the sequence $I'_j \frown \langle \bar{c}'_j, \bar{d}'_j \rangle \frown I''_j$ is indiscernible over $A$.

Let us check that $\bar{c}', \bar{d}'$ are as required. To show this it is enough to see that for every $j < \delta$, $\bar{c}_{\mathbf{x}_j}\bar{d}_{\mathbf{x}_j}\bar{c}'_j\bar{d}'_j \equiv_A \bar{c}_{\mathbf{x}_j}\bar{d}_{\mathbf{x}_j}\bar{c}_j\bar{d}_j$. Suppose $\varphi(\bar{c}_{\mathbf{x}_j}, \bar{d}_{\mathbf{x}_j}, \bar{c}_j, \bar{d}_j, \bar{a})$ holds, where $\bar{a}$ is a finite tuple from $A$. By indiscernibility, $\varphi(\bar{c}_{\mathbf{x}_j}, \bar{d}_{\mathbf{x}_j}, \bar{c}_{\delta \cdot \kappa + j}, \bar{d}_{\delta \cdot \kappa + j}, \bar{a})$ holds as well. By choice of $\bar{c}_{\delta \cdot \kappa + j + 1}, \bar{d}_{\delta \cdot \kappa + j + 1}$, it follows that

$$(*) \quad r_{\mathbf{x}_j}(\bar{c}_{\mathbf{x}_j}, \bar{x}_{\bar{d}_{\mathbf{x}_j}}, \bar{c}_{\delta \cdot \kappa + j + 1} \restriction \beta_j, \bar{d}_{\delta \cdot \kappa + j + 1} \restriction \gamma_j) \vdash \varphi(\bar{c}_{\mathbf{x}_j}, \bar{x}_{\bar{d}_{\mathbf{x}_j}}, \bar{c}_{\delta \cdot \kappa + j}, \bar{d}_{\delta \cdot \kappa + j}, \bar{a}).$$

By indiscernibility,

$$(\bar{c}_{\delta\cdot\kappa+j+1} \restriction \beta_j)(\bar{d}_{\delta\cdot\kappa+j+1} \restriction \gamma_j)\bar{c}_{\delta\cdot\kappa+j}\bar{d}_{\delta\cdot\kappa+j} \\ \equiv_A (\bar{c}_{\delta\cdot\kappa+j+1} \restriction \beta_j)(\bar{d}_{\delta\cdot\kappa+j+1} \restriction \gamma_j)(\bar{c}' \restriction \beta_j)(\bar{d}' \restriction \gamma_j),$$

and as $\mathrm{tp}(\bar{c}_{\mathbf{x}}/M)$ does not split over $A$,

$$\bar{c}_{\mathbf{x}_j}(\bar{c}_{\delta\cdot\kappa+j+1} \restriction \beta_j)(\bar{d}_{\delta\cdot\kappa+j+1} \restriction \gamma_j)\bar{c}_{\delta\cdot\kappa+j}\bar{d}_{\delta\cdot\kappa+j} \\ \equiv_A \bar{c}_{\mathbf{x}_j}(\bar{c}_{\delta\cdot\kappa+j+1} \restriction \beta_j)(\bar{d}_{\delta\cdot\kappa+j+1} \restriction \gamma_j)\bar{c}'_j\bar{d}'_j.$$

Applying the last equation to $(*)$, we get that

$$(**) \qquad r_{\mathbf{x}_j}(\bar{c}_{\mathbf{x}_j}, \bar{x}_{\bar{d}_{\mathbf{x}_j}}, \bar{c}_{\delta\cdot\kappa+j+1} \restriction \beta_j, \bar{d}_{\delta\cdot\kappa+j+1} \restriction \gamma_j) \vdash \varphi(\bar{c}_{\mathbf{x}_j}, \bar{x}_{\bar{d}_{\mathbf{x}_j}}, \bar{c}'_j, \bar{d}'_j, \bar{a}).$$

