ISRAEL JOURNAL OF MATHEMATICS 112 (1999), 327–356

ON THE p-RANK OF Ext

BY

ALAN MEKLER†*
Department of Mathematics, Simon Fraser University, B.C., Canada

AND

ANDRZEJ ROSŁANOWSKI
Institute of Mathematics, The Hebrew University of Jerusalem
91904 Jerusalem, Israel
and
Mathematical Institute of Wroclaw University, 50384 Wroclaw, Poland
e-mail: roslanow@math.idbsu.edu
URL: http://math.idbsu.edu/~roslanow

AND

SAHARON SHELAH**
Institute of Mathematics, The Hebrew University of Jerusalem
91904 Jerusalem, Israel
and
Department of Mathematics, Rutgers University
New Brunswick, NJ 08854, USA
e-mail: shelah@math.huji.ac.il
URL: http://www.math.rutgers.edu/~shelah

ABSTRACT

Assume $V = L$ and λ is regular smaller than the first weakly compact cardinal. Under those circumstances and with arbitrary requirements on the structure of $\operatorname{Ext}(G, \mathbb{Z})$ (under well known limitations), we construct an abelian group G of cardinality λ such that for no $G' \subsetneq G$, $|G'| < \lambda$ is G/G' free and $\operatorname{Ext}(G, \mathbb{Z})$ realizes our requirements.

† Deceased.
* Partially supported by NSERC.
** Partially supported by the United States — Israel Binational Science Foundation. Publication No. 314.
Received December 5, 1994 and in revised form November 24, 1998

1. Introduction

In section 2 we give a principle true if $\mathbf{V} = \mathbf{L}$ which is stronger than $\diamondsuit^+$ (which was enough for building Kurepa tree); of course the proof follows proofs of Jensen for diamonds. It seems we do not use its full strength—we seem to actually need only the cases $\mathfrak{z}(M^1_\delta) = \mathrm{cf}(\delta)$ (see 2.1). The principle should be helpful for building models on λ with Σ^1_2-properties (on λ). Also there should be cases where we can prove impossibility (by playing with those cofinalities).

In the third section we apply the principle to construct abelian groups (we drop "abelian"). For a torsion free group G the group $\mathrm{Ext}\,(G, \mathbb{Z})$ is divisible and therefore its structure is determined by ranks $\nu_0(G)$, $\nu_p(G)$ (the numbers of copies of $\mathbb{Q}$ and $Z(p^\infty)$ in the decomposition of the divisible group, where p ranges over primes) and $\nu_p(G) \leq 2^{\|G\|}$. By [HHSh91], if $\mathbf{V} = \mathbf{L}$ and a group G is not of the form $G_1 \oplus G_2$, $\|G_1\| < \|G\|$ and G_2 free then $\nu_0(G) = 2^{\|G\|}$. If λ is a regular cardinal smaller than the first weakly compact cardinal, $\lambda_p \leq \lambda^+$ (for p prime), then assuming $\mathbf{V} = \mathbf{L}$ we construct an abelian torsion free group G such that $\nu_p(G) = \lambda_p$ for each p, $\nu_0(G) = \lambda^+$ and $\|G\| = \lambda$. This result can be considered as a generalization of a result of Sageev and Shelah which states the same for $\lambda = \aleph_1$ but under the assumption of CH only (see [SgSh 138]; for an alternative proof see Eklof and Huber [EH] or Theorem XII.2.10 of [EM]).

No advanced knowledge of Group Theory is required; constructions of the third section are purely combinatorial applications of the principle of the second section. On the other hand, no advance tools of Set Theory are used — if one accepts 2.1, the rest is elementary.

SET THEORETICAL NOTATION. For a cardinal κ, $\mathcal{H}(\kappa)$ is the family of sets with transitive closure of cardinality $< \kappa$. If e is a set of ordinals then $\mathrm{acc}\,(e)$ denotes the set of accumulation points of e (i.e. limits of e) and $\mathrm{otp}(e)$ is the order type of e.

GROUP THEORETICAL NOTATION. $\mathbf{P}$ is the set of prime numbers. As all the groups we shall deal with are abelian, we omit this adjective. G, H, K denote (abelian) groups, $\oplus$ denotes a direct sum. $\mathbb{Z}$ is the additive group of integers. $\mathrm{Hom}(G, H)$ is the group of homomorphisms from G to H (with the pointwise addition, i.e. $(f+g)(x) = f(x) + g(x)$). If $f \in \mathrm{Hom}(G, \mathbb{Z})$ and $p \in \mathbf{P}$ then $f/p\mathbb{Z}$ is the following member of $\mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})$:

$$(f/p\mathbb{Z})(x) = f(x)/p\mathbb{Z} \quad \text{(also called } f(x) + p\mathbb{Z}\text{)}.$$

For a group G and its subset $\{x_n : n \in I\} \subseteq G$, $\langle x_n : n \in I\rangle_G$ denotes the subgroup of G generated by $\{x_n : n \in I\}$.

HISTORY. The study of the structure of Ext has a long history already. For a review of the main results in the area we refer the reader to [EM].

The results of this paper were proved in 1986/87 and a preliminary version of the paper was ready in 1992. The illness and death of the first author[1] stopped his work on the paper. Later, the second author joined in finishing the paper.

ACKNOWLEDGEMENT: We would like to thank Paul C. Eklof for helpful comments on earlier versions of the paper.

2. A construction principle in L

THEOREM 2.1 (**V=L**): *Assume*

(A) *λ a regular uncountable not weakly compact[2] cardinal.*

(B) *$S \subseteq \lambda$ is stationary.*

(C) *$\bar{P}, \bar{Q}, \bar{R}$ are finite pairwise disjoint sequences of predicates and function symbols (so a $\bar{P}$-model M is $(A, \bar{P}^M) = (A, \dots, P_i^M, \dots)$).*

(D) *Let $\varphi = \varphi(\bar{P}, \bar{Q}, \bar{R})$ be a first order sentence.*

(E) *M^0 is a $\bar{P}$-model with universe λ.*

(F) *E is a club of λ such that $\delta \in E \Rightarrow M^0_\delta \overset{\text{def}}{=} M^0 \restriction \delta \prec M^0$.*

(G) *For $\delta \in E \cup \{\lambda\}$ let $\mathbf{k}_\delta = \{M^1_\delta \colon M^1_\delta$ a $(\bar{P}^\frown\bar{Q})$-model expanding $M^0_\delta\}$ and for $M^1_\delta \in \mathbf{k}_\delta$ let $\mathbf{k}^+_\delta(M^1_\delta) = \{M^2_\delta \colon M^2_\delta$ is a $(\bar{P}^\frown\bar{Q}^\frown\bar{R})$-model expanding M^1_δ and satisfying $\varphi = \varphi(\bar{P}, \bar{Q}, \bar{R})\}$. Lastly $\mathbf{k}^-_\delta = \{M^1_\delta \in \mathbf{k}_\delta : \mathbf{k}^+_\delta(M^1_\delta) \neq \emptyset\}$.*

Then we can find a well ordering $<^*$ of $\mathcal{H}(\lambda^+)$ of order type λ^+, a sequence $\bar{e} = \langle e_\delta : \delta < \lambda$ is a singular ordinal $\rangle$ and functions $\mathfrak{z}, \mathfrak{N}, \mathfrak{B}_\varepsilon$ (for $\varepsilon < \lambda$) such that:

(a) *The domain of the functions $\mathfrak{z}$ and $\mathfrak{N}$ is $\bigcup_{\delta \in E} \mathbf{k}^-_\delta$; superscript δ means the restriction of the function to $\mathbf{k}^-_\delta$.*

(b) *For $\delta \in E$ and $M^1_\delta \in \mathbf{k}^-_\delta$ we have: $\mathfrak{z}(M^1_\delta)$ is zero or a limit ordinal $< |\delta|^+$.*

(c) *For $\delta \in E$, $\mathfrak{B}^\delta_\varepsilon$ is a function with domain $\{M^1_\delta \in \mathbf{k}^-_\delta : \mathfrak{z}(M^1_\delta) > \varepsilon\}$, and $\mathfrak{B}^\delta = \bigcup_\varepsilon \mathfrak{B}^\delta_\varepsilon$.*

(d) *For $\delta \in E$, $M^1_\delta \in \mathbf{k}^-_\delta$ we have $\mathfrak{N}(M^1_\delta) \in \mathbf{k}^+_\delta(M^1_\delta)$.*

(e) *For $\delta \in E$ and $M^1_\delta \in \mathbf{k}^-_\delta$ we have: $\langle \mathfrak{B}^\delta_\varepsilon(M^1_\delta) : \varepsilon < \mathfrak{z}(M^1_\delta) \rangle$ is an increasing sequence of models, isomorphic to some elementary submodels of $(\mathcal{H}(\lambda^+), \in, <^*)$ but we do not require it to be an elementary chain nor continuous but we do require the following:*

   (*) $x \in \mathfrak{B}^\delta_\zeta(M^1_\delta) \setminus \mathfrak{B}^\delta_\varepsilon(M^1_\delta), y \in \mathfrak{B}^\delta_\varepsilon(M^1_\delta) \Rightarrow \mathfrak{B}^\delta_\zeta(M^1_\delta) \models y <^* x$.

---

1 Professor Alan H. Mekler died in 1992.

2 Can be weakened to the existence of Y (see the proof).

(f) *For* $M^1_\delta \in \mathbf{k}^-_\delta$, $\varepsilon < \mathfrak{z}(M^1_\delta)$ *the universe of* $\mathfrak{B}^\delta_\varepsilon(M^1_\delta)$ *is a transitive set to which* δ *belongs.*

(g) *For* $M^1_\delta \in \mathbf{k}^-_\delta$ *for* $\zeta < \mathfrak{z}(M^1_\delta)$ *we have* $\langle \mathfrak{B}^\delta_\varepsilon(M^1_\delta)\colon \varepsilon \le \zeta\rangle \in \mathfrak{B}^\delta_{\zeta+1}(M^1_\delta)$ *(remember:* $\mathfrak{z}(M^1_\delta)$*, if not zero, is a limit ordinal) and:*

$$\mathfrak{B}^\delta_{\zeta+1}(M^1_\delta) \models \text{“}\|\mathfrak{B}^\delta_\zeta(M^1_\delta)\| = \|\delta\|\text{”}.$$

(h) *If* $M^1_\delta \in \mathbf{k}^-_\delta$ *then* $\mathfrak{N}(M^1_\delta) \notin \bigcup\{\mathfrak{B}^\delta_\varepsilon(M^1_\delta)\colon \varepsilon < \mathfrak{z}(M^1_\delta)\}$ *and* $M^1_\delta \in \mathfrak{B}^\delta_0(M^1_\delta)$ *when* $\mathfrak{z}(M^1_\delta) > 0$.

(i) *If* $M^1_\lambda \in \mathbf{k}^-_\lambda$ *then for some* $M^2_\lambda \in \mathbf{k}^+_\lambda$ *we have:*

   (∗) for some regular $\sigma \le \lambda$ *(possibly* $\sigma = 0$*)* for every regular $\theta < \lambda$ *(the most interesting case is:* $0 < \sigma < \lambda \Rightarrow \theta = \sigma$*) such that* $\{\delta \in S\colon \mathrm{cf}(\delta) = \theta\}$ *is stationary and* for stationarily[3] many $\delta \in S$ *we have:*

   (α) $\mathrm{cf}(\delta) = \theta$,

   (β) $M^2_\lambda\restriction\delta \prec M^2_\lambda$,

   (γ) $\mathfrak{N}(M^1_\lambda\restriction\delta) = M^2_\lambda\restriction\delta$,

   (δ) $[\sigma < \lambda \Rightarrow \mathfrak{z}(M^1_\lambda\restriction\delta) = \sigma]$ *and* $[\sigma = \lambda \Rightarrow \mathfrak{z}(M^1_\lambda\restriction\delta) = \mathrm{cf}(\delta)]$.

(j) *Suppose* $M^1_\lambda \in \mathbf{k}_\lambda \setminus \mathbf{k}^-_\lambda$ *but* $M^1_\lambda\restriction\delta \in \mathbf{k}^-_\delta$ *for* $\delta \in E$. If M^3_λ *is an expansion of* M^1_λ *with a finite vocabulary* then[4] *for some club* $E' \subseteq E$ *we have*

$$\delta \in E' \Rightarrow \{E' \cap \delta, M^3_\lambda\restriction\delta\} \in \bigcup\{\mathfrak{B}^\delta_\varepsilon(M^1_\lambda\restriction\delta)\colon \varepsilon < \mathfrak{z}(M^1_\lambda\restriction\delta)\}.$$

Moreover, if $M \prec (\mathcal{H}(\lambda^+), \in, <^*)$ *is a proper* $<^*$*-initial segment,* $\lambda \subseteq M$, $M = \bigcup_{i<\lambda} M_i$ *where* M_i *increasing continuous,* $\|M_i\| < \lambda$ *for* $i < \lambda$

then *for some club* $E' \subseteq E$ *for every* $\delta \in E'$:

if $\mathbf{j}_\delta$ *is the Mostowski collapse of* M_δ *then*

$$\{E' \cap \delta, \mathbf{j}_\delta[M_\delta]\} \in \bigcup_{\varepsilon < \mathfrak{z}(M^1_\lambda\restriction\delta)} \mathfrak{B}^\delta_\varepsilon(M^1_\lambda\restriction\delta) \text{ and}$$

$\mathbf{j}_\delta[M_\delta]$ *is a proper initial segment of*

$$\bigcup_{\varepsilon < \mathfrak{z}(M^1_\lambda\restriction\delta)} \mathfrak{B}^\delta_\varepsilon(M^1_\lambda\restriction\delta)$$

(also the order).

(Remark: *Many* M *has a tower* $\langle B^\lambda_\varepsilon\colon \varepsilon < \varepsilon^*\rangle$ *which collapses to an initial segment.)*

---

3 We can get a club $E_0 \subseteq \lambda$ such that every $\delta \in S \cap E_0$ is OK.

4 We can also assign stationary $S' \subseteq S$ where M^3_λ was guessed but this is not what we need.

(k) $\bar{e}$ *is a square, i.e.* e_δ *a club of* δ *of order type* $< \delta$ *and*

$$\alpha \in \mathrm{acc}\,(e_\delta) \Rightarrow e_\alpha = e_\delta \cap \alpha;$$

if $\mathfrak{z}(M^1_\delta) > 0$ & $\alpha \in \mathrm{acc}\,(e_\delta)$ *then* $M^1_\alpha \stackrel{\mathrm{def}}{=} M^1_\delta \restriction \alpha \prec M^1_\delta$ *and* $\mathfrak{z}(M^1_\alpha) > 0$.

(l) *If* $\delta \in E$, $\alpha < \delta$ *and* $\mathfrak{z}(M^1_\delta) > 0$ *then* $e_\delta \cap \alpha \in \mathfrak{B}^\delta_0(M^1_\delta)$.

Remark 2.2: The interesting case is when the set S satisfies:

(α) for every $\theta = \mathrm{cf}(\theta) < \lambda$, $\{\delta \in S\colon \mathrm{cf}(\delta) = \theta\}$ is stationary,

(β) S a set of singular ordinals,

(γ) $\lambda = \mu^+ \Rightarrow S \subseteq [\mu + 1, \lambda)$ and if λ is inaccessible then S is a set of strong limit singular cardinals, and

(δ) S does not reflect.

Proof: Let $Y \subseteq \lambda$ be such that for every $\alpha < \lambda$ the set

$$\{\beta > \alpha\colon (L_\beta[Y \cap \alpha], \in, Y \cap \alpha) \equiv (L_{\lambda^+}, \in, Y)\}$$

is bounded in $|\alpha|^+$ (e.g. Y is a non-reflecting stationary subset of λ).

