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The number of infinite substructures

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Abstract

Given a relational structure M and a cardinal $\lambda < |M|$, let ϕ_λ denote the number of isomorphism types of substructures of M of size λ. It is shown that if $\mu < \lambda$ are cardinals, and $|M|$ is sufficiently larger than λ, then $\phi_\mu \leqslant \phi_\lambda$. A description is also given for structures with few substructures of given infinite cardinality.

Introduction

In this paper we shall consider relational structures, that is, structures over languages containing relation symbols but no function or constant symbols. The word 'structure' will always mean 'relational structure'. Suppose that M is a structure and that λ is a cardinal, and let ϕ_λ denote the number of isomorphism types of substructures of M of cardinality λ. It is well-known that if M is an infinite structure in a finite language and $n < m \leqslant \aleph_0$, then $\phi_n \leqslant \phi_m$. A proof of this using Ramsey's theorem is due to Pouzet and can be found in [**3**] and in more detail in [**4**], and a linear algebraic proof can be obtained by an easy adaptation of an argument in Cameron [**1**] and a different linear algebraic proof appears in [**5**]. In this paper we extend this result to infinite cardinals, under certain set-theoretic assumptions, thereby partially answering a question of Pouzet. We also obtain a structure theorem (Theorem 4·5) for structures M for which ϕ_λ is small. The work was stimulated by results of Pouzet and Woodrow [**6**] on relational structures M with ϕ_λ finite for infinite λ. We thank them for communicating these results, which preceded our Theorem 4·5. We remark that results of this kind will not in general hold over languages with function symbols.

The conjectured answer about the behaviour of the values of ϕ_λ, is that if M is a structure of cardinality κ and $\mu < \lambda < \kappa$, then $\phi_\mu \leqslant \phi_\lambda$. The example $(\kappa, <)$ shows that the conjecture cannot be extended all the way to ϕ_κ. In this structure, $\phi_n = 1$ for all $n < \omega$, $\phi_\lambda = \lambda^+$ for $\lambda < \kappa$, and $\phi_\kappa = 1$. This counter-example leads naturally to a subsidiary conjecture that any other counter-example will be similar to this one. The main conjecture is true, if we assume a weak form of GCH, which we call WGCH;

namely that for all $\mu < \lambda$, $2^\mu < 2^\lambda$. We will not prove the full theorem here but instead will concentrate on the case of regular cardinals. The other case will be dealt with in a paper by Mekler and Shelah which is in preparation. The case of singular cardinals is more complicated. The following two theorems are typical of the results we prove here.

Theorem 4·2. *Assume* GCH *holds. Let M be a structure in a relational language. If $\rho < \mu \leqslant \lambda < |M|$, where ρ is a regular cardinal, then $\phi_\rho \leqslant \phi_\mu$.*

Theorem 4·4. *Assume* WGCH *holds. Let M be a structure in a relational language. If $\mu < \lambda < |M|$, where λ is a regular cardinal, then $\phi_\mu \leqslant \phi_\lambda$.*

Given infinite cardinals $\mu < \lambda$ and a sufficiently large structure M, our basic strategy to show $\phi_\mu \leqslant \phi_\lambda$ is as follows: for each substructure $A \subseteq M$ with $|A| = \mu$, find a set $I \subseteq M$ of indiscernibles over A with $|I| = \lambda$, and form the substructure $A \cup I$ of size λ. It is possible that this will produce ϕ_μ non-isomorphic substructures of cardinality λ. If it does not, then we will analyse the failure and show how to produce the necessary number of substructures. The method of the proof in Theorem 4·5 is analogous.

There remains the difficult question of the existence of the indiscernibles. In the first version of this paper the first two authors considered structures in a language of bounded arity and used the Erdös–Rado theorem to produce indiscernibles. This approach restricted both the languages which could be used and the cardinals to which the argument could be applied. Shelah supplied the argument which shows, under some set-theoretic hypotheses, that if the desired indiscernibles do not occur then $\phi_\mu \geqslant 2^\mu$ for all $\mu \leqslant \lambda$.

Before beginning the proof we need to review some facts and definitions. In a structure M a *μ-substructure* is a subset of M of cardinality μ endowed with the restriction of the relations. A *μ-set* will be a set of size μ. Let $A \subseteq M$. Suppose Δ is a set of quantifier-free formulae. An ordered set $(J, <)$ is *Δ-indiscernible* over a set A if for any formula $\phi(y_1, \ldots, y_m, x_1, \ldots, x_n) \in \Delta$ and $a_1, \ldots, a_m \in A$ and $j_1 < \ldots < j_n$ and $i_1 < \ldots < i_n \in J$, $\phi(a_1, \ldots, a_m, j_1, \ldots, j_n)$ holds if and only if $\phi(a_1, \ldots, a_m, i_1, \ldots, i_n)$ holds. A set J is *totally Δ-indiscernible* over A if and only if it is Δ-indiscernible over A under all orderings of J. When Δ is the set of all quantifier-free formulas we shall write 'indiscernible' in place of 'Δ-indiscernible'. The *quantifier-free type* of a tuple $\bar{a}$ over A is the set of quantifier-free formulae with parameters from A which are satisfied by a. Since all the types we will deal with are quantifier-free, we shall use *type* to mean quantifier-free type. Note as well that we shall use type to mean 1-type and will refer to a type in m variables as an m-type.

A key technical tool in our proof is the following theorem of Hodges, Lachlan and Shelah [2].

Theorem. *Suppose $(J, <)$ is an infinite set of order indiscernibles over a set A in a structure of bounded arity. Then exactly one of the following possibilities holds.*

(a) J is totally indiscernible.

(b) An ordering $(J, \prec)$ is order indiscernible if and only if there are disjoint Y, $Z \subseteq J$ and $i \in \{1, -1\}$ such that

$$(J, <) = (Y, < \restriction Y) + (Z, < \restriction Z),$$
$$(J, \prec^i) = (Z, < \restriction Z) + (Y, < \restriction Y).$$

(*c*) *There are finite $Y, Z \subseteq J$ such that, writing $J' = J\backslash(Y \cup Z)$, we have*

$$(J, <) = (Y, < \restriction Y) + (J', < \restriction J') + (Z, < \restriction Z),$$

and an ordering $\prec$ of J is indiscernible if and only if there is an $i \in \{1, -1\}$ such that

$$(J, \prec^i) = (Y, \prec^i \restriction Y) + (J', < \restriction J') + (Z, \prec^i \restriction Z),$$

Here $<^1$ denotes $<$ and $<^{-1}$ denotes the reverse ordering. In the theorem if the arity of the language is n, then it is only necessary that $|J| \geqslant \max(2n-3, n+2)$ and in case (*c*) we can stipulate that $|Y \cup Z| \leqslant n$. Note as well that in case (*c*) $(J, \prec)$ or its reverse ordering is isomorphic to $(J, <)$.

1. *Language and the number of types*

Given a relational structure M, the *canonical language* for M consists of an m-ary relation for every (quantifier-free) m-type. In the language of permutations, the canonical language consists of the relations which are invariant under all partial isomorphisms. Since the relations in the original language and in the canonical language can be defined as an infinitary Boolean combination of relations in the other language, we are free to view M as a structure in whichever language is convenient.

The following proposition is obvious.

Proposition 1·1. *Suppose the canonical language for M has cardinality χ. Then $\phi_\mu \leqslant \chi^\mu$ for all μ.*

The size of the canonical language can also be used to give lower bounds for the number of substructures.

Proposition 1·2. *Suppose the canonical language for M has cardinality χ. If $\aleph_0 \leqslant \mu < \chi$, then $\phi_\mu \geqslant \chi$.*

Proof. For each relation R in the canonical language choose a substructure A_R of cardinality μ so that for some $a_1, \ldots, a_m \in A$, $R(a_1, \ldots, a_m)$ holds. Since in any structure of size μ at most μ m-types are realized for any m, there are χ pairwise non-isomorphic structures among $\{A_R : R$ is in the canonical language$\}$.

Using these two propositions we can easily extend the result for finite languages to every language. The result is probably well-known.

Theorem 1·3. *Suppose M is an infinite structure and $n < m \leqslant \aleph_0$. Then $\phi_n \leqslant \phi_m$.*

Proof. First note that ϕ_n is bounded by the number of n-ary relations in the canonical language. If this number is finite then the theorem for finite languages applies. If this number is infinite then ϕ_n is the number of n-ary relations in the canonical language. We distinguish two cases. If the number of n-ary relations in the canonical language is $\aleph_0$, then applying the finite theorem to finite sublanguages, we have $\phi_m \geqslant \aleph_0$. If the number of n-ary relations is μ an uncountable cardinal, then Proposition 1·2 implies $\phi_m \geqslant \mu (= \phi_n)$.