As $\bar{d}_{\mathbf{x}_j}$ satisfies the left hand side of $(**)$, it also satisfies the right side, and we are done. ∎

*Remark 5.6:* The proof of Theorem 5.5 as above can be simplified in the case where $D$ is trivial (i.e., the usual first order case). There, we would not need to introduce $\kappa$ (i.e., we can choose $\kappa = 1$), and we would not have to use dependence (which we used in applying Lemma 2.12 which states that the average type of an indiscernible sequence exists and is a $D$-type). To make the proof work, we only needed to find $\bar{c}', \bar{d}'$ such that the sequence $I'_j \frown \langle \bar{c}' \restriction \beta_j, \bar{d}' \restriction \gamma_j \rangle \frown I''_j$ is indiscernible over $A$, and this can easily be done by compactness.

We now move on to points (8) and (9) of Definition 3.2.

Suppose $\mathbf{x}$ is a $\lambda$-tree-type decomposition. Let $L_{\bar{c}_{\mathbf{x}}}$ be the set of formulas $\varphi(\bar{x}_{\bar{c}_{\mathbf{x}}}, \bar{y})$ where $\bar{x}_{\bar{c}_{\mathbf{x}}}$ is a tuple of variables in the length of $\bar{c}_{\mathbf{x}}$ (of course only finitely many of them appear in $\varphi$). For $B \subseteq M_{\mathbf{x}}$ over which $\mathrm{tp}(\bar{c}_{\mathbf{x}}/M_{\mathbf{x}})$ does not split, define $\Phi_{\mathbf{x},B} : L_{\bar{c}_{\mathbf{x}}} \to \mathcal{P}(S_D^{<\omega}(B))$ by

$$\Phi_{\mathbf{x},B}(\varphi(\bar{x}_{\bar{c}_{\mathbf{x}}}, \bar{y})) = \{p(\bar{y}) \in S_D(B) |\, \exists \bar{e} \in M_{\mathbf{x}}^{\lg(\bar{y})}(\bar{e} \models p \wedge \mathfrak{C}_D \models \varphi(\bar{c}_{\mathbf{x}}, \bar{e}))\}.$$

As $\mathrm{tp}(\bar{c}_{\mathbf{x}}/M)$ does not split over $B$, we can also replace $\exists$ with $\forall$ in the definition of $\Phi_{\mathbf{x},B}$. This implies that for $B' \supseteq B$ and $p \in S_D(B')$,

$$(\dagger) \qquad p \in \Phi_{\mathbf{x},B'}(\varphi) \Leftrightarrow p|_B \in \Phi_{\mathbf{x},B}(\varphi).$$

Suppose that $\mathbf{y}$ is another $\lambda$-tree-type decomposition. When $h$ is an elementary map from $B_{\mathbf{x}}$ to $B_{\mathbf{y}}$, then it induces a well defined map from $S_D(B_{\mathbf{x}})$ to $S_D(B_{\mathbf{y}})$ which we will also call $h$. So if $\bar{c}_{\mathbf{x}}$ has the same length as $\bar{c}_{\mathbf{y}}$, it makes sense to ask that $h \circ \Phi_{\mathbf{x},B_{\mathbf{x}}} = \Phi_{\mathbf{y},B_{\mathbf{y}}}$. When $r_{\mathbf{x}} = r_{\mathbf{y}}$, a partial elementary map $h$

whose domain is $B_{\mathbf{x}} \cup \bigcup \bar{c}_{\mathbf{x}} \cup \bigcup \bar{d}_{\mathbf{x}}$ which maps $(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}, B_{\mathbf{x}})$ onto $(\bar{d}_{\mathbf{y}}, \bar{c}_{\mathbf{y}}, B_{\mathbf{y}})$ and satisfies $h \circ \Phi_{\mathbf{x},B_{\mathbf{x}}} = \Phi_{\mathbf{y},B_{\mathbf{y}}}$ is called a **pseudo isomorphism** between $\mathbf{x}$ and $\mathbf{y}$.