Let $\bar{e} = \langle \bar{e}_\delta\colon \delta < \lambda$ singular limit ordinal$\rangle$ be as defined by Jensen [Jn].

Let $<^*$ be the canonical well ordering of L.

Supose now that $\delta \in E \cup \{\lambda\}$ and $M^1_\delta \in \mathbf{k}_\delta$. If $\delta < \lambda$ and $M^1_\delta \in \mathbf{k}^-_\delta$ then we let $\mathfrak{N}(M^1_\delta)$ be the $<^*$-first member of $\mathbf{k}^+_\delta(M^1_\delta)$. Let

$$\begin{aligned} W^1_\delta(M^1_\delta) \stackrel{\mathrm{def}}{=} \{\alpha > \delta\colon\ & L_\alpha[M^1_\delta, Y \cap \delta] \cap \mathbf{k}^+_\delta(M^1_\delta) = \emptyset \text{ and} \\ & (L_\alpha[M^1_\delta, Y \cap \delta], \in, Y \cap \delta) \text{ is elementarily equivalent} \\ & \text{to } (L_{\lambda^+}, \in, Y), \text{ moreover it is isomorphic to some} \\ & \text{elementary submodel of it (demand } \delta = \sup(Y \cap \delta))\}. \end{aligned}$$

Let $W^2_\delta(M^1_\delta) = \mathrm{acc}\,(W^1_\delta(M^1_\delta))$. If $\delta < \lambda$, $W^2_\delta(M^1_\delta) = \emptyset$ then we let $\mathfrak{z}(M^1_\delta) = 0$. Otherwise

$$\mathfrak{z}(M^1_\delta) = \mathrm{cf}\Big(W^1_\delta(M^1_\delta) \cap \sup W^2_\delta(M^1_\delta)\Big)$$

(so we loose at most finitely many members of $W^1_\delta(M^1_\delta)$), and let

$$a_\delta = a_\delta[M^1_\delta] \subseteq W^1_\delta(M^1_\delta) \cap \sup W^2_\delta(M^1_\delta)$$

be unbounded of order type $\mathfrak{z}(M^1_\delta)$ and $\gamma \in a_\delta \Rightarrow a_\delta \cap \gamma \in L_\gamma[M^1_\delta, Y \cap \delta]$ (use the definition of the square in Jensen [Jn] for $L[Y \cap \delta]$ and the ordinal $\sup(W^1_\delta(M^1_\delta) \cap \sup W^2_\delta(M^1_\delta))$. Let $\mathfrak{B}^\delta_\varepsilon(M^1_\delta)$ be $(L_\alpha[M^1_\delta, Y \cap \delta], \in, <^*)$ where α is the ε-th member of a_δ.

Let us show that it works, i.e. that the clauses **(a)**–**(l)** are satisfied.

Clause (a) Directly from the choice.

Clause (b) By the use of $\sup W^2_\delta(M^1_\delta)$.

Clause (c) Directly from the choice of $\mathfrak{B}^\delta_\varepsilon$ and $\mathfrak{z}(M^1_\delta)$.

Clause (d) By the choice of $\mathfrak{N}(M^1_\delta)$.

Clause (e) Since $\mathfrak{B}^\delta_\varepsilon(M^1_\delta)$ is $L_\gamma[M^1_\delta, Y\cap\delta]$ for γ the ε-th member of $a_\delta = a_\delta[M^1_\delta]$ and $a_\delta[M^1_\delta]$ is a subset of $W^1_\delta(M^1_\delta)$ (see its definition) we are sure that $\mathfrak{B}^\delta_\varepsilon(M^1_\delta)$ is OK: they are increasing with ε as the ε-th member of $a_\delta[M^1_\delta]$ increases with ε; **(*)** is satisfied as $<^*$ is the canonical well ordering of L (or $L[Y]$, in our model not a big difference).

Clause (f) See the choice of $W^1_\delta(M^1_\delta)$.

Clause (g) It follows from $L_\alpha[M^1_\delta, Y\cap\delta] \in L_{\alpha+1}[M^1_\delta, Y\cap\delta]$, the definition of $\mathfrak{B}^\delta_\varepsilon(M^1_\delta)$ (and the presence of $a_\delta[M^1_\delta]$), etc.

Clause (h) It is a consequence of the first clause in the definition of $W^1_\delta(M^1_\delta)$: $L_\alpha[M^1_\delta, Y\cap\delta]\cap \mathbf{k}^+_\delta(M^1_\delta) = \emptyset$.

Clause (i) Let $M^2_\lambda \in \mathbf{k}^+_\lambda(M^1_\lambda)$ be $<^*$-minimal. Define $W^1_\lambda(M^1_\lambda)$ as above. It is bounded in λ^+ as $M^2_\lambda \in L_{\lambda^+}$. Then define $W^2_\lambda(M^1_\lambda)$, $a_\lambda[M^1_\lambda]$ as above and let $\gamma^* = \sup(W^2_\lambda(M^1_\lambda))$ (so it is a limit ordinal). Let $\sigma = \mathrm{cf}(\gamma^*)$. Let $\gamma < \lambda^+$ be such that

$$(L_\gamma, \in, M^1_\lambda, Y, E, S, M^2_\lambda) \prec (L_{\lambda^+}, \in, M^1_\lambda, Y, E, S, M^2_\lambda).$$

Let $L_\gamma = \bigcup_{i<\lambda} N_i$, $\|N_i\| < \lambda$, $\langle N_i : i<\lambda\rangle$ increasing continuous and such that

- $\{E, S, M^1_\lambda, M^2_\lambda, Y, W^1_\lambda(M^1_\lambda), W^2_\lambda(M^1_\lambda), \gamma^*, a[M^1_\gamma]\} \in N_0$,
- $N_i \prec (L_\gamma, \in)$,
- $\langle N_j : j \le i\rangle \in N_{i+1}$.

Let $E' = \{i<\lambda\colon N_i\cap\lambda = i\}$, $\delta \in S\cap E'$ and let $\mathbf{j}_\delta$ be the Mostowski collapse of N_δ. Note: $\mathbf{j}_\delta(<^*\restriction N_\delta) = <^*_{\mathbf{j}_\delta[N_\delta]}$, etc. Now clearly $\mathbf{j}_\delta$ maps M^2_λ to $\mathfrak{N}(M^1_\delta)$, $W^1_\lambda(M^1_\lambda)$ to $W^1_\delta(M^1_\delta)$, $W^2_\lambda(M^1_\lambda)$ to $W^2_\delta(M^1_\delta)$ and $a[M^1_\lambda]$ to $a[M^1_\delta]$. If $\sigma<\lambda$ then necessarily

$$a[M^1_\delta] = \mathbf{j}_\delta[a[M^1_\lambda]\cap N_\delta] = \{\mathbf{j}(\beta)\colon \beta\in a[M^1_\lambda]\cap N_\delta\} = \{\mathbf{j}(\beta)\colon \beta\in a[M^1_\lambda]\}$$

and if $\sigma = \lambda$ then

$$a[M^1_\delta] = \mathbf{j}_\delta[a[M^1_\lambda]\cap N_\delta] = \{\mathbf{j}_\delta(\beta)\colon \beta\in a[M^1_\lambda],\ \beta \text{ is } {<}\delta\text{-th member of } a[M^1_\lambda]\}.$$

Similarly we can check (j), (k), (l). This finishes the proof. ∎

Sh:314

Remark 2.3: More generally we can phrase parallels of the squared diamond and/or diamond$^+$.

Discussion 2.4: What is the point of this principle?

You can just read the next section to see how it works. Still let us try to explain it. Diamonds on λ have been very good in helping to build a structure M with universe λ satisfying some Π^1_2 statement (like being Souslin).

We are given $\bar{P}$. We build here by induction on $\delta \in S$ an increasing sequence of models $M^1_\delta = (\delta, \bar{P} \restriction \delta, \bar{Q}^\delta)$ carrying some induction hypothesis. We want to have at the end that there is no $\bar{R}$ such that $(\lambda, \bar{P}, \bigcup_\alpha \bar{Q}^\alpha, \bar{R}) \models \varphi$, so at stage δ we look at $\mathfrak{N}(M^1_\delta)$ as a candidate for the bad phenomena. For $\varepsilon < \mathfrak{z}(M^1_\delta)$ (if $\mathfrak{z}(M^1_\delta) = 0$ then our life is easier) in $\mathfrak{B}^\delta_{\varepsilon+1}(M^1_\delta)$ we know

$$\mathfrak{B}^\delta_{\varepsilon+1}(M^1_\delta) \models \text{“}\mathfrak{B}^\delta_\varepsilon(M^1_\delta) \text{ is a transitive set of cardinality } \delta\text{”}.$$

So we can list all elements of $\mathfrak{B}^\delta_\varepsilon(M^1_\delta)$ in $\mathfrak{B}^\delta_{\varepsilon+1}(M^1_\delta)$, i.e. we have $f \in \mathfrak{B}^\delta_{\varepsilon+1}(M^1_\delta)$, $f\colon \delta \longrightarrow \mathfrak{B}^\delta_\varepsilon(M^1_\delta)$ which is one-to-one and onto. From the outside point of view δ has small cofinality and by (k)+(l) one can find a sequence $\langle \alpha_\xi \colon \xi < \mathrm{cf}(\delta)\rangle$ cofinal in δ and such that every proper initial segment is in $\mathfrak{B}^\delta_{\varepsilon+1}(M^1_\delta)$ (even in $\mathfrak{B}^\delta_0(M^1_\delta)$, usually even in L_γ). So we have a fair chance to diagonalize over those sets to fulfill the obligation in the inductive construction of $\bar{Q}^\delta$, while “destroying” the possibility of $\mathfrak{N}(M^1_\delta)$.

But doing it for one ε does not suffice. However, if $\mathrm{cf}(\mathfrak{z}(M^1_\delta)) = \mathrm{cf}(\delta)$ then we can do better. We can find $f\colon \delta \longrightarrow \mathfrak{B}^\delta(M^1_\delta)$, one-to-one and onto and such that $(\forall \alpha < \delta)(f \restriction \alpha \in \mathfrak{B}^\delta(M^1_\delta))$ (remember: $\mathfrak{B}^\delta(M^1_\delta) = \bigcup_{\varepsilon < \mathfrak{z}(M^1_\delta)} \mathfrak{B}^\delta_\varepsilon(M^1_\delta)$). This is possible as $\langle \mathfrak{B}^\delta_\varepsilon(M^1_\delta) \colon \varepsilon \le \zeta\rangle \in \mathfrak{B}^\delta_{\zeta+1}(M^1_\delta)$, so we can choose $f_\varepsilon \in \mathfrak{B}^\delta_{\varepsilon+1}(M^1_\delta)$ as the first one-to-one mapping from δ onto $\mathfrak{B}^\delta_{\varepsilon+1}(M^1_\delta)$. So by a demand $\langle f_\varepsilon \colon \varepsilon \le \zeta\rangle \in \mathfrak{B}^\delta_{\varepsilon+1}(M^1_\delta)$. Now by an easy manipulation we can combine them (using $\langle \beta_\varepsilon : \varepsilon < \mathrm{cf}(\delta)\rangle$).

In the proof of 3.9 and 3.4, to make $\nu_p(G) = \lambda_p$, we build together with G_α also $f^{p,\zeta}$ (for $\zeta < \lambda_p$). We need that all non-trivial combinations $\sum_{\ell<n} a_\ell f^{p,\zeta^\ell} \in \mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ are not of the form $f/p\mathbb{Z}$. This could be a typical application of the diamond. But we also need that for every $f \in \mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ there will be $f' = \sum_{\ell<n} a_\ell f^{p,\zeta^\ell}$ and $f^* \in \mathrm{Hom}(G, \mathbb{Z})$ such that $f - f' = f^*/p\mathbb{Z}$. For this the normal thing is to apply $\diamondsuit^+_\lambda$ and to choose $\langle a_\ell : \ell < n\rangle$ and $f^* \restriction G_\delta$ by giving them to approximations of f. But the two demands seem to be hard to go together without what was said above.

3. Building abelian groups

One can think of Ext (G, K) as essentially the family of isomorphism types of models $(K, H, G, g, h, c)_{c \in K \cup G}$ such that (in our case K, H, G are abelian groups; we will not mention this usually, and) h is an embedding from K into H, g a homomorphism from H onto G with the range of h being the kernel of g (i.e. $0 \to K \xrightarrow{h} H \xrightarrow{g} G \to 0$ being exact) up to isomorphism over $K \cup G$. Moreover, it has a natural additive structure. So Ext $(G, K) = 0$ if and only if for any $0 \to K \xrightarrow{h} H \xrightarrow{g} G \to 0$ as above, the range of h is a direct summand of H.

We shall not define Ext $(G, \mathbb{Z})$ fully as below we shall quote theorems characterizing it in a convenient way in the relevant cases.

In this section we show how to construct a group G such that Ext $(G, \mathbb{Z})$ satisfies pre-given requirements (within well-known limitations, see below). The main tools in the construction are 2.1 and, as a kind of a single step, 3.4 below.

Definition 3.1: The quantifier $(\forall^* i < \lambda)$ means "for every large enough $i < \lambda$", so this is an abbreviation for "$(\exists j < \lambda)(\forall i \in (j, \lambda))$".

Definition 3.2: 1. For a sequence $\bar{\lambda} = \langle \lambda_\ell : \ell < n \rangle$ $(n < \omega)$ of pairwise distinct infinite regular cardinals, $I_{\bar{\lambda}}$ is the ideal on Dom $(I_{\bar{\lambda}}) = \prod_{\ell < n} \lambda_\ell$ (called its domain) such that

$$A \in I_{\bar{\lambda}} \quad \text{iff} \quad (\forall^* i_0 < \lambda_0)(\forall^* i_1 < \lambda_1) \cdots (\forall^* i_{n-1} < \lambda_{n-1})[\langle i_0, \ldots, i_{n-1} \rangle \notin A].$$

2. For any $\lambda \geq \aleph_0$ we define $\mathfrak{J}_\lambda$ as the set of ideals of the form $I_{\bar{\lambda}}$ with $\lambda = \max\{\lambda_\ell : \ell < n\}$. Let $\mathfrak{J}_{\leq \lambda} = \bigcup_{\mu \leq \lambda} \mathfrak{J}_\mu$.