Proposition 1·4. *Suppose the canonical language for M has cardinality χ and μ is infinite. If $2^\mu < \chi^\mu$, then $\phi_\mu = \chi^\mu$. If μ is regular and $2^{<\mu} < \chi$, then $\phi_\mu \geqslant \chi^{<\mu}$.*

Proof. First assume $2^\mu < \chi^\mu$. Since $\phi_\mu \leqslant \chi^\mu$, we only have to show that $\phi_\mu \geqslant \chi^\mu$. For each subset X of the canonical language of cardinality μ, choose A_X of size μ so that for all $R \in X$ there are $a_1, \ldots, a_n \in A_X$ so that $R(a_1, \ldots, a_n)$ holds. We say that A_X realizes X. Notice that each A_X is isomorphic to at most 2^μ A_Y. So there are χ^μ pairwise non-isomorphic substructures among $\{A_X : X$ is a subset of the canonical language and $|X| = \mu\}$.

Now suppose $2^{<\mu} < \chi$. Notice that $\mu^\rho \leqslant 2^{<\mu}$ for all $\rho < \mu$. Fix $\rho < \mu$. We now define a sequence $(A_\alpha : \alpha < \chi^\rho)$ of non-isomorphic μ-substructures. Suppose $\{A_\beta : \beta < \alpha\}$ has been defined. Since $\{X : |X| = \rho$, X is a subset of the canonical language, and there is $\beta < \alpha$ so that X is realized in $A_\beta\}$ has cardinality at most $|\alpha| . 2^{<\lambda}$, there is Y a subset of the canonical language of cardinality ρ so that Y is not in this set. Choose A_α a μ-substructure which realizes Y.

Proposition 1·5. *If $\mu < \lambda \leqslant |M|$ and $\phi_\lambda \geqslant 2^\lambda$, then $\phi_\mu \leqslant \phi_\lambda$.*

Proof. Let χ be the cardinality of the canonical language of M. We can assume that λ is infinite. If $2^\lambda < \chi^\lambda$ then $\phi_\lambda = \chi^\lambda \geqslant \chi^\mu \geqslant \phi_\mu$. Otherwise we have $\phi_\lambda = 2^\lambda = \chi^\lambda \geqslant \chi^\mu \geqslant \phi_\mu$.

Next let $S_m(B)$ denote the set of (quantifier-free) m-types realized in M over B.

Lemma 1·6. *Suppose λ is a regular uncountable cardinal, $\aleph_0 \leqslant \mu \leqslant \lambda$, and there is a set B so that for some m, $|S_m(B)| \geqslant \lambda$ and $\mu^{|B|} < \lambda^\mu$. Then $\phi_\mu \geqslant \lambda^\mu$.*

Proof. Without loss of generality we can assume m is minimal so that there is a set C with $|S_m(C)| \geqslant \lambda$, where C is finite if B is finite and $|C| \leqslant |B|$ otherwise. Now choose a set $\{p_i : i < \lambda\}$ of distinct m-types over B. Next choose $\bar{c}_i$ realizing p_i. Define

$$Q_i = \{j < \lambda : \text{there is } \bar{d} \text{ realizing } p_j \text{ with } \bar{d} \cap (\bar{c}_i \cup B) \neq \varnothing\}.$$

By the minimality of m each set Q_i has cardinality less than λ. Since λ is regular and uncountable, the finest equivalence relation such that each Q_i is contained in an equivalence class, has equivalence classes all of cardinality less than λ. So this equivalence relation has λ classes. Hence by passing to a subset of $\{p_i : i < \lambda\}$ we can assume that each Q_i has cardinality 1.

For any $X \subseteq \lambda$, let $M(X) = B \cup \{\bar{c}_i : i \in X\}$. By our assumption that $|Q_i| = 1$, we have that p_i is realized in $M(X)$ if and only if $i \in X$. Hence if $X \neq Y$ then $M(X)$ is not isomorphic to $M(Y)$ over B. Since $\mu^{|B|} < \lambda^\mu$ the lemma is proved.

Since the cardinality of the canonical language is the same as the number of types over the empty set, we have the following result.

Theorem 1·7. *Suppose λ is a regular uncountable cardinal and the canonical language has cardinality at least λ. Then for every infinite $\mu \leqslant \lambda$, we have $\phi_\mu \geqslant \lambda^\mu$.*

2. *Existence of indiscernibles*

We begin this section with a result on the number of suborderings of linear orderings. This result will be needed in the proof of the main theorem of this section (Theorem 2·4). As well, the method of proof of Theorem 2·3 is a simplified version of some of the arguments we will use in Theorem 2·4.

Lemma 2·1. *Suppose μ is a regular cardinal and $(I, <)$ is an ordered set of cardinality μ. Then one of the following possibilities holds.*

(i) *I contains a subset of order type μ.*

(ii) *I contains a subset of order type μ^*. (If τ is a linear order then τ^* denotes the reverse order.)*

(iii) *I contains a dense linear order of cardinality μ.*

Proof. Define an equivalence relation $\equiv$ on I by $i \equiv j$ if and only if $[i,j]$ is scattered (cf. [7], p. 33; a linear order is *scattered* if it contains no dense linear order). If some equivalence class has cardinality μ, then that equivalence class is a scattered subordering of I of cardinality μ. So by corollary 5.29 of [7], either (i) or (ii) holds. If no equivalence class has cardinality μ, then $I/\equiv$ is a dense linear ordering of cardinality μ. By taking a set of representatives for the equivalence classes of $\equiv$, we obtain a dense linear subordering of I which has cardinality μ.

In order to apply the next theorem in many contexts, we define the following concept. Suppose M is a structure of size κ and $\lambda < \kappa$. A set $A \subseteq M$ is said to be *complete* for λ if whenever $B \subseteq A$ and $|B| < \lambda$ then every type over B which is realized in M is realized in A and if some type over B has at most $|A|$ realizations in M then all the realizations are in A. The following proposition gives two sufficient conditions for the existence of complete sets.

PROPOSITION 2·2. *Suppose M is a structure of cardinality κ and λ is a cardinal less than κ.*

(*a*) *If λ is regular and for every $\theta < \lambda$ there are at most λ types realized over any set of size θ, then every subset of cardinality λ is contained in a set of cardinality λ which is complete for λ.*

(*b*) *Suppose $2^{<\lambda} < \kappa$ and λ is regular. Then either every subset of cardinality λ is contained in a set of size $2^{<\lambda}$ which is complete for λ, or $\phi_\lambda \geqslant 2^\lambda$ and if $\lambda = \mu^+$ then $\phi_\mu \geqslant 2^\mu$ as well.*

Proof. Since the proofs of (*a*) and (*b*) are similar, we will only prove (*b*). If over every set of cardinality less than λ, there are at most $2^{<\lambda}$ types then the complete set can be built as the union of $2^{<\lambda}$ sets of cardinality $2^{<\lambda}$, where at each stage we add realizations of all types over all subsets of cardinality less than λ.

On the other hand, suppose there is a set B with $|B| < \lambda$ such that there are at least $(2^{<\lambda})^+$ types over B. Let θ denote $(2^{<\lambda})^+$. Then

$$\lambda^{|B|} = \sum_{\nu<\lambda} \nu^{|B|} \leqslant \sum_{\nu<\lambda} 2^{\nu|B|} = 2^{<\lambda} < \theta.$$

So $\lambda^{|B|} < \theta^\lambda$. Also, if $\lambda = \mu^+$ then, as $\mu^{|B|} \leqslant 2^\mu < \theta$, we have $\mu^{|B|} < \theta^\mu$. The result now follows from Lemma 1·6.

THEOREM 2·3. *Suppose L is a dense linear order of cardinality κ. Then for every infinite $\rho < \kappa$, L has 2^ρ pairwise non-isomorphic suborderings of cardinality ρ.*

Proof. We first note the following.

Claim 1. Suppose I is a dense linear order and μ is an infinite cardinal. If there is a sequence of order type μ or μ^* in I, then I has 2^μ pairwise non-isomorphic suborders of power μ.

Proof of Claim 1. Let $\{a_\alpha : \alpha < \mu\}$ be a strictly increasing sequence in I (the case where the sequence is decreasing is similar). For each ordinal α, choose $J_\alpha \subseteq I$ a set of order type $\mathbb{Q}$ so that $a_\alpha < J_\alpha < a_{\alpha+1}$. Such a choice is possible, since I is a dense linear order. Fix $\rho \leqslant \lambda$. For $X \subseteq \rho$ such that X consists of odd ordinals, let

$$M(X) = \{a_\alpha : \alpha < \rho\} \cup \bigcup_{\alpha \in X} J_\alpha.$$

Notice that, in $M(X)$, $\{a_\alpha : \alpha < \rho\}$ is the set of elements which either have a unique immediate successor or a unique immediate predecessor. It is easy to verify that if $X, Y \subseteq \rho$ consist of odd ordinals and $X \neq Y$, then $M(X)$ is not isomorphic to $M(Y)$. So the claim is proved.