Note that if $h$ is a pseudo isomorphism, then for any two tuples $\bar{a}$, $\bar{b}$, from $M_{\mathbf{x}}$, $M_{\mathbf{y}}$ respectively, if $h \restriction B_{\mathbf{x}}$ can be extended to witness that $B_{\mathbf{x}}\bar{a} \equiv B_{\mathbf{y}}\bar{b}$, then $\bar{c}_{\mathbf{x}} B_{\mathbf{x}} \bar{a} \equiv \bar{c}_{\mathbf{y}} B_{\mathbf{y}} \bar{a}$.

PROPOSITION 5.7: *Suppose $\mathbf{x}, \mathbf{y} \in \mathfrak{F}$ are such that $r_{\mathbf{x}} = r_{\mathbf{y}}$, and suppose that $h : \mathbf{x} \to \mathbf{y}$ is a pseudo isomorphism. Then $h$ is a weak isomorphism, i.e., it extends to an isomorphism $h^+ : \mathbf{x} \to \mathbf{y}$. Conversely, if $h$ is a weak isomorphism, then it is a pseudo isomorphism.*

*Proof.* We will do a back and forth argument. In each successor step we will add an element to either $B_{\mathbf{x}}$ or $B_{\mathbf{y}}$ and increase $h$. In doing so, the new $\mathbf{x}$ and $\mathbf{y}$'s will still remain in $\mathfrak{F}$ (by point (5) of Definition 3.2 which is easily true for $\mathfrak{F}$). In addition, the increased $h$'s will still be pseudo isomorphisms by (†). In order to do this, it is enough to do a single step, so assume that $h : \mathbf{x} \to \mathbf{y}$ is a pseudo isomorphism, and $a \in M_{\mathbf{x}}$. We want to find $b \in M_{\mathbf{y}}$ such that $h \cup \{(a, b)\}$ is a pseudo isomorphism from $\mathbf{x}' = (M_{\mathbf{x}}, B_{\mathbf{x}} \cup \{a\}, \bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}, r_{\mathbf{x}})$ to $\mathbf{y}' = (M_{\mathbf{y}}, B_{\mathbf{y}} \cup \{b\}, \bar{d}_{\mathbf{y}}, \bar{c}_{\mathbf{y}}, r_{\mathbf{y}})$.

Let $A = B_{\mathbf{x}} \cup \{a\}$, and let $\bar{c}^{\mathbf{x}}_A, \bar{d}^{\mathbf{x}}_A$ be as in Definition 4.15 for $\mathbf{x}$. Let $\bar{c}^{\mathbf{y}}_{B_{\mathbf{y}}}, \bar{d}^{\mathbf{y}}_{B_{\mathbf{y}}}$ be the parallel tuples for $\mathbf{y}$ and $B_{\mathbf{y}}$. By (3) of Definition 4.15,

$$B_{\mathbf{x}} \bar{c}^{\mathbf{x}}_A \bar{d}^{\mathbf{x}}_A \equiv B_{\mathbf{y}} \bar{c}^{\mathbf{y}}_{B_{\mathbf{y}}} \bar{d}^{\mathbf{y}}_{B_{\mathbf{y}}},$$

as witnessed by expanding $h \restriction B_{\mathbf{x}}$ to $B_{\mathbf{x}} \bar{c}^{\mathbf{x}}_A \bar{d}^{\mathbf{x}}_A$. Hence as $M_{\mathbf{y}}$ is $D$-saturated there is some $b \in M_{\mathbf{y}}$ such that $B_{\mathbf{x}} a \bar{c}^{\mathbf{x}}_A \bar{d}^{\mathbf{x}}_A \equiv B_{\mathbf{y}} b \bar{c}^{\mathbf{y}}_{B_{\mathbf{y}}} \bar{d}^{\mathbf{y}}_{B_{\mathbf{y}}}$. So we have found our $b$.