LEMMA 3.3: *Suppose that $\bar{\lambda} = \langle \lambda_\ell : \ell < n \rangle$ is a sequence of pairwise distinct infinite regular cardinals and $\bar{\lambda}^{\mathrm{dec}} = \langle \lambda'_\ell : \ell < n \rangle$ is the re-enumeration of $\bar{\lambda}$ in the decreasing order. Let $\pi: \prod_{\ell < n} \lambda_\ell \longrightarrow \prod_{\ell < n} \lambda'_\ell$ be the canonical bijection (i.e. $\pi(\eta)(\ell_0) = \eta(\ell_1)$ provided $\lambda'_{\ell_0} = \lambda_{\ell_1}$). Then*

$$A \in I_{\bar{\lambda}} \quad \Rightarrow \quad \pi[A] \in I_{\bar{\lambda}^{\mathrm{dec}}}.$$

Proof: This is an iterated application of the following observation:

CLAIM 3.3.1: *Let $\lambda_0 < \lambda_1$ be regular infinite cardinals, $\psi(x, y, \bar{z})$ be a formula. Then*

$$(\forall^* i_0 < \lambda_0)(\forall^* i_1 < \lambda_1)\psi(i_0, i_1, \bar{\imath}) \Rightarrow (\forall^* i_1 < \lambda_1)(\forall^* i_0 < \lambda_0)\psi(i_0, i_1, \bar{\imath}).$$

The claim should be clear and so the lemma. ∎

THEOREM 3.4 (V=L): *Assume λ is a regular cardinal smaller than the first uncountable weakly compact cardinal. Suppose that $I_k \in \mathfrak{J}_{\leq\lambda}$, for $k < k^* < \omega$, H is a free group with the free basis*

$$\{x^k_t \colon t \in \mathrm{Dom}\,(I_k) \text{ and } k < k^*\}.$$

Further, let $p \in \mathbf{P}$ and let $f^* \in \mathrm{Hom}(H, \mathbb{Z})$ be a homomorphism such that for some $\ell_0 < k^*$

$$\{t \in \mathrm{Dom}\,(I_{\ell_0}) \colon f^*(x^{\ell_0}_t) = 0\} \in I_{\ell_0}.$$

Then *there is a free group G, $H \subseteq G$ such that $\|G\| = \lambda$, G/H is λ-free and:*

(α) *there is no $f \in \mathrm{Hom}(G, \mathbb{Z})$ extending f^*,*

(β) *if $k' < k^*$, $A \in I_{k'}$ then $G/\langle x^k_t \colon [k = k' \;\&\; t \in A] \text{ or } k \neq k'\rangle_H$ is free,*

(γ) *if a homomorphism $g^* \in \mathrm{Hom}(H, \mathbb{Z})$ is such that for every $k' < k^*$*

$$\{t \in \mathrm{Dom}\,(I_{k'}) \colon g^*(x^{k'}_t) \neq 0\} \in I_{k'}$$

and $g^+ \in \mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ extends $g^*/p\mathbb{Z}$, then there exists $g \in \mathrm{Hom}(G, \mathbb{Z})$ such that $g/p\mathbb{Z} = g^+$ and $g^* \subseteq g$,

(δ) *if $q \in \mathbf{P}$, $h \in \mathrm{Hom}(H, \mathbb{Z}/q\mathbb{Z})$ is such that for every $k' < k^*$*

$$\{t \in \mathrm{Dom}\,(I_{k'}) \colon h(x^{k'}_t) \neq 0\} \in I_{k'},$$

then h can be extended to an element of $\mathrm{Hom}(G, \mathbb{Z}/q\mathbb{Z})$.

Proof: Due to Lemma 3.3 it is enough to prove the theorem under the assumption that the ideals I_k are determined by decreasing sequences $\bar\lambda^k$ of regulars. The proof is by induction on λ. To carry out the induction we need the existence of stationary non-reflecting sets and $\diamondsuit^+_\lambda$ only. However, we will use this opportunity to show a simpler application of 2.1 and instead of the diamonds we will use our principle. The construction of 3.9, though more complicated, will be similar to the one here.

For $k < k^*$ let $I_k = I_{\bar\lambda^k}$, $\bar\lambda^k = \langle \lambda^k_\ell \colon \ell < n_k\rangle$, $\lambda^k_\ell \leq \lambda$. Thus $\mathrm{Dom}\,(I_k) = \prod_{\ell<n_k} \lambda^k_\ell$ and according to what we noted earlier we assume that the sequences $\bar\lambda^k$ are decreasing.

If $\lambda = \aleph_0$ then $\bigwedge_k n_k = 1$: this case is easy and can be concluded from [EM], pp. 362–363. However for the sake of the completeness we will sketch the construction (skipping only some technical details). The following claim gives us slightly more than needed:

CLAIM 3.4.1: *Suppose that H is a free group with basis $\{x_n\colon n \in \omega\}$, $f^* \in \mathrm{Hom}(H,\mathbb{Z})$ is a homomorphism such that $(\forall N \in \omega)(\exists n > N)(f^*(x_n) \neq 0)$. Then there is a free group $G \supseteq H$ such that $G/H \cong \mathbb{Q}$ and*

1. *there is no $f \in \mathrm{Hom}(G,\mathbb{Z})$ extending f^*,*
2. *if $A \subseteq \omega$ is infinite and $h \in \mathrm{Hom}(H,\mathbb{Z}/q\mathbb{Z})$ is such that*

$$\mathrm{Ker}(h) \supseteq \langle x_n\colon n \in A\rangle_G,$$

   then $G/\langle x_n : n \notin A\rangle_G$ is free and h can be extended to a homomorphism from G to $\mathbb{Z}/q\mathbb{Z}$,
3. *if $g^* \in \mathrm{Hom}(H,\mathbb{Z})$ is such that $(\exists N \in \omega)(\forall n > N)(g^*(x_n) = 0)$ and $g^+ \in \mathrm{Hom}(G,\mathbb{Z}/p\mathbb{Z})$ extends $g^*/p\mathbb{Z}$, then there is $g \in \mathrm{Hom}(G,\mathbb{Z})$ such that $g^* \subseteq g$ and $g/p\mathbb{Z} = g^+$.*

Proof of the Claim: Let $A_0 = \{n \in \omega\colon f^*(x_n) \neq 0\}$ and let $\{r_n\colon n \in A_0\}$ enumerate $\mathbb{Z}$. Choose inductively positive integers s_n and integers m_n such that

(a) $m_n \in \{-1,1\}$,
(b) if $n \notin A_0$ then $m_n = 1$, $s_n = (n+p)!$,
(c) if $n \in A_0$ then
$r_n + s_0 \cdots s_{n-1} m_n f^*(x_n) + s_0 \cdots s_{n-2} m_{n-1} f^*(x_{n-1}) + \cdots + s_0 m_1 f^*(x_1) + m_0 f^*(x_0) \neq 0$,
$s_n = (n+p)! \cdot |r_n + s_0 \cdots s_{n-1} m_n f^*(x_n) + s_0 \cdots s_{n-2} m_{n-1} f^*(x_{n-1}) + \cdots + s_0 m_1 f^*(x_1) + m_0 f^*(x_0)|$.

Now, let G be the group generated freely by $\{y_n\colon n \in \omega\} \cup \{x_n\colon n \in \omega\}$ except that

$(*)$ $s_n y_{n+1} = y_n + m_n x_n$.

Note that the condition $(*)$ implies that for each $k > 0$, $n \in \omega$

$(**)^k_n$ $y_n = s_n s_{n+1} \cdots s_{n+k} y_{n+k+1} - [s_n s_{n+1} \cdot \ldots \cdot s_{n+k-1} m_{n+k} x_{n+k} + s_n \cdot \ldots \cdot s_{n+k-2} m_{n+k-1} x_{n+k-1} + \ldots + s_n m_{n+1} x_{n+1} + m_n x_n]$.

0. G is freely generated by $\{y_n : n \in \omega\}$ and $G/H \cong \mathbb{Q}$.

1. There is no $f \in \mathrm{Hom}(G,\mathbb{Z})$ extending f^*.

Why? By $(**)^n_0$ the value of f at y_0 determines $f(y_{n+1})$ in the way that is excluded by the choice of s_n for $n \in A_0$ such that $f(y_0) = r_n$ (clause (c)).

2. If $A \subseteq \omega$ is infinite then $G/\langle x_n\colon n \notin A\rangle$ is free.

Why? Let $\{n_k\colon k \in \omega\} = A$ be the increasing enumeration. Let

$$G_i = \langle y_n\colon n_{i-1} < n \leq n_i\rangle_G, \quad H_i = \langle x_n\colon n_{i-1} < n < n_i\rangle_G$$

(with a convention that $n_{-1} = -1$). Then $G = \bigoplus_{i\in\omega} G_i$, $H_i = G_i \cap \langle x_n\colon n \notin A\rangle_G$ and $\langle x_n\colon n \notin A\rangle_G = \bigoplus_{i<\omega} H_i$. The groups G_i/H_i are (freely) generated by

y_{n_i}/H_i. Hence G/H is free. Extending suitable homomorphisms into $\mathbb{Z}/q\mathbb{Z}$ should be clear now.

3. If $g^* \in \mathrm{Hom}(H,\mathbb{Z})$ is such that $(\exists N)(\forall n > N)(g^*(x_n) = 0)$ and $g^+ \in \mathrm{Hom}(G,\mathbb{Z}/p\mathbb{Z})$ extends $g^*/p\mathbb{Z}$, then there is $g \in \mathrm{Hom}(G,\mathbb{Z})$ such that $g^* \subseteq g$ and $g/p\mathbb{Z} = g^+$.

Why? First note that there is at most one homomorphism $g^+ \in \mathrm{Hom}(G,\mathbb{Z}/p\mathbb{Z})$ such that $g^+ \supseteq g^*/p\mathbb{Z}$. This is because $G/H \cong \mathbb{Q}$: if $g_1^+, g_2^+ \in \mathrm{Hom}(G,\mathbb{Z}/p\mathbb{Z})$ agree on H then $g_1^+ - g_2^+ \in \mathrm{Hom}(G,\mathbb{Z}/p\mathbb{Z})$, $\mathrm{Ker}(g_1^+ - g_2^+) \supseteq H$ and hence $(g_1^+ - g_2^+)/H \in \mathrm{Hom}(G/H,\mathbb{Z}/p\mathbb{Z})$. But the only homomorphism of $\mathbb{Q}$ into $\mathbb{Z}/p\mathbb{Z}$ is the trivial one.

Hence it is enough to show that there is an extension g of g^* to a member of $\mathrm{Hom}(G,\mathbb{Z})$ (as then necessarily $g/p\mathbb{Z} = g^+$ by the uniqueness).

Now let N be such that $(\forall n > N)(g^*(x_n) = 0)$. Define

$$\begin{aligned}
g(y_n) &= 0 \text{ for } n > N,\\
g(y_N) &= -\, m_N g^*(x_N),\\
g(y_n) &= -\,[s_n s_{n+1}\cdots s_{N-1} m_N g^*(x_N) + s_n \cdots s_{N-2} m_{N-1} g^*(x_{N-1}) + \cdots +\\
&\qquad s_n m_{n+1} g^*(x_{n+1}) + m_n g^*(x_n)] \text{ for } n < N,
\end{aligned}$$

and extend it to a homomorphism from $\mathrm{Hom}(G,\mathbb{Z})$. Clearly this g satisfies $g^* \subseteq g$. This finishes the proof of the claim. ∎

Assume now that λ is smaller than the first weakly compact uncountable cardinal, $\lambda > \aleph_0$ and below λ the theorem holds.

We may think that for some $k_0 \leq k^*$ we have

$$k < k_0 \;\Rightarrow\; \lambda_0^k < \lambda \quad\text{and}\quad k_0 \leq k < k^* \;\Rightarrow\; \lambda_0^k = \lambda.$$

Of course we may assume that $k_0 < k^*$ (otherwise the inductive hypothesis applies directly).

Recall that $\ell_0 < k^*$ is such that

$$\{t \in \mathrm{Dom}\,(I_{\ell_0})\colon f^*(x_t^{\ell_0}) = 0\} \in I_{\ell_0}.$$

Let α_0 be defined as follows:

if $\ell_0 < k_0$ then $\alpha_0 = 0$,

if $k_0 \leq \ell_0 < k^*$ and $n_{\ell_0} > 1$, then $\alpha_0 < \lambda$ is (the first) such that

$$(\forall \alpha > \alpha_0)(\forall^* i_1 < \lambda_1^{\ell_0})\ldots(\forall^* i_{n_{\ell_0}-1} < \lambda_{n_{\ell_0}-1}^{\ell_0})(f^*(x^{\ell_0}_{\langle \alpha, i_1,\ldots,i_{n_{\ell_0}-1}\rangle}) \neq 0),$$

if $k_0 \le \ell_0 < k^*$ and $n_{\ell_0} = 1$, then $\alpha_0 < \lambda$ is the first ordinal such that $(\forall i_0 > \alpha_0)(f^*(x^{\ell_0}_{\langle i_0\rangle}) \neq 0)$.

We may assume that the group H has universe $\{2i\colon i < \lambda\}$. Moreover, we may have an increasing continuous sequence $\langle \gamma_\alpha : \alpha < \lambda\rangle$ of limit ordinals such that

$\boxtimes_1$ $\{x^k_t\colon t \in \mathrm{Dom}\,(I_k)\ \&\ k < k_0\} \subseteq \gamma_0$, $\alpha_0 < \gamma_0$ and $\lambda^k_1 < \gamma_0$ if $k_0 \le k < k^*$, $1 < n_k$, and

$\boxtimes_2$ $H_\alpha \stackrel{\mathrm{def}}{=} H\restriction\{2i\colon i < \gamma_\alpha\}$ is the subgroup of H generated by

$$\{x^k_t\colon k < k_0 \ \text{ or } \ [k_0 \le k < k^*\ \&\ t(0) < \gamma_\alpha]\}.$$

For $k < k^*$ we define the reduction I^{red}_k of the ideal I_k by:

if $k < k_0$ then $I^{\mathrm{red}}_k = I_k$,

if $k_0 \le k < k^*$ and $n_k > 1$ then $I^{\mathrm{red}}_k = I_{\langle \lambda^k_1,\dots,\lambda^k_{n_k-1}\rangle}$, and

if $k_0 \le k < k^*$ and $n_k = 1$ then $I^{\mathrm{red}}_k = I_{\langle \aleph_0\rangle}$.

Next we define $y^k_t[\gamma_\alpha]$ (for $k < k^*$, $t \in \mathrm{Dom}\,(I^{\mathrm{red}}_k)$) as

$$\begin{array}{ll} x^k_t & \text{if } \ k < k_0, \\ x^k_{\langle\gamma_\alpha\rangle^\frown t} & \text{if } \ k_0 \le k < k^*,\ n_k > 1, \\ x^k_{\langle\gamma_\alpha + t(0)\rangle} & \text{if } \ k_0 \le k < k^*, n_k = 1. \end{array}$$

It follows from $\boxtimes_1$, $\boxtimes_2$ and the fact that γ_α are limit ordinals that $y^k_t[\gamma_\alpha] \in H_{\alpha+1}$ for all $k < k^*$, $t \in \mathrm{Dom}\,(I^{\mathrm{red}}_k)$. The subgroup generated by these elements with some side elements will be the one to which we will apply the inductive hypothesis.

Let $E \subseteq \mathrm{acc}\,(\{\alpha < \lambda : \alpha = \gamma_\alpha\})$ be a thin enough club of λ. By our assumptions we find a stationary set $S \subseteq E$ such that

(α) for every $\theta = \mathrm{cf}(\theta) < \lambda$, $\{\delta \in S : \mathrm{cf}(\delta) = \theta\}$ is stationary,

(β) S a set of singular limit ordinals,

(γ) $\lambda = \mu^+ \Rightarrow S \subseteq [\mu+1, \lambda)$ and if λ is inaccessible then S is a set of strong limit singular cardinals, and

(δ) S does not reflect.

We will use the principle formulated in 2.1 to choose by induction on $\alpha < \lambda$ a group G_α with the universe γ_α and extending H_α. For this we have to define finite vocabularies $\bar P$, $\bar Q$, $\bar R$ and a formula φ. Thus we declare that $\bar P$ is $\langle P_0, P_1, \dots\rangle$, P_0 a unary predicate and P_1 a unary function symbol, $\bar Q = \langle Q_0, \dots\rangle$ where Q_0 is a binary function symbol, and $\langle M, \bar Q^M, \bar R^M\rangle \models \varphi$ means:

(a) $\langle M, Q^M_0\rangle$ is a group, P^M_0 is its subgroup (intension: H), and $P^M_1\restriction P^M_0 \in \mathrm{Hom}(P^M_0, \mathbb{Z})$ (intension: f^*)

[we should use some additional predicates to encode $\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}, \dots$],

(b) $\bar{R}^M$ encodes a homomorphism $f \in \mathrm{Hom}(M, \mathbb{Z})$ extending f^*.

Let functions $\mathfrak{N}^\delta, \mathfrak{B}^\delta_\varepsilon, \mathfrak{z}$ be given by 2.1 for $\bar{P}$, $\bar{Q}$, $\bar{R}$ and φ as defined above.