Since every singular cardinal is a limit of regular cardinals it is enough to prove the theorem under the assumption that κ is regular. By Claim 1, we can assume that L has no increasing or decreasing sequence of order type κ. Define the equivalence relation $\equiv$ on L by $a \equiv b$ if and only if the interval $[a, b]$ has cardinality less than κ. If some equivalence class has cardinality κ then, as κ is regular, L has either an increasing or decreasing sequence of order type κ. Hence $L/\equiv$ has cardinality κ. So we can assume:

(*) L is a dense linear order of cardinality κ and every interval of L has cardinality κ.

It suffices to show

Claim 2. If a linear order L satisfies (*) then for all cardinals $\mu < \kappa$ L has 2^μ suborderings of cardinality μ.

Proof of Claim 2. Before we continue notice that if L satisfies (*) then so does every interval of L. Let $K = \{\mu \leqslant \kappa : 2^{<\mu} < 2^\mu\}$. It suffices to prove the claim just for elements of K. To see this suppose $\lambda \leqslant \kappa$ and $\lambda \notin K$. Choose $\mu < \lambda$ minimal so that $2^\mu = 2^\lambda$. Consider any $a \in L$. Let L_1 be a subset of $(-\infty, a]$ of cardinality λ so that every interval of L_1 has cardinality λ. Now choose a maximal set $\mathscr{A}$ of pairwise non-isomorphic suborders of size μ of (a, ∞). The suborders $L_1 + A$ for $A \in \mathscr{A}$ are pairwise non-isomorphic.

We prove the claim by induction on $\mu \in K$. There are two cases to consider.

Case 1. There is B of cardinality less than μ so that L makes at least μ cuts in B. In this case there are 2^μ suborders of cardinality μ which are non-isomorphic over B. To finish the proof, note that $\mu \leqslant 2^{|B|}$, since there are at most $2^{|B|}$ cuts in B. Hence $\mu^{|B|} \leqslant 2^{|B|} < 2^\mu$. So we have 2^μ pairwise non-isomorphic suborders.

Case 2. Not Case 1. Let $\lambda = cf(\mu)$. We first show that there is a sequence of order type or reverse order type λ. Choose an increasing sequence $(B_i : i < \lambda)$ of subsets of L so that for all i, $|B_i| < \mu$ and every cut in B_i which L makes is realized in B_{i+1}. Choose $a \notin \bigcup \{B_i : i < \lambda\}$. Now consider a sequence $(a_i : i < \lambda)$, where $a_i \in B_{i+1}$ and makes the same cut in B_i as a does. Without loss of generality we can assume that either for all i, $a_i < a$ or for all i, $a < a_i$. In the first case the sequence is increasing and in the second the sequence is decreasing. Notice that, by Claim 1, this finishes the proof in the case where μ is regular.

Now assume that μ is singular. Let $\mu = \sum_{i<\lambda} \mu_i$, where $\mu_i < \mu$. Since $2^\mu = \prod_{i<\lambda} 2^{\mu_i}$, the 2^{μ_i} cannot be eventually constant. It follows that μ is a limit of members of K. For each odd i, choose by the inductive hypothesis a set $\mathscr{A}_i$ of $2^{<\mu}$ pairwise non-isomorphic suborders of (a_i, a_{i+1}) each of cardinality less than μ. For i even choose L_i a suborder of (a_i, a_{i+1}) so that $|L_i| = \mu$ and every interval of L_i has cardinality μ. For even i, let $\mathscr{A}_i = \{L_i\}$. Consider two orders

$$C_j = \sum_{i<\lambda} I_{i,j}$$

for $j = 0, 1$ where each $I_{i,j} \in \mathscr{A}_i$. Then $C_0 \cong C_1$ if and only if for all i, $I_{i,0} = I_{i,1}$. As $2^\mu = \prod_{i<\lambda} 2^{\mu_i}$, the result follows.

The main theorem in this section guarantees the existence of indiscernibles or maximally many substructures.

THEOREM 2·4. *Suppose M is a structure in a relational language and λ is an uncountable regular cardinal so that every set of size λ is contained in a set of cardinality less than $|M|$ which is complete for λ. If $B \subseteq M$ and $|B| < \lambda$ then one of the following holds:*

(A) *there is a set $I \subseteq M$ of indiscernibles of cardinality λ over B;*

(B) *for all μ such that $\aleph_0 \leqslant \mu \leqslant \lambda$, M has at least 2^μ pairwise non-isomorphic substructures of cardinality μ;*

(C) *there is an indiscernible sequence of order type λ^+ over B.*

Proof. To begin choose A^* so that $B \subseteq A^* \subseteq M$, $|A^*| < |M|$ and A^* is complete for λ. Next choose $a^* \in M \backslash A^*$. For every $\bar{c} \in A^*$ and quantifier-free formula $\phi(\bar{y}, x_1, \ldots, x_n)$, we define a game $\mathrm{GM}_{\phi(\bar{c},\bar{x})}$ of length n as follows.

Players I and II play alternately. Player I chooses subsets of A^* of cardinality $< \lambda$ and Player II chooses elements of A^*. Further for all $i < n-1$, Player I must choose $A_{i+1} \supseteq A_i \cup \{a_i\}$. Player II is required to choose $a_i \in A^* \backslash A_i$ so that a_i and a^* realize the same type over A_i. Player I wins a play of the game if $\phi(\bar{c}, a_0, \ldots, a_{n-1})$ holds.

There are actually two suppressed parameters in the description of the game, namely the structure M and the cardinal λ. Later we shall have occasion to refer to the game for $\phi(\bar{c}, \bar{x})$ in N relative to ρ, where N is a substructure of M and ρ is an infinite cardinal. More exactly Player I plays subsets of $A^* \cap N$ of cardinality $< \rho$ and Player II plays elements of $A^* \cap N$.

Clearly, this game is determined. If Player I has a winning strategy for $\mathrm{GM}_{\phi(\bar{c},\bar{x})}$ we will denote this fact by $\Vdash \phi(\bar{c}, \bar{x})$. Notice that it is impossible that both $\Vdash \phi(\bar{c}, \bar{x})$ and $\Vdash \neg\phi(\bar{c}, \bar{x})$ hold, but it is possible that neither holds. There are two main cases.

Case I. For every quantifier-free $\phi(\bar{y}, x_1, \ldots, x_n)$ and $\bar{c} \in A$ either $\Vdash \phi(\bar{c}, \bar{x})$ or $\Vdash \neg\phi(\bar{c}, \bar{x})$.

Fix winning strategies for Player I in all the games for which such strategies exist. Since λ is regular and uncountable, for every $C \subseteq A^*$ if $|C| < \lambda$ then there is C_1 such that $C \subseteq C_1 \subseteq A^*$, $|C_1| < \lambda$ and C_1 is closed under Player I's winning strategies. That is for any ϕ and $\bar{c} \in C_1$ so that $\Vdash \phi(\bar{c}, x_1, \ldots, x_n)$ if $A_0, a_0, A_1, a_1, \ldots, A_l$ is a play of the game according to Player I's strategy where all the a_i are in C_1, then $A_l \subseteq C_1$.

If we were only interested in getting indiscernibles of order type λ, then we could finish easily. Namely we could take an increasing sequence $B \subseteq C_0, c_0, C_1, c_1, \ldots, C_\alpha, c_\alpha, \ldots$ for $\alpha < \lambda$ such that: for all $\alpha < \beta$, $C_\alpha \cup \{c_\alpha\} \subseteq C_\beta$; C_α is closed under Player I's winning strategies; $c_\alpha \in A^*$ and realizes the type of a^* over C_α. It is easy to see that $\{c_\alpha : \alpha < \lambda\}$ is an indiscernible sequence of order type λ over B. The easiest case to dispose of occurs when this sequence is actually a set of indiscernibles, i.e. Case Ia occurs.

Case Ia. For every formula $\phi(\bar{y}, x_1, \ldots, x_n)$ and all $\bar{c} \in A^*$ and permutation σ, $\Vdash \phi(\bar{c}, x_1, \ldots, x_n)$ if and only if $\Vdash \phi(\bar{c}, x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

In this case we can construct, as above, a set of indiscernibles of cardinality λ over B, i.e. possibility (A) holds.