As noted above, as $h$ is a pseudo isomorphism, we get that

$$(\dagger\dagger) \qquad B_{\mathbf{x}} a \bar{c}^{\mathbf{x}}_A \bar{d}^{\mathbf{x}}_A \bar{c}_{\mathbf{x}} \equiv B_{\mathbf{y}} b \bar{c}^{\mathbf{y}}_{B_{\mathbf{y}}} \bar{d}^{\mathbf{y}}_{B_{\mathbf{y}}} \bar{c}_{\mathbf{y}}.$$

Suppose now that $\varphi(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}, a, \bar{e})$ holds, where $\bar{e}$ is a finite tuple from $B_{\mathbf{x}}$. By the choice of $\bar{c}^{\mathbf{x}}_A, \bar{d}^{\mathbf{x}}_A$, $r_{\mathbf{x}}(\bar{c}_{\mathbf{x}}, \bar{x}_{\bar{d}_{\mathbf{x}}}, \bar{c}^{\mathbf{x}}_A, \bar{d}^{\mathbf{x}}_A) \vdash \varphi(\bar{x}_{\bar{d}_{\mathbf{x}}}, \bar{c}_{\mathbf{x}}, a, \bar{e})$. Applying (††), we get that $r_{\mathbf{x}}(\bar{c}_{\mathbf{y}}, \bar{x}_{\bar{d}_{\mathbf{y}}}, \bar{c}^{\mathbf{y}}_{B_{\mathbf{y}}}, \bar{d}^{\mathbf{y}}_{B_{\mathbf{y}}}) \vdash \varphi(\bar{x}_{\bar{d}_{\mathbf{y}}}, \bar{c}_{\mathbf{y}}, b, h(\bar{e}))$. As $r_{\mathbf{x}} = r_{\mathbf{y}}$, $\bar{d}_{\mathbf{y}}$ realizes the left hand side, so also the right hand side and so $\mathfrak{C}_D \models \varphi(\bar{d}_{\mathbf{y}}, \bar{c}_{\mathbf{y}}, b, h(\bar{e}))$.

For the limit stages, note that if $\langle h_i | \, i < \delta \rangle$ is an increasing sequence of pseudo isomorphisms $h_i : \mathbf{x}_i \to \mathbf{y}_i$ where $\mathbf{x}_i = (M_{\mathbf{x}}, B_{\mathbf{x}_i}, \bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}, r_{\mathbf{x}})$ and $\mathbf{y}_i = (M_{\mathbf{y}}, B_{\mathbf{y}_i}, \bar{d}_{\mathbf{y}}, \bar{c}_{\mathbf{y}}, r_{\mathbf{y}})$ are increasing, and $\delta < \lambda$, then $\bigcup \{h_i | \, i < \delta\}$ is a pseudo isomorphism from $\sup_{i<\delta} \mathbf{x}_i$ to $\sup_{i<\delta} \mathbf{y}_i$.

The other direction is immediate. ∎

Corollary 5.8: *Clause* (8) *in Definition 3.2 holds for* $\mathfrak{F}$.

*Proof.* We are given two increasing sequences of decompositions $\langle \mathbf{x}_i | \, i < \delta \rangle$ and $\langle \mathbf{y}_i | \, i < \delta \rangle$ in $\mathfrak{F}$, and we assume that for each $i < \delta$ there is a weak isomorphism $g_i : \mathbf{x}_i \to \mathbf{y}_i$ such that $g_i \subseteq g_i$ whenever $i < j$. We need to show that the union $g = \bigcup_{i<\delta} g_i$ is also a weak isomorphism from $\mathbf{x} = \sup_{i<\delta} \mathbf{x}_i$ to $\mathbf{y} = \sup_{i<\delta} \mathbf{y}_i$. We already know by Theorem 5.5 that $\mathbf{x}, \mathbf{y} \in \mathfrak{F}$, so by Proposition 5.7, we only need to show that $g$ is a pseudo isomorphism and that $r_{\mathbf{x}} = r_{\mathbf{y}}$. The latter is clear, as

$$r_{\mathbf{x}} = \bigcup_{i<\delta} r_{\mathbf{x}_i} = \bigcup_{i<\delta} r_{\mathbf{y}_i} = r_{\mathbf{y}}.$$

Also, it is clear that $g$ is an elementary map taking $(\bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}, B_{\mathbf{x}})$ to $(\bar{d}_{\mathbf{y}}, \bar{c}_{\mathbf{y}}, B_{\mathbf{y}})$.