Now, by induction on $\alpha < \lambda$, we choose models M^1_α (i.e. groups G_α, their subgroups $P_0^{M^1_\alpha} = H_\alpha$ and homomorphisms f^+_α, f^*_α) and T_α, R_α, $h_{(g^*,g^+,C)}$ (for $(g^*,g^+,C) \in R_\alpha$), R^q_α and $h^+_{(h,C)}$ (for $(h,C) \in R^q_\alpha$) such that:

1. $\langle G_\alpha \colon \alpha < \lambda\rangle$ is an increasing continuous sequence of free groups,
2. $P_0^{M^1_\alpha} = H_\alpha \subseteq G_\alpha$, G_α is a (free) group on γ_α,
3. if $\beta < \alpha$ then G_α/G_β is free,
4. G_α/H_α is free,
5. if $\alpha \notin S$, $\alpha < \beta$ then $G_\beta/(G_\alpha + H_\beta)$ is a free group with a basis of size $\|\gamma_\beta\|$,
6. if $\alpha \in S$, $k' < k^*$, $A \in I^{\mathrm{red}}_{k'}$ then $G_{\alpha+1}/(G_\alpha + H^A_{\alpha+1})$ is free, where $H^A_{\alpha+1}$ is the group generated by all elements x^k_t such that $k < k^*$, $t \in \mathrm{Dom}\,(I_k)$, $x^k_t \in H_{\alpha+1}$ but $(\forall s \in \mathrm{Dom}\,(I^{\mathrm{red}}_{k'}))(x^k_t = y^{k'}_s[\gamma_\alpha] \ \Rightarrow\ s \in A)$,
7. $f^*_\alpha = f^*{\restriction}H_\alpha$.
8. For $\alpha \in E$ let $N^\alpha = \mathfrak{B}^\alpha(M^1_\alpha)$, $\mathfrak{z}_\alpha = \mathfrak{z}(M^1_\alpha)$, $N^\alpha_i = \mathfrak{B}^\alpha_i(M^1_\alpha)$ for $i < \mathfrak{z}_\alpha$.
   Remark: Since the group G_α/H_α is free we have $M^1_\alpha \in \mathbf{k}^-_\alpha$. If $\mathfrak{z}(M^1_\alpha) = 0$ then N^α is empty, and below $T_\alpha = R_\alpha = \emptyset$.
   Assume $\alpha \in E$ (so $\gamma_\alpha = \alpha$). Then T_α is the family of all pairs (g^*, g^+) of homomorphisms $g^* \in \mathrm{Hom}(H_\alpha, \mathbb{Z}) \cap N^\alpha$, $g^+ \in \mathrm{Hom}(G_\alpha, \mathbb{Z}/p\mathbb{Z}) \cap N^\alpha$ such that $g^*/p\mathbb{Z} \subseteq g^+$.
9. R_α is the family of all triples (g^*, g^+, C) such that:
   (a) $C \in N^\alpha$ is a non-empty closed subset of $\alpha \cap E$, $(g^*, g^+) \in T_\alpha$,
   (b) for $\beta \in C$: $\quad (g^*{\restriction}H_\beta, g^+{\restriction}G_\beta, C \cap \beta) \in R_\beta$,
   (c) for $\beta < \gamma$ in C: $h_{(g^*{\restriction}H_\beta, g^+{\restriction}G_\beta, C\cap\beta)} \subseteq h_{(g^*{\restriction}H_\gamma, g^+{\restriction}G_\gamma, C\cap\gamma)}$,
   (d) if $\beta \in [\min C, \alpha) \cap E$ then for all $k' < k^*$:
   $$\{t \in \mathrm{Dom}\,(I^{\mathrm{red}}_{k'}) \colon g^*(y^{k'}_t[\gamma_\beta]) \neq 0\} \in I^{\mathrm{red}}_{k'}.$$

   R^q_α (for $q \in \mathbf{P}$) is the family of all pairs (h, C) such that:
   (a) $C \in N^\alpha$ is a nonempty closed subset of $\alpha \cap E$, $h \in \mathrm{Hom}(H_\alpha, \mathbb{Z}/q\mathbb{Z}) \cap N^\alpha$,
   (b) for $\beta \in C$: $\quad (h{\restriction}H_\beta, C \cap \beta) \in R^q_\beta$,
   (c) for $\beta < \gamma$ in C: $h^+_{(h{\restriction}H_\beta, C\cap\beta)} \subseteq h^+_{(h{\restriction}H_\gamma, C\cap\gamma)}$,
   (d) if $\beta \in [\min C, \alpha) \cap E$ then for all $k' < k^*$:
   $$\{t \in \mathrm{Dom}\,(I^{\mathrm{red}}_{k'}) \colon h(y^{k'}_t[\gamma_\beta]) \neq 0\} \in I^{\mathrm{red}}_{k'}.$$

10. If $(g^*, g^+, C) \in R_\alpha$ *then* $h_{(g^*,g^+,C)} \in \mathrm{Hom}(G_\alpha, \mathbb{Z}) \cap N^\alpha$, $h_{(g^*,g^+,C)}/p\mathbb{Z} = g^+$, $g^* \subseteq h_{(g^*,g^+,C)}$ and

$$\beta \in C \quad \Rightarrow \quad h_{(g^* \restriction H_\beta, g^+ \restriction G_\beta, C \cap \beta)} \subseteq h_{(g^*,g^+,C)};$$

if $(h, C) \in R^q_\alpha$ then $h^+_{(h,C)} \in \mathrm{Hom}(G_\alpha, \mathbb{Z}/q\mathbb{Z})$ extends $h \cup \bigcup_{\beta \in C} h^+_{h \restriction H_\beta, C \cap \beta}$.

11. If $(g^*, g^+, C) \in R_\alpha$, $g^*_0 \in \mathrm{Hom}(H_{\alpha+1}, \mathbb{Z})$ is such that $g^* \subseteq g^*_0$ and for every $k' < k^*$:

$$\{t \in \mathrm{Dom}\,(I^{\mathrm{red}}_{k'}) \colon g^*_0(y^{k'}_t[\gamma_\alpha]) \neq 0\} \in I^{\mathrm{red}}_{k'}$$

and if $g^+_0 \in \mathrm{Hom}(G_{\alpha+1}, \mathbb{Z}/p\mathbb{Z})$ is such that $g^*_0/p\mathbb{Z} \subseteq g^+_0$, $g^+ \subseteq g^+_0$
then there is $h' \in \mathrm{Hom}(G_{\alpha+1}, \mathbb{Z})$ extending $h_{(g^*,g^+,C)}$ and such that $g^*_0 \subseteq h'$, $h'/p\mathbb{Z} = g^+_0$;
if $(h, C) \in R^q_\alpha$, $h_0 \in \mathrm{Hom}(H_{\alpha+1}, \mathbb{Z}/q\mathbb{Z})$ is such that $h \subseteq h_0$ and for every $k' < k^*$

$$\{t \in \mathrm{Dom}\,(I^{\mathrm{red}}_{k'}) \colon h_0(y^{k'}_t[\gamma_\alpha]) \neq 0\} \in I^{\mathrm{red}}_{k'}$$

then there is $h' \in \mathrm{Hom}(G_{\alpha+1}, \mathbb{Z}/q\mathbb{Z})$ extending $h^+_{(h,C)} \cup h_0$.

12. Assume that $\alpha \in S$ and

$$\mathfrak{N}(M^1_\alpha) = \langle G_\alpha, H_\alpha, f^*_\alpha, \cdots, f \rangle,$$

where $f \in \mathrm{Hom}(G_\alpha, \mathbb{Z})$.
If there is a free group G^*, $G_\alpha \cup H_{\alpha+1} \subseteq G^*$ such that: $\|G^*\| = \|\gamma_{\alpha+1}\|$, G^* satisfies (2)—(6), (11) (with G^* playing the role of $G_{\alpha+1}$) and
(∗) there is no $g' \in \mathrm{Hom}(G^*, \mathbb{Z})$ extending $f \cup f^*_{\alpha+1}$
then $G_{\alpha+1}$ satisfies (∗) too.

The limit stages of the construction are actually determined by the continuity demands of (1), (7). Concerning the requirements (2)—(5) note that (2) is preserved because of (3) at previous stages, (3) is preserved because of (2), (4) is kept due to (5) and the fact that the set S is not reflecting, and finally (5) holds at the limit because of (4) at previous stages and non-reflection of S (see e.g. Proposition IV.1.7 of [EM]). There is some uncertainty in defining $h_{(g^*,g^+,C)}$ for $(g^*, g^+, C) \in R_\alpha$ (for $\alpha \in E$). However it is possible to find a suitable $h_{(g^*,g^+,C)}$ since in the most difficult case when $\sup C < \alpha$, $\sup C \in S$ we may apply first (11) and then (5). Similarly we handle $h^+_{(h,C)}$.

If $\alpha \notin S$, then we choose a group $G_{\alpha+1} \supseteq H_{\alpha+1} \cup G_\alpha$ such that $G_{\alpha+1}/(H_{\alpha+1} + G_\alpha)$ is a free group with a basis of size $\|\gamma_{\alpha+1}\|$.

If $\alpha \in S$, then condition (12) of the construction describes $G_{\alpha+1}$. (We will see later that this condition is not empty, i.e. that there is a group G^* as there.)

Thus we have carried out the definition and we may put $G = G_\lambda = \bigcup_{\alpha<\lambda} G_\alpha$. Let us check that the G satisfies the desired properties (the main point will be the requirement (α) of the theorem).

By (2) and (3) the group G is free of cardinality λ (and the set of elements is λ) and it extends each G_α. Due to (4) the quotient G/H is λ-free.

Clause (α) of the assertion. This is a consequence of the condition (12) of the construction. Suppose that the homomorphism f^* has an extension to a homomorphism in $\mathrm{Hom}(G,\mathbb{Z})$. This means that

$$M^1 = \langle G, H, f^*, \ldots\rangle \in \mathbf{k}^-_\lambda.$$

By condition **(i)** of 2.1 we find a regular cardinal $\sigma \le \lambda$ and

$$M^2 = \langle G, H, f^*, \ldots, f\rangle \in \mathbf{k}^+_\lambda(M^1) \quad \text{and } \alpha \in S$$

such that $\mathfrak{N}(M^1_\alpha) = M^2_\alpha \prec M^2$ and

$$\begin{array}{ll} \mathrm{cf}(\alpha) = \omega_1 & \text{if } \lambda = \aleph_2, \\ \mathrm{cf}(\alpha) \ne \mathrm{cf}(\mu) & \text{if } \lambda = \mu^+,\ \mu \text{ is an uncountable limit cardinal}, \\ \mathrm{cf}(\alpha) \notin \{\mu, \mathrm{cf}(\theta)\} & \text{if } \lambda = \mu^+,\ \mu = \theta^+,\ \theta > \aleph_0 \end{array}$$

(remember (i)(α)). Look now at the stage α of the construction.

Before we continue with the proof we give a claim which helps us to apply the inductive hypothesis.

Claim 3.4.2: (1) If $R \subseteq \mathrm{Hom}(G_\alpha,\mathbb{Z}) \cup \bigcup_{q\in\mathbf{P}} \mathrm{Hom}(G_\alpha,\mathbb{Z}/q\mathbb{Z})$, $2^{\|R\|} < \|\alpha\|$ then *for every $\beta \in \alpha \setminus S$ large enough there is $x \in G_\alpha$ such that:*

(a) $(\forall h \in R)(h(x) = 0)$,

(b) *$x \in G_{\beta+1}$ is a member of a basis of $G_{\beta+1}$ over $H_{\beta+1} + G_\beta$.*

(2) *Suppose that $\mu < \alpha < \mu^+$, μ is an uncountable limit cardinal (so we are in the case $\lambda = \mu^+$), $R \subseteq \mathrm{Hom}(G_\alpha,\mathbb{Z}) \cup \bigcup_{q\in\mathbf{P}} \mathrm{Hom}(G_\alpha,\mathbb{Z}/q\mathbb{Z})$ and $\|R\| = \mu$. Then for each $\beta \in (\mu,\alpha) \setminus S$ there exist $x_j \in G_{\beta+1}$ for $j < \mathrm{cf}(\mu)$ such that:*

(a) *if $h \in R$ then the set $\{j < \mathrm{cf}(\mu)\colon h(x_j) \ne 0\}$ is bounded in $\mathrm{cf}(\mu)$,*

(b) *$\{x_j\colon j < \mathrm{cf}(\mu)\}$ can be extended to a basis of $G_{\beta+1}$ over $H_{\beta+1} + G_\beta$.*

(3) *Suppose that $\theta^+ = \mu < \alpha < \mu^+$, $R \subseteq \mathrm{Hom}(G_\alpha,\mathbb{Z}) \cup \bigcup_{q\in\mathbf{P}} \mathrm{Hom}(G_\alpha,\mathbb{Z}/q\mathbb{Z})$, $\|R\| = \mu$. Then for every $\beta \in (\mu,\alpha) \setminus S$ there exists a sequence $\langle x_{j,k}\colon j < \mu, k < \mathrm{cf}(\theta)\rangle \subseteq G_{\beta+1}$ such that*

(a) *if $h \in R$ then $\{(j,k) \in \mu \times \mathrm{cf}(\theta)\colon h(x_{j,k}) \ne 0\} \in I_{\langle\mu,\mathrm{cf}(\theta)\rangle}$,*

(b) *$\langle x_{j,k} : j < \mu, k < \mathrm{cf}(\theta)\rangle$ can be extended to a basis of $G_{\beta+1}$ over $H_{\beta+1} + G_\beta$.*

(4) *Suppose* $\aleph_1 < \alpha < \aleph_2$ *(so* $\lambda = \aleph_2$*),*

$$R \subseteq \mathrm{Hom}(G_\alpha, \mathbb{Z}) \cup \bigcup_{q\in\mathbf{P}} \mathrm{Hom}(G_\alpha, \mathbb{Z}/q\mathbb{Z}),$$

$\|R\| = \aleph_0$. *Then for each* $\beta \in (\aleph_1, \alpha) \setminus S$ *there are* $x_\ell \in G_{\beta+1}$ *(for* $\ell < \omega$*) such that*

(a) *if* $h \in R$ *then the set* $\{\ell < \omega\colon h(x_\ell) \neq 0\}$ *is finite,*

(b) $\{x_\ell\colon \ell < \omega\}$ *can be extended to a basis of* $G_{\beta+1}$ *over* $H_{\beta+1} + G_\beta$.

Proof of the Claim: (1) Let $\beta_0 < \alpha$ be such that $\|\gamma_{\beta_0}\| > 2^{\|R\|}$ (remember $\alpha \in S \subseteq E$, see the choice of S). Let $\beta \in \alpha \setminus S$, $\beta > \beta_0$. Let $\{y_\xi\colon \xi < \gamma_{\beta+1}\}$ be a free basis of $G_{\beta+1}$ over $H_{\beta+1} + G_\beta$ (exists by condition (5) of the construction). If R is finite, then considering first $\|R\|+1$ elements of the basis we find a respective point x in the group generated by them. If R is infinite, so $2^{\|R\|} = \|{}^R\mathbb{Z}\|$, then we find $\xi_0 < \xi_1 < \gamma_{\beta+1}$ such that $(\forall h \in R)(h(y_{\xi_0}) = h(y_{\xi_1}))$ and we may put $x = y_{\xi_0} - y_{\xi_1}$.