Case Ib. There is a formula ϕ and $\bar{c} \in A^*$ so that $\Vdash \phi(\bar{c}, x_1, x_2, x_3, \ldots, x_n)$ and $\Vdash \neg\phi(\bar{c}, x_2, x_1, x_3, \ldots, x_n)$. (It requires a slight argument to see that this is equivalent to not Case Ia.)

The proof in this case is subsumed under the argument in Case IId. So we delay the proof until then.

Case II. Not Case I. Choose $\phi(\bar{c}, x_1, \ldots, x_n)$ so that neither $\Vdash \phi(\bar{c}, x_1, \ldots, x_n)$ nor $\Vdash \neg\phi(\bar{c}, x_1, \ldots, x_n)$ and n is minimal. It is clear that $n \geqslant 2$. Also, for every A with $B \subseteq A \subseteq A^*$ and $|A| < \lambda$, there are $a, d \in A^* \backslash A$ realizing the type of a^* over A such that $\Vdash \phi(\bar{c}, a, x_2, \ldots, x_n)$ and $\Vdash \neg\phi(\bar{c}, d, x_2, \ldots, x_n)$.

Case IIa. For every A if $B \subseteq A \subseteq A^*$ and $|A| < \lambda$ then there is $a \in A^* \backslash A$ realizing the type of a^* over A such that

$$\Vdash \phi(\bar{c}, a, x_2, \ldots, x_n),$$
$$\Vdash \neg\phi(\bar{c}, x_1, a, x_3, \ldots, x_n).$$

Further there is $d \in A^* \backslash A$ realizing the type of a^* over A such that

$$\Vdash \neg\phi(\bar{c}, d, x_2, \ldots, x_n),$$
$$\Vdash \neg\phi(\bar{c}, x_1, d, x_3, \ldots, x_n).$$

We choose a sequence $\{a_\alpha : \alpha < \lambda\}$ of elements of A^* so that for all $\alpha < \beta_2 < \ldots < \beta_n$ if α is even then

$$\neg\phi(\bar{c}, a_\alpha, a_{\beta_2}, \ldots, a_{\beta_n}) \quad \text{and} \quad \neg\phi(\bar{c}, a_{\beta_2}, a_\alpha, a_{\beta_3}, \ldots, a_{\beta_n})$$

and if α is odd then

$$\phi(\bar{c}, a_\alpha, a_{\beta_2}, \ldots, a_{\beta_n}) \quad \text{and} \quad \neg\phi(\bar{c}, a_{\beta_2}, a_\alpha, a_{\beta_3}, \ldots, a_{\beta_n}).$$

Suppose now that $\mu \leqslant \lambda$ is an infinite cardinal and X is a subset of μ consisting of even ordinals. Let

$$M(X) = \bar{c} \cup \{a_\alpha : \alpha < \mu \text{ and } \alpha \text{ is odd}\} \cup \{a_\alpha : \alpha \in X\}.$$

We shall show that X is an isomorphism invariant of $M(X)$ over $\bar{c}$. For any $a \in M(X) \backslash \bar{c}$ let $a \in U$ if and only if Player I has a winning strategy in the game for $\phi(\bar{c}, a, x_2, \ldots, x_n)$ in $M(X)$ relative to μ. Notice that $a_\alpha \in U$ if and only if α is odd. Define a relation $<_1 \subseteq U \times (M(X) \backslash \bar{c})$ by $a <_1 b$ if and only if Player I has winning strategies in the game for $\phi(\bar{c}, a, b, x_3, \ldots, x_n)$ and the game for $\neg\phi(\bar{c}, b, a, x_3, \ldots, x_n)$ in $M(X)$ relative to μ. Notice $a_\alpha <_1 a_\beta$ if and only if α is odd and $\alpha < \beta$. Let $<_2$ be the relation defined on $M(X) \backslash \bar{c}$ to be the unique total order extending $<_1$ and satisfying the following: if a, $b \in M(X) \backslash \bar{c}$ and $a \neq b$, then $a <_2 b$ if and only if $\{x : x <_1 a\} \subseteq \{x : x <_1 b\}$. It is easy to reconstruct X from the information given. Since there are 2^μ choices for X, case (B) holds.

Case II*b*. For every A if $B \subseteq A \subseteq A^*$ and $|A| < \lambda$ then there is $a \in A^* \setminus A$ realizing the type of a^* over A such that

$$\Vdash \phi(\bar{c}, a, x_2, \ldots, x_n),$$
$$\Vdash \neg\phi(\bar{c}, x_1, a, x_3, \ldots, x_n).$$

Further there is $d \in A^* \setminus A$ realizing the type of a^* over A such that

$$\Vdash \phi(\bar{c}, d, x_2, \ldots, x_n),$$
$$\Vdash \phi(\bar{c}, x_1, d, x_3, \ldots, x_n).$$

This case is just like II*a*.

Case II*c*. For every A if $B \subseteq A \subseteq A^*$ and $|A| < \lambda$ then there is $a \in A^* \setminus A$ realizing the type of a^* over A such that

$$\Vdash \phi(\bar{c}, a, x_2, \ldots, x_n),$$
$$\Vdash \phi(\bar{c}, x_1, a, x_3, \ldots, x_n).$$

Further there is $d \in A^* \setminus A$ realizing the type of a^* over A such that

$$\Vdash \neg\phi(\bar{c}, d, x_2, \ldots, x_n),$$
$$\Vdash \neg\phi(\bar{c}, x_1, d, x_3, \ldots, x_n).$$

We choose a sequence $\{a_\alpha : \alpha < \lambda\}$ so that if α is even and $\alpha < \beta_2 < \ldots < \beta_n$ then

$$\phi(\bar{c}, a_\alpha, a_{\beta_2}, \ldots, a_{\beta_n}) \quad \text{and} \quad \phi(\bar{c}, a_{\beta_2}, a_\alpha, a_{\beta_3}, \ldots, a_{\beta_n})$$

and if α is odd and $\alpha < \beta_2 < \ldots < \beta_n$ then

$$\neg\phi(\bar{c}, a_\alpha, a_{\beta_2}, \ldots, a_{\beta_n}) \quad \text{and} \quad \neg\phi(\bar{c}, a_{\beta_2}, a_\alpha, a_{\beta_3}, \ldots, a_{\beta_n}).$$

Fix $\mu \leqslant \lambda$, and infinite cardinal. For X a subset of the limit ordinals less than μ let $M(X)$ be the structure consisting of $\bar{c}$ together with

$$\{a_{\delta+k} : k < \omega, \quad \delta \in X\} \cup \{a_{\delta+k} : 0 < k < \omega, \ \delta \text{ a limit ordinal and } \delta \notin X\}.$$

We claim that X is an isomorphism invariant of $M(X)$ over $\bar{c}$. First note that we can partition $M(X) \setminus \bar{c}$ into two pieces U and V by letting $a \in U$ if Player I can win the game for $\phi(\bar{c}, a, x_2, \ldots, x_n)$ in $M(X)$ relative to μ. Notice that $a_\alpha \in U$ if and only if α is even. Next for $a \in U$ and $b \in V$ we can define $a <_1 b$ if and only if Player I has a winning strategy for the game for $\phi(\bar{c}, a, b, x_3, \ldots, x_n)$ in $M(X)$ relative to μ otherwise let $b <_1 a$. It is easy to check that $a_\alpha <_1 a_\beta$ for α and β of different parity if and only if $\alpha < \beta$. Finally if we define $<_2$ on $U \cup V$ to be the transitive closure of $<_1$, then $a_\alpha <_2 a_\beta$ if and only if $\alpha < \beta$. Thus $<_2$ is an ordering of order type μ. For limit ordinals δ, the δth element of $<_2$ is in U if and only if $\delta \in X$.

Case II*d*. Not II*a*, II*b*, II*c*. Let A_1, A_2, A_3 be subsets of A^* which witness respectively that Cases II*a*, II*b* and II*c* do not hold. Put $A = A_1 \cup A_2 \cup A_3$. Then $|A| < \lambda$, and by the remark before Case II*a* we have: for every $a \in A^* \setminus A$ realizing the type of a^* over A

$$\text{if } \Vdash \phi(\bar{c}, a, x_2, \ldots, x_n) \text{ then } \Vdash \neg\phi(\bar{c}, x_1, a, x_3, \ldots, x_n)$$

and

$$\text{if } \Vdash \neg\phi(\bar{c}, a, x_2, \ldots, x_n) \text{ then } \Vdash \phi(\bar{c}, x_1, a, x_3, \ldots, x_n).$$

Recall that we have promised that the proof in this case will include the proof of Case Ib. The proofs will be identical except for a few exceptions which we will note.