Note that $L_{\bar{c}_{\mathbf{x}}} = \bigcup_{i<\delta} L_{\bar{c}_{\mathbf{x}_i}}$ and that for $\varphi \in L_{\bar{c}_{\mathbf{x}_i}}$, $\Phi_{\mathbf{x},B_{\mathbf{x}}}(\varphi) = \Phi_{\mathbf{x}_i,B_{\mathbf{x}}}(\varphi)$. The same is true for $\mathbf{y}$. Hence, for such $i < \delta$, $\varphi$ and for any $p \in S_D(B_{\mathbf{y}})$,

$$\begin{aligned} p \in g(\Phi_{\mathbf{x},B_{\mathbf{x}}}(\varphi)) &\Leftrightarrow p \in g(\Phi_{\mathbf{x}_i,B_{\mathbf{x}}}(\varphi)) \\ &\Leftrightarrow p|_{B_{\mathbf{y}_i}} \in g(\Phi_{\mathbf{x}_i,B_{\mathbf{x}_i}}(\varphi)) \\ &\Leftrightarrow p|_{B_{\mathbf{y}_i}} \in g_i(\Phi_{\mathbf{x}_i,B_{\mathbf{x}_i}}(\varphi)) \\ &\Leftrightarrow p|_{B_{\mathbf{y}_i}} \in \Phi_{\mathbf{y}_i,B_{\mathbf{y}_i}}(\varphi) \\ &\Leftrightarrow p \in \Phi_{\mathbf{y}_i,B_{\mathbf{y}}}(\varphi). \qquad \blacksquare \end{aligned}$$

*Definition 5.9:* For a model $M \prec \mathfrak{C}$ and $B \subseteq \mathfrak{C}$, we let $M_{[B]}$ be $M$ with predicates for all $B$-definable subsets. More precisely, for each formula $\varphi(x_1, \ldots, x_n, \bar{b})$ over $B$, we add a predicate $R_{\varphi(\bar{x},\bar{b})}(\bar{x})$ and we interpret it as $\varphi(\mathfrak{C}^n, \bar{b}) \cap M^n$. If $B \subseteq M$, then this is definably equivalent to adding names for elements of $B$.

For a $\lambda$-decomposition $\mathbf{x}$, denote by $M_{[\mathbf{x}]}$ the structure $M_{[\bar{c}_{\mathbf{x}}+\bar{d}_{\mathbf{x}}+B_{\mathbf{x}}]}$.

Theorem 5.10: *Suppose* $\mathbf{x} \in \mathfrak{F}$. *Then* $M_{[\mathbf{x}]}$ *is homogeneous.*

*Proof.* We have to show that if $A \subseteq M$ is of cardinality $< \lambda$, and $f$ is a partial elementary map of $M_{[\mathbf{x}]}$ with domain $A$, then we can extend it to an automorphism. We may assume that $B_{\mathbf{x}} \subseteq A$ and that $f \restriction B_{\mathbf{x}} = \mathrm{id}$, as $f$ preserves all $B_{\mathbf{x}}$-definable sets. It follows that $\mathbf{x}' = (M_{\mathbf{x}}, A, \bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}, r_{\mathbf{x}})$ and $\mathbf{x}'' = (M_{\mathbf{x}}, f(A), \bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}, r_{\mathbf{x}})$ are both in $\mathfrak{F}$. By definition, $f$ extends to an elementary map $f' : (A, \bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}}) \to (f(A), \bar{d}_{\mathbf{x}}, \bar{c}_{\mathbf{x}})$, but moreover $f$ is a pseudo isomorphism. This follows easily by (†) above. Hence we are done by Proposition 5.7. $\blacksquare$

Corollary 5.11: *Clause* (9) *in Definition 3.2 holds for* $\mathfrak{F}$.