(2) We follow exactly the lines of (1), but first we have to choose an increasing sequence $\langle R_j\colon j < \mathrm{cf}(\mu)\rangle$ such that $\bigcup_{j<\mathrm{cf}(\mu)} R_j = R$, $\|R_j\| < \|R\|$ (and hence $2^{\|R_j\|} < \mu$ as μ is a limit cardinal). Now if $\beta \in (\mu, \alpha) \setminus S$ then we find $\langle x_j\colon j < \mathrm{cf}(\mu)\rangle \subseteq G_{\beta+1}$ which can be extended to a basis of $G_{\beta+1}$ over $H_{\beta+1} + G_\beta$ and such that $(\forall h \in R_j)(h(x_j) = 0)$.

(3) Similarly: first find $\langle R_{j,k}\colon j < \mu, k < \mathrm{cf}(\theta)\rangle$ such that $\|R_{j,k}\| < \theta$, the sequence $\langle \bigcup_{k<\mathrm{cf}(\theta)} R_{j,k}\colon j < \mu\rangle$ is increasing, for each $j < \mu$ the sequence $\langle R_{j,k}\colon k < \mathrm{cf}(\theta)\rangle$ is increasing and $\bigcup_{j<\mu}\bigcup_{k<\mathrm{cf}(\theta)} R_{j,k} = R$. Next follow as in (2).

(4) Represent R as an increasing (countable) union of finite sets and follow as in (2) above. The claim is proved. ∎

Now we are going back to the proof of clause (α). The following claim will finish it.

CLAIM 3.4.3: *Suppose that* $\alpha, f, \ldots$ *are as chosen earlier. Then there exists a free group* $G^* \supseteq H_{\alpha+1} \cup G_\alpha$ *such that* $\|G^*\| = \|\gamma_{\alpha+1}\|$, G^* *satisfies the conditions (2)–(6), (11) of the construction as* $G_{\alpha+1}$ *and there is no* $g' \in \mathrm{Hom}(G^*, \mathbb{Z})$ *extending* $f{\restriction}G_\alpha \cup f^*_{\alpha+1}$.

Proof of the Claim: Let $R = \{h_{(g^*,g^+,C)}\colon (g^*, g^+, C) \in R_\alpha\} \cup \{h^+_{(h,C)}\colon (h, C) \in R^q_\alpha,\ q \in \mathbf{P}\}$. By clauses (g) and (b) of 2.1 we have $\|R\| \le \|\alpha\|$ (of course R may be empty). Let $\langle \alpha_i : i < \mathrm{cf}(\alpha)\rangle$ be an increasing continuous sequence cofinal in α and disjoint from S (possible by the choice of S).

CASE A: α is a strongly limit singular cardinal (so we are in the case when λ is inaccessible).

We find an increasing sequence $\langle R_i^*: i < \mathrm{cf}(\alpha)\rangle$ such that $\bigcup_{i<\mathrm{cf}(\alpha)} R_i^* = R$ and $\|R_i^*\| < \|\alpha\|$. But in this case we have

$$(\forall i < \mathrm{cf}(\alpha))(2^{\|R_i^*\|} < \alpha).$$

So we may apply Claim 3.4.2(1) to choose by induction on $i < \mathrm{cf}(\alpha)$ an increasing sequence $\langle j_i: i < \mathrm{cf}(\alpha)\rangle \subseteq \mathrm{cf}(\alpha)$ and $x_i^{k^*} \in G_{\alpha_{j_i}+1}$ such that:

(a) $h \in R_i^* \Rightarrow h(x_i^{k^*}) = 0$,

(b) $x_i^{k^*}$ is a member of a basis of $G_{\alpha_{j_i}+1}$ over $H_{\alpha_{j_i}+1} + G_{\alpha_{j_i}}$.

Since $\langle \alpha_i: i < \mathrm{cf}(\alpha)\rangle \subseteq \alpha \setminus S$ is increasing continuous (and cofinal in α) we get that $\{x_i^{k^*}: i < \mathrm{cf}(\alpha)\}$ can be extended to a basis of G_α over H_α. Now we apply the inductive hypothesis to k^*+1, I_k^{red} (for $k < k^*$), $I_{\langle \mathrm{cf}(\alpha)\rangle}$, the group H^* generated by

$$\{y_t^k[\gamma_\alpha]: k < k^*, t \in \mathrm{Dom}\,(I_k^{\mathrm{red}})\} \cup \{x_i^{k^*}: i < \mathrm{cf}(\alpha)\}$$

and the function $(f \cup f_{\alpha+1}^*)\restriction H^*$. This gives us a group $G_0^* \supseteq H^*$. Let H' be such that $G_\alpha + H_{\alpha+1} = H^* \oplus H'$. Then put $G^* = G_0^* \oplus H'$. It satisfies the requirements of the claim: condition (3) follows from the presence of the $y_t^k[\gamma_\alpha]$'s part of H^* (remember the inductive assumption 3.4(β)), condition (4) holds due to the $x_i^{k^*}$. It follows from the fact that the α_{j_i} are cofinal in α (and from the choice of $x_i^{k^*} \in G_{\alpha_{j_i}+1}$) that (5) is satisfied. Similarly, (11) is a consequence of the choice of $x_i^{k^*}$ and the inductive hypothesis 3.4(γ, δ). Finally clause (6) follows from the inductive assumption 3.4(β)

CASE B: $\aleph_0 < \alpha < \aleph_1$ (so $\lambda = \aleph_1$).

Thus R is at most countable, so let $R = \bigcup_{\ell<\omega} R_\ell$, where R_ℓ are finite increasing with ℓ. Apply 3.4.2(1) to find an increasing sequence $\langle j_\ell: \ell < \omega\rangle \subseteq \omega$ and $x_\ell^{k^*} \in G_{\alpha_{j_\ell}+1}$ such that

(a) $h \in R_\ell \quad \Rightarrow \quad h(x_\ell^{k^*}) = 0$,

(b) $x_\ell^{k^*}$ is a member of a basis of $G_{\alpha_{j_\ell}+1}$ over $H_{\alpha_{j_\ell}+1} + G_{\alpha_{j_\ell}}$.

Proceed as in Case A (so apply the inductive hypothesis to I_k^{red} (for $k < k^*$) and $I_{\langle \aleph_0\rangle}$).

CASE C: $\alpha \in (\mu, \mu^+)$ for some limit cardinal $\mu > \aleph_0$ (so $\lambda = \mu^+$).

Then we have $\|R\| \leq \mu$ and by Claim 3.4.2(2) we can choose $x_{i,j}^{k^*} \in G_{\alpha_i+1}$ (for $i < \mathrm{cf}(\alpha)$, $j < \mathrm{cf}(\mu)$) such that:

(a) for each $h \in R$ for every $i < \mathrm{cf}(\alpha)$ the set $\{j < \mathrm{cf}(\mu)\colon h(x^{k^*}_{i,j}) \neq 0\}$ is bounded,

(b) for each $i < \mathrm{cf}(\alpha)$ the set $\{x^{k^*}_{i,j}\colon j < \mathrm{cf}(\mu)\}$ extends to a basis of G_{α_i+1} over $H_{\alpha_i+1} + G_{\alpha_i}$.

Now apply the inductive hypothesis for k^*+1, I^{red}_k (for $k < k^*$) and $I_{\langle \mathrm{cf}(\alpha),\mathrm{cf}(\mu)\rangle}$ (remember that $\mathrm{cf}(\alpha) \neq \mathrm{cf}(\mu)$ in this case).

CASE D: $\aleph_1 < \alpha < \aleph_2$ (so $\lambda = \aleph_2$ and $\mathrm{cf}(\alpha) = \omega_1$).

Write R as an increasing union $\bigcup_{i<\omega_1} R_i$ of countable sets. Using 3.4.2(4) choose $x^{k^*}_{i,\ell}$ (for $i < \omega_1$, $\ell < \omega$) such that

(a) for each $h \in R$, for every sufficiently large $i < \omega_1$ the set $\{\ell < \omega\colon h(x^{k^*}_{i,\ell}) \neq 0\}$ is finite,

(b) for each $i < \omega_1$ the set $\{x^{k^*}_{i,\ell}\colon \ell < \omega\}$ can be extended to a basis of G_{α_i+1} over $H_{\alpha_i+1} + G_{\alpha_i}$.

Proceed as above (using I^{red}_k (for $k < k^*$) and $I_{\langle \aleph_1,\aleph_0\rangle}$).

CASE E: $\alpha \in (\mu, \mu^+)$ for some cardinal μ such that $\mu = \theta^+ > \aleph_1$ (so $\lambda = \mu^+$).

Using Claim 3.4.2(3) we choose a sequence $\langle x^{k^*}_{l,j,i}\colon l < \mathrm{cf}(\alpha),\ j < \mu,\ i < \mathrm{cf}(\theta)\rangle$ such that

(a) for each $h \in R$ and for every $l < \mathrm{cf}(\alpha)$, for every $j < \mu$ large enough for every $i < \mathrm{cf}(\theta)$ large enough, $h(x^{k^*}_{l,j,i}) = 0$.

(b) for every $l < \mathrm{cf}(\alpha)$ the set $\{x^{k^*}_{l,j,i}\colon j < \mu,\ i < \mathrm{cf}(\theta)\}$ can be extended to a basis of G_{α_l+1} over $H_{\alpha_l+1} + G_{\alpha_l}$.

Now apply the inductive hypothesis to k^*+1, I^{red}_k $(k < k^*)$ and $I_{\langle \mathrm{cf}(\alpha),\mu,\mathrm{cf}(\theta)\rangle}$ (remember $\mathrm{cf}(\alpha) \notin \{\mu, \mathrm{cf}(\theta)\}$ in this case).

This completes the proof of Claim 3.4.3. ∎

It follows from the above claim that at the stage α of the construction we had a non-trivial application of the condition (12) "killing" the function f. This gives a contradiction proving the clause (α).

CLAUSE (β) OF THE ASSERTION. It follows from conditions (6) and (5) of the construction.

CLAUSE (γ) OF THE ASSERTION. Assume that g^*, g^+ are as there. Then by the clause (j) of 2.1 we have a club $C \subseteq E$ such that for each $\alpha \in C$

$$(g^* \restriction H_\alpha, g^+ \restriction G_\alpha, C \cap \alpha) \in R_\alpha.$$

Consequently we may use the functions $h_{(g^*\restriction H_\alpha, g^+\restriction G_\alpha, C\cap\alpha)}$ for $\alpha \in C$.

CLAUSE (δ) OF THE ASSERTION. Like clause (γ). ∎

Before we state the main result let us recall basic properties of Ext. First note that

> if G is an (abelian) group satisfying $G \models (\forall x)(px = 0)$ then G is a vector space over $\mathbb{Z}/p\mathbb{Z}$.

Definition 3.5: 1. For a group G and $p \in \mathbf{P}$ let $\nu_p(G)$ be the dimension of $\mathrm{Ext}_p(G, \mathbb{Z})$ as a vector space over $\mathbb{Z}/p\mathbb{Z}$ where

$$\mathrm{Ext}_p(G, \mathbb{Z}) = \{x \in \mathrm{Ext}(G, \mathbb{Z}) : \mathrm{Ext}(G, \mathbb{Z}) \models px = 0\}.$$

2. For a group G let $\nu_0(G)$ be the rank (=maximal cardinality of an independent subset) of the torsion free group $\mathrm{Ext}(G, \mathbb{Z})/\mathrm{tor}(\mathrm{Ext}(G, \mathbb{Z}))$ where for a group G':

$$\mathrm{tor}(G') = \{x \in G' \colon \text{ for some } n,\ 0 < n \in \mathbb{Z} \text{ we have } G' \models nx = 0\}.$$

LEMMA 3.6 (see Fuchs [Fu] or Eklof and Mekler [EM, Ch. XII]): *Let G be an abelian torsion-free group. Then:*

1. *For $p \in \mathbf{P}$, $\nu_p(G)$ is the dimension of the vector space*

$$\mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})/\mathrm{Hom}^-(G, \mathbb{Z}/p\mathbb{Z})$$

over the field $\mathbb{Z}/p\mathbb{Z}$, where $\mathrm{Hom}^-(G, \mathbb{Z}/p\mathbb{Z}) \stackrel{\mathrm{def}}{=} \{f/p \colon f \in \mathrm{Hom}(G, \mathbb{Z})\}$.

2. *$\mathrm{Ext}(G, \mathbb{Z})$ is a divisible group, hence characterized up to isomorphism by cardinals $\nu_0(G), \nu_p(G)$ (for $p \in \mathbf{P}$).*

THEOREM 3.7 (Hiller, Huber, Shelah [HHSh 91]] ($\mathbf{V}$=$\mathbf{L}$)): *If a group G is not free, moreover it is not $G_1 \oplus G_2$ with G_2 free, $\|G_1\| < \|G\|$, then $\nu_0(G) = 2^{\|G\|}$.*

Remark 3.8: If $G = G_1 \oplus G_2$ and G_2 is free then $\mathrm{Ext}(G, \mathbb{Z}) \cong \mathrm{Ext}(G_1, \mathbb{Z})$, so the demand is reasonable.

MAIN THEOREM 3.9 (V=L): *Suppose that λ is an uncountable regular cardinal which is smaller than the first weakly compact cardinal. Let $\lambda_p \leq \lambda^+$ for $p \in \mathbf{P}$.* Then *there exists a (torsion free) strongly λ-free group G such that $\|G\| = \lambda$, $\nu_p(G) = \lambda_p$,* and *$\nu_0(G) = \lambda^+$.*

Proof: During the proof we will use consequences of the assumption $\mathbf{V} = \mathbf{L}$ like GCH, the principle proved in 2.1 etc. without recalling the main assumption.

The construction is much easier if $\lambda_p = \lambda^+$ for some $p \in \mathbf{P}$ and $\lambda_q = 0$ for all $q \neq p$ (remember that $\mathrm{Ext}(\bigoplus_{n\in\omega} G_n, \mathbb{Z}) = \prod_{n\in\omega} \mathrm{Ext}(G_n, \mathbb{Z})$). Therefore we assume that we are done with this particular case and we assume that $\lambda_p \leq \lambda$ for all $p \in \mathbf{P}$.

We shall build a λ-free group $G = G_\lambda = \bigcup_{\alpha<\lambda} G_\alpha$ with universe λ (the sequence $\langle G_\alpha\colon \alpha < \lambda\rangle$ increasing continuous, G_α a group on an ordinal $\gamma_\alpha < \lambda$ for $\alpha < \lambda$). As witnesses for $\nu_p(G) \geq \lambda_p$ there will be also homomorphisms $f^{p,\zeta}_\lambda \in \mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ for $\zeta < \lambda_p$, $f^{p,\zeta}_\lambda = \bigcup_{\alpha<\lambda} f^{p,\zeta}_\alpha$. For the witnesses to work we need:

$(*)_1$ *If* $p \in \mathbf{P}$, $0 < n < \omega$, $\zeta^0 < \cdots < \zeta^{n-1} < \lambda_p$, $a_\ell \in \{1/p\mathbb{Z}, \ldots, (p-1)/p\mathbb{Z}\}$ (for $\ell < n$),
then $\sum_{\ell<n} a_\ell f^{p,\zeta^\ell}_\lambda \notin \mathrm{Hom}^-(G, \mathbb{Z}/p\mathbb{Z})$
(of course $\sum_{\ell<n} a_\ell f^{p,\zeta^\ell}_\lambda \in \mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})$).

This is equivalent to

$(*)_2$ there are no $p \in \mathbf{P}$, $0 < n < \omega$, $\zeta^0 < \cdots < \zeta^{n-1} < \lambda_p$, $a_\ell \in \{1/p\mathbb{Z}, \ldots, (p-1)/p\mathbb{Z}\}$ (for $\ell < n$) and $g \in \mathrm{Hom}(G, \mathbb{Z})$ such that $g/p\mathbb{Z} = \sum_{\ell<n} a_\ell f^{p,\zeta^\ell}_\lambda$.