For any $C \subseteq A^*$ let

$$\mathrm{Can}\,(C) = \{a \in M : a \text{ and } a^* \text{ have the same type over } C\}.$$

Note that by our choice of a^* and A^* if $|C| < \lambda$ then $|\mathrm{Can}\,(C)| \geqslant \lambda^+$. We consider two subsubcases.

Subsubcase (i). There is some A so that $A \subseteq C \subseteq A^*$, $|C| < \lambda$ and $\mathrm{Can}\,(C)$ can be ordered and partitioned into two sets D_0, D_1 so that each D_i is order-indiscernible over B.

We can assume that $|D_0| = \lambda^+$. If D_0 is a set of total indiscernibles we are done. In the case where it is not a set of indiscernibles, fix a linear ordering $<$ of D_0 so that D_0 is order-indiscernible with respect to $<$ (by the theorem of [**2**] which has been quoted there are a limited number of ways to pick this ordering). Then, by Lemma 2·1, either $<$ contains a sequence of order type λ^+ (or the reverse order) or it contains a dense linear ordering of cardinality λ^+. Furthermore if there is no increasing or decreasing sequence of order type λ^+ then for any m we can find $d_0, \ldots, d_m \in D_0$ so that the interval from d_i to d_{i+1} has cardinality λ^+ for all $i < m$. In this case we can find a finite set over which we can define the ordering on a subset of D_0 of cardinality λ^+. If we fix a finite set necessary to define the ordering on a dense linear ordering of cardinality λ^+ then, by Theorem 2·3, we find that in every infinite power $\mu \leqslant \lambda$, there are 2^μ structures no two of which are isomorphic via any map which respects the finite set. Since there are only μ finite subsets of a set of size μ, there are 2^μ non-isomorphic structures. So we have shown that in this subsubcase one of the three possibilities holds.

Subsubcase (ii). Not Subsubcase (i).

First, note that by Theorem 1·7 we may assume that the canonical language has size less than λ.

By the compactness theorem if $\mathrm{Can}\,(C)$ cannot be ordered and partitioned into two sets order-indiscernible over B then there are finite subsets $J_C \subseteq \mathrm{Can}\,(C)$ and $B_C \subseteq B$ and Δ_C a finite subset of the canonical language so that under no ordering can J_C be partitioned into two sets Δ_C-order-indiscernible over B_C. By the choice of A^* there is $I_C \subseteq A^* \cap \mathrm{Can}\,(C)$ so that the Δ_C-type of I_C over B_C is the same as the Δ_C-type of J_C over B_C. So there is no ordering of I_C which can be partitioned into two sets Δ_C-indiscernible over B_C. Denote the cardinality of I_C by n_C. Define sequences $\{C_\alpha : \alpha < \lambda\}$ of subsets of A^* and $\{a_{\alpha+1} : \alpha < \lambda\}$ of points of A^* (and also in case IId a sequence $\{d_{\alpha+1} : \alpha < \lambda\}$ of points of A^*) so that for all $\alpha < \beta$: $|C_\alpha| < \lambda$; $B \cup \bar{c} \subseteq C_\alpha \subseteq C_\beta$; C_α is closed under Player I's winning strategies; if α is a limit ordinal, then $C_\alpha \cup I_{C_\alpha} \subseteq C_\beta$; $a_{\alpha+1}(d_{\alpha+1}) \in \mathrm{Can}\,(C_\alpha) \cap C_{\alpha+1}$; $\Vdash \phi(\bar{c}, a_{\alpha+1}, x_2, \ldots, x_n)$ ($\Vdash \neg\, \phi(\bar{c}, d_{\alpha+1}, x_2, \ldots, x_n)$); and in Case II$d$, $A \subseteq C_0$. By the pigeon-hole principle we can assume for all α, β that $B_{C_\alpha} = B_{C_\beta}$, $\Delta_{C_\alpha} = \Delta_{C_\beta}$, and $n_{C_\alpha} = n_{C_\beta}$. To simplify notation we will write I_α, B_0, Δ and n_0 for $I_{C_\alpha}, B_{C_\alpha}, \Delta_{C_\alpha}$ and n_{C_α}. By Ramsey's theorem we can assume that for all limit ordinals $\delta \{a_{\delta+n} : 0 < n < \omega\}$ (and $\{d_{\delta+n} : 0 < n < \omega\}$) are Δ-order-indiscernible over B_0.

For any infinite cardinal $\mu \leqslant \lambda$ and a set of limit ordinals $X \subseteq \mu$, let $M(X)$ be the substructure whose universe is $\bar{c} \cup B_0 \cup \bigcup_{\alpha \in X} I_\alpha \cup \bigcup_{\alpha < \mu} \{a_{\alpha+1}\} (\cup \bigcup_{\alpha < \mu} \{d_{\alpha+1}\})$. To complete the proof that there are 2^μ models in power μ we will show that X can be defined in the structure $(M(X), \bar{c}, B_0)$; i.e. X is an isomorphism invariant of $M(X)$ over $\bar{c} \cup B_0$.

We can partition $M(X) \backslash (\bar{c} \cup B_0)$ into two pieces by letting $a \in U$ if Player I can win the game for $\phi(\bar{c}, a, x_2, \ldots, x_n)$ in $M(X)$ relative to μ and letting $a \in V$ if Player I can win the game for $\neg \phi(\bar{c}, a, x_2, \ldots, x_n)$ in $M(X)$ relative to μ. The fact that this is a partition follows from the minimality of n. In case Ib, V will be the empty set. From now on we will just present the proof in case IId. The modifications needed in Case Ib, all consist of simplifying the proof. Define a relation R_U on U by letting $aR_U b$ if and only if Player I can win the games for $\phi(\bar{c}, a, b, x_3, \ldots, x_n)$ and $\neg \phi(\bar{c}, b, a, x_3, \ldots, x_n)$ in $M(X)$ relative to μ. Similarly define R_V on V but reversing the use of ϕ and $\neg \phi$. By the assumptions of case IId, if $a, b \in U$ and there is no $\delta \in X$ with $\{a, b\} \subseteq I_\delta$ for some limit ordinal $\delta \in X$, then $aR_U b$ if and only if one of the following possibilities occurs for some $\beta < \alpha : a \in I_\beta$ and $b \in I_\alpha$; $a \in I_\beta$ and $b = a_{\alpha+1}$; $a = a_{\beta+1}$ and $b \in I_\alpha$; or $a = a_{\beta+1}$ and $b = a_{\alpha+1}$. Similar comments apply to V and R_V. Let θ_U be the least congruence on (U, R_U) so that $(U, R_U)/\theta_U$ is well-ordered. Similarly define θ_V. Note that the only non-singleton classes of θ_U are subsets of some I_δ. Enumerate the classes of θ_U in increasing R_U/θ_U-order as $\{u_\alpha : \alpha < \mu\}$. Also enumerate the classes of θ_V in increasing R_V/θ_V-order as $\{v_\alpha : \alpha < \mu\}$. By the construction and our comments it should be clear that for $\delta \in X$, $I_\delta \subseteq \bigcup_{i < n_0} u_{\delta+i} \cup \bigcup_{i < n_0} v_{\delta+i}$. Also for any limit ordinal $\delta \notin X$, $\bigcup_{i < n_0} u_{\delta+i}$ and $\bigcup_{i < n_0} v_{\delta+i}$ are both sets of Δ-order-indiscernibles over B_0. So X is defined as the set of limit ordinals so that $\bigcup_{i < n_0} u_{\delta+i} \cup \bigcup_{i < n_0} v_{\delta+i}$ cannot be ordered and partitioned into two Δ-order indiscernible sets.

This last case completes the proof of the theorem.

3. *Number of substructures with indiscernibles*

Suppose now that $\mu < \lambda < \kappa = |M|$ where λ, κ are infinite. In this section, we will study the number of substructures if there is a sufficient supply of indiscernibles. We will assume that if A is any μ-substructure (i.e. a substructure of cardinality μ) then either there is a set of total indiscernibles over A of cardinality λ or there is a set of order indiscernibles of order type λ^+ over A. We can also assume that if a relation is in our language then any relation which can be obtained by permutation of the variables or negation is also present in the language. Let $\mathscr{A}$ be a set of μ-substructures of M so that each μ-substructure of M is isomorphic to exactly one element of $\mathscr{A}$. For each $A \in \mathscr{A}$, let I_A be a set of order indiscernibles over A of order type λ. Further choose I_A to be totally indiscernible if possible. Then let $S_A = A \cup I_A$.