*Proof.* Suppose $\{\mathbf{x}_i | \, i < \lambda^+\}$ is a set of pairwise non-isomorphic elements of $\mathfrak{F}$ with $M_{\mathbf{x}_i} = M$ for all $i$. We may assume that for some $\beta, \gamma < \lambda$ and all $i < \lambda^+$, $\bar{c}_{\mathbf{x}_i}$ is of length $\beta$ and $\bar{d}_{\mathbf{x}_i}$ is of length $\gamma$. We may also assume, as $\lambda^{<\lambda} = \lambda$, that $B_{\mathbf{x}_i} = B$ for all $i < \lambda^+$. Let $L'$ be the common language of the structures $M_{[\mathbf{x}_i]}$ (which we may assume is constant as it only depends on the length of $\bar{c}_{\mathbf{x}_i}$, $\bar{d}_{\mathbf{x}_i}$ and $B_{\mathbf{x}_i}$). Let $D_i = D(M_{[\mathbf{x}_i]})$ in the language $L'$ (recall that $D(A)$ consists of all types of finite tuples from $A$ over $\emptyset$). The language $L'$ has size $< \lambda$, so the number of possible $D$'s is $\leq 2^{2^{|L'|}} < \lambda$, so we may assume that $D_i = D_0$ for all $i < \lambda$ (it follows that $M_{\mathbf{x}_i} \equiv M_{\mathbf{x}_0}$). Finally, we are done by Lemma 2.3, Corollary 2.4 and Theorem 5.10. ∎

*Remark 5.12:* One can also prove Corollary 5.11 directly, showing that the number of $\lambda$-decompositions in $\mathfrak{F}$ up to pseudo isomorphism is $\leq \lambda$, and then use Proposition 5.7.

Finally, we have proved that $\mathfrak{F}$ is a good family of $\lambda$-decompositions, so by Theorem 3.11 we get:

Corollary 5.13: *Conjecture 3.1, and the conclusion of Theorem 3.6 hold when $\lambda$ is measurable.*

*Problem 5.14:* To what extent can we generalize [She12, Theorem 7.3] to dependent finite diagrams? For instance, is the generic pair conjecture for dependent finite diagrams also true when $\lambda$ is weakly compact?

## References

[GL02] R. Grossberg and O. Lessmann, *Shelah's stability spectrum and homogeneity spectrum in finite diagrams*, Archive for Mathematical Logic **41** (2002), 1–31.

[Hod93] W. Hodges, *Model Theory*, Encyclopedia of Mathematics and its Applications, Vol. 42, Cambridge University Press, Cambridge, 1993.

[Kan09] A. Kanamori, *The Higher Infinite*, second edition, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2009.

[She71] S. Shelah, *Finite diagrams stable in power*, Annals of Mathematical Logic **2** (1970/1971), 69–118.

[She90] S. Shelah, *Classification Theory and the Number of Nonisomorphic Models*, second edition, Studies in Logic and the Foundations of Mathematics, Vol. 92, North-Holland Publishing Co., Amsterdam, 1990.

[She06] S. Shelah, *Dependent T and existence of limit models*, 2006, submitted.

[She11] S. Shelah, *No limit model in inaccessibles*, in *Models, Logics, and Higher-Dimensional Categories*, CRM Proceedings & Lecture Notes, Vol. 53, American Mathematical Society, Providence, RI, 2011, pp. 277–290.

[She12] S. Shelah, *Dependent dreams: recounting types*, 2012.

[She13] S. Shelah, *Dependent theories and the generic pair conjecture*, Communications in Contemporary Mathematics, 2013, to appear.