We shall also have to take care showing that $\nu_p(G)$ is not $> \lambda_p$ (if $\lambda_p < 2^\lambda$) and for this it suffices to show that $\{f^{p,\zeta}_\lambda : \zeta < \lambda_p\}$ generates $\mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ over $\mathrm{Hom}^-(G, \mathbb{Z}/p\mathbb{Z})$. For this we shall use the $h_{(g,C)}$ (and T^p_α) below.

By induction on $\alpha < \lambda$ choose an increasing continuous sequence $\langle \gamma_\alpha\colon \alpha < \lambda\rangle \subseteq \lambda$ such that $\lambda_p < \lambda \Rightarrow \lambda_p + \omega < \gamma_0$, $\gamma_{\alpha+1} = \gamma_\alpha + \gamma_\alpha$.

For our given λ, we want to use 2.1; we use a club $E \subseteq \mathrm{acc}(\{\alpha < \lambda\colon \gamma_\alpha = \alpha\})$ thin enough. As $\mathbf{V} = \mathbf{L}$ we find a stationary set $S \subseteq E$ such that:

(α) for every $\theta = \mathrm{cf}(\theta) < \lambda$, $\{\delta \in S\colon \mathrm{cf}(\delta) = \theta\}$ is stationary,

(β) S a set of singular limit ordinals,

(γ) $\lambda = \mu^+ \Rightarrow S \subseteq [\mu + 1, \lambda)$ and if λ is inaccessible then S is a set of strong limit singular cardinals, and

(δ) S does not reflect.

Now, $\bar{P}$ is empty, $\bar{Q} = \langle Q_0, Q_1, Q_2, Q_3, Q_4 \ldots\rangle$ where Q_0 is a binary function symbol, Q_1, Q_4 are 3-place ones and Q_2, Q_3 are binary predicates and $\langle M, \bar{Q}^M, \bar{R}^M\rangle \models \varphi$ means:

(a) $\langle M, Q^M_0\rangle$ is a group, $Q^M_1(p, \zeta, \cdot)$ is a homomorphism from the group to $\mathbb{Z}/p\mathbb{Z}$ with p, ζ variable (it corresponds to $f^{p,\zeta}$); also $\mathbb{Z}, \mathbf{P}, \ldots$ are coded in some way (see below),

(b) $\bar{R}^M$ codes a counterexample to $(*)_2$, i.e. $p, n, \zeta^0, \ldots, \zeta^{n-1}, a_0, \ldots, a_{n-1}, f$, g such that $g \in \mathrm{Hom}(M, \mathbb{Z})$, $f = \sum_{\ell<n} a_\ell f^{p,\zeta^\ell} = g/p\mathbb{Z} \in \mathrm{Hom}(M, \mathbb{Z}/p\mathbb{Z})$.

Let $\mathfrak{N}^\delta, \mathfrak{B}^\delta_\varepsilon, \mathfrak{z}$ be as obtained in 2.1. Choose a sequence $\langle A_\alpha\colon \alpha < \lambda\rangle$ such that $A_\alpha \subseteq \alpha$ for $\alpha < \lambda$ and

if $A \subseteq \beta$, $\beta < \lambda$

then there is $\alpha \in (\beta, (\|\beta\| + \aleph_0)^+)$ such that $A = A_\alpha$
(remember we have GCH).

We now choose by induction on $\alpha < \lambda$ the following objects: M^1_α (i.e. a group G_α and homomorphisms $f^{p,\zeta}_\alpha$ (for $p \in \mathbf{P}, \zeta \in \lambda_p \cap \gamma_\alpha$) and $Q_2^{M^1_\alpha}$, $Q_3^{M^1_\alpha}$, $Q_4^{M^1_\alpha}, \dots$), T^p_α, R^p_α, $h_{(g,C)}$ (for $(g,C) \in R^p_\alpha$) such that:

1. G_α is a free group with universe γ_α,
2. G_α is increasing continuous in α,
3. if $\beta < \alpha, \beta \notin S$ then G_α / G_β is a free group of size $\|\gamma_\alpha\|$,
4. $f^{p,\zeta}_\alpha \in \mathrm{Hom}(G_\alpha, \mathbb{Z}/p\mathbb{Z})$, M^1_α is $(G_\alpha, f^{p,\zeta}_\alpha)$ considering $f^{p,\zeta}_\alpha(x)$ a function with three places (so $f^{p,\zeta}_\alpha$ is not defined for $\zeta \geq \gamma_\alpha$),
5. if $\beta < \alpha$ then $f^{p,\zeta}_\beta \subseteq f^{p,\zeta}_\alpha$ (so that $f^{p,\zeta}_\alpha$ is increasing continuous in α),
6. if $\alpha \notin S$ then there is a basis Y_α of $G_{\alpha+1}$ over G_α such that:
   if $p \in \mathbf{P}$, $n < \omega$, $\zeta^0 < \cdots < \zeta^{n-1} < \lambda_p \cap \gamma_\alpha$ and $a_0, \dots, a_{n-1} \in \mathbb{Z}/p\mathbb{Z}$
   then for $\|\gamma_\alpha\|$ members $y \in Y_\alpha$, $f^{p,\zeta^\ell}_{\alpha+1}(y) = a_\ell$ (for $\ell < n$) and $f^{q,\zeta}_{\alpha+1}(y) = 0$ if $q \in \mathbf{P}$, $\zeta < \lambda_q \cap \gamma_\alpha$, $(q,\zeta) \notin \{(p,\zeta^0), \dots, (p,\zeta^{n-1})\}$.
7. Let $\alpha \in E$, $N^\alpha = \mathfrak{B}^\alpha(M^1_\alpha)$, $\mathfrak{z}_\alpha = \mathfrak{z}(M^1_\alpha)$, $N^\alpha_i = \mathfrak{B}^\alpha_i(M^1_\alpha)$ for $i < \mathfrak{z}_\alpha$.
   Note: $M^1_\alpha \in \mathbf{k}^-_\alpha$ by clause (1), the universe of N^α is a transitive set (see (f) of 2.1) so ord $\cap N^\alpha$ is an ordinal greater than α (if non-zero).
   Assume $\alpha \in S$ (so $\gamma_\alpha = \alpha$) and G_α, $\langle f^{p,\zeta}_\alpha : \zeta < \lambda_p \cap \alpha\rangle$ belong to N^α. Then we choose by induction on $\zeta \in \mathrm{ord} \cap N^\alpha \setminus (\lambda_p \cap \alpha)$ the function $f^{p,\zeta}_\alpha \in N^\alpha$ as the $<^*_{N^\alpha}$-first member of $\mathrm{Hom}(G_\alpha, \mathbb{Z}/p\mathbb{Z})$ (as a vector space over the field $\mathbb{Z}/p\mathbb{Z}$) which does not depend on $\{f^{p,\xi}_\alpha : \xi < \zeta\}$.
   Let $f^{p,\zeta}_\alpha$ be defined if and only if $\zeta < \zeta(p,\alpha)$.
   Note: if $\mathfrak{z}(M^1_\alpha) = 0$ then $\zeta(p,\alpha) = \lambda_p \cap \alpha$, N^α is empty, and below $T^p_\alpha = R^p_\alpha = \emptyset$.
8. $T^p_\alpha = \{f^{p,\xi}_\alpha : \lambda_p \cap \alpha \leq \xi < \zeta(p,\alpha)\}$.
9. R^p_α is the family of all pairs $(f^{p,\xi}_\alpha, C)$ such that:
   (a) $f^{p,\xi}_\alpha \in T^p_\alpha$ and $C \in N^\alpha$ is a closed subset of $\alpha \cap E$,
   (b) for $\beta \in C$: $f^{p,\xi}_\alpha \restriction G_\beta \in T^p_\beta$ and $C \cap \beta \in N^\beta$,
   (c) for $\beta < \gamma$ in C, $h_{(f^{p,\xi}_\alpha \restriction G_\beta, C\cap\beta)} \subseteq h_{(f^{p,\xi}_\alpha \restriction G_\gamma, C \cap \gamma)}$,
10. if $(g,C) \in R^p_\alpha$ then $h_{(g,C)} \in \mathrm{Hom}(G_\alpha, \mathbb{Z}) \cap N^\alpha$ and $h_{(g,C)}/p\mathbb{Z} = g$ and $\bigcup_{\beta \in C} h_{(g\restriction G_\beta, C\cap\beta)} \subseteq h_{(g,C)}$,
11. if $(g,C) \in R^p_\alpha$, $g \subseteq g' \in \mathrm{Hom}(G_{\alpha+1}, \mathbb{Z}/p\mathbb{Z})$
    then there is $h' \in \mathrm{Hom}(G_{\alpha+1}, \mathbb{Z})$ extending $h_{(g,C)}$ and $h'/p\mathbb{Z} = g'$.
12. Assume that $\alpha \in S$, $0 < n < \omega$ and

$$\mathfrak{N}(M^1_\alpha) = \langle G_\alpha, f^{p,\zeta}_\alpha, \dots, p, n, \zeta^0, \dots, \zeta^{n-1}, a_0, \dots, a_{n-1}, f, g\rangle,$$

where $f = \sum_{\ell<n} a_\ell f^{p,\zeta^\ell}_\alpha \in \mathrm{Hom}(G_\alpha, \mathbb{Z}/p\mathbb{Z})$, $a_\ell \in \mathbb{Z}/p\mathbb{Z} \setminus \{0\}$, $f = g/p\mathbb{Z}$, $g \in \mathrm{Hom}(G_\alpha, \mathbb{Z})$.
If there is a free group H, $G_\alpha \subseteq H$ such that: for every $q \in \mathbf{P}$, $\zeta < \lambda_q \cap \gamma_\alpha$ the homomorphism $f^{q,\zeta}_\alpha$ can be extended to a member of $\mathrm{Hom}(H, \mathbb{Z}/q\mathbb{Z})$, and the quotient H/G_β is free for $\beta \in (\alpha \setminus S)$, $\|H\| = \|\gamma_{\alpha+1}\|$, and H satisfies (11) (with H playing the role of $G_{\alpha+1}$) and there are some $f^{p,\zeta}_*$ satisfying (4), (5) (with H as $G_{\alpha+1}$ and $f^{p,\zeta}_*$ as $f^{p,\zeta}_{\alpha+1}$) such that for $f' = \sum_{\ell<n} a_\ell f^{p,\zeta}_* \in \mathrm{Hom}(H, \mathbb{Z}/p\mathbb{Z})$ we have:
(*) $\quad \neg(\exists g')[g \subseteq g' \in \mathrm{Hom}(H, \mathbb{Z})\ \&\ g'/p\mathbb{Z} = f']$
then $H = G_{\alpha+1}$, $f' = \sum_{\ell<n} a_\ell f^{p,\zeta^\ell}_{\alpha+1}$ satisfy (*) too;
13. if $\bigwedge_{p\in\mathbf{P}} \lambda_p = 0$ (so (12) is an empty demand), $\alpha \in S$, and there is a group H such that $G_\alpha \subseteq H$ and for each $\beta \in \alpha \setminus S$ the quotient H/G_β is free, and $\|H\| = \|\gamma_{\alpha+1}\|$, and it satisfies (11) (with H playing the role of $G_{\alpha+1}$) and H/G_α is not free
then $G_{\alpha+1}/G_\alpha$ is not free.
14. $Q_2^{M^1_\alpha}, Q_3^{M^1_\alpha} \subseteq \gamma_\alpha \times \gamma_\alpha$ are such that for each $\beta < \gamma_\alpha$ we have:
$A_\beta = \{i < \gamma_\alpha\colon M^1_\alpha \models Q_2(\beta, i)\}$ and
if β is limit then $\{i < \gamma_\alpha : M^1_\alpha \models Q_3(\beta, i)\}$ is a cofinal subset of β of the order type $\mathrm{cf}(\beta)$.
$Q_4^{M^1_\alpha}$ is such that if $\zeta, \xi < \gamma_\alpha$, $\|\zeta\| = \|\xi\|$ then the function

$$Q_4^{M^1_\alpha}(\zeta, \xi, \cdot){\restriction}\zeta\colon \zeta \longrightarrow \xi$$

is one-to-one and onto. We require that $Q_2^{M^1_\alpha}, Q_3^{M^1_\alpha}$ and $Q_4^{M^1_\alpha}$ are increasing with α, of course.

The conditions (6), (12), (13) and (14) fully describe what happens at successor stages of the construction. Limit cases are determined by the continuity demands (2) and (5). Note that the demands (1), (3) are preserved at the limit stages as the set S is not reflecting (see e.g. [EM, Proposition IV.1.7]). Hence there is no problem to carry out the definition and let $G = G_\lambda = \bigcup_{\alpha<\lambda} G_\alpha$ (though it is not so immediate that G is not free!).

Some of the desired properties are clear:

$(\otimes)_1$ G is a group of cardinality λ (and the set of elements is λ) extending each G_α

(by (1)+(2)),

$(\otimes)_2$ G is λ-free and even strongly λ-free

(by (1) each G_α is free so G is λ-free; by (3) if $\beta \in \lambda \setminus S$, $\beta < \alpha < \lambda$ then G_α/G_β is free, so G is strongly λ-free, see e.g. [EM, pp. 87–88]).

Let $f^{p,\zeta} = f^{p,\zeta}_\lambda = \bigcup_{\alpha<\lambda} f^{p,\zeta}_\alpha$ for $p \in \mathbf{P}$, $\zeta < \lambda_p$.
$(\otimes)_3$ $f^{p,\zeta} \in \mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ (extending each $f^{p,\zeta}_\alpha$ for $\alpha < \lambda$)
(by (4)+(5)). Before checking the main properties of G let us note the following two facts (which explain the condition (14) of the construction).

CLAIM 3.9.1: *If $\delta \in E$, $\mathfrak{z}(M^1_\delta) > 0$ and $\kappa < \delta$, $\|\kappa\| < \|\delta\|$ then $\mathcal{P}(\kappa) \subseteq N^\delta$.*

Proof of the Claim: By the clauses (f), (h) of 2.1 and condition (14) of the construction we have $A_i \in N^\delta_0$ for all $i \in (\kappa, \|\kappa\|^+)$. By the choice of the sequence $\langle A_i : i < \lambda\rangle$ we are done. ∎

CLAIM 3.9.2: *If $\delta \in E$, $\mathfrak{z}(M^1_\delta) > 0$ then there is an increasing cofinal in δ sequence $\langle \beta^\delta_i : i < \mathrm{cf}(\delta)\rangle$ such that for every $i^* < \mathrm{cf}(\delta)$ we have*

$$\langle \beta^\delta_i : i < i^*\rangle \in N^\delta.$$

Proof of the Claim: By the clauses (k), (l) of 2.1 we have a club $e_\delta \subseteq \delta$ such that $\mathrm{otp}(e_\delta) < \delta$ and for each $\alpha < \delta$ the intersection $e_\delta \cap \alpha$ is in N^δ. The set $b^* \stackrel{\mathrm{def}}{=} \{i < \gamma_\delta : M^1_\delta \models Q_3(\mathrm{otp}(e_\delta), i)\}$ is an increasing cofinal subset of $\mathrm{otp}(e_\delta)$ of the order type $\mathrm{cf}(\delta) = \mathrm{cf}(\mathrm{otp}(e_\delta))$. It follows from the condition (h) of 2.1 that $b^* \in N^\delta$. But with e_δ and b^* in hand we may easily build $\langle \beta^\delta_i : i < \mathrm{cf}(\delta)\rangle$ as required. ∎

Now comes the main point:
$(\otimes)_4$ if $p \in \mathbf{P}$, $0 < n < \omega$, $\zeta^0 < \cdots < \zeta^{n-1} < \lambda_p$, $a_0, \ldots, a_{n-1} \in \mathbb{Z}/p\mathbb{Z} \setminus \{0\}$,
$f = \sum_{\ell<n} a_\ell f^{p,\zeta^\ell} \in \mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})$
then $f \notin \mathrm{Hom}^-(G, \mathbb{Z}/p\mathbb{Z})$.