It is useful to keep in mind two examples where A and B are not isomorphic but $S_A \cong S_B$. Suppose the language of M has a single unary predicate P and $|P(M)| = \mu$ and $|M \backslash P(M)| = \kappa$. Choose A and B so that $|P(A)| = |P(B)| = \mu$ but $|A \backslash P(A)| \neq |B \backslash P(B)|$. If I_A, I_B are chosen to be subsets of $M \backslash P(M)$, then $S_A \cong S_B$. In this example we cannot get the required λ-substructures by varying the cardinality of $C \backslash P(C)$, but we can by varying the cardinality of $P(C)$. Consider next the case where M is the ordinal κ and $<$ an ordering of order type κ. If A and B are initial segments of κ and I_A and I_B have order type λ, then S_A and S_B will be both isomorphic to λ. However we can get the required number of λ-substructures by looking at different order types. The proof of our main theorem shows that these examples capture all the ways in which S_A might become isomorphic to S_B.

LEMMA 3·1. *Suppose A is an infinite substructure of M and $J_1, J_2 \subseteq A$ are such that for all i, $|A\backslash J_i| < |A|$ and J_i is order indiscernible over $A\backslash J_i$. Then there is an ordering of $J_1 \cup J_2$ so that $J_1 \cup J_2$ is order indiscernible over $A\backslash(J_1 \cup J_2)$ and J_1 with its ordering is an interval in the new ordering.*

Proof. There are two cases to consider. First consider the case where J_2 is a set of total indiscernibles. Since $J_1 \cap J_2$ is infinite, J_1 is also a set of total indiscernibles. It is easy to see that $J_1 \cup J_2$ is totally indiscernible over $A\backslash(J_1 \cup J_2)$.

Suppose now that J_1 is not a set of total indiscernibles. Let $<_1$ denote the ordering of J_1 and let $<_2$ denote the ordering of J_2. Let $\phi(x_1, x_2, \ldots, x_n)$ be a quantifier-free formula with parameters from $A\backslash J_2$ such that $\phi(j_1, \ldots, j_k, j_{k+1}, \ldots, j_n)$ holds for $j_1 <_2 j_2 <_2 \ldots <_2 j_n$ but does not hold for $\phi(j_1, \ldots, j_{k+1}, j_k, \ldots, j_n)$. We claim that $J_2\backslash J_1$ makes at most n cuts in $J_1 \cap J_2$ (in the sense of the ordering $<_2$). If there are more than n cuts made in $J_1 \cap J_2$ we can find $j_1 <_2 \ldots <_2 j_{k-1} <_2 i <_2 j <_2 i' <_2 j_{k+2} <_2 \ldots <_2 j_n$. Here the j's are in $J_2\backslash J_1$ and the i's are in $J_1 \cap J_2$. So $\phi(j_1, \ldots, j_{k-1}, i, j, j_{k+1}, \ldots, j_n)$ holds but $\phi(j_1, \ldots, j_{k-1}, i', j, j_{k+1}, \ldots, j_n)$ does not hold. This contradicts the indiscernibility of J_1 over $A\backslash J_1$. Now $J_1 \cap J_2$ must have the same cardinality as A and so there is some infinite interval X in J_2 consisting of elements of J_1. By Ramsey's theorem we can find an infinite $Y \subseteq X$ such that either for all $y_1, y_2 \in Y$, $y_1 <_1 y_2$ implies $y_1 <_2 y_2$ or for all $y_1, y_2 \in Y$, $y_1 <_1 y_2$ implies $y_2 <_2 y_1$. (It is also possible to use the theorem in [**2**].) By reversing the ordering of J_2 if necessary we can assume that the first possibility holds. (This new ordering also orders J_2 as a set of indiscernibles.) Now define an ordering $<'$ on $J_1 \cup J_2$ so that $x_1 <' x_2$ if and only if

$$\begin{cases} x_1 <_1 x_2 & \text{if} \quad x_1, x_2 \in J_1 \\ x_1 \in J_2\backslash J_1, \quad x_2 \in J_1 & \text{and for some} \quad y \in Y, \quad x_1 <_2 y \\ x_2 \in J_2\backslash J_1, \quad x_1 \in J_1 & \text{and for some } y \in Y, \quad y <_2 x_2 \\ x_1 <_2 x_2 & \text{and} \quad x_1, x_2 \in J_2\backslash J_1. \end{cases}$$

This makes $J_1 \cup J_2$ into a set of order indiscernibles over $A\backslash(J_1 \cup J_2)$ with J_1 as an interval.

The following two facts are immediate consequences of Lemma 3·1.

COROLLARY 3·2. *Let $A \in \mathscr{A}$. There is a unique $J \subseteq S_A$ which is maximal with respect to being order indiscernible over $A\backslash J$ with $|S_A\backslash J| < \lambda$.*

COROLLARY 3·3. *Let J be a maximal set of order indiscernibles such that $|S_A\backslash J| < \lambda$. Then there is an ordering of J as a set of order indiscernibles so that I_A is an interval of J and the ordering of J coincides with the ordering of I_A.*

We note here a corollary of the lemma which will be used to produce λ^+ λ-substructures.

COROLLARY 3·4. *Let $\mu < \lambda$. Suppose A is a μ-set and I is a set of order indiscernibles over A of order type $\alpha < \lambda^+$ such that $\mu < |I|$. Further suppose that I is not a set of total indiscernibles over A. There is an ordinal $\beta < \lambda^+$ such that if J is a set of well ordered indiscernibles in $A \cup I$ over $(A \cup I)\backslash J$ then the order type of $J < \beta$.*

Proof. By [2] the order type of $J \cap I$ (in the ordering of J) has order type $< \alpha+\alpha$. Since $|J \backslash I| \leqslant \mu$, the order type of J is $< \alpha+\alpha+\mu^+$.

From the above corollary we immediately obtain that if A is a μ-set, $\mu < \lambda$ and I is a set of order indiscernibles over A of order type λ^+ then $\phi_\lambda \geqslant \lambda^+$.

THEOREM 3·5. *Suppose $\mu < \lambda < \kappa = |M|$. Further suppose that if A is a μ-substructure of M than either there is a set of total indiscernibles over A of cardinality λ or there is a set of order indiscernibles over A of order type λ^+. Then $\phi_\mu \leqslant \phi_\lambda$.*

Proof. Choose ρ so that $\mu = \aleph_\rho$. We use the notation above. Define an equivalence relation $\equiv$ on $\mathscr{A}$ by $A \equiv B$ if and only if $S_A \cong S_B$. Let γ be the supremum of the cardinalities of the $\equiv$-classes. So $\phi_\lambda . \gamma \geqslant \phi_\mu$. We will prove the following claim.

Claim (i). If I_A is totally indiscernible and $|A/\equiv| > 1$, then $|A/\equiv| \leqslant |\rho| + \aleph_0$ and $|\rho| + \aleph_0 \leqslant \phi_\lambda$.

(ii) If I_A is not totally indiscernible and $|A/\equiv| > 1$, then $|A/\equiv| \leqslant \mu^+$ and $\lambda^+ \leqslant \phi_\lambda$.

Assume for the moment that we have proved the claim. Then we have shown that if $\gamma > 1$, then ϕ_λ is infinite and $\phi_\lambda \geqslant \gamma$. So the inequality $\phi_\lambda . \gamma \geqslant \phi_\mu$ simplifies to $\phi_\lambda \geqslant \phi_\mu$. This is the required conclusion.

Proof of Claim (i). Suppose now that I_{A_1} and I_{A_2} are totally indiscernible and $S_{A_1} \cong S_{A_2}$. So by Corollary 3·2 there are unique $K_i \subseteq A_i$ and $J_i \supseteq I_{A_i}$ a maximal set of indiscernibles over K_i so that S_{A_i} equals the disjoint union of K_i and J_i. Since the isomorphism of S_{A_1} with S_{A_2} carries K_1 to K_2, the only way that A_1 might not be isomorphic to A_2 is for $|A_1 \cap J_1| \neq |A_2 \cap J_2|$. So $|A_1/\equiv| < |\rho| + \aleph_0$ (the number of cardinals $\leqslant \mu$). Since one of these cardinals must be less than μ, $|K_1| = \mu$. It suffices to prove the following.

Subclaim. $\phi_\lambda \geqslant |\rho| + \aleph_0$.

Proof of subclaim. Hold J_1 fixed. For each cardinal $\delta < \rho$, choose $X \subseteq K_1$ so that $|X| = \delta$. Now choose H_δ so that $X \subseteq H_\delta \subseteq K_1$ and in $H_\delta \cup J_1$ no element of X is in the maximal set of indiscernibles. We can choose H_δ so that $|H_\delta| = \delta$ if δ is infinite, and $|H_\delta|$ is finite if δ is finite. In particular this means that there is a set of non-isomorphic λ-structures consisting of $\aleph_0$ structures of the form $H_\delta \cup J_1$ where H_δ and δ are finite, together with all structures $H_\delta \cup J_1$ where δ is an infinite cardinal.

Proof of Claim (ii). If there exists A so that I_A is not totally indiscernible then by the choice of I_A for every $\alpha < \lambda^+$ there is a set of indiscernibles over A of order type α which is not totally indiscernible. So by Corollary 3·4, $\phi_\lambda \geqslant \lambda^+$ (we use here the fact that successor cardinals are regular).

Suppose that $\{A_\alpha : \alpha < \delta\}$ is the enumeration of an $\equiv$-class and that each of the I_α is not totally indiscernible. Here we let I_α denote I_{A_α}. By Corollary 3·2, we can assume each $S_{A_\alpha} = K \cup J_\alpha$ where J_α is the (unique) maximal set of indiscernibles whose complement has cardinality $< \lambda$. Since $J_\alpha \supseteq I_\alpha$, we can assume $K \subseteq A_\alpha$. (Of course we are making an identification here.) For each α, let $<_\alpha$ be an indiscernible ordering of J_α so that $J_\alpha = L_\alpha + I_\alpha + U_\alpha$ where the ordering of I_α coincides with $<_\alpha \restriction I_\alpha$. (Such a decomposition is possible by Corollary 3·3.) For all α, β there is an isomorphism from S_{A_α} to S_{A_β}. This map takes K onto K and J_α onto J_β. This induces another ordering of J_β indiscernible over K. By [2] there are two possibilities:

(*a*) there are disjoint Y, $Z \subseteq J_\beta$ so that

$$(J_\beta, <_\beta) = (Y, <_\beta \restriction Y) + (Z, <_\beta \restriction Z),$$
$$(J_\alpha, <_\alpha) \cong (Z, <_\beta \restriction Z) + (Y, <_\beta \restriction Y)$$

and for any disjoint $Y_1, Z_1 \subseteq J_\beta$ if $J_\beta = Y_1 + Z_1$ then $Z_1 + Y_1$ is also an indiscernible ordering over K;

or

(*b*) not (*a*) and $(J_\alpha, <_\alpha) \cong (J_\beta, <_\beta)$.

We will prove the following lemma.

LEMMA 3·6. *Fix orderings τ and σ of cardinality μ. Then there are at most μ^+ pairs of orderings (τ', σ') so that $\tau + \lambda + \sigma \cong \tau' + \lambda + \sigma'$.*

We complete the proof by analyzing the Cases (*a*) and (*b*). In case (*b*) there is one subtlety to consider. Why if $L_\alpha + U_\alpha \cong L_\beta + U_\beta$ does the order isomorphism induce an isomorphism from A_α to A_β? The isomorphism from J_α to J_β is order preserving on an infinite set and extends to an isomorphism from S_{A_α} to S_{A_β} which fixes K pointwise; hence the order isomorphism $L_\alpha + U_\alpha \cong L_\beta + U_\beta$ extends to an isomorphism from A_α to A_β. Using Lemma 3·6 we see that there are at most μ^+ ordinals α for which J_α is isomorphic to an ordering obtained by Case (*b*). Suppose now that J_α cannot be obtained as in Case (*b*) and Y and Z are as in Case (*a*). There are three possibilities for the position of the cut which Y makes in J_β. Assume it occurs in L_β. So $J_\alpha \cong V + \lambda + U_\beta + Y$ where $Y + V = L_\beta$. The order J_α has a unique first cut C with the property that there are λ elements to the left of the cut. Hence $V + \lambda \cong L_\alpha + \lambda$ and $U_\beta + Y \cong U_\alpha$. Hence $U_\beta + L_\beta + \lambda = U_\beta + Y + V + \lambda \cong U_\alpha + L_\alpha + \lambda$. In the other cases we can also show that $U_\beta + L_\beta + \lambda \cong U_\alpha + L_\alpha + \lambda$. Suppose now that $U_\beta + L_\beta \cong U_\alpha + L_\alpha$. Since we are not in case (*b*) these are also sets of indiscernibles. So the map which extends the isomorphism from K to K by taking $U_\alpha + L_\alpha$ onto $U_\beta + L_\beta$ is an isomorphism of A_α with A_β. This is a contradiction as $\alpha \neq \beta$. So there can be at most μ^+ α's obtained by Case (*a*). Hence $\delta \leqslant \mu^+$.

Proof (*of Lemma* 3·6). In this case we must have $\sigma \cong \sigma'$ and $\tau + \lambda \cong \tau' + \lambda$. Since any isomorphism takes τ to an initial segment of $\tau' + \lambda$ and $|\tau| \leqslant \mu$, there is an ordinal $\alpha < \mu^+$ such that either $\tau + \alpha = \tau'$ or $\tau' + \alpha = \tau$. Clearly there are at most μ^+ orderings τ' such that for some $\alpha < \mu^+$, $\tau + \alpha = \tau'$. For the other possibility, let I_0 be any well ordered end segment of τ. For any n, let I_{n+1} be a well ordered end segment properly containing I_n if one exists. Otherwise, let $I_{n+1} = I_n$. Notice that any well ordered end segment of τ is contained in some I_n. So for any isomorphism from $\tau' + \alpha$ to τ, there must be an n so that α is taken to an end segment of I_n. Of course τ' is taken onto the complement. Since I_n is well ordered with order type $< \mu^+$ there are only μ possibilities for the end segment. Hence there are only μ possibilities for τ'.

Remark. It is curious to note that we have proved a weak version of the result for finite substructures. If we increase two non-isomorphic finite substructures of the same size by adding on sufficiently large sets of indiscernibles then the resulting structures cannot be isomorphic. Hence, if M is an infinite structure in a finite relational language then for all $m < \aleph_0$ there is $N < \aleph_0$ so that for all k, if $N \leqslant k \leqslant \aleph_0$ then $\phi_m \leqslant \phi_k$. (Here we can use Ramsey's theorem to provide the indiscernibles.) If the collection $\mathscr{A}$ of m-substructures is chosen carefully, N can be taken to be $m+1$.

This line of argument is close to the proof due to Pouzet (sketched on p. 310 of [**4**]) that if $m, n \leqslant \aleph_0$ with $m < n$, and M is a relational structure over a finite language with domain infinite or finite and large relative to n, then $\phi_m \leqslant \phi_n$.

4. *Conclusions*

The results in the previous sections give a great deal of information about the possible behaviour of the sequence $(\phi_\mu : \mu < |M|)$. Under various cardinal arithmetic hypotheses, it can be deduced that (parts of) the sequence is non-decreasing. We will state some of these conclusions here, although we will not capture the full force of what has been proved in the previous sections.

THEOREM 4·1. *Suppose M is a structure in a relational language and λ is a regular cardinal such that $2^{<\lambda} < |M|$. Then for all $\rho < \mu \leqslant \lambda$, $\phi_\rho \leqslant \phi_\mu$.*

Proof. By Theorem 1·3 we can assume that ρ is infinite. We prove the result by induction on λ. By the induction hypothesis we only have to consider the values $\mu = \lambda$ and $\mu = \gamma$, if $\lambda = \gamma^+$. By Lemma 2·2 (b) either every subset of cardinality λ is contained in a set of cardinality $2^{<\lambda}$ which is complete for λ or $\phi_\lambda \geqslant 2^\lambda$ (and $\phi_\gamma \geqslant 2^\gamma$). In the latter case, we have (see Proposition 1·5), that $\phi_\rho \leqslant \phi_\mu$. In the first case, Theorems 2·4 and 3·5 imply the theorem.

THEOREM 4·2. *Assume* GCH *holds. Let M be a structure in a relational language. If $\rho < \mu \leqslant \lambda < |M|$, where λ is a regular cardinal, then $\phi_\rho \leqslant \phi_\mu$.*

THEOREM 4·3. *Suppose M is a structure in a relational language and $\lambda < |M|$ is a regular cardinal such that $2^{<\lambda} < 2^\lambda$. Then for all $\mu < \lambda$, $\phi_\mu \leqslant \phi_\lambda$.*

Proof. If there is a set B of size less than λ such that there are λ types realized over B, then by Lemma 1·6 $\phi_\lambda \geqslant 2^\lambda$, so we can apply Proposition 1·5. Otherwise, by Lemma 2·2, Theorems 2·4 and 3·5 apply.

Note that WGCH guarantees $2^{<\lambda} < 2^\lambda$ for all cardinals. For λ a successor cardinal this is just the statement of WGCH. If λ is a limit cardinal then WGCH implies that the cofinality of $2^{<\lambda}$ is the cofinality of λ. But by König's theorem the cofinality of 2^λ is greater than the cofinality of λ. From Lemma 1·6 and Lemma 2·2 (a) we also obtain the following result.