Why $(\otimes)_4$?

Assume that $(\otimes)_4$ fails, so there are $p \in \mathbf{P}$, $\zeta^0 < \cdots < \zeta^{n-1}$, $a_0, \ldots, a_{n-1} \in \mathbb{Z}/p\mathbb{Z} \setminus \{0\}$ and $g \in \mathrm{Hom}(G, \mathbb{Z})$ with $f = \sum_{\ell<n} a_\ell f^{p,\zeta^\ell} = g/p\mathbb{Z}$. Let

$$M^2 = \langle G, f^{p,\zeta}, \ldots, p, n, \zeta^0, \ldots, \zeta^{n-1}, a_0, \ldots, a_{n-1}, f, g\rangle, \quad M^2_\delta = M^2 \restriction \delta.$$

By 2.1 condition (i) without loss of generality (i.e. possibly replacing

$$p, n, \zeta^0, \ldots, \zeta^{n-1}, a_0, \ldots, a_{n-1}, f, g$$

by some other

$$p^*, n^*, \zeta^0_*, \ldots, \zeta^{n^*-1}_*, a^*_0, \ldots, a^*_{n^*-1}, f^*, g^*$$

with the same properties) we have: the set

$$S^* \stackrel{\mathrm{def}}{=} \{\delta \in S : \mathfrak{N}(M^1_\delta) = M^2_\delta \text{ and } M^2_\delta \prec M^2 \text{ and } \mathfrak{z}(M^1_\delta) = \mathrm{cf}(\delta) \text{ or } \mathfrak{z}(M^1_\delta) = 0\}$$

is stationary. (Just applying (i) choose $\theta = \sigma$ when $0 < \sigma < \lambda$ and take arbitrary regular $\theta < \lambda$ in other cases.) Choose $\delta \in S^*$. Remember that $S^* \subseteq S$, so e.g. $\delta = \gamma_\delta$, $\mathrm{cf}(\delta) < \delta$ and

$\lambda = \mu^+ \Rightarrow \delta \in [\mu + 1, \lambda)$ and

if λ is inaccessible then δ is a strongly limit singular cardinal.

Let us first consider the case $\mathfrak{z}(M^1_\delta) \neq 0$ (so $\mathrm{cf}(\delta) = \mathfrak{z}(M^1_\delta)$). To show $(\otimes_4)$ we will need the following technical but useful claims.

CLAIM 3.9.3: If $R \in N^\delta$, $R \subseteq \mathrm{Hom}(G_\delta, \mathbb{Z}) \cup \bigcup_{q \in \mathbf{P}} \mathrm{Hom}(G_\delta, \mathbb{Z}/q\mathbb{Z})$, $2^{\|R\|} < \|\delta\|$ *(so $2^{2^{\|R\|}} \leq \delta$) then for every $\beta \in \delta \setminus S$ large enough there is $x \in G_\delta$ such that:*

(a) $h \in R \Rightarrow h(x) = 0$,

(b) $x \in G_{\beta+1}$, *moreover $G_\beta \oplus (\mathbb{Z}x)$ is a direct summand of $G_{\beta+1}$,*

(c) $g(x) \neq 0$.

Proof of the Claim: First assume that R is infinite, so $2^{\|R\|} = \|{}^R\mathbb{Z}\|$.

As $2^{\|R\|} < \|\delta\|$ clearly there are $x \in G_\delta \setminus \{0\}$ satisfying (a). If x satisfies (a)+(c), $x \in G_\beta$ and $\beta \in \delta \setminus S$ large enough then we can find $y \in G_{\beta+1}$ which is a member of a basis of $G_{\beta+1}$ over G_β and which satisfies (a) and $g(y) = 0$. Then the element $x + y$ satisfies (b) (and (a), (c)). So it suffices to find $x \in G_\delta$ satisfying (a)+(c). If this fails then for every $x_1, x_2 \in G_\delta$ we have

$$[\bigwedge_{h \in R} h(x_1) = h(x_2)] \;\Rightarrow\; g(x_1) = g(x_2).$$

So there is a function $F\colon {}^R\mathbb{Z} \longrightarrow \mathbb{Z}$ such that $g(x) = F(\ldots, h(x), \ldots)_{h \in R}$. Take $i < \mathfrak{z}(M^1_\delta)$ such that $R \in N^\delta_i$. Then $N^\delta_{i+1} \models \|R\| \leq \|\delta\|$ (remember 2.1(f, g)) and necessarily $N^\delta_{i+1} \models \|R\| < \|\delta\|$ (as $\|R\| < \|\delta\|$ in $\mathbf{V}$). Applying 2.1(h) we get that $Q_4^{M^1_\delta} \in N^\delta_{i+1}$ and therefore $N^\delta_{i+1} \models \|R\| = \kappa_0$, where $\kappa_0 = \|R\|$ (in $\mathbf{V}$). Let $\kappa_1 = 2^{\kappa_0}$ (so $\kappa_1 < \|\delta\|$). Then, by 3.9.1, we get $\mathcal{P}(\kappa_0) \subseteq N^\delta_{i+1}$ and $N^\delta_{i+1} \models \|{}^{\kappa_0}\mathbb{Z} \times \mathbb{Z}\| = \kappa_1$. Again by 3.9.1, we get $\mathcal{P}(\kappa_1) \subseteq N^\delta_{i+1}$. But this implies that $\mathcal{P}({}^{\kappa_0}\mathbb{Z} \times \mathbb{Z}) \subseteq N^\delta_{i+1}$ and $\mathcal{P}({}^R\mathbb{Z} \times \mathbb{Z}) \subseteq N^\delta_{i+1}$. In particular $F \in N^\delta$. Since $\{G_\delta, R\} \in N^\delta$ (G_δ by clause (h) of 2.1, R by the assumption) we conclude that $g \in N^\delta$ — a contradiction to condition (h) of 2.1. The case when R is finite is much easier. We start as above, but getting $x \in G_\delta$ with (a)+(c) we give purely algebraical arguments. The claim is proved. ∎

CLAIM 3.9.4: (1) *Suppose that $\mathrm{cf}(\delta) \neq \mathrm{cf}(\mu) < \mu < \delta < \mu^+$ (so we are in the case $\lambda = \mu^+$), $R \in N^\delta_i$, $R \subseteq \mathrm{Hom}(G_\delta, \mathbb{Z}) \cup \bigcup_{q \in \mathbf{P}} \mathrm{Hom}(G_\delta, \mathbb{Z}/q\mathbb{Z})$, $i < \mathfrak{z}(M^1_\delta)$ and $N^\delta_i \models \|R\| = \mu$. Then for each sufficiently large $\beta \in (\mu, \delta) \setminus S$ there exist $x_j \in G_{\beta+1}$ for $j < \mathrm{cf}(\mu)$ such that:*

(a) *if* $h \in R$ *then the set* $\{j < \mathrm{cf}(\mu): h(x_j) \neq 0\}$ *is bounded in* $\mathrm{cf}(\mu)$,
(b) $G_\beta \oplus \langle x_j: j < \mathrm{cf}(\mu)\rangle_{G_{\beta+1}}$ *is a direct summand of* $G_{\beta+1}$ *(and so of* G_δ*)*,
(c) $g(x_j) \neq 0$ *for all* $j < \mathrm{cf}(\mu)$.

(2) *In* (1), *if we change the assumptions to:*

$$\mathrm{cf}(\delta) = \theta^+ = \mu < \delta < \mu^+, \qquad \|R\| = \theta$$

then the assertion holds true after replacing $\mathrm{cf}(\mu)$ *by* $\mathrm{cf}(\theta)$ *(so* x_j *are being chosen for* $j < \mathrm{cf}(\theta)$*).*

(3) *Suppose that* $\mathrm{cf}(\delta) \neq \mathrm{cf}(\theta)$, $\mathrm{cf}(\delta) < \theta^+ = \mu < \delta < \mu^+$, $R \in N_i^\delta$, $R \subseteq \mathrm{Hom}(G_\delta, \mathbb{Z}) \cup \bigcup_{q \in \mathbf{P}} \mathrm{Hom}(G_\delta, \mathbb{Z}/q\mathbb{Z})$, $i < \mathfrak{z}(M_\delta^1)$ *and* $N_i^\delta \models \|R\| = \mu$. *Then for sufficiently large* $\beta \in (\mu, \delta) \setminus S$ *there exists a sequence* $\langle x_{j,k}: j < \mu, k < \mathrm{cf}(\theta)\rangle \subseteq G_{\beta+1}$ *such that*
(a) *if* $h \in R$ *then* $\{(j,k) \in \mu \times \mathrm{cf}(\theta): h(x_{j,k}) \neq 0\} \in I_{\langle \mu, \mathrm{cf}(\theta)\rangle}$,
(b) $G_\beta \oplus \langle x_{j,k}: j < \mu, k < \mathrm{cf}(\theta)\rangle_{G_{\beta+1}}$ *is a direct summand of* $G_{\beta+1}$,
(c) $g(x_{j,k}) \neq 0$ *for all* $j < \mu$, $k < \mathrm{cf}(\theta)$.

Proof of the Claim: (1) We follow exactly the lines of the proof of 3.9.3, but first we have to choose an increasing sequence $\langle R_j: j < \mathrm{cf}(\mu)\rangle \in N_i^\delta$ such that $\bigcup_{j<\mathrm{cf}(\mu)} R_j = R$, $\|R_j\| < \|R\|$ (and hence $2^{\|R_j\|} < \mu$ as μ is a limit cardinal). To find the R_j use condition (14) of the construction (and Q_3, Q_4). Then use 3.9.3 to find $\beta_0 \in (\mu, \delta)$ such that there are $x_j^* \in G_{\beta_0}$ (for $j < \mathrm{cf}(\mu)$) with

$$(\forall h \in R_j)(h(x_j^*) = 0) \ \text{ and } \ g(x_j^*) \neq 0$$

(remember that $\mathrm{cf}(\delta) \neq \mathrm{cf}(\mu)$). Now if $\beta \in (\beta_0, \delta) \setminus S$ then we find a sequence $\langle y_j : j < \mathrm{cf}(\mu)\rangle \subseteq G_{\beta+1}$ which can be extended to a basis of $G_{\beta+1}$ over G_β and such that for all $j < \mathrm{cf}(\mu)$

$$(\forall h \in R_j)(h(y_j) = 0) \ \text{ and } \ g(y_j) = 0.$$

Put $x_j = y_j + x_j^*$.
(2) Similarly (note that if $R \in N_i^\delta$, $\|R\| = \theta$ then $N_i^\delta \models \|R\| = \theta$).
(3) Similarly: first find $\langle R_{j,k}: j < \mu, k < \mathrm{cf}(\theta)\rangle \in N_i^\delta$ such that $\|R_{j,k}\| < \theta$, the sequence $\langle \bigcup_{k<\mathrm{cf}(\theta)} R_{j,k}: j < \mu\rangle$ is increasing, for each $j < \mu$ the sequence $\langle R_{j,k}: k < \mathrm{cf}(\theta)\rangle$ is increasing and $\bigcup_{j<\mu} \bigcup_{k<\mathrm{cf}(\theta)} R_{j,k} = R$. Next follow as in (1). ∎

Now we are going to finish the proof of $(\otimes)_4$ (in the case $\mathfrak{z}(M_\delta^1) \neq 0$). Note that by $(\otimes)_3$, $f^{p,\zeta^\ell} \restriction G_\delta = f_\delta^{p,\zeta^\ell}$, so we can try to apply condition (12). But condition

(12) says "help only those who can help themselves". More specifically we have to prove that there are $H, f', f^{p,\zeta}_{\delta+1}$ as required there (in particular (*)) and then by (12), $g\restriction G_{\delta+1}$ gives the desired contradiction. But this is done by the following claim.

CLAIM 3.9.5: *Suppose that δ, $g, \ldots$ are as chosen earlier. Then there exists a free group H such that $G_\delta \subseteq H$, H/G_β is free for $\beta \in \delta \setminus S$, $\|H\| = \|\delta\|$, the homomorphism g cannot be extended to a homomorphism $g' \in \mathrm{Hom}(H, \mathbb{Z})$ and*

- (α) *for every $h \in \mathrm{Hom}(G_\delta, \mathbb{Z}) \cap N^\delta$ and $h^+ \in \mathrm{Hom}(H, \mathbb{Z}/p\mathbb{Z})$ such that $h/p\mathbb{Z} \subseteq h^+$ there is $h^* \in \mathrm{Hom}(H, \mathbb{Z})$ with $h \subseteq h^*$ and $h^*/p\mathbb{Z} = h^+$;*
- (β) *if $q \in \mathbf{P}$, $\zeta < \lambda_q \cap \delta$ then $f^{q,\zeta}_\delta$ extends to an element of $\mathrm{Hom}(H, \mathbb{Z}/q\mathbb{Z})$.*

Proof of the Claim: Let us recall that $\mathfrak{z}(M^1_\delta) = \mathrm{cf}(\delta)$. For $\varepsilon < \mathfrak{z}(M^1_\delta)$ let $R_\varepsilon = \mathfrak{B}^\delta_\varepsilon(M^1_\delta) \cap \big(\mathrm{Hom}(G_\delta, \mathbb{Z}) \cup \bigcup_{q\in\mathbf{P}} \mathrm{Hom}(G_\delta, \mathbb{Z}/q\mathbb{Z})\big)$ and let $\langle f_\varepsilon : \varepsilon < \mathfrak{z}(M^1_\delta)\rangle$ be a sequence of functions such that for each $\varepsilon < \mathfrak{z}(M^1_\delta)$ we have:

$$f_\varepsilon : \delta \xrightarrow{\text{onto}} R_\varepsilon \quad \text{and} \quad \langle f_\zeta : \zeta \leq \varepsilon\rangle \in N^\delta_{\varepsilon+1}$$

(see clauses (e) and (g) of 2.1). Let $\langle \beta^\delta_i : i < \mathrm{cf}(\delta)\rangle$ be an increasing sequence with limit δ such that $\beta^\delta_0 > \mathrm{cf}(\delta)$ and $\langle \beta^\delta_i : i < i^*\rangle \in N^\delta$ for all $i^* < \mathrm{cf}(\delta)$ (see Claim 3.9.2). Finally for $i < \mathrm{cf}(\delta)$ let

$$R^*_i \stackrel{\text{def}}{=} \{f_\varepsilon(\zeta) : \zeta < \beta^\delta_i \ \&\ \varepsilon \leq i\}.$$

Then R^*_i are increasing with i and

$$\bigcup_{i<\mathrm{cf}(\delta)} R^*_i = \bigcup_{\varepsilon<\mathfrak{z}(M^1_\delta)} R_\varepsilon$$

and for each $i^* < \mathrm{cf}(\delta)$ the sequence $\langle R^*_i : i < i^*\rangle$ belongs to $N^\delta_{\zeta_{i^*}}$ (for some $\zeta_{i^*} < \mathfrak{z}(M^1_\delta)$). Moreover $N^\delta_{\zeta_{i^*}} \models \|R^*_i\| \leq \|\beta^\delta_i\|$ for each $i < i^*$.

Let $\langle \alpha^\delta_i : i < \mathrm{cf}(\delta)\rangle$ be an increasing continuous sequence cofinal in δ and disjoint from S (possible by the choice of S).

CASE A: δ is a strongly limit singular cardinal.

In this case we have

$$(\forall i < \mathrm{cf}(\delta))(2^{\|R^*_i\|} < \delta).$$

Thus we may apply Claim 3.9.3 and choose by induction on $i < \mathrm{cf}(\delta)$ an increasing sequence $\langle j_i : i < \mathrm{cf}(\delta)\rangle \subseteq \mathrm{cf}(\delta)$ and $x^\delta_i \in G_{\alpha^\delta_{j_i}+1}$ such that:

- (a) $h \in R^*_i \Rightarrow h(x^\delta_i) = 0$;

(b) $G_{\alpha^\delta_{j_i}} \oplus (\mathbb{Z}x^\delta_i)$ is a direct summand of G_δ;

(c) $g(x^\delta_i) \neq 0$.

Since $\langle \alpha^\delta_i \colon i < \mathrm{cf}(\delta)\rangle \subseteq \delta \setminus S$ is increasing continuous (and cofinal in δ) we get that the subgroup $H_\delta = \langle x^\delta_i \colon i < \mathrm{cf}(\delta)\rangle_{G_\delta}$ is a direct summand of G_δ, say $G_\delta = H_\delta \oplus H^*_\delta$ (and $\{x^\delta_i \colon i < \mathrm{cf}(\delta)\}$ is a free basis of H_δ). Let $I = \{A \subseteq \mathrm{cf}(\delta) \colon A$ is bounded$\}$ and apply 3.4 for $\mathrm{cf}(\delta)$, g, p, I and H_δ and get the respective free group $H' \supseteq H_\delta$ ($H' \cap G_\delta = H_\delta$). We claim that the group $H = H' \oplus H^*_\delta$ is as required. For this, first note that if $\beta \in \delta \setminus S$, $\alpha^\delta_{j_{i_0}} > \beta$, $A = [0, j_{i_0})$ then $H'/\langle x^\delta_i \colon i \in A\rangle$ is free and hence $H/G_{\alpha^\delta_{j_{i_0}}}$ is free. But $G_{\alpha^\delta_{j_{i_0}}}/G_\beta$ is free so we conclude that H/G_β is free.

As g cannot be extended to a member of $\mathrm{Hom}(H', \mathbb{Z})$ it has no extension in $\mathrm{Hom}(H, \mathbb{Z})$. Suppose now that $h \in \mathrm{Hom}(G_\delta, \mathbb{Z}) \cap N^\delta$, so $h \in R^*_{i_0}$ for some $i_0 < \mathrm{cf}(\delta)$. Let $h^+ \in \mathrm{Hom}(H, \mathbb{Z}/p\mathbb{Z})$ extend $h/p\mathbb{Z}$. Since for all $i \geq i_0$ we have $h(x^\delta_i) = 0$, we may apply clause (γ) of 3.4 to get a suitable lifting $h^* \in \mathrm{Hom}(H, \mathbb{Z})$ of h^+. Similarly, we use 3.4(δ) to show that $f^{q,\xi}_\delta$ can be extended onto H (remember $f^{q,\xi}_\delta \in N^\delta$).

Case B: $\delta \in (\mu, \mu^+)$ for some cardinal μ such that $\mathrm{cf}(\mu) = \mathrm{cf}(\delta) < \mu$.

By condition (14) of the construction and the use of Q_3, Q_4 we have that, letting $\alpha = \mathrm{cf}(\delta)$, for each $i < \mathrm{cf}(\delta)$

$$N^\delta_{\zeta_{i+1}} \models \text{“ } \|\beta^\delta_i\| = \|\mu\| \ \& \ \mathrm{cf}(\mu) = \alpha \text{ ”}.$$

This allows us to build R^{**}_i such that

$$i < j < \mathrm{cf}(\delta) \quad \Rightarrow \quad R^{**}_i \subseteq R^{**}_j \in N^\delta,$$

$$\bigcup_{i<\mathrm{cf}(\delta)} R^{**}_i = \bigcup_{i<\mathrm{cf}(\delta)} R^*_i \quad \text{and} \quad \|R^{**}_i\| < \mu.$$

Now we can continue as in the previous case.

Case C: $\delta \in (\mu, \mu^+)$ for some cardinal number μ such that $\mathrm{cf}(\delta) \neq \mathrm{cf}(\mu) < \mu$.

By Claim 3.9.4(1) we can choose $x^\delta_{i,j}$ (for $i < \mathrm{cf}(\delta)$, $j < \mathrm{cf}(\mu)$) such that:

(a) for each $h \in R^*_i$, for every $j < \mathrm{cf}(\mu)$ large enough $h(x^\delta_{i,j}) = 0$;

(b) $\{x^\delta_{i,j} \colon i < \mathrm{cf}(\delta), j < \mathrm{cf}(\mu)\}$ is a free basis of a direct summand of G_δ, moreover for some increasing sequence $\langle j_i \colon i < \mathrm{cf}(\delta)\rangle \subseteq \mathrm{cf}(\delta)$, for each $i^* < \mathrm{cf}(\delta)$, the family $\{x^\delta_{i,j} \colon i^* \leq i < \mathrm{cf}(\delta), j < \mathrm{cf}(\mu)\}$ is a free basis of a subgroup $H \subseteq G_\delta$ such that $G_{\alpha^\delta_{j_{i^*}}} \oplus H$ is a direct summand of G_δ;

(c) $g(x^\delta_{i,j}) \neq 0$.

Let

$$I = I_{\langle \mathrm{cf}(\delta), \mathrm{cf}(\mu)\rangle} = \{A \subseteq \mathrm{cf}(\delta) \times \mathrm{cf}(\mu) \colon (\forall^* i < \mathrm{cf}(\delta))(\forall^* j < \mathrm{cf}(\mu))((i,j) \notin A)\}.$$

Again apply 3.4 (with λ there standing for $\mathrm{cf}(\delta) + \mathrm{cf}(\mu)$).

Case D: $\delta \in (\mu, \mu^+)$ for some cardinal number μ such that $\mathrm{cf}(\delta) = \mu = \mathrm{cf}(\mu)$ is inaccessible.

Similar to Case B.

Case E: $\delta \in (\mu, \mu^+)$ for some cardinal number μ such that $\mu = \theta^+ = \mathrm{cf}(\delta)$.

First find an increasing sequence $\langle R_i^{**} \colon i < \theta^+\rangle$ such that $R_i^{**} \in N^\delta$, $\bigcup_{i<\theta^+} R_i^{**} = \bigcup_{i<\mathrm{cf}(\delta)} R_i^*$ and $\|R_i^{**}\| \leq \theta$. Then apply Claim 3.9.4(2) to choose a sequence $\langle x_{i,j}^\delta \colon i < \mathrm{cf}(\delta), j < \mathrm{cf}(\theta)\rangle$ similarly as in case C.

Case F: $\delta \in (\mu, \mu^+)$ for some cardinal μ such that $\mathrm{cf}(\delta) < \mu = \theta^+$, $\mathrm{cf}(\delta) \neq \mathrm{cf}(\theta)$.

Using Claim 3.9.4(3) we choose an increasing sequence $\langle j_i \colon i < \mathrm{cf}(\delta)\rangle \subseteq \mathrm{cf}(\delta)$ and a sequence $\langle x_{i,j,k}^\delta \colon i < \mathrm{cf}(\delta),\ j < \mu,\ k < \mathrm{cf}(\theta)\rangle$ such that

(a) for each $h \in R_i^*$, for every $j < \mu$ large enough for every $\kappa < \mathrm{cf}(\theta)$ large enough, $h(x_{i,j,k}^\delta) = 0$;
(b) $\{x_{i,j,k}^\delta \colon i < \mathrm{cf}(\delta),\ j < \mu,\ \kappa < \mathrm{cf}(\theta)\}$ is a free basis of a direct summand of G_δ; moreover for each $i^* < \mathrm{cf}(\delta)$ the set
$$\{x_{i,j,k}^\delta \colon i^* \leq i < \mathrm{cf}(\delta), j < \mu,\ k < \mathrm{cf}(\theta)\}$$
is a free basis of a subgroup $H \subseteq G_\delta$ such that $G_{\alpha^\delta_{j_{i^*}}} \oplus H$ is a direct summand of G_δ;
(c) $g(x_{i,j,k}^\delta) \neq 0$.

Let $I = I_{\langle \mathrm{cf}(\delta), \mu, \mathrm{cf}(\theta)\rangle}$ and apply 3.4.

Case G: $\delta \in (\mu, \mu^+)$ for some cardinal μ such that $\mu = \theta^+$, $\mathrm{cf}(\delta) = \mathrm{cf}(\theta)$.

This is similar to case F though we have to modify the application of 3.4. First we choose increasing sequences $\langle R_{i,j}^{**} \colon j < \mu\rangle \in N^\delta$ (for $i < \mathrm{cf}(\delta)$) such that

$$\|R_{i,j}^{**}\| < \mu, \qquad \bigcup_{j<\mu} R_{i,j}^{**} = R_i^* \quad \text{and}$$

$$\langle\langle R_{i,j}^{**} \colon j < \mu\rangle \colon i < i^*\rangle \in N^\delta \qquad \text{for each } i^* < \mathrm{cf}(\delta).$$

Then we choose increasing sequences $\langle R_{i,j,k}^{***} \colon k < \mathrm{cf}(\theta)\rangle$ (for $i < \mathrm{cf}(\delta) = \mathrm{cf}(\theta)$, $j < \mu$) such that

$$\|R_{i,j,k}^{***}\| < \theta, \qquad \bigcup_{k<\mathrm{cf}(\theta)} R_{i,j,k}^{***} = R_{i,j}^{**} \quad \text{and}$$

$$\langle\langle R^{***}_{i,j,k}\colon j<\mu, k<\mathrm{cf}(\theta)\rangle\colon i<i^*\rangle\in N^\delta.$$

Now, for $\ell<\mathrm{cf}(\theta)=\mathrm{cf}(\delta)$, $j<\mu$ put $R^+_{j,\ell}=\bigcup_{i,k<\ell}R^{***}_{i,j,k}$. Note that $R^+_{j,\ell}\in N^\delta$ and $\|R^+_{j,\ell}\|<\theta$. Moreover if $h\in\bigcup_{i<\mathrm{cf}(\delta)}R^*_i$ then $(\forall^* j<\mu)(\forall^*\ell<\mathrm{cf}(\delta))$ $(h\in R^+_{j,\ell})$. Next, as in the proof of 3.9.4 we choose $x^*_{j,\ell}$, $y_{j,\ell}$ such that

$(\forall h\in R^+_{j,\ell})(h(x^*_{j,\ell})=h(y_{j,\ell})=0)$, $g(x^*_{j,\ell})\neq 0$, $g(y_{j,\ell})=0$,

if $\rho_{j,\ell}\overset{\mathrm{def}}{=}\min\{\alpha^\delta_{\ell_0}\colon x^*_{j,\ell}\in G_{\alpha^\delta_{\ell_0}}\}$ then $\ell<\ell_0$ and

$\{y_{j,\ell}\colon \rho_{j,\ell}=\beta\}\subseteq G_{\beta+1}$ can be extended to a basis of $G_{\beta+1}$ over G_β.

Then we put $x_{j,\ell}=x^*_{j,\ell}+y_{j,\ell}$ (for $j<\mu$, $\ell<\mathrm{cf}(\delta)$) and we apply 3.4 as earlier.

CASE H: $\delta\in(\mu,\mu^+)$ for some inaccessible cardinal μ such that $\mathrm{cf}(\delta)<\mathrm{cf}(\mu)=\mu$.

This is similar to case C.

This completes the proof of Claim 3.9.5. ∎

The case $\mathfrak{z}(M^1_\delta)=0$ is much easier and can be done similarly. We do not have N^δ and we have to take care of extending homomorphisms $f^{q,\zeta}_\delta$ only. We basically follow the lines of the previous case, but proving the suitable variants of 3.9.3, 3.9.4 instead of the fact that $g\notin N^\delta$ we use clause (6) of the inductive construction.

This completes the proof of $(\otimes)_4$.

To finish the proof of the theorem we have to show

$(\otimes)_5$ if $p\in\mathbf{P}$, $f\in\mathrm{Hom}(G,\mathbb{Z}/p\mathbb{Z})$ then there are $n<\omega$, $\zeta^0,\dots,\zeta^{n-1}<\lambda_p$, $a_0,\dots,a_{n-1}\in\mathbb{Z}/p\mathbb{Z}$ such that $f-\sum_{\ell<n}a_\ell f^{p,\zeta^\ell}\in\mathrm{Hom}^-(G,\mathbb{Z}/p\mathbb{Z})$ (i.e. the difference can be lifted to a homomorphism to $\mathbb{Z}$).

For this we inductively define a sequence $\langle f_\xi\colon\xi<\xi(*)\rangle\subseteq\mathrm{Hom}(G,\mathbb{Z}/p\mathbb{Z})$ by:

f_ξ is the $<^*$-first member of the vectór space $\mathrm{Hom}(G,\mathbb{Z}/p\mathbb{Z})$ over the field $\mathbb{Z}/p\mathbb{Z}$ which does not depend on

$$\{f^{p,\zeta}\colon\zeta<\lambda_p\}\cup\{f_\zeta\colon\zeta<\xi\}.$$

(So $\xi(*)$ is the maximal length of a sequence with the property stated above.) It is enough to show that all the homomorphisms f_ξ (for $\xi<\xi(*)$) can be lifted to homomorphisms $f^*_\xi\in\mathrm{Hom}(G,\mathbb{Z})$. But by $(\otimes)_4$ we know that $M^1_\lambda\in\mathbf{k}_\lambda\setminus\mathbf{k}^-_\lambda$, so we may apply (j). Thus we have a club $C\subseteq E$ such that for each $\delta\in C$:

$$\{C\cap\delta,\langle f_\zeta{\restriction}\delta\colon\zeta<\delta\rangle\}\in N^\delta.$$

Applying the "moreover" part of (j) of 2.1 we may make use of conditions (8), (9) of the construction. (Remember that in (7) the sequence $\langle f^{p,\xi}_\alpha\colon\lambda_p\cap\alpha<\xi<\zeta(p,\alpha)\rangle$ has the same definition as our sequence $\langle f_\xi:\xi<\xi(*)\rangle$.) ∎

Remark 3.10: The main theorem 3.9 can be proved for all regular cardinals which are not weakly compact. This requires some changes in the construction (and 3.4).

References

[EM] P. C. Eklof and A. H. Mekler, *Almost Free Modules: Set-theoretic Methods*, North-Holland Mathematical Library, 1990.

[EH] P. C. Eklof and M. Huber, *On the p-ranks of* Ext (A, G), *assuming CH,* in *Abelian Group Theory (Oberwolfach, 1981)* (R. Gobel and A. E. Walker, eds.), Lecture Notes in Mathematics **874**, Springer-Verlag, Berlin, 1981, pp. 93–108.

[Fu] L. Fuchs, *Infinite Abelian Groups,* Academic Press, New York, 1970, 1973.

[HHSh 91] H. L. Hiller, M. Huber and S. Shelah, *The structure of Ext*$(A, \mathbb{Z})$ *and* $V = L$, Mathematische Zeitschrift **162** (1978), 39–50.

[Jn] R. B. Jensen, *The fine structure of the constructible hierarchy,* Annals of Mathematical Logic **4** (1972), 229–308.

[SgSh 138] G. Sageev and S. Shelah, *On the structure of* $\mathrm{Ext}(A, \mathbb{Z})$ *in* ZFC^+, The Journal of Symbolic Logic **50** (1985), 302–315.

[SgSh 148] G. Sageev and S. Shelah, *Weak compactness and the structure of* Ext $(A, \mathbb{Z})$, in *Abelian Group Theory (Oberwolfach, 1981)* (R. Gobel and A. E. Walker, eds.), Lecture Notes in Mathematics **874**, Springer-Verlag, Berlin, 1981, pp. 87–92.