THEOREM 4·4. *Assume* WGCH *holds. Let M be a structure in a relational language. If $\mu < \lambda < |M|$, where λ is a regular cardinal, then $\phi_\mu \leqslant \phi_\lambda$.*

As was mentioned in the Introduction it is possible to eliminate the requirement that λ be regular.

Next we obtain some information about the structures with few μ-substructures.

Theorem 4·5. *Suppose M is a structure in a relational language. Further suppose that for some infinite $\mu = \aleph_\rho$, $\phi_\mu < |\rho| + \aleph_0$. Then $M = K \cup I$, where $|K| < \mu$ and I is a set of total indiscernibles over K, provided any of the following conditions holds: μ is regular, $\mu < |M|$ and $2^{<\mu} < 2^\mu$; μ is regular, uncountable and $2^{<\mu} < |M|$; or $2^\mu < |M|$. Further if I is chosen maximally then I and K are unique.*

Proof. There are various cases to consider. Since the proofs are largely the same we will do one of the more complicated cases. Assume that μ is a regular uncountable cardinal and that either $\mu < |M|$ and $2^{<\mu} < 2^\mu$ or $2^{<\mu} < |M|$. The assumptions, together with Lemma 1·6 and Proposition 2·2, imply that the hypotheses of Theorem

2·4 are satisfied. So using 2·4, we have that over every set of size less than μ there is a set of total indiscernibles of cardinality μ or a sequence of indiscernibles of order type μ^+. If for some set of cardinality μ there was a sequence of indiscernibles of order type μ^+ which was not totally indiscernible then by Corollary 3·4 (see the remark after the corollary), $\phi_\mu \geqslant \mu^+$. So over every set of cardinality less than μ there is a set of total indiscernibles of cardinality μ. Note that by Theorems 4·1 and 4·3, $\phi_\gamma \leqslant \phi_\mu$ for all $\gamma < \mu$. Let $\theta < \mu$ be an infinite cardinal such that there are more than ϕ_μ cardinals less than θ. (To see such a θ exists, note that since μ is regular either μ is a successor cardinal or μ is a regular limit cardinal and $\rho = \mu$.)

For the rest of this proof, 'indiscernibles' will mean 'totally indiscernible set'. If a substructure A is the disjoint union of H and J where $|H| < |J|$, we shall say that J is a *maximal set of indiscernibles in A* if it is indiscernible over its complement in A and is maximal subject to this. The fundamental fact we shall use is that under these circumstances, if $\theta < |J|$ then $|H| < \theta$. Otherwise by the subclaim in Claim (i) of Theorem 3·5 there would be more than ϕ_μ pairwise non-isomorphic substructures of some cardinality less than or equal to μ, contradicting Theorem 4·1 or Theorem 4·3.

To begin fix J a set of size μ which is totally indiscernible over the empty set. Notice that for any A and $L_1 \subseteq L \subseteq A$ if L is totally indiscernible over $A \backslash L$ then L_1 is totally indiscernible over $A \backslash L_1$.

Claim (i). For any finite $H \subseteq M$ there is a unique maximal subset $J(H) \subseteq J$ so that $J(H)$ is indiscernible over $(J \cup H)\backslash J(H)$ and $|J \backslash J(H)| < \theta$.

Proof of Claim (i). Write J as the union of an increasing continuous chain $(J_\alpha: \alpha < \mu)$ where for all α, $\theta \leqslant |J_\alpha| < \mu$. For each α, choose L_α a set of total indiscernibles of cardinality μ over $H \cup J_\alpha$. Expand L_α to a maximal set of indiscernibles K_α in $H \cup J_\alpha \cup L_\alpha$. By the second paragraph of the proof $J_\alpha \backslash K_\alpha$ has cardinality less than θ. Expand $J_\alpha \cap K_\alpha$ to a maximal set of indiscernibles E_α in $J_\alpha \cup H$, and let $F_\alpha = J_\alpha \backslash E_\alpha$. We claim that for $\alpha < \beta$, $F_\alpha \subseteq F_\beta$. To see this consider $E_\beta \cap J_\alpha$. The complement of this set in $H \cup J_\alpha$ has cardinality less than θ. So $E_\beta \cap J_\alpha$ can be expanded to a unique maximal set of indiscernibles in $J_\alpha \cup H$. But this set is E_α. So $E_\beta \cap F_\alpha = \varnothing$.

For all α, $|F_\alpha| < \theta$. So there is some α_0 such that for all $\beta > \alpha_0$, $F_\beta = F_{\alpha_0}$. So we can let $J(H) = J \backslash F_{\alpha_0}$.

Notice in the proof of the claim that for $a \in J_{\alpha_0}$ if there is $K \subseteq J_{\alpha_0}$ so that $|K| \geqslant \theta$, $a \in K$, and K is indiscernible over $(H \cup J_{\alpha_0})\backslash K$, then $a \in J(H)$.

Claim (ii). There exists $J_1 \subseteq J$ so that $|J \backslash J_1| < \theta$ and J_1 is totally indiscernible over $M \backslash J_1$.

Proof of Claim (ii). Let $J_1 = \bigcap J(H)$, where H ranges over the finite subsets of $M \backslash J$. It suffices to show that $|J \backslash J_1| < \theta$. Assume not. Then there is a set R of cardinality θ which is disjoint from J so that if we let $J_2 = \bigcap J(H)$ where H ranges over the finite subsets of R, then $|J \backslash J_2| \geqslant \theta$. We use the notation of Claim (i). Since μ is regular there is an ordinal $\alpha < \mu$ so that $|J_\alpha \backslash J_2| \geqslant \theta$ and for all finite subsets $H \subseteq R$ and $a \in J_\alpha$ if there is $K \subseteq J_\alpha$ so that $|K| \geqslant \theta$, $a \in K$, and K is indiscernible over $(H \cup J_\alpha)\backslash K$, then $a \in J(H)$.

Let L be a set of total indiscernibles of cardinality μ over $R \cup J_\alpha$. Let L_1 be a maximal set of indiscernibles extending L. Note that $|J_\alpha \backslash L_1| < \theta$. But by the choice of α, $J_\alpha \cap L_1 \subseteq J_2$, a contradiction.

Using the two claims above, we can prove the theorem. Choose $J_2 \subseteq J_1$ with $|J_2| = \aleph_0$. Let

$$I = \{a \in M : J_2 \cup \{a\} \text{ is indiscernible over } M \backslash (J_2 \cup \{a\})\}.$$

Since J_2 is infinite, I is indiscernible over $M \backslash I$. If the theorem is false then there exists a subset K of $M \backslash I$ of size θ. Now for all $k \in K$, $J_2 \cup \{k\}$ is not indiscernible over $M \backslash (J_2 \cup \{k\})$. But now we can choose $H \supseteq K$ so that $|H| = \theta$ and for all $k \in K$, $J_2 \cup \{k\}$ is not indiscernible over $H \backslash (J_2 \cup \{k\})$. If we let L be a set of indiscernibles of cardinality μ over $H \cup J_2$, then we can obtain a contradiction just as in Claim (i). This completes the proof in the case where μ is regular and uncountable.

In the case μ is singular, the hypotheses, 2·2 (b) and 2·4 guarantee the existence of a set of total indiscernibles of cardinality μ^+ over every set of size μ. The proof can be simplified. We may take θ to be μ, and when we choose the sets of total indiscernibles they should have cardinality μ^+.

In the case where $\mu = \aleph_0$, there is a finite bound n for the cardinality of a finite set H so that there is an infinite set J disjoint from H so that in $H \cup J$, J is a maximal set of indiscernibles. Fix some such choice of H and J and a finite sublanguage so that J is a maximal set of indiscernibles in the sublanguage. Let m be the arity of the sublanguage and assume that $m > n$. We claim that $I = M \backslash H$ is the required set of indiscernibles. It is enough to show that for all $a \notin H$, $J \cup \{a\}$ is indiscernible over H. Consider any finite $J_0 \subseteq J$ so that $|J_0| > 2m$. Choose L an infinite set of indiscernibles over $H \cup J_0 \cup \{a\}$. Let L_1 be the maximal set of indiscernibles containing L. Since the complement of L_1 has cardinality at most n, $|L_1 \cap J_0| > m$. So $H \cap L_1 = \varnothing$. Hence $J_0 \cup \{a\} \subseteq L_1$.

Remark. In [6], the case where $|M| = \lambda$ and ϕ_λ is finite is analyzed. It is shown that any such structure is the union of a finite set and a set of order indiscernibles. The indiscernibles will be totally indiscernible if $\phi_\mu < \mu^+$ for some infinite $\mu < \lambda$.